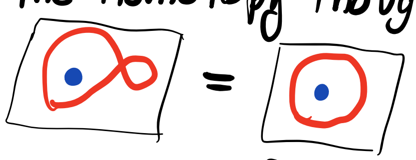
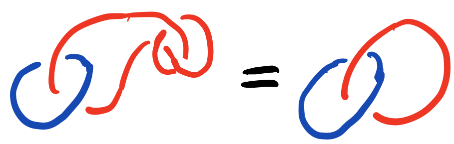
# Gauß' Linking Number

(I) Let $M, N$ be manifolds immersed (but disjoint) in $\mathbb{R}^{\overbrace{m+n+1}^{p}}$. Want an invariant of $M, N$ upto homotopies where they do not intersect each other

(we do allow $M$ or $N$ to pass thru themselves during the homotopy though:





)

What is this information?

We can take maps $K : M \to \mathbb{R}^p$ $\quad L : N \to \mathbb{R}^p$

Together these make a map

$$K \times L : M \times N \to \mathbb{R}^p \times \mathbb{R}^p$$

But since the two manifolds never intersect in $\mathbb{R}^p$ we never have $K(x) = L(y)$ so $K \times L$ avoids the diagonal

$$\Delta = \{ (x,x) \mid x \in \mathbb{R}^p \}$$

Any homotopy where they remain disjoint induces a homotopy

$$M \times N \longrightarrow \mathbb{R}^p \times \mathbb{R}^p \setminus \Delta$$

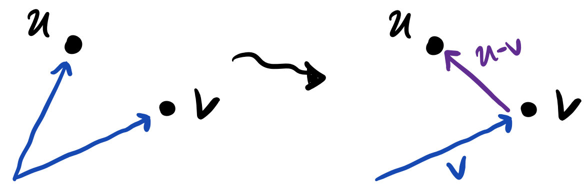
Thus, the space we are interested in is the set of homotopy classes of maps

$$\left[ M \times N , \mathbb{R}^p \times \mathbb{R}^p \setminus \Delta \right]$$

And the linking invariant of embeddings $K, L$ is the class $[K \times L]$ in this space.

## II What is this space?

Given $\mathbb{R}^p \times \mathbb{R}^p \setminus \Delta = \{(u,v)\}$ we can change coordinates to $(v, u-v)$




This is a homeomorphism $\mathbb{R}^p \times \mathbb{R}^p \setminus \Delta \cong \mathbb{R}^p \times (\mathbb{R}^p \setminus 0)$
And $\mathbb{R}^p$ is contractible so projecting it off is a htpy equivalence.

Now $\mathbb{R}^p \setminus 0$ deformation retracts to $\mathbb{S}^{p-1}$ via

$$\pi : x \longmapsto x/|x|$$

And the composition is a homotopy equivalence

$$\mathbb{R}^p \times \mathbb{R}^p \setminus \Delta \longrightarrow \mathbb{S}^{\underbrace{p-1}_{m+n}}$$

$$(u,v) \longrightarrow (v, u-v) \rightarrow u-v \rightarrow \frac{u-v}{|u-v|}$$

This homotopy equivalence induces a bijection

$$[M \times N, \mathbb{R}^p \times \mathbb{R}^p \setminus \Delta] \xrightarrow{\sim} [M \times N, \mathbb{S}^{m+n}]$$

<u>But</u> this is now the space of homotopy classes of maps of an $m+n$-manifold $(M \times N)$ into the $m+n$-sphere $[M \times N, \mathbb{S}^{m+n}]$, which are completely classified by degree:

<u>Thm</u> (Hopf): If $X^N$ is an orientable $N$-manifold then two maps $f, g : X \to \mathbb{S}^N$ are homotopic <u>if and only if</u> $\deg f = \deg g$.

Thus; $[M \times N, \mathbb{S}^{m \times n}]$ is in bijective correspondence with $\mathbb{Z}$

## III The Degree of A Map.

Degree is captured by the top homology classes of the manifolds involved: if $X, Y$ are $N$ manifolds and $f: X \to Y$ then $f$ induces a map $f_*: H_\bullet(X,\mathbb{Z}) \to H_\bullet(Y,\mathbb{Z})$ and in top dimension $N$, each of $H_N(X)$, $H_N(Y)$ are one dimensional, generated by a choice of fundamental class $[X]$, $[Y]$:

$$H_N(X,\mathbb{Z}) = \mathbb{Z}\cdot[X] \qquad H_N(Y,\mathbb{Z}) = \mathbb{Z}\cdot[Y]$$

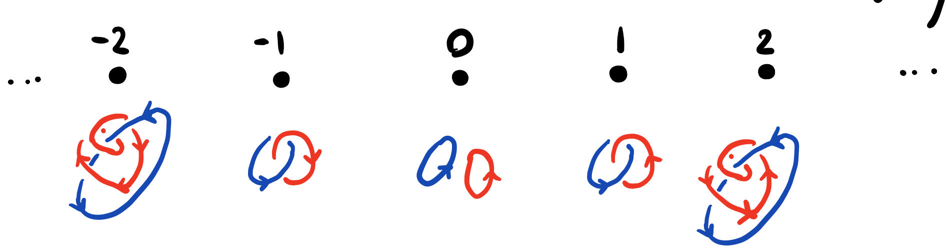
The induced map $f_*$ then is identified with an endomorphism of $\mathbb{Z}$: this is just multiplication by a number, which we call $\deg(f) \in \mathbb{Z}$:

$$\begin{aligned} f_*: H_N(X,\mathbb{Z}) &\longrightarrow H_N(Y,\mathbb{Z}) \\ [X] &\longmapsto \deg(f)[Y] \end{aligned}$$

## ⊛ The Linking Number as Degree:

Putting all of this together, we have a precise description of the invariant of $m$ & $n$-manifolds in $\mathbb{R}^{m+n+1}$ up to non-intersecting homotopy: Given $K: M \to \mathbb{R}^p$ and $L: N \to \mathbb{R}^p$

$$\mathrm{Link}(K,L) := \deg\left( (x,y) \xmapsto{f} \frac{L(x)-K(y)}{|L(x)-K(y)|} \right)$$



## IV Accessing this number via Cohomology

General theory gives us the existence of degree, but does not provide an accessible way to compute it: we need to first compute the application of $f_*$ on $[M\times N]$ (probably not bad) but then find a way to write $f_*[M\times N]$ in terms of our chosen generator $[\mathbb{S}^{m+n}]$

$$f_*[M\times N] = \underbrace{[\mathbb{S}^{m+n}] + [\mathbb{S}^{m+n}] + \cdots + [\mathbb{S}^{m+n}]}_{\deg f} \quad \text{(this looks hard!)}$$

To help, we turn to cohomology:

Coho classes induce maps $H_n \to \mathbb{Z}$
Since everything is 1-D im
nonzero cohomology class induces
iso $H_n \to \mathbb{Z}$

Should write out details for expository note.

# V The Smooth World.

Actually computing $H_{m+n}(M\times N) \xrightarrow{f_*} H_{m+n}(S^{m+n}) \xrightarrow{\alpha} \mathbb{Z}$

$$[M\times N] \longmapsto \deg(f)$$

Requires choosing some type of cohomology that we actually know how to work with:
When $f$ is smooth we can use de Rham cohomology for the actual computation:

<u>smooth Chains</u> In singular homology of a smooth mfld every class has a (smooth chain) representative.

<u>deRham's theorem</u>

de Rham cohomology of a smooth manifold is isomorphic to its singular cohomology.

As linear maps on homology, an element $[\alpha]\in H^n$ acts on a smooth chain $[C]\in H_n$ by

$$[\alpha]([C]) := \int_C \alpha$$

Thus, to set up an isomorphism $H_n(S^{n+m};\mathbb{R}) \to \mathbb{R}$ we need only select a cohomology class $[\alpha]$ with

$$\int_{S^{n+m}} \alpha = 1$$

There are many such forms ( $\frac{1}{\mathrm{Vol}(S^{n+m})}$ vol works, for vol the volume form of any Riemannian metric ) but of course they are all cohomologous.

# The Abstract Computation

For specificity let $\tilde{\alpha}$ be any volume form on $\mathbb{S}^{n+m}$ and define $V = \int_{\mathbb{S}^{n+m}} \tilde{\alpha}$. Then define $\alpha = \frac{1}{V}\tilde{\alpha}$.

Now if $K: M \to \mathbb{R}^{m+n+1}$ and $L: N \to \mathbb{R}^{m+n+1}$ are disjoint embeddings of $M, N$; form the map

$$f: M \times N \longrightarrow \mathbb{S}^{m+n}$$
$$(s,t) \longmapsto \frac{K(s)-L(t)}{|K(s)-L(t)|}$$

If $[M\times N]$ is a choice of fundamental class (orientation) for $M\times N$, then as smooth chains,

$$f_*([M\times N]) = [f(M\times N)] \in H_{m+n}(\mathbb{S}^{m+n})$$

And, as $H_{m+n}(\mathbb{S}^{n+m}) = \mathbb{Z}\cdot[\mathbb{S}^{m+n}]$ for a choice of fundamental class for $\mathbb{S}^{n+m}$, we know abstractly

$$[f(M\times N)] = [\mathbb{S}^{m+n}] + \cdots + [\mathbb{S}^{m+n}] = \deg(f)[\mathbb{S}^{m+n}]$$

Thus, integration against $\alpha$ yelds

$$\int_{f(M\times N)} \alpha = \int_{\mathbb{S}^{m+n} \sqcup \cdots \sqcup \mathbb{S}^{m+n}} \alpha = \int_{\mathbb{S}^{m+n}} \alpha + \cdots + \int_{\mathbb{S}^{m+n}} \alpha$$

$$= \deg(f) \int_{\mathbb{S}^{m+n}} \alpha = \deg(f) = \mathrm{Link}(K, L)$$

# Working in $\mathbb{R}^{n+m+1}$

So we've laid out a complete & viable computation strategy for Link$(K, L)$, and the next step is to put this into practice. With care, the integral $\int_{f(M\times N)} \alpha$ can be made explicitly computable; and by the above abstract computation, we know this evaluates to the linking number.

The first practical difficulty is that $\alpha$ is defined on $\mathbb{S}^{m+n}$, and writing things down in coordinates will require multiple charts and make the integral all but impossible to practically evaluate. One solution is to consider instead the map $F: M\times N \to \mathbb{R}^{m+n+1} \setminus 0$ given by

$$F(s,t) = K(s) - L(t).$$

If $\pi: \mathbb{R}^{m+n+1} \to \mathbb{S}^{m+n}$ is the projection $x \mapsto x/|x|$, then

$$f = \pi F, \text{ and}$$

$$\int_{f(M\times N)} \alpha = \int_{\pi F(M\times N)} \alpha = \int_{F(M\times N)} \pi^* \alpha$$

Now $\pi^*\alpha$ is an $m+n$ form on $\mathbb{R}^{n+m+1} \setminus 0$ where we can use a single coordinate chart

$$(x_1 \ldots\ldots x_{n+m+1})$$

avoiding coordinate-concerns on the codomain. Of course, the next obstacle is to compute $\pi^*\alpha$ explicitly.

# Computing $\pi^*\alpha$

Given a volume form $\alpha$ of total volume 1 on $S^{m+n}$, we wish to compute its pullback in standard coordinates on $\mathbb{R}^{m+n+1}$.
Unfortunately this is itself difficult as the whole trouble began with writing down $\alpha$ explicitly!

Instead - remember $\alpha$ itself has no special importance, only its cohomology class. Similarly $\pi^*\alpha$ is not important, any representative of $[\pi^*\alpha] \in H^{m+n}(\mathbb{R}^{m+n+1} \setminus 0)$ is equally good.

What are the abstract properties of $[\pi^*\alpha]$?

1) It generates $H^{n+m}(\mathbb{R}^{m+n+1} \setminus 0)$
2) It evaluates to 1 on $S^{m+n} \subseteq \mathbb{R}^{m+n+1}$,

Thus the real question is how do we produce a form $\omega \in \Lambda^{n+m}(\mathbb{R}^{n+m+1} \setminus 0)$ so that $[\omega]$ has these properties?

## New Goal: Construct $\omega$ such that

- $\omega \in \Lambda^{n+m}(\mathbb{R}^{n+m+1} \setminus 0)$
- $\omega$ is closed
- $\omega$ is not exact
  - ↳ in particular $\int_{S^{n+m}} \omega = 1$

# Conditions on $[\omega] = [\pi^* \alpha]$

Here we look to turn the set of conditions on $\omega$ above into a set of reasonable ansatzes, and then solve for a $\omega$ with these properties.

(i) $\omega$ is a "codimension-1" form since we pulled back a volume form along $\mathbb{R}^{n+m+1} \setminus 0 \to S^{m+n}$. Thus its reasonable to seek $\omega$ as the hodge dual of a 1-form $\eta$

$$\omega = \star \eta$$

(ii) If $\omega$ is to be closed, then $d\omega = 0 \cdot \mathrm{vol}$ this puts a constraint on $\eta$; $d \star \eta = 0 \cdot \mathrm{vol}$. Applying the hodge star once more takes this from a multiple of the volume form to a scalar function

$$\star d \star \eta = 0$$

(iii) $\omega$ is to be non-exact, its sufficient (by Stokes) that $\int_S \omega \neq 0$. An idea is to use a one-form $\eta$ whose kernels fit together into concentric spheres. Then, on any sphere level set, $\star\eta$ is a multiple of the volume form of the sphere, so has nonzero integral: This implies

$$\eta = d\rho \quad \text{for } \rho = \rho(r) \quad r = \sqrt{x_1^2 + \cdots}$$

# Solving for $\omega$

We have managed to translate the abstract requirements on $\omega$ to sufficient concrete constraints:

$$① \ \omega = \star\eta \text{ for } \eta \in \Lambda^1$$

$$② \ \star d \star \eta = 0$$

$$③ \ \eta = d\rho \text{ for } \rho = \rho(r)$$

$$r = \sqrt{x_1^2 + \cdots x_N^2}$$

Putting together ② and ③ shows

$$\star d \star d\rho = 0$$

<u>But</u> $(\star d)^2 = \Delta$ is the Laplacian on functions

$$\therefore \Rightarrow \boxed{\Delta\rho = 0 \text{ on } \mathbb{R}^{m+n+1} \setminus 0}$$

This is the right condition!

If $\omega = \star d\rho$ for $\rho = \rho(r)$ then:

$$d\rho = \rho'(r)\,dr = \rho'(r)\frac{1}{2\sqrt{x_1^2 + \cdots}}\,d(x_1^2 + \cdots)$$

$$= \rho'(r)\frac{1}{r}\sum x_i\,dx_i$$

$$\Rightarrow \omega = \star d\rho = \star\left(\frac{\rho'(r)}{r}\sum x_i\,dx_i\right) = \frac{\rho'(r)}{r}\sum_i x_i \star dx_i$$

Thus, restricted to $\mathbb{S}^{n+m}$ (where $r=1$) we have

$$\omega\big|_{\mathbb{S}^{n+m}} = \frac{\rho'(1)}{1} \sum_i x_i \star dx_i$$

so $$\int_{\mathbb{S}^{n+m}} \omega = \int_{\mathbb{S}^{n+m}} \rho'(1) \sum_i x_i \star dx_i$$

Thus, we need to compute $\int \sum x_i \star dx_i$ on $\mathbb{S}^{n+m}$.
Note this form extends smoothly (by same formula) to all of $\mathbb{R}^{n+m+1}$:
and, on $\mathbb{R}^{n+m+1}$ its derivative is easy to compute (note this is <u>NOT</u> the derivative of $\omega$, but a <u>different</u> smooth extension of $\omega|_{\mathbb{S}^{n+m}}$).
We use the definition of the Hodge star explicitly, where if $\beta$ is a nonzero 1-form then $\beta \wedge \star\beta = \text{vol}$ is the volume form:

$$d\Big(\sum_i x_i \star dx_i\Big) = \sum_i d(x_i \star dx_i) = \sum dx_i \wedge \star dx_i + x_i\, d(\star dx_i)$$

and as $\star dx_i = \pm dx_1 \wedge dx_{\hat{i}} \wedge \dots dx_{n+m+1}$ is the wedge of closed forms, $\star dx_i$ is closed so $d(\star dx_i) = 0$

$$\therefore\ d\Big(\sum_i x_i dx_i\Big) = \sum dx_i \wedge \star dx_i$$

Thus;

$$= \sum \text{vol} = (n+m+1)\cdot \text{vol}$$

$$\int_{\mathbb{S}^{n+m}} \omega = \int_{\mathbb{S}^{n+m}} \omega\big|_{\mathbb{S}^{n+m}} = \int_{\mathbb{S}^{n+m}} \rho'(1) \sum x_i \star d i = \rho'(1) \int_{\partial B^{n+m+1}} \sum x_i \star dx_i$$

$$\overset{\text{stokes}}{=} \rho'(1) \int_{B^{n+m+1}} d\Big(\sum x_i \star dx_i\Big) = \rho'(1) \int_{B^{n+m+1}} (n+m+1)\text{vol} = \rho'(1) \underbrace{(n+m+1)\,\text{Vol}(B^{n+m+1})}_{= \text{Vol}(\mathbb{S}^{n+m})} \neq 0$$

Putting this all together we get:

<u>Thm</u> | If $\rho$ is harmonic, radially symmetric on $\mathbb{R}^{n+m-1}$ and $\rho'(1) \neq 0$ then $\omega = \star d\rho$ generates $H^{m+n}(\mathbb{R}^{n+m+1} \setminus 0)$.

Furthermore, if $\rho'(1) = 1/\mathrm{vol}(S^{n+m})$ then $[\omega]$ sends $[S^{n+m}]$ to 1.

Thus, all we need is to produce <u>explicit</u> harmonic functions on $\mathbb{R}^r \setminus 0$: We've reduced our abstract differential equation

$$\pi^* \mathrm{vol} = \omega$$

To an explicit ODE for $\rho(r)$:
The solutions in each dimension are well-known:

$\mathbb{R}^2$: $\rho(r) = \ln(r)$ $\qquad \frac{1}{2\pi} \ln r$

$\mathbb{R}^3$: $\rho(r) = \frac{-1}{r}$ $\qquad \frac{-1}{4\pi} \frac{1}{r}$

$\vdots$

$\mathbb{R}^n$: $\rho(r) = \frac{-1}{(n-2) r^{n-2}}$ $\qquad \frac{-1}{\mathrm{vol}(S^{n-1})} \frac{1}{(n-2) r^{n-2}}$

In each of these standardized normalizations have $\rho'(1) = 1$ so our purposes require we divide each by $\frac{1}{\mathrm{vol}}$

This gives an explicit form in coordinates since

$$\omega = \star d\rho = \frac{\rho'(r)}{r} \sum x_i \star dx_i$$

$\mathbb{R}^2$: $$\omega = \frac{1}{2\pi}\left(\frac{d}{dr}\ln r\right)\frac{1}{r}\left(x \star dx + y \star dy\right)$$

$$= \frac{1}{2\pi}\frac{1}{r^2}\left(x \star dx + y \star dy\right) = \frac{1}{2\pi}\frac{x \star dx + y \star dy}{x^2+y^2}$$

$$= \frac{1}{2\pi}\frac{x\,dy - y\,dx}{x^2+y^2}$$

$\mathbb{R}^3$: $$\omega = \frac{1}{4\pi}\left(\frac{d}{dr}\frac{-1}{r}\right)\frac{1}{r}\left(x \star dx + y \star dy + z \star dz\right)$$

$$= \frac{1}{4\pi}\frac{1}{r^3}\left(x \star dx + y \star dy + z \star dz\right)$$

$$= \frac{1}{4\pi}\frac{1}{(x^2+y^2+z^2)^{3/2}}\left(x\,dy \wedge dz - y\,dx \wedge dz + z\,dx \wedge dy\right)$$

What can we say explicitly now?

**In $\mathbb{R}^2$** If $n+m+1=2$ then only option is for one of our manifolds to be 0 dim and one to be 1-dim.

↳ Only connected, compact 0-mfld is a point $\{\bullet\}$

↳ Only connected compact 1-mfld is a circle $S^1$

$\Rightarrow$ We are looking at

$$K : \{\bullet\} \to \mathbb{R}^2$$
$$L : S^1 \to \mathbb{R}^2$$

Call $K(\bullet) = a = (a_x\ a_y)$, and write

$L(t) = (x(t), y(t))$

Then pulling back $\omega = \frac{1}{2\pi} \frac{x\,dy - y\,dx}{x^2+y^2}$

under $t \mapsto L(t) - K(\bullet) = (x(t)-a_x, y(t)-a_y)$ gives

$$\frac{1}{2\pi} \frac{(x(t)-a_x)\,d(y(t)-a_y) - (y(t)-a_y)\,d(x(t)-a_x)}{(x(t)-a_x)^2 + (y(t)-a_y)^2}$$

$$= \frac{1}{2\pi} \frac{(x(t)-a_x)\,y'(t)\,dt - (y(t)-a_y)\,x'(t)\,dt}{(x(t)-a_x)^2 + (y(t)-a_y)^2}$$

$$= \frac{1}{2\pi} \frac{(x(t)-a_x, y(t)-a_y) \cdot (y'(t), -x'(t))\,dt}{(x(t)-a_x)^2 + (y(t)-a_y)^2}$$

Let $R: \mathbb{R}^2 \to \mathbb{R}^2$ be the "rotate by $i$" map

$$(x,y) \longmapsto (y,-x)$$

Then this is

$$\frac{1}{2\pi} \frac{(L(t)-a) \cdot R(L'(t))\, dt}{|L(t)-a|^2}$$

And the Linking Number of "$L$" with "$a$" is

$$Lk(\vec{a}, L) = \frac{1}{2\pi} \int_{S^1} \frac{(L(t)-a) \cdot R(L'(t))}{|L(t)-a|^2}\, dt$$

Whats the Next Nontrivial Case:

In $\mathbb{R}^3$: $n+m+1=3$ so either

① $n=m=1$, both are circles, or

② $n=2$ $m=0$ so one is a compact surface and one is a point

(will do ① first! This is Gauss').

$\boxed{\text{In } \mathbb{R}^3}$

Pull back $\omega = \frac{1}{4\pi} \frac{1}{r^3}\left(x \star dx + y \star dy + z \star dz\right)$

Under the map $\mathbb{R}^3 \times \mathbb{R}^3 \setminus \Delta \to \mathbb{R}^3 \setminus 0$

$$(P_L, P_K) \longmapsto (P_L - P_K)$$

I $\quad \frac{1}{r^3} = \frac{1}{|L-K|^3}$

II $\quad x \star dx = x\, dy \wedge dz$

$$= (x_L - x_K)\, \underbrace{d(y_L - y_K) \wedge d(z_L - z_K)}$$

$$= (dy_L - dy_K) \wedge (dz_L - dz_K)$$

$$= dy_L \wedge dz_L - dy_L \wedge dz_K - dy_K \wedge dz_L + dy_K \wedge dz_K$$

$$= (dy_L \wedge dz_L + dy_K \wedge dz_K) - (dy_L \wedge dz_K + dy_K \wedge dz_L)$$

$$= \underbrace{(dy_L \wedge dz_L + dy_K \wedge dz_K)} - \underbrace{(dy_L \wedge dz_K - dz_L \wedge dy_K)}$$

This will be $= 0$ on $T^2 \to \mathbb{R}^3 \setminus \Delta$ b/c $dy_L \wedge dz_L = y_L'(s)\, z_L'(s) \underbrace{ds \wedge ds}_{=0}$

This is part of cross product: ie oriented area patch of $T^2$...

To Simplify this mess...

(*) Somehow need to use that my surface $T^2 \to \mathbb{R}^3 \times \mathbb{R}^3 \setminus \Delta$ always has tangent planes in direction span( vec from $\mathbb{R}^3_1$, vec from $\mathbb{R}^3_2$ ) and anything w/ 2 form "in a single $\mathbb{R}^3$" will evaluate to 0 on our map.

$$\text{II}\quad y\star dy = y\,dz\wedge dx$$

$$= (y_L - y_k)\underbrace{d(z_L - z_k)\wedge d(x_L - x_k)}$$

$$= (dz_L - dz_k)\wedge(dx_L - dx_k)$$

$$= dz_L\wedge dx_L - dz_L\wedge dx_k - dz_k\wedge dx_L + dz_k\wedge dx_k$$

$$= (dz_L\wedge dx_L + dz_k\wedge dx_k) - (dz_L\wedge dx_k + dz_k\wedge dx_L)$$

$$\text{III}\quad z\star dz = z\,dx\wedge dy$$

$$= (z_L - z_k)\underbrace{d(x_L - x_k)\wedge d(y_L - y_k)}$$

$$= (dx_L - dx_k)\wedge(dy_L - dy_k)$$

$$= dx_L\wedge dy_L - dx_L\wedge dy_k - dx_k\wedge dy_L + dx_k\wedge dy_k$$

$$= (dx_L\wedge dy_L + dx_k\wedge dy_k) - (dx_L\wedge dy_k + dx_k\wedge dy_L)$$

Putting all these terms together we have 3 of the "first kind" minus 3 of the "second kind"

$$\begin{aligned} = \underbrace{(x_L - x_k)(dy_L\wedge dz_L + dy_k\wedge dz_k)}_{} &- \underbrace{(x_L - x_k)(dy_L\wedge dz_k + dy_k\wedge dz_L)}_{} \\ + (y_L - y_k)(dz_L\wedge dx_L + dz_k\wedge dx_k) &- (y_L - y_k)(dz_L\wedge dx_k + dz_k\wedge dx_L) \\ + (z_L - z_k)(dx_L\wedge dy_L + dx_k\wedge dy_k) &- (z_L - z_k)(dx_L\wedge dy_k + dx_k\wedge dy_L) \end{aligned}$$

First kind

Second kind

The FIRST KIND will all be zero. we'll deal w/ these later. ] They all end up w/ $ds \wedge ds = 0$ or $dt \wedge dt = 0$...

The SECOND KIND: we switch around orders to get a convention: "L before K"

$$= -(x_L - x_K)(dy_L \wedge dz_K - dz_L \wedge dy_K)$$
$$+ (y_L - y_K)(dx_L \wedge dz_K - dz_L \wedge dx_K)$$
$$- (z_L - z_K)(dx_L \wedge dy_K - dy_L \wedge dx_K)$$

To get rid of minus signs out front, absorb into coefs

$$= (x_K - x_L)(dy_L \wedge dz_K - dz_L \wedge dy_K)$$
$$- (y_K - y_L)(dx_L \wedge dz_K - dz_L \wedge dx_K)$$
$$+ (z_K - z_L)(dx_L \wedge dy_K - dy_L \wedge dx_K)$$

Thus, evaluating the homology class $K \times L : S^1 \times S^1 \to \mathbb{R}^3 \times \mathbb{R}^3 \setminus \Delta$ against the generator of $H^2(\mathbb{R}^3 \times \mathbb{R}^3 \setminus \Delta)$ gives

$$\boxed{\frac{1}{4\pi} \iint_{K(S^1) \times L(S^1)} \frac{(x_K - x_L)(dy_L \wedge dz_K - dz_L \wedge dy_K) - (y_K - y_L)(dx_L \wedge dz_K - dz_L \wedge dx_K) + (z_K - z_L)(dx_L \wedge dy_K - dy_L \wedge dx_K)}{\left|(x_L - x_K)^2 + (y_L - y_K)^2 + (z_L - z_K)^2\right|^{3/2}}}$$

Almost there (Sheesh!!)
Now we just need to pull back along $K \times L$ to get an integral on $S^1 \times S^1$

If $s$ is the coordinate on the first circle and $t$ is the coord on the second, then the length forms are $ds, dt$ and a choice of area form on the torus $T^2 = S^1 \times S^1$ is

$$\alpha = ds \wedge dt$$

Along $(s,t) \mapsto K(s) - L(t) = \begin{pmatrix} x_K(s) - x_L(t) \\ y_K(s) - y_L(t) \\ z_K(s) - z_L(t) \end{pmatrix}$
each of the basis one forms pulls back as

$$(K-L)^* dx_K = d(x_K(s)) = x'_K(s)\,ds$$

etc...

Thus, pulling back the numerator gives

$$(K-L)^*\Big((x_K - x_L)(dy_L \wedge dz_K - dz_L \wedge dy_K) - (y_K - y_L)(dx_L \wedge dz_K - dz_L \wedge dx_K) + (z_K - z_L)(dx_L \wedge dy_K - dy_L \wedge dx_K)\Big)$$

$$= (x_K - x_L)(y'_L z'_K - z'_L y'_K) - (y_K - y_L)(x'_L z'_K - z'_L x'_K) + (z_K - z_L)(x'_L y'_K - y'_L x'_K)\, ds \wedge dt$$

$$= \begin{pmatrix} x_K - x_L \\ y_K - y_L \\ z_K - z_L \end{pmatrix} \bullet \Big( (x'_L, y'_L, z'_L) \times (x'_K\ y'_K\ z'_K) \Big)\, ds \wedge dt$$

This is

$$\Big(K(s)-L(t)\Big)\bullet\Big(L'(t)\times K'(s)\Big)\; ds\wedge dt$$ !! ← OMG! :)

Thus all together on $S^1\times S^1$ we have

$$\operatorname{Link}(K,L)=\frac{1}{4\pi}\int_{S^1\times S^1}\frac{\big(K(s)-L(t)\big)\bullet\big(L'(t)\times K'(s)\big)}{|L(t)-K(s)|^3}\,ds\wedge dt$$

Finally, on the product manifold $S^1\times S^1$ with volume form split as a product we apply Fubini:

$$\boxed{\operatorname{Link}(K,L)=\frac{1}{4\pi}\int_{S^1}\int_{S^1}\frac{\big(K(s)-L(t)\big)\bullet\big(L'(t)\times K'(s)\big)}{|L(t)-K(s)|^3}\,ds\,dt}$$

↑ for surface linking w/ point $\vec{a}$: integral is similar

$$=\frac{1}{4\pi}\int_{\Sigma}\frac{\big(K(p)-\vec{a}\big)}{|K(p)-\vec{a}|^3}\bullet K^{*}dA$$

← $= |K_s\times K_t|\,ds\,dt$ in a coord patch...