Why You Can't Tie Your Shoes in the Fourth Dimension

(but you can knot a basketball)

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Theorem:

You can't tie a string in a knot in 4 dimensional space

How do we picture 4 dimensions?

How do we prove things about 'floppy' objects like ropes?

...this result is only cool if you first know that you *can* tie knots in 3D!

Knots in 3 Dimensions

Proving they actually exist

Defining a Knot

A *knot* is an embedding of a compact 1-dimensional curve into 3-dimensional space.

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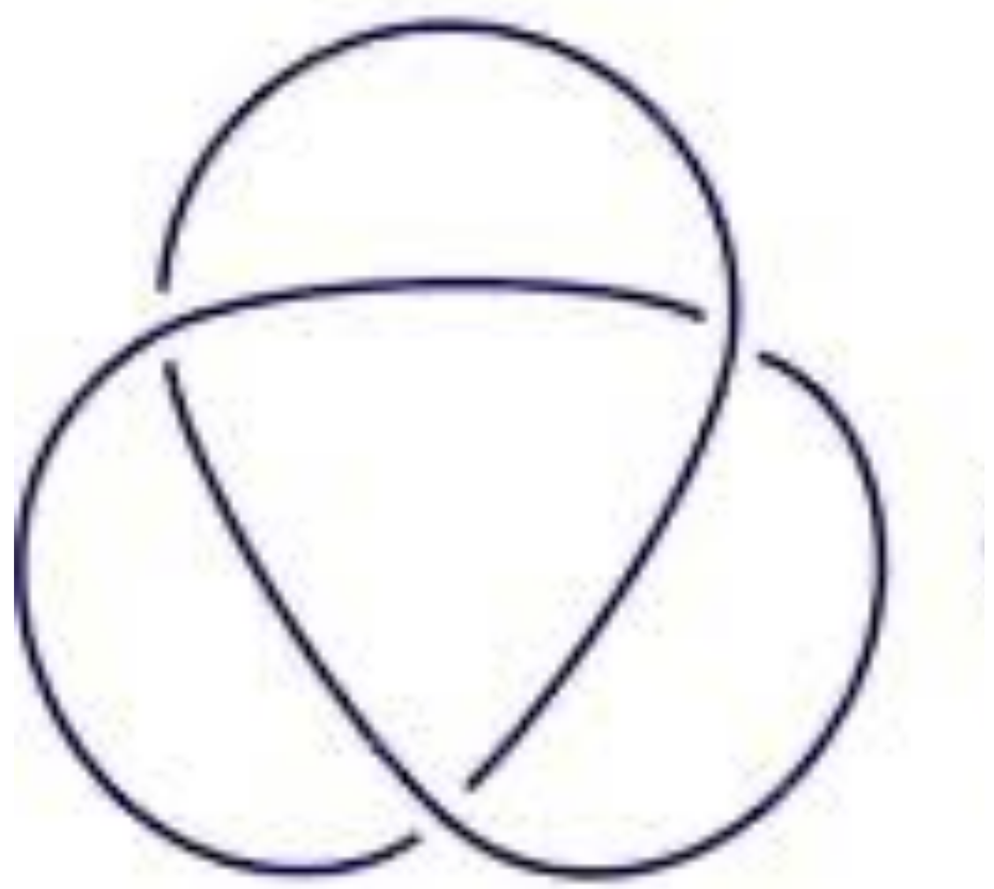
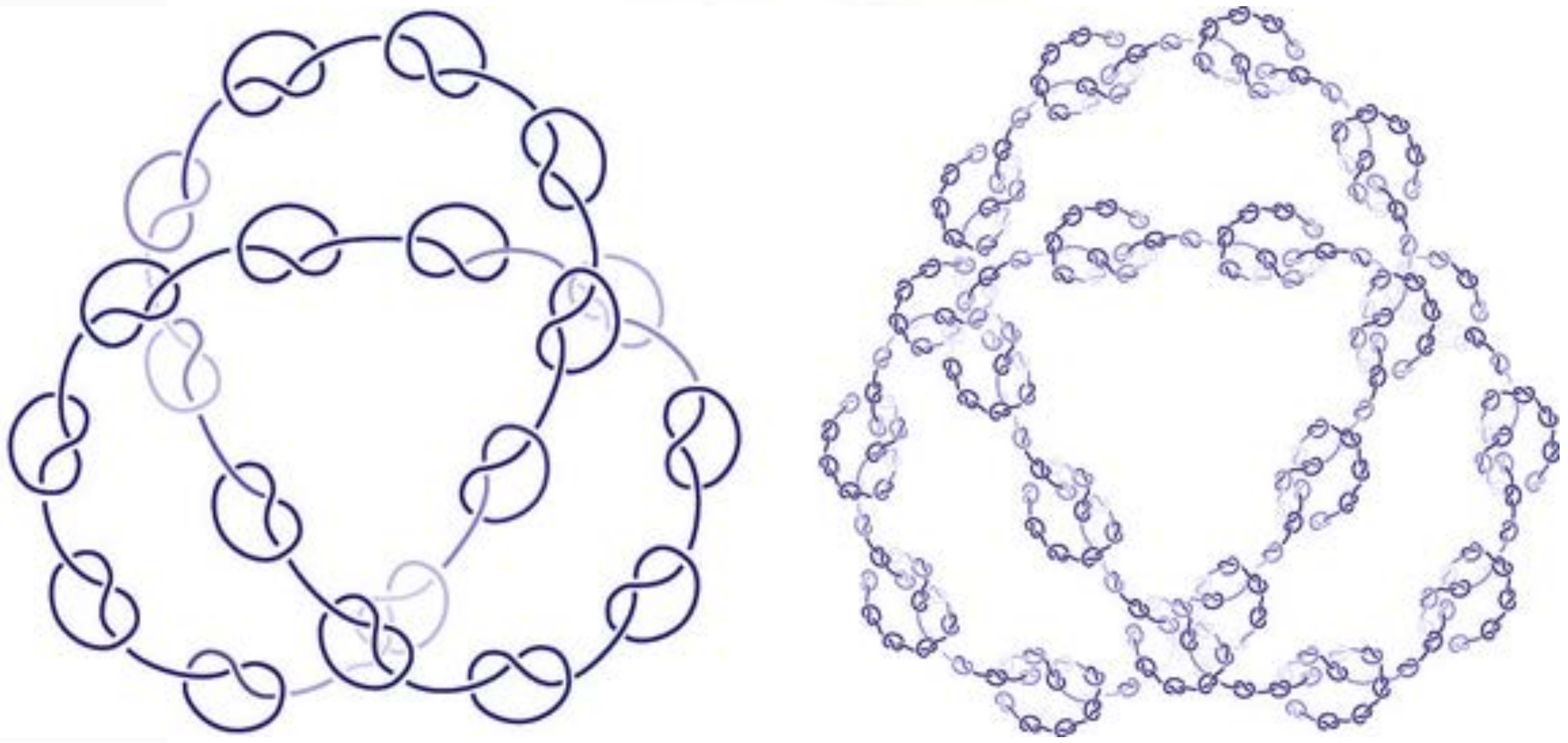
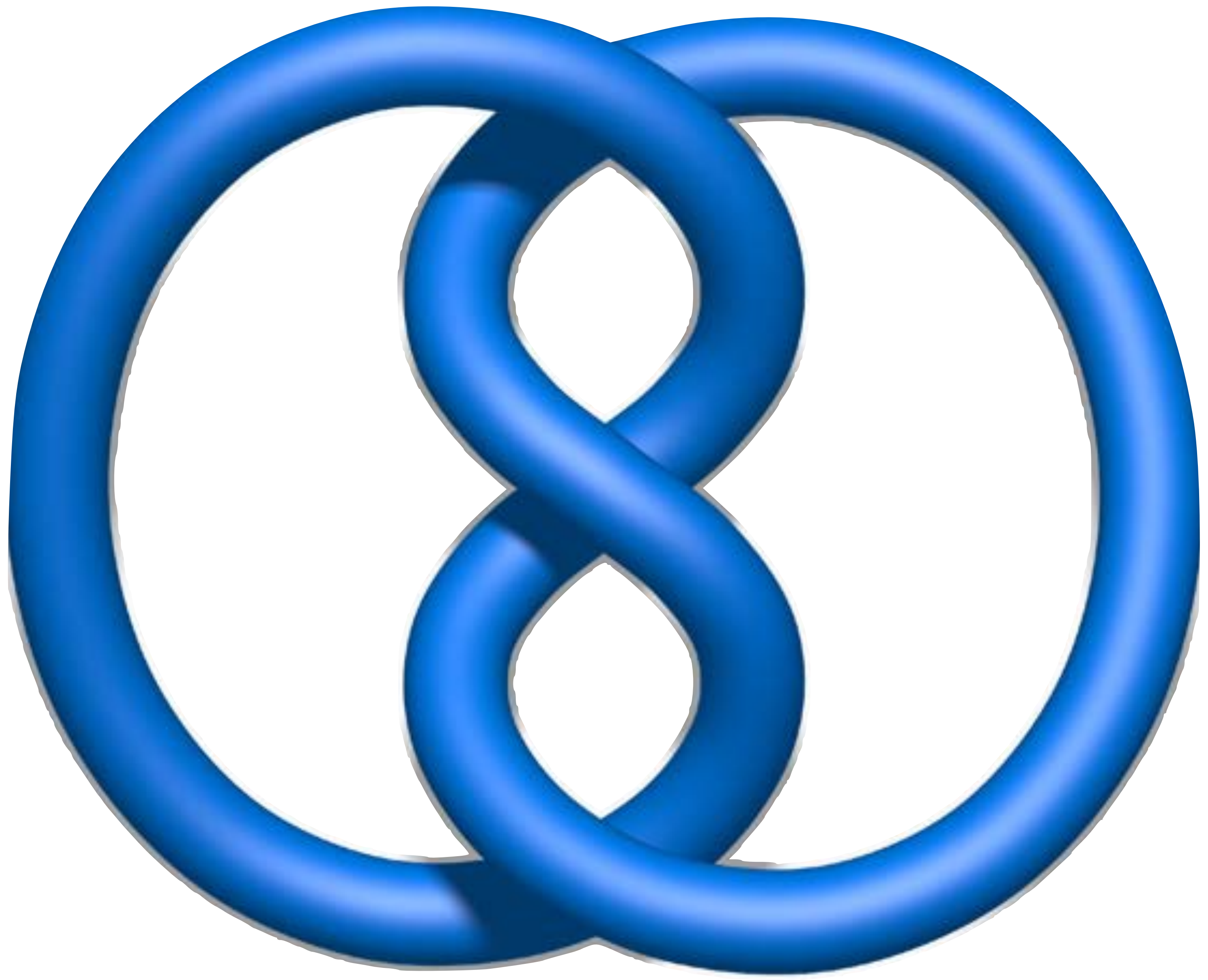




Make smooth, or piecewise linear (avoid "wild knots")

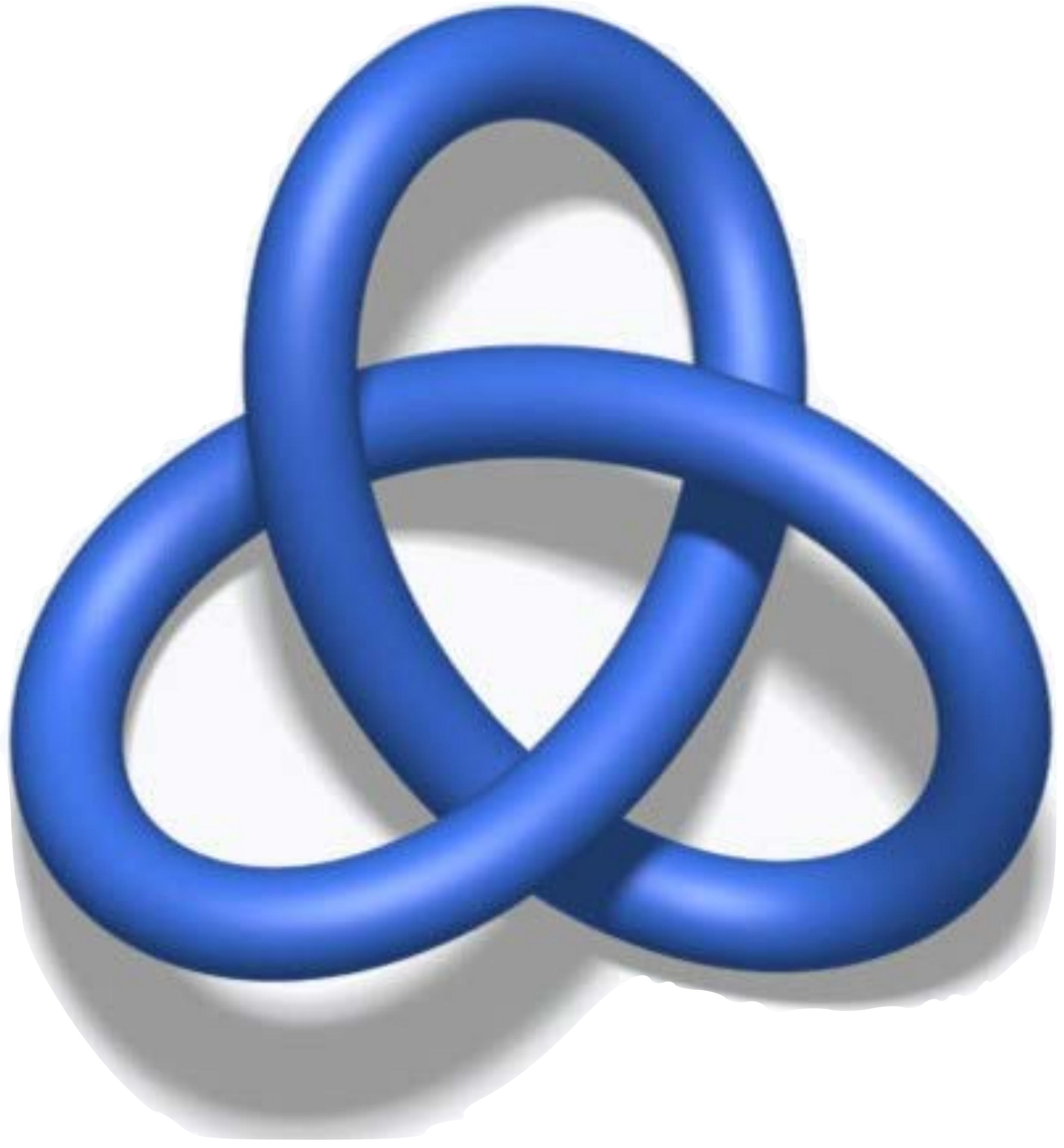
Knot Equivalence

Two knots K, L are equivalent if there is a homeomorphism $h\colon \mathbb{R}^3 \to \mathbb{R}^3$ taking K to L.



Theorem:

This is equivalent to "ambient isotopy": a continuous path of homeomorphisms of $\mathbb{R}^3$ taking K to L



Types of Knots

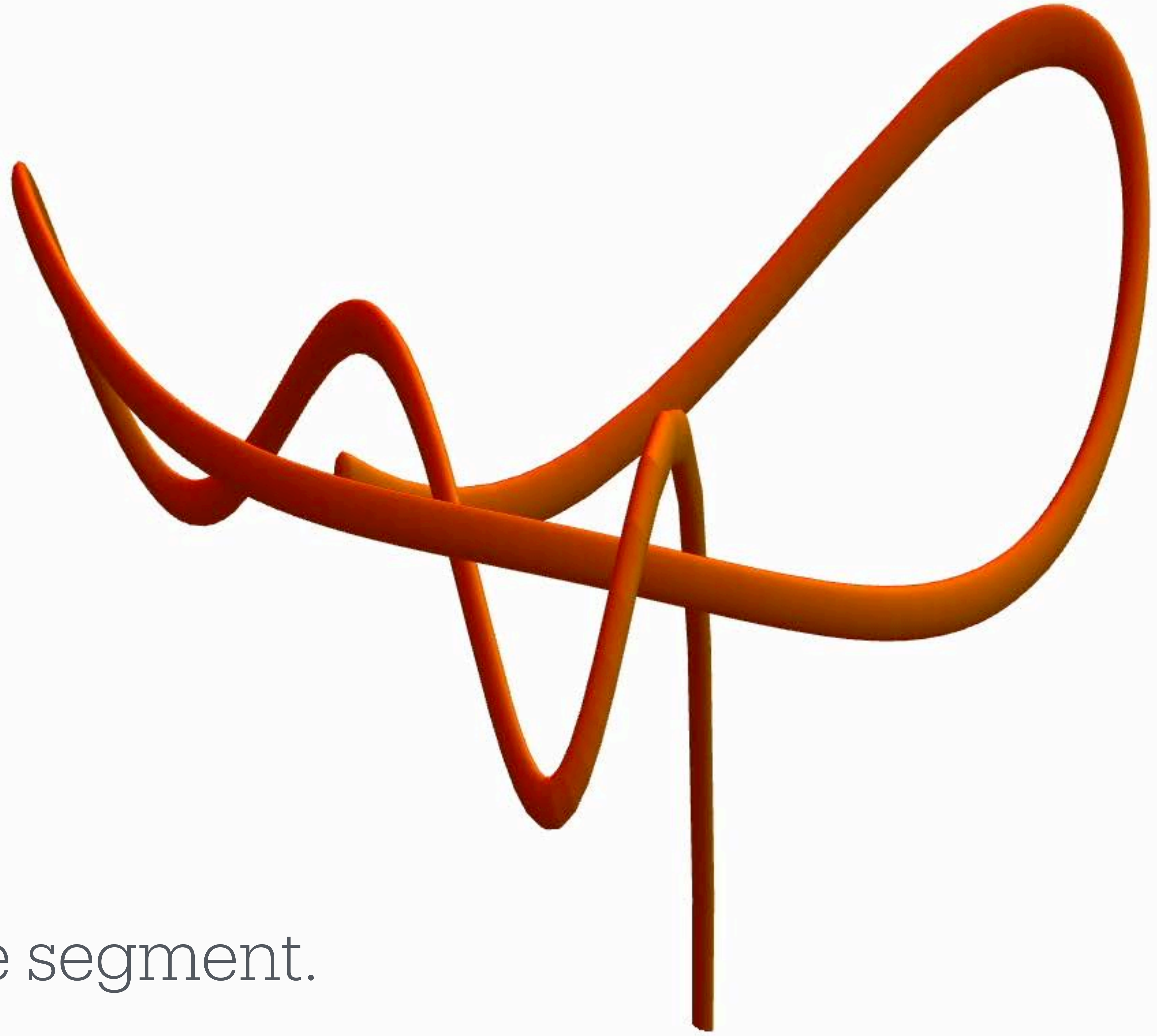
Two compact one manifolds: intervals and circles

Theorem:

There are no (nontrivially) knotted intervals

Given any $\gamma: [0,1] \to \mathbb{R}^3$, ambient isotopic to shrinking the interval

Eventually, its short enough that its approximately its tangent line

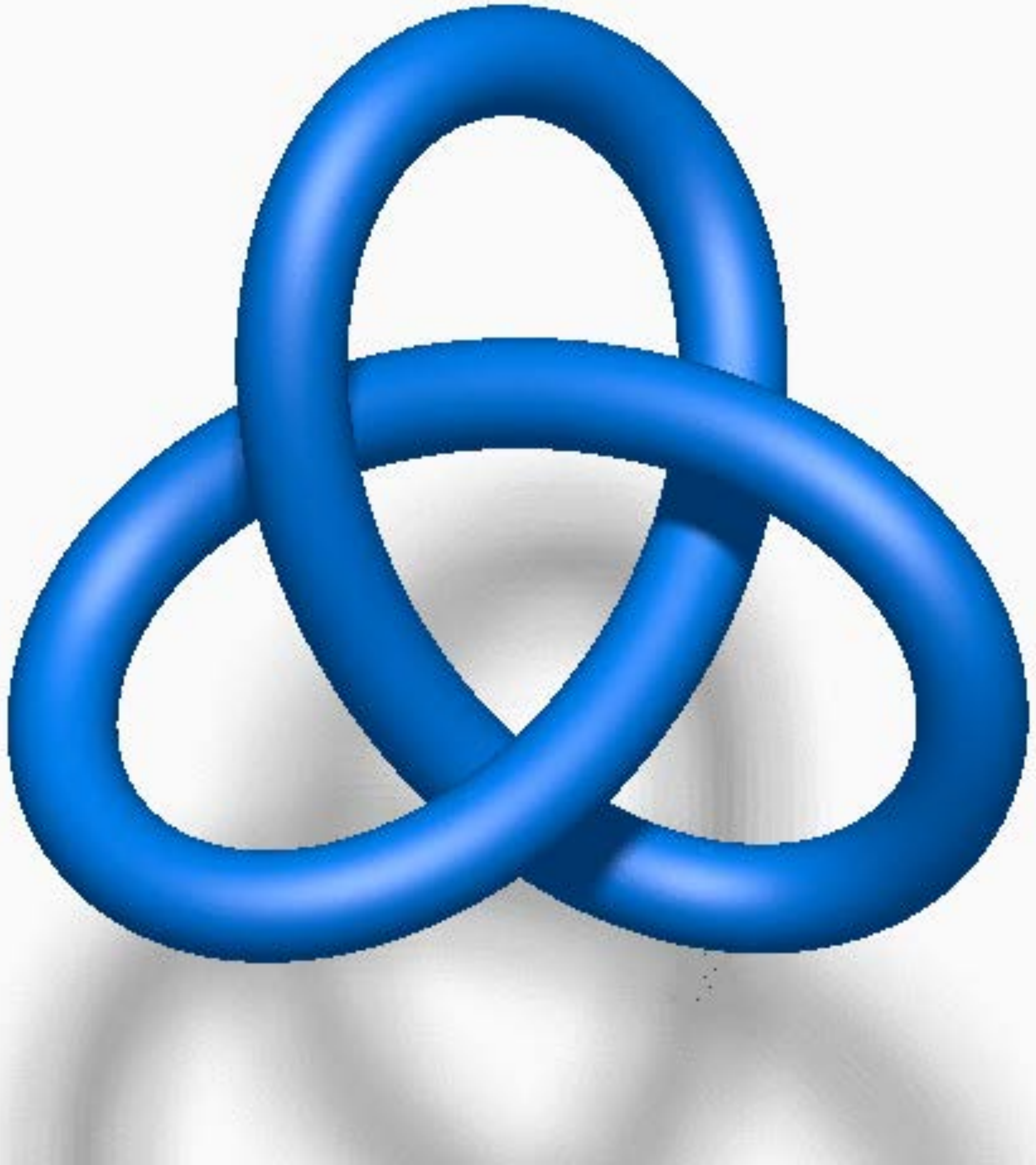


Thus all such are isotopic to a line segment.

But there must be....

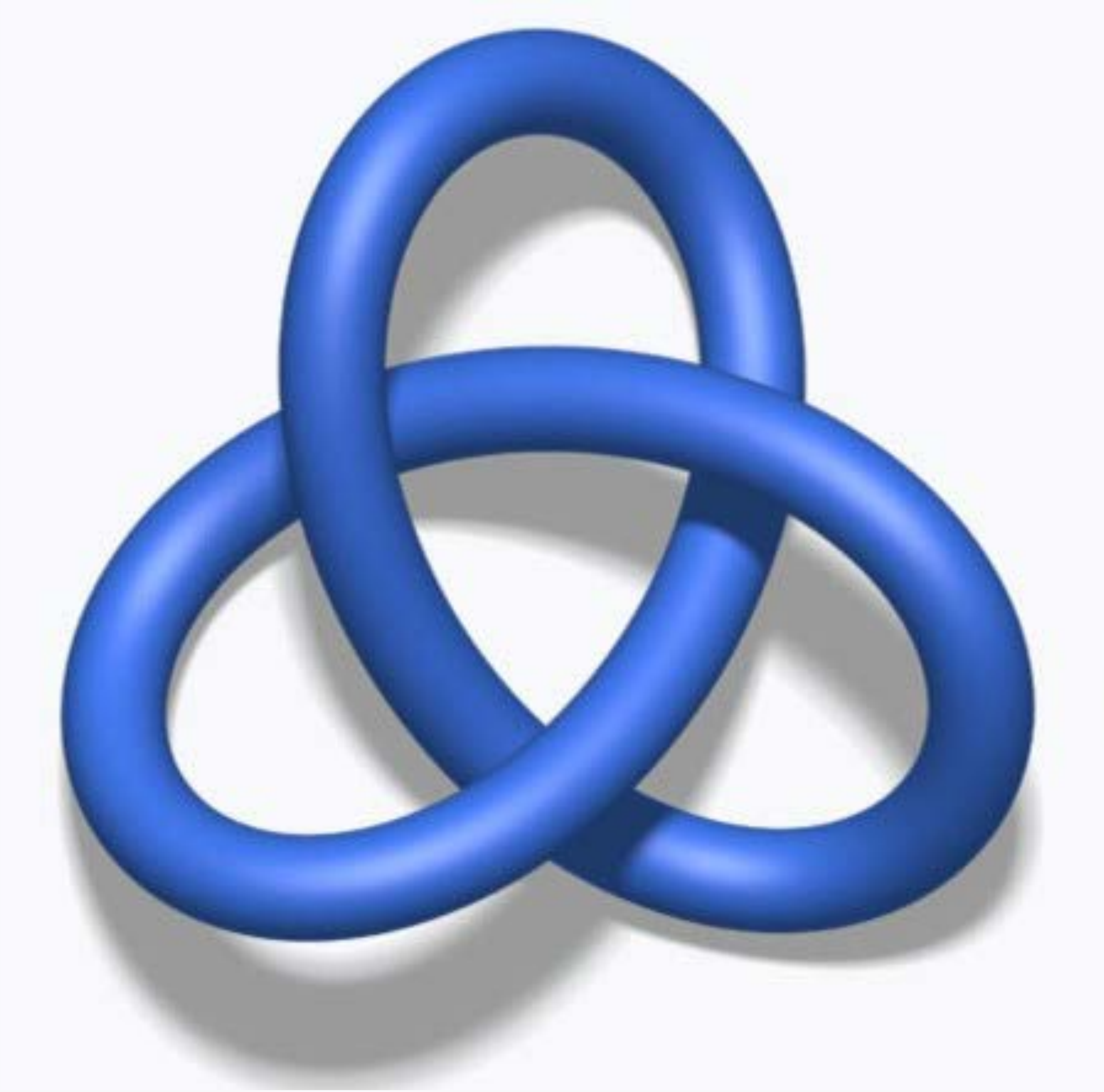
Knotted Circles?

We need to somehow get a handle on ”all possible ambient isotopies”

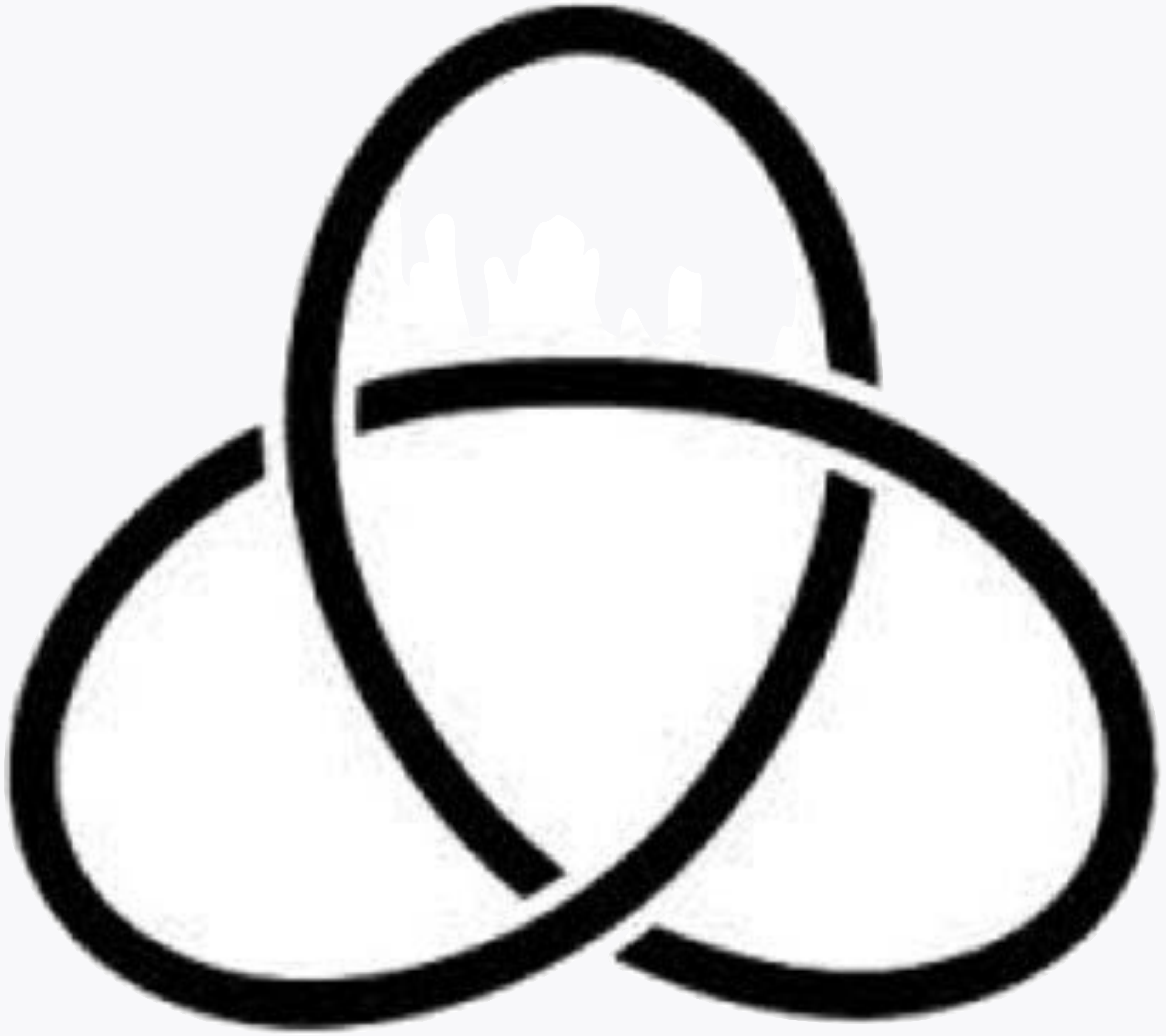


Knot Projections

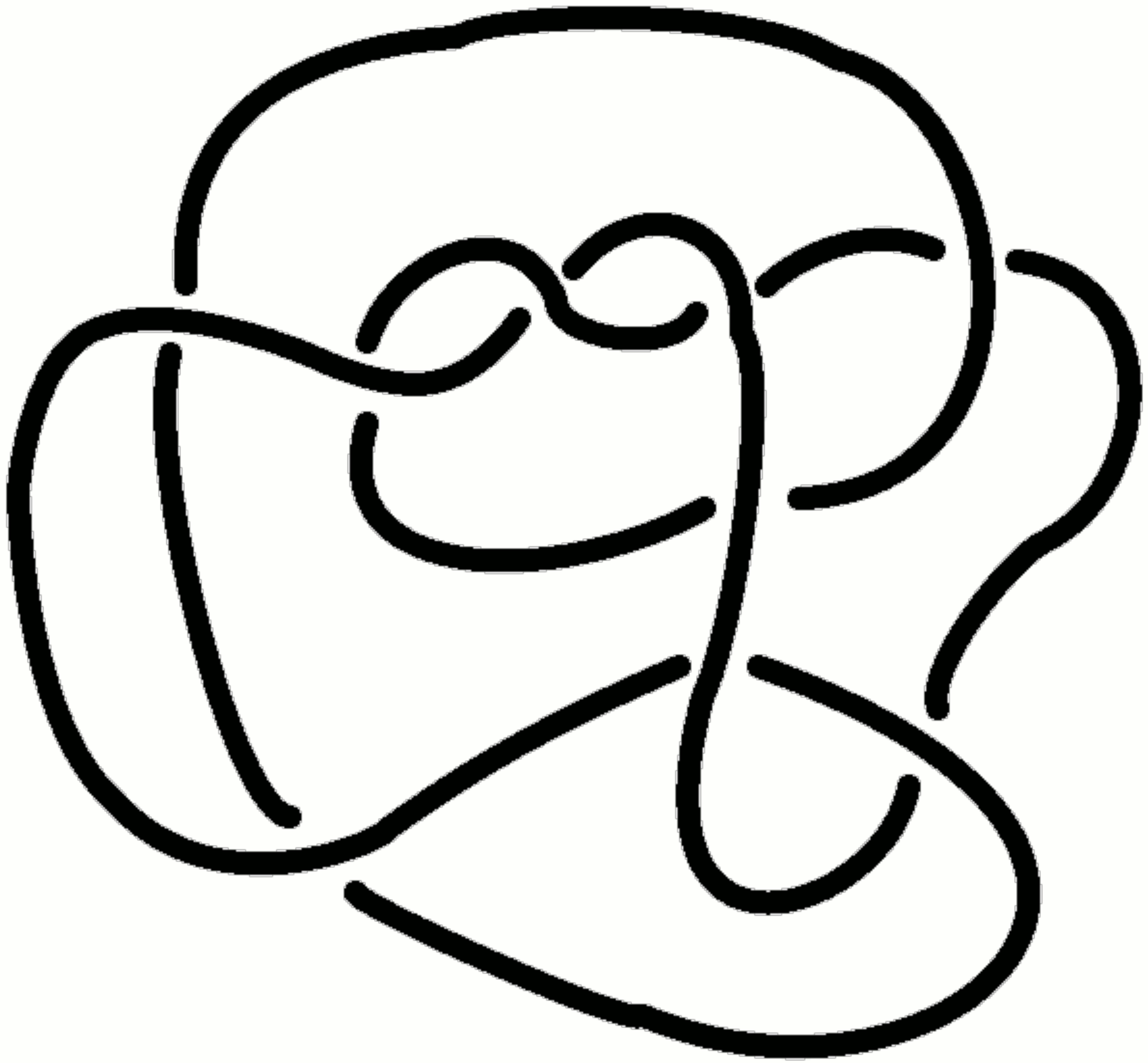
Draw a 2D view of a knot via orthogonal projection $\mathbb{R}^3 \to \mathbb{R}^2$

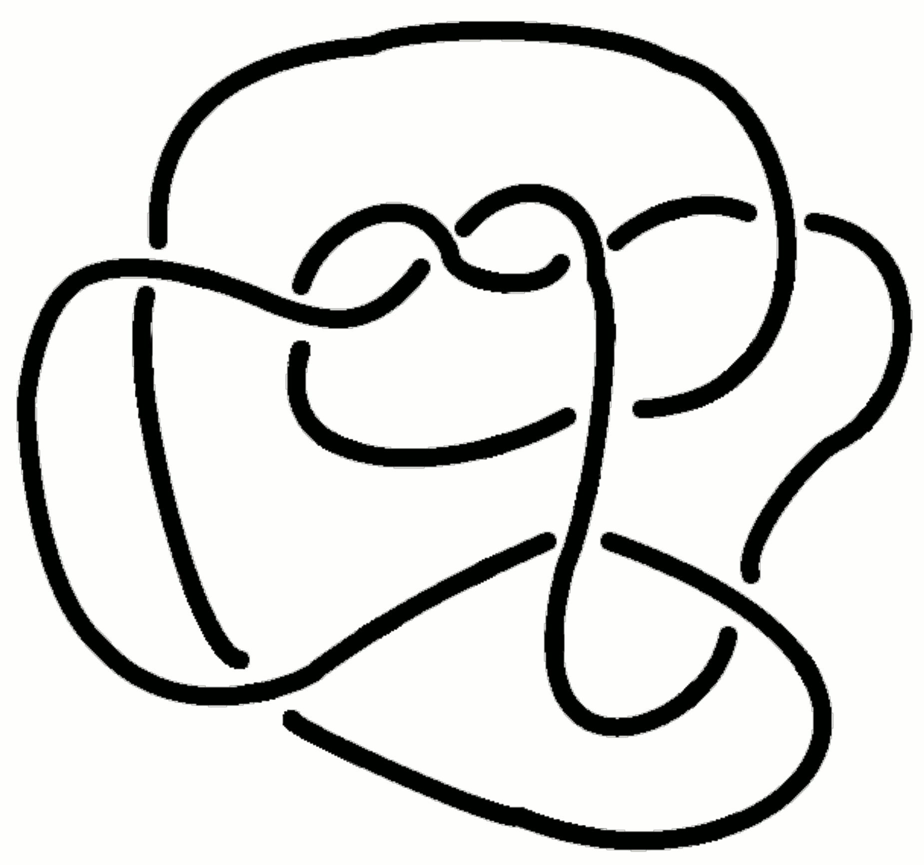


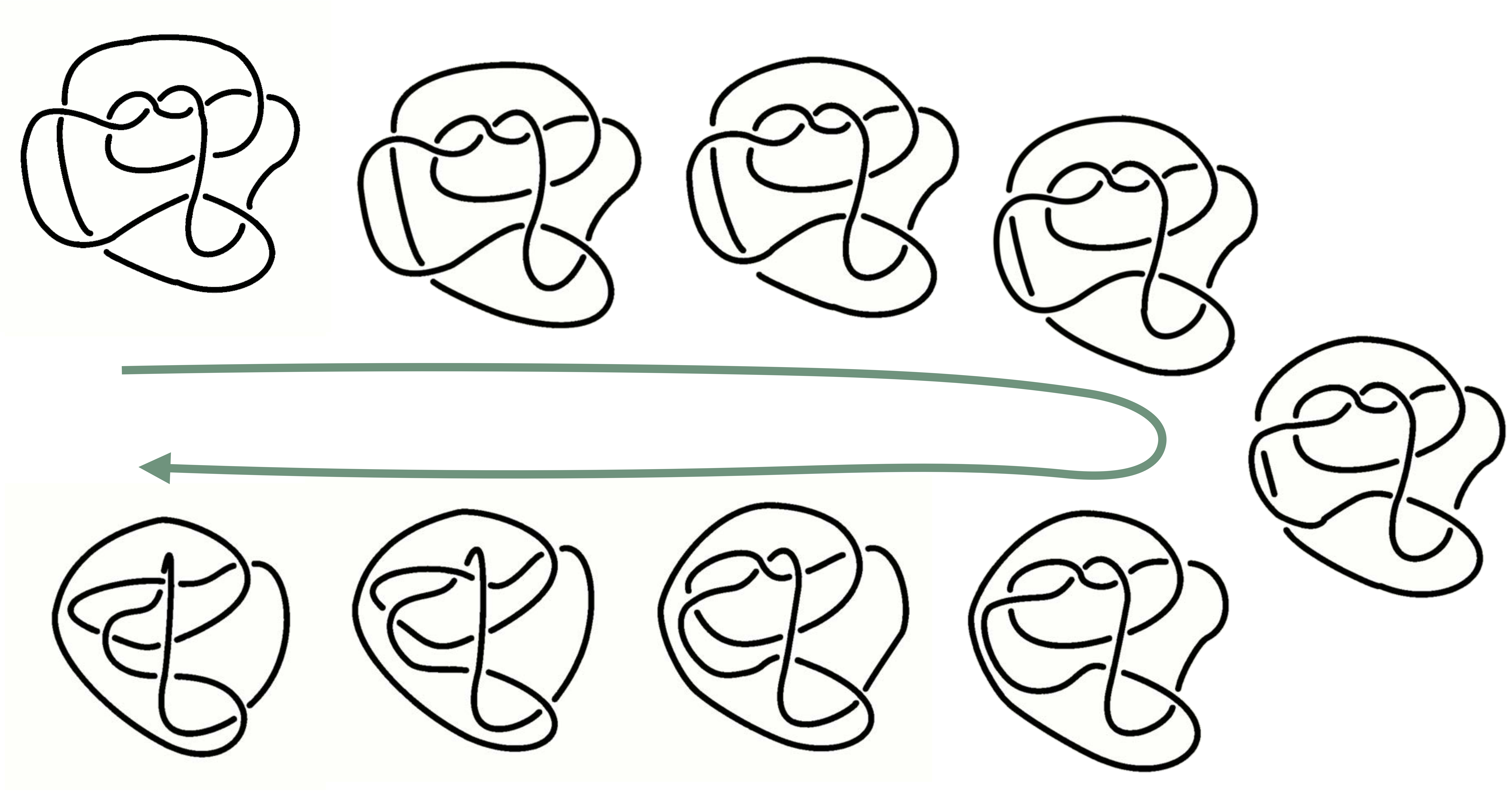
Broken strands = under crossings

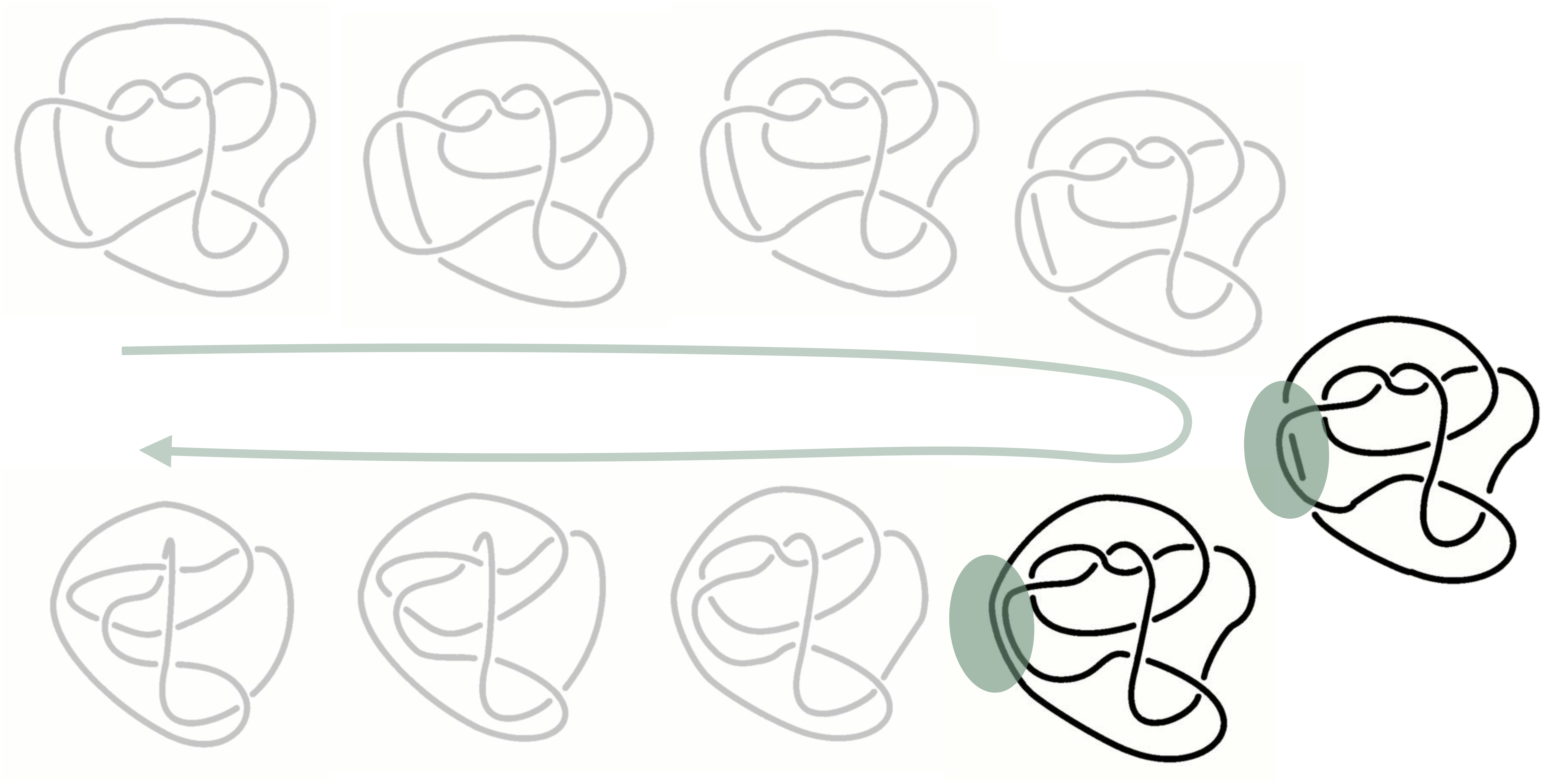


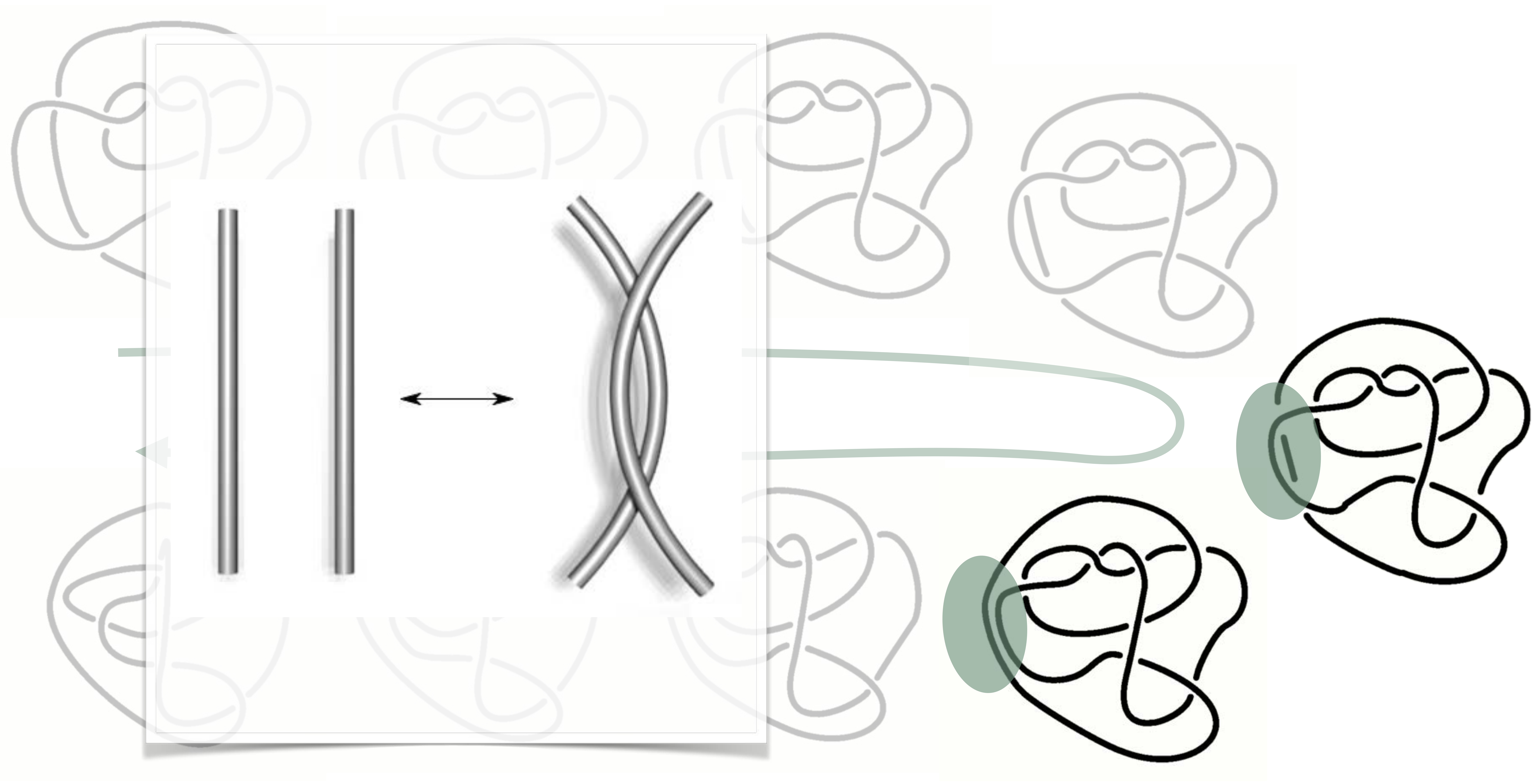
What do we “see” when we watch an isotopy unfold?

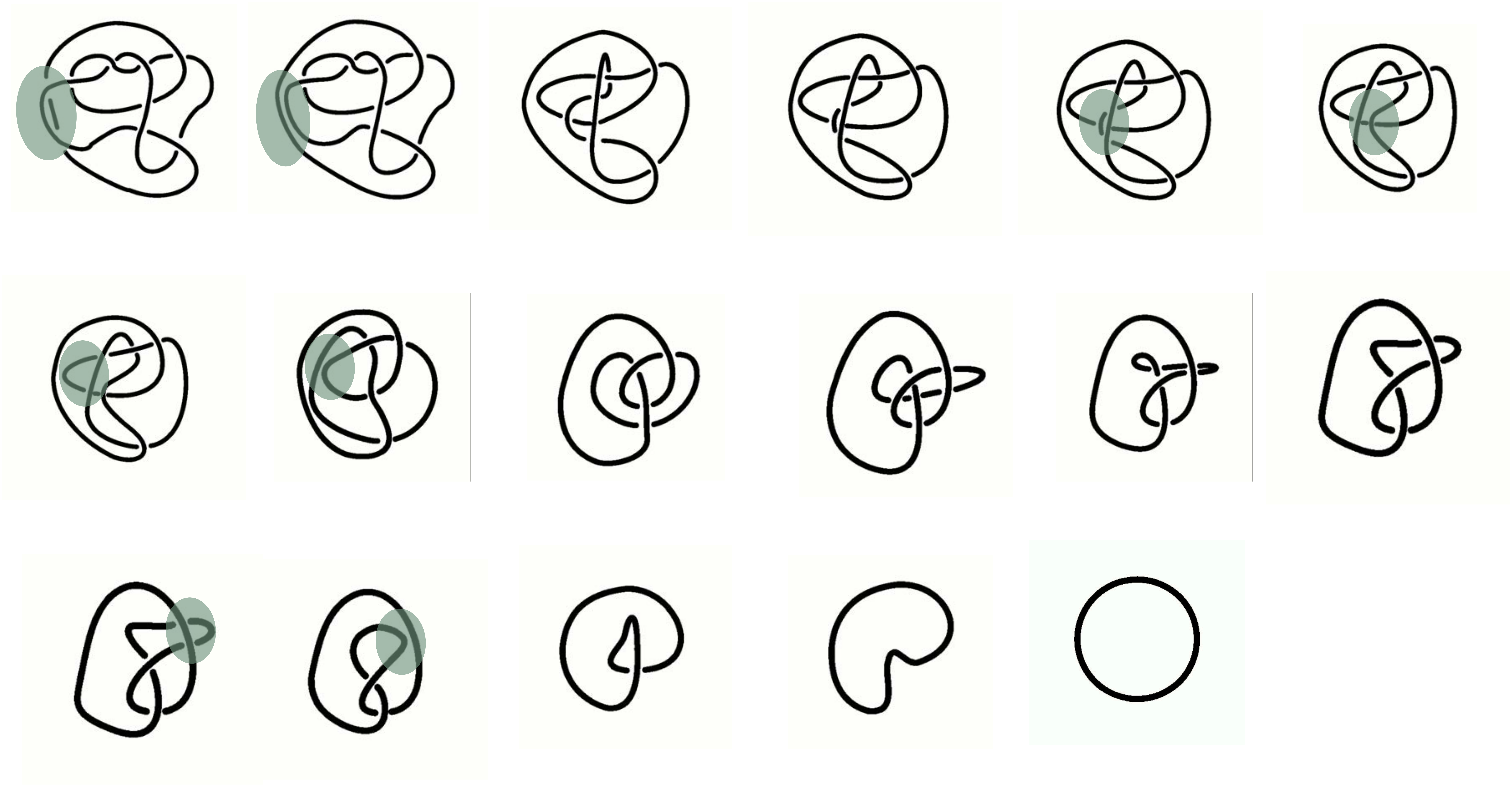


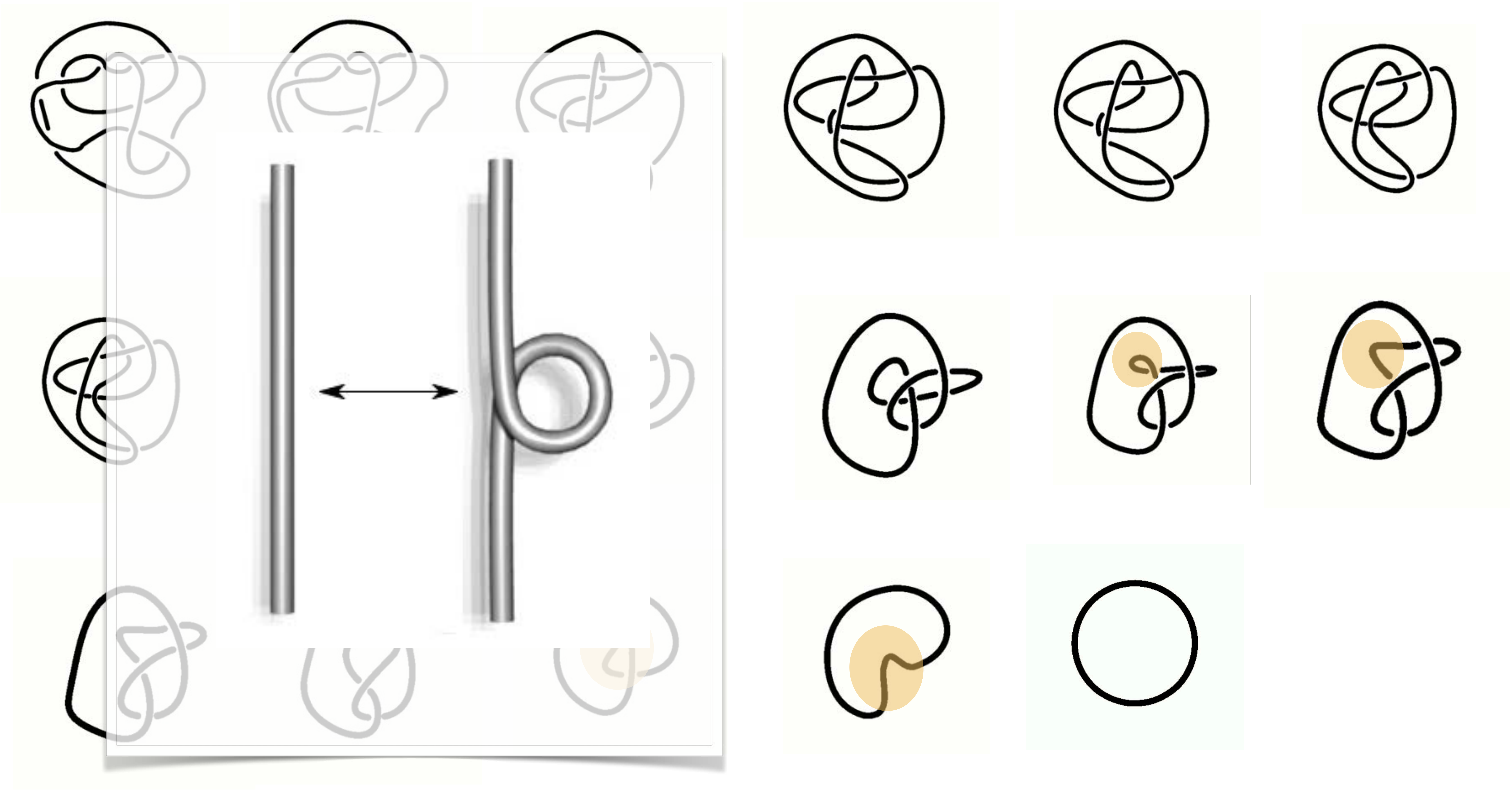


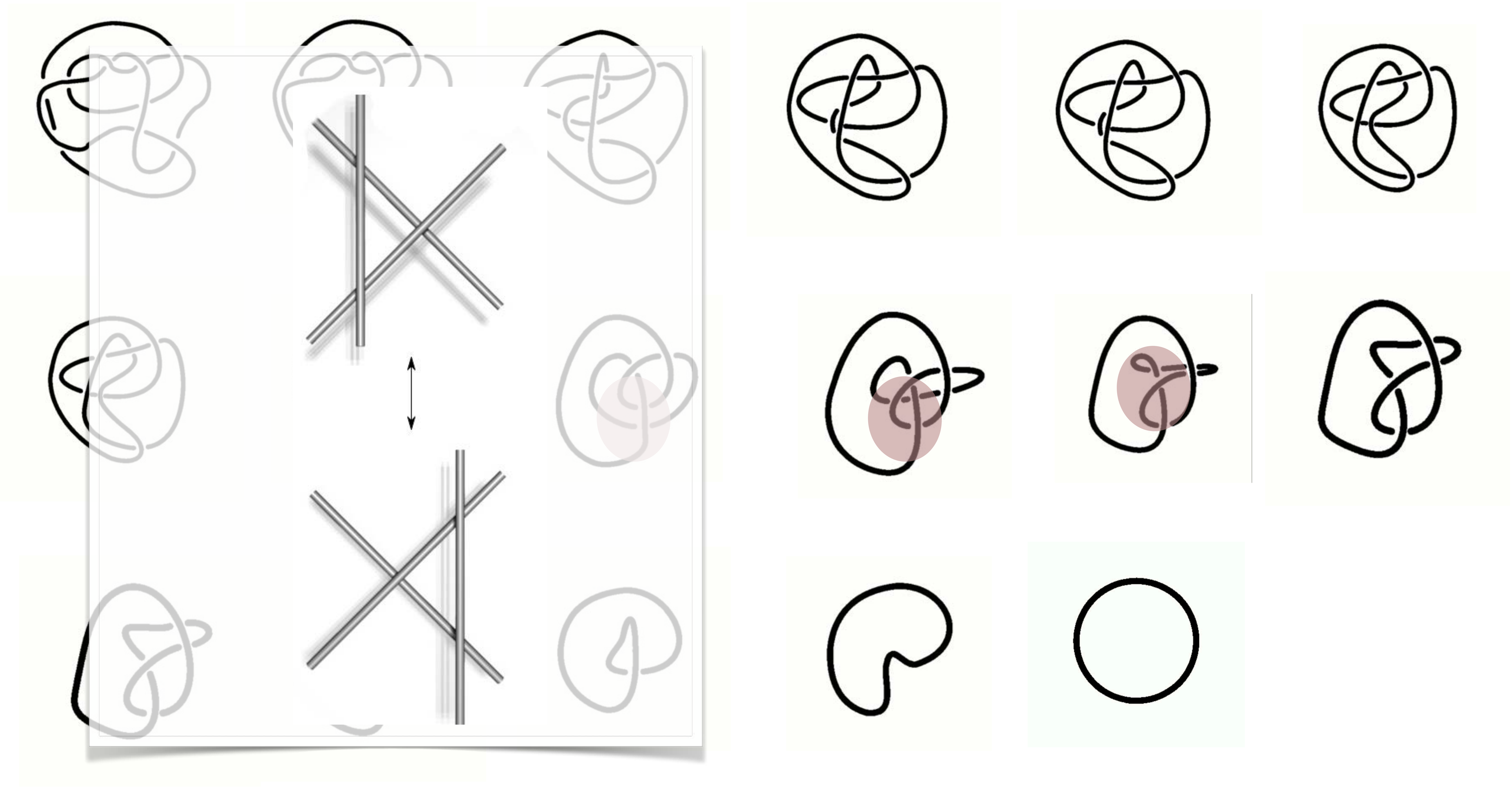








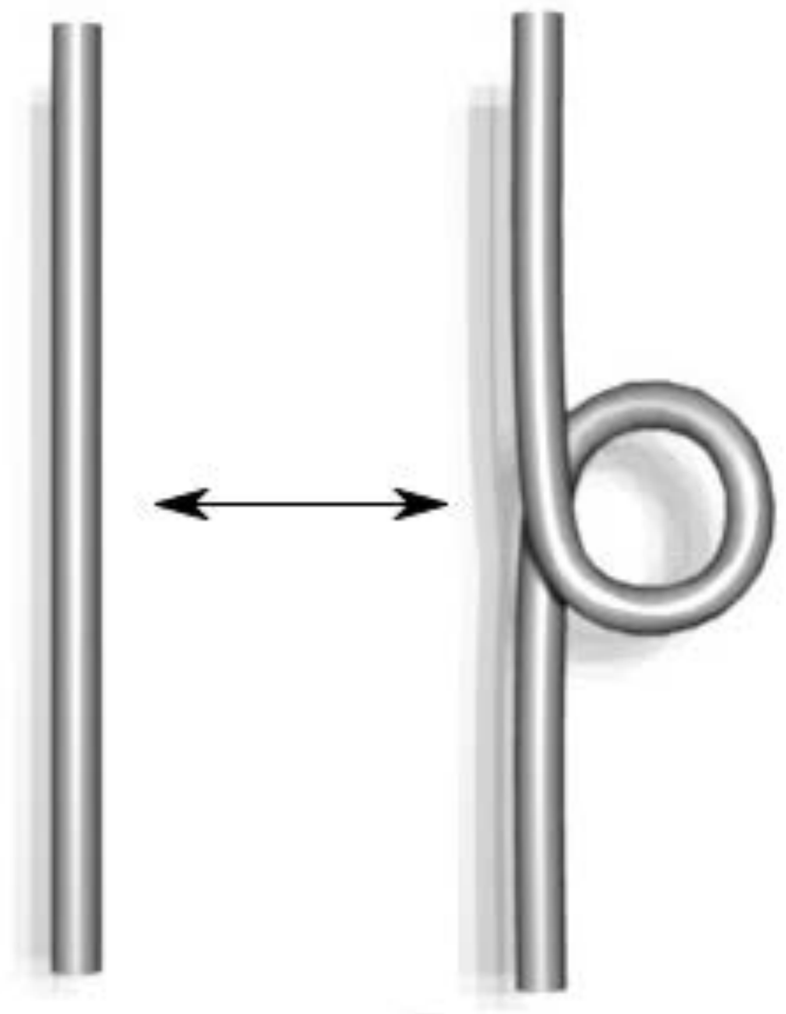


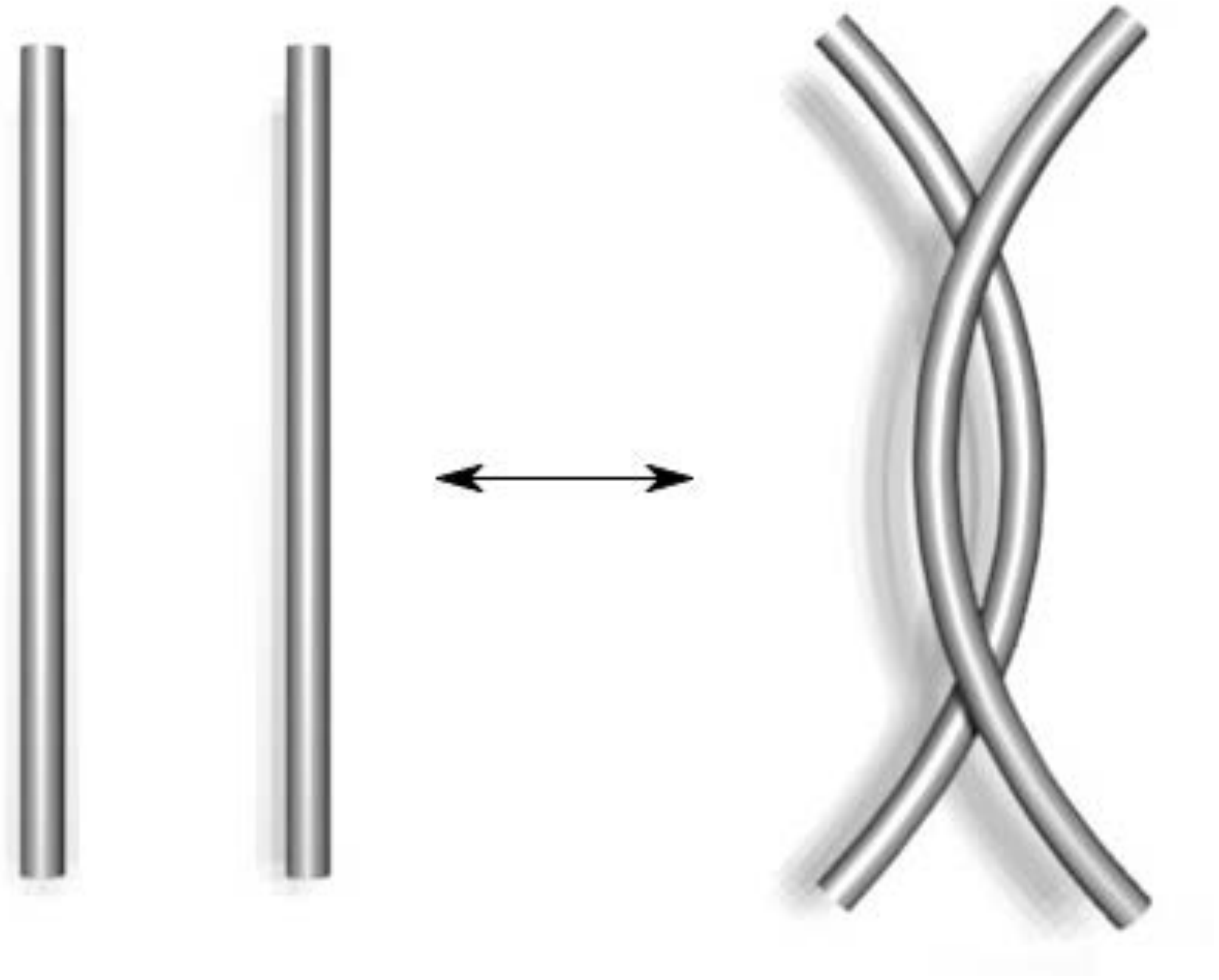


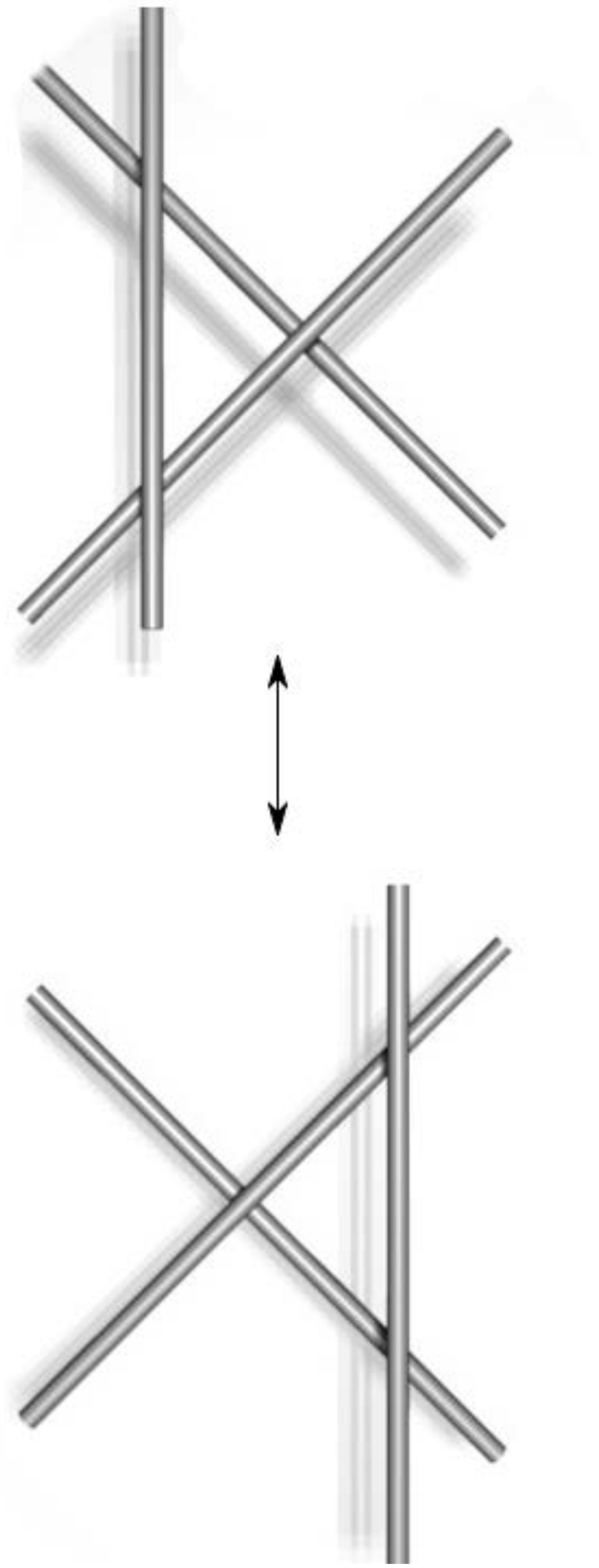
(1927)

Reidemeister's Theorem

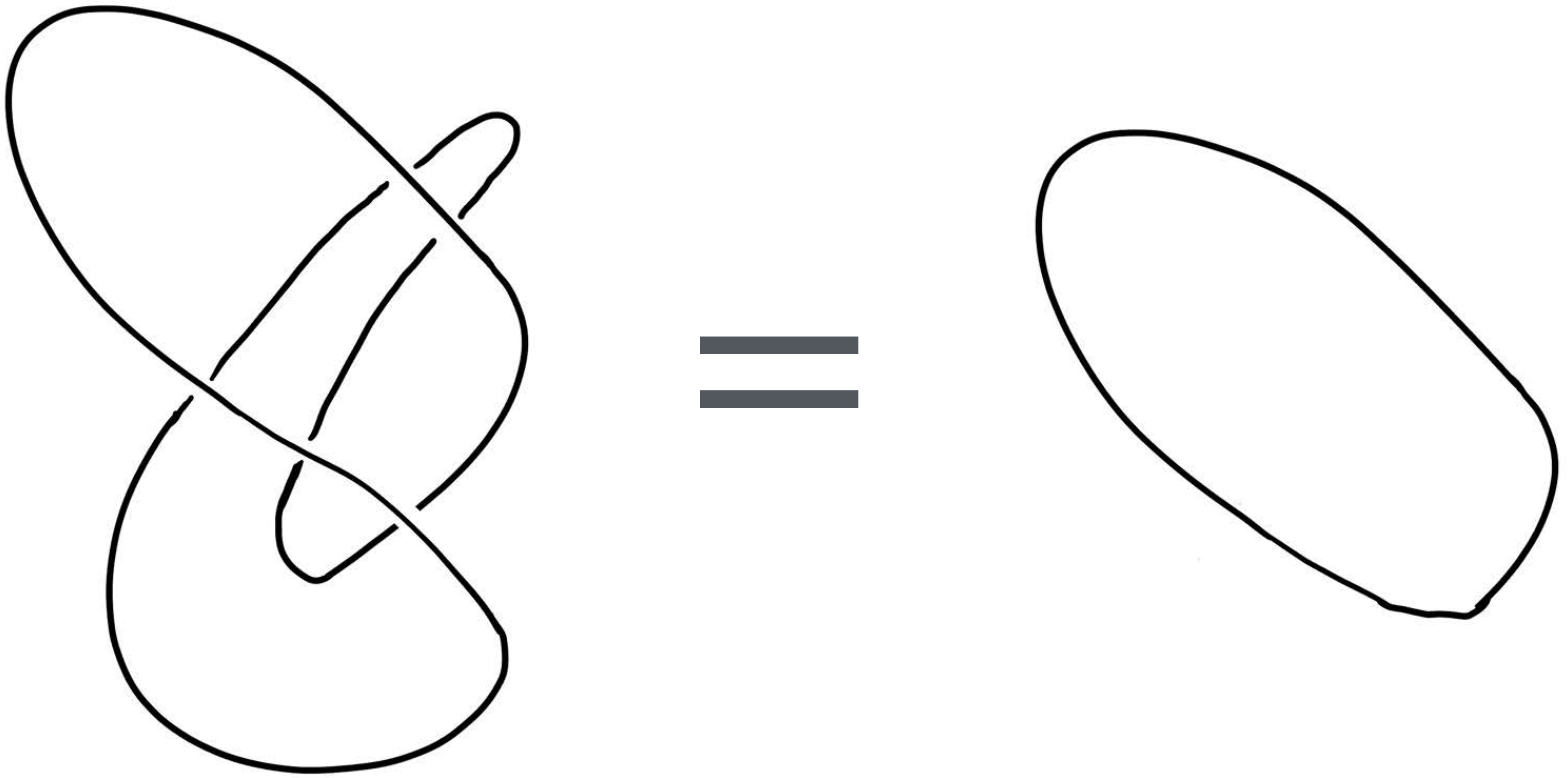
Every equivalence of knots is achievable using only these three moves.







Example: Unknotting the unknot



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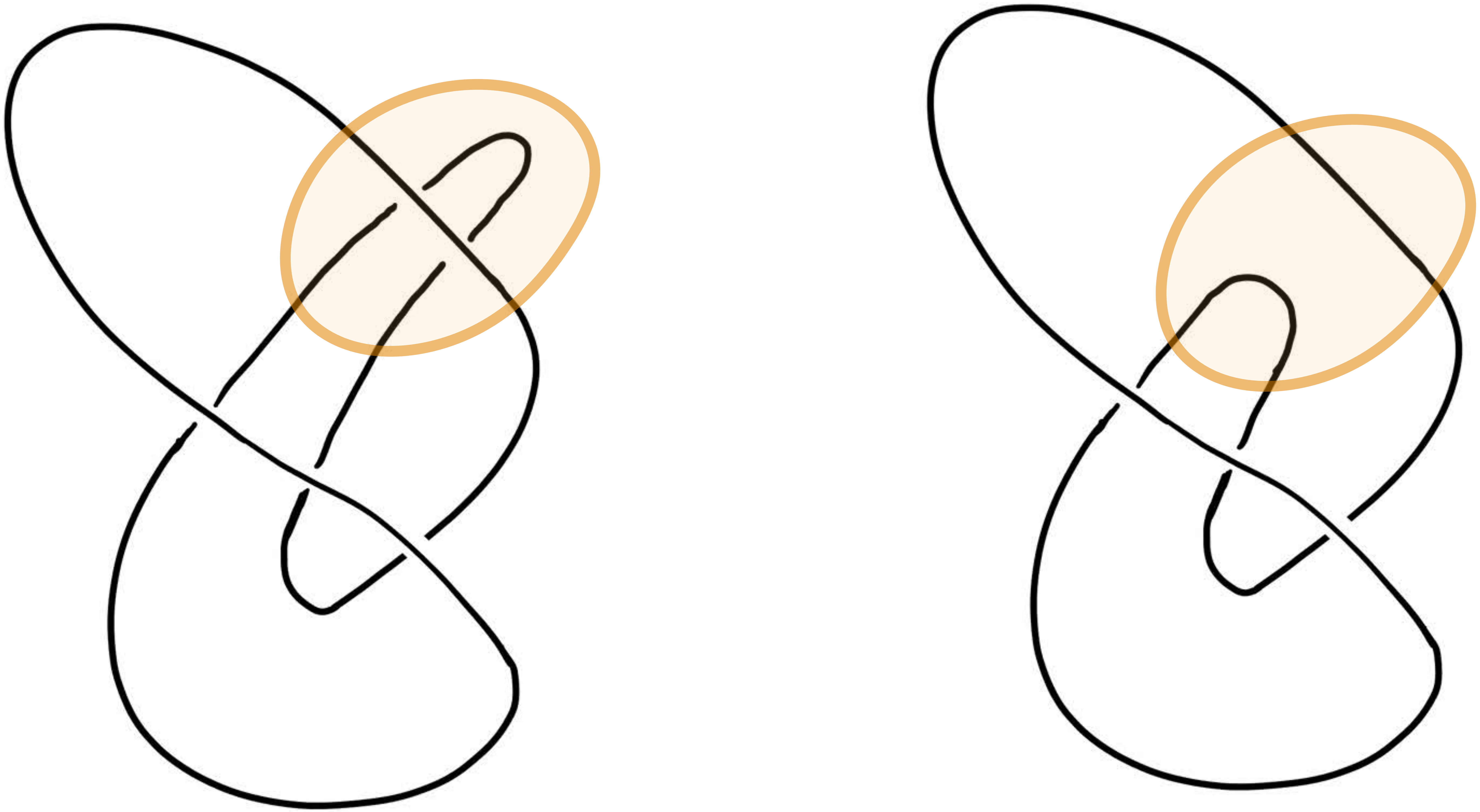

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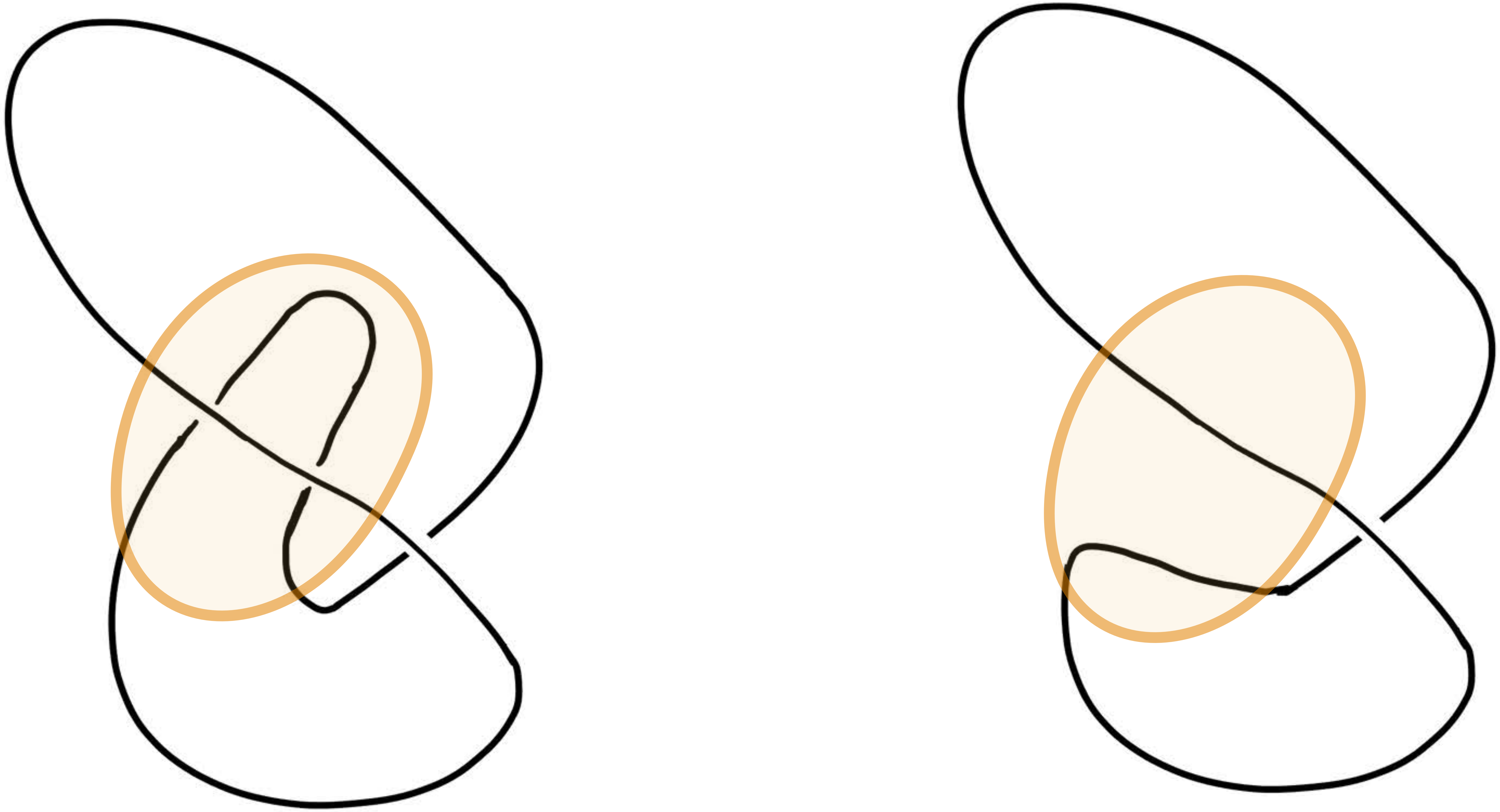

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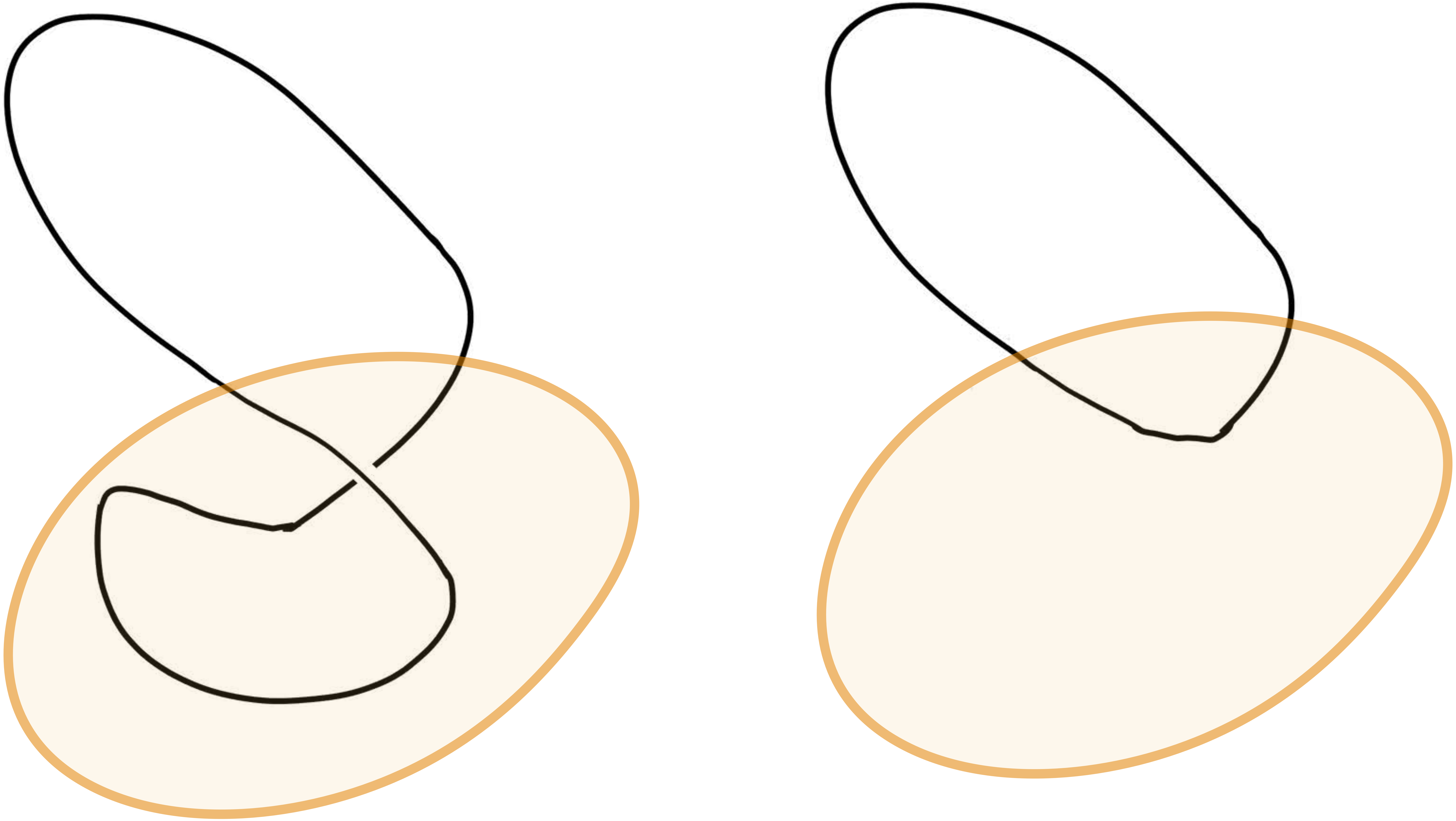
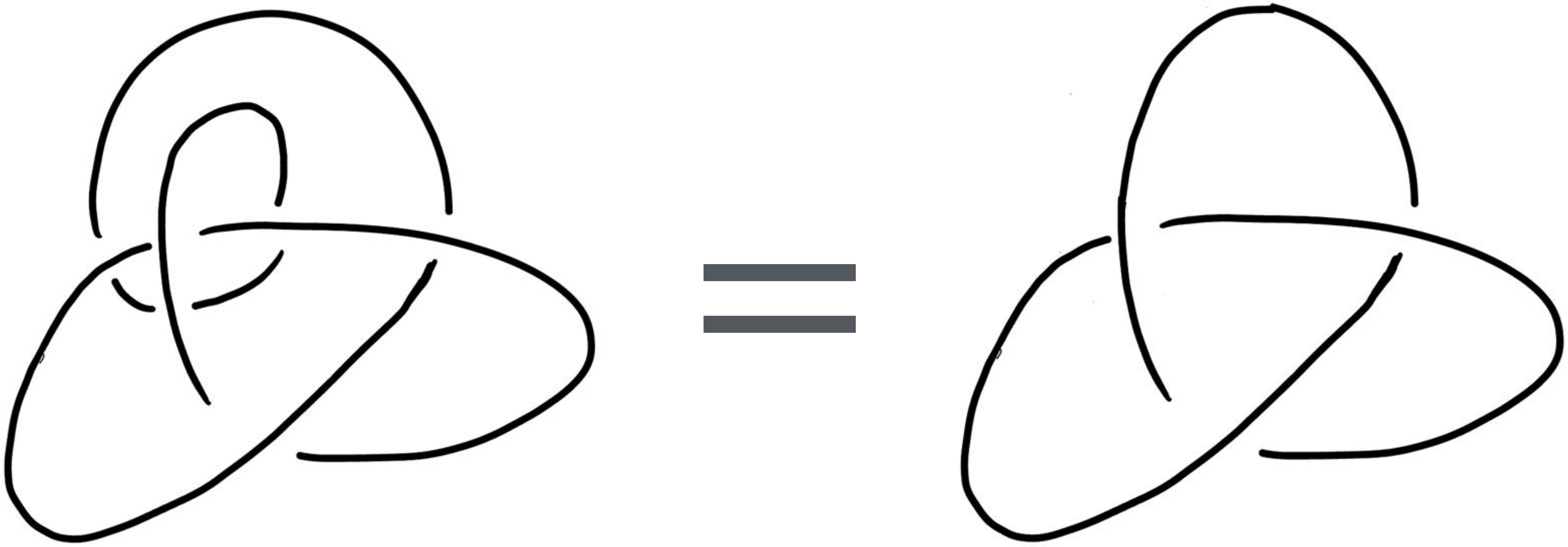


Example: These are both the trefoil



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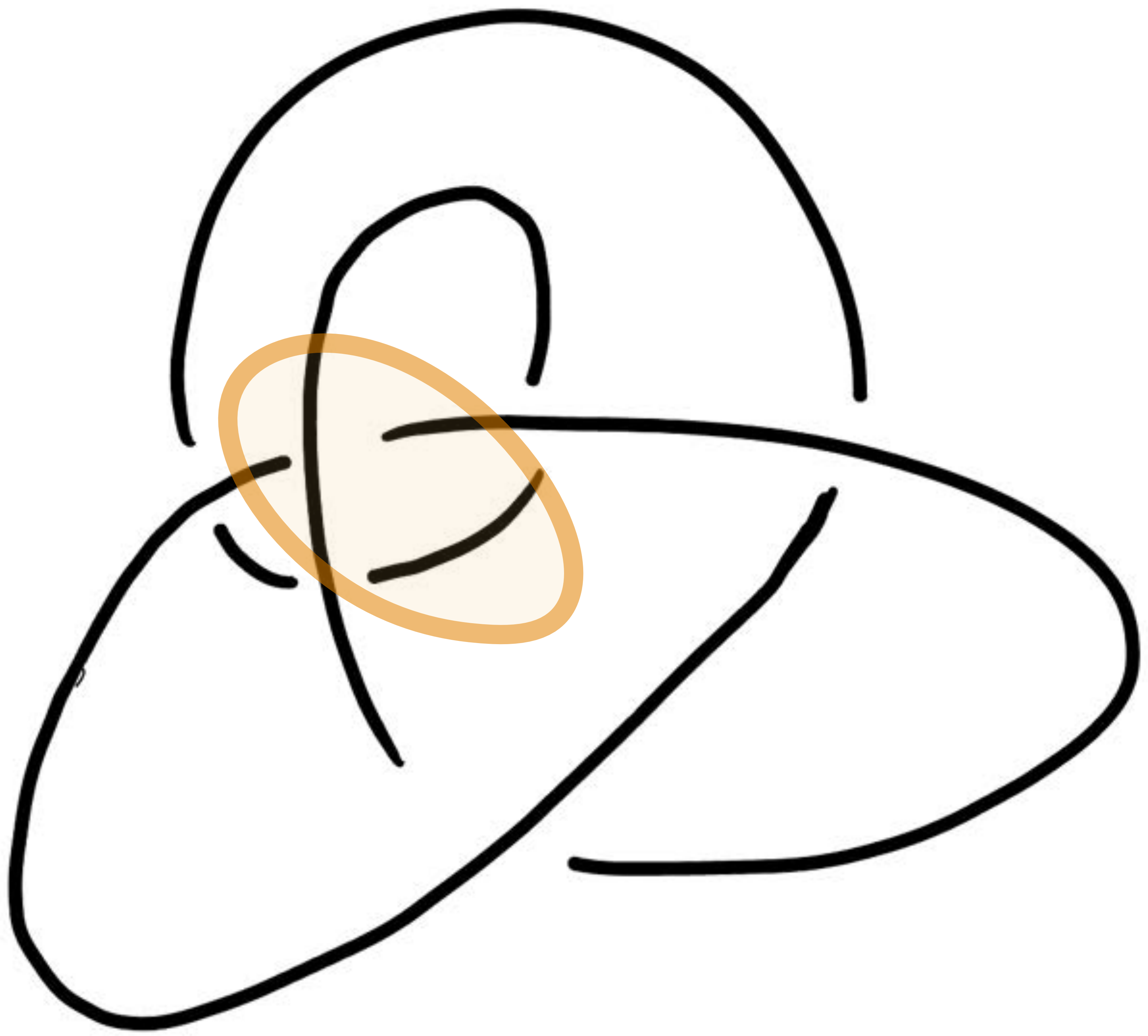

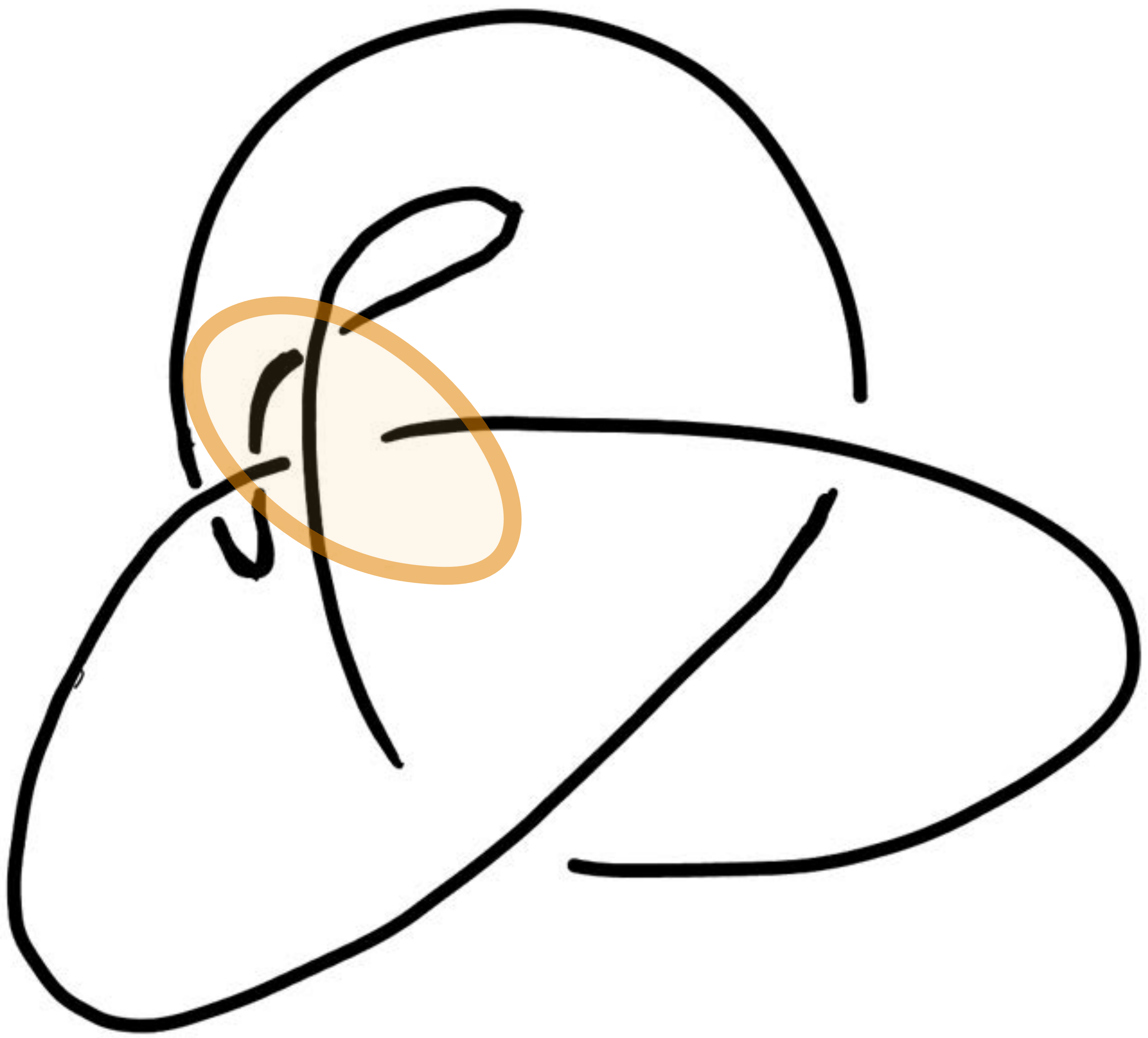

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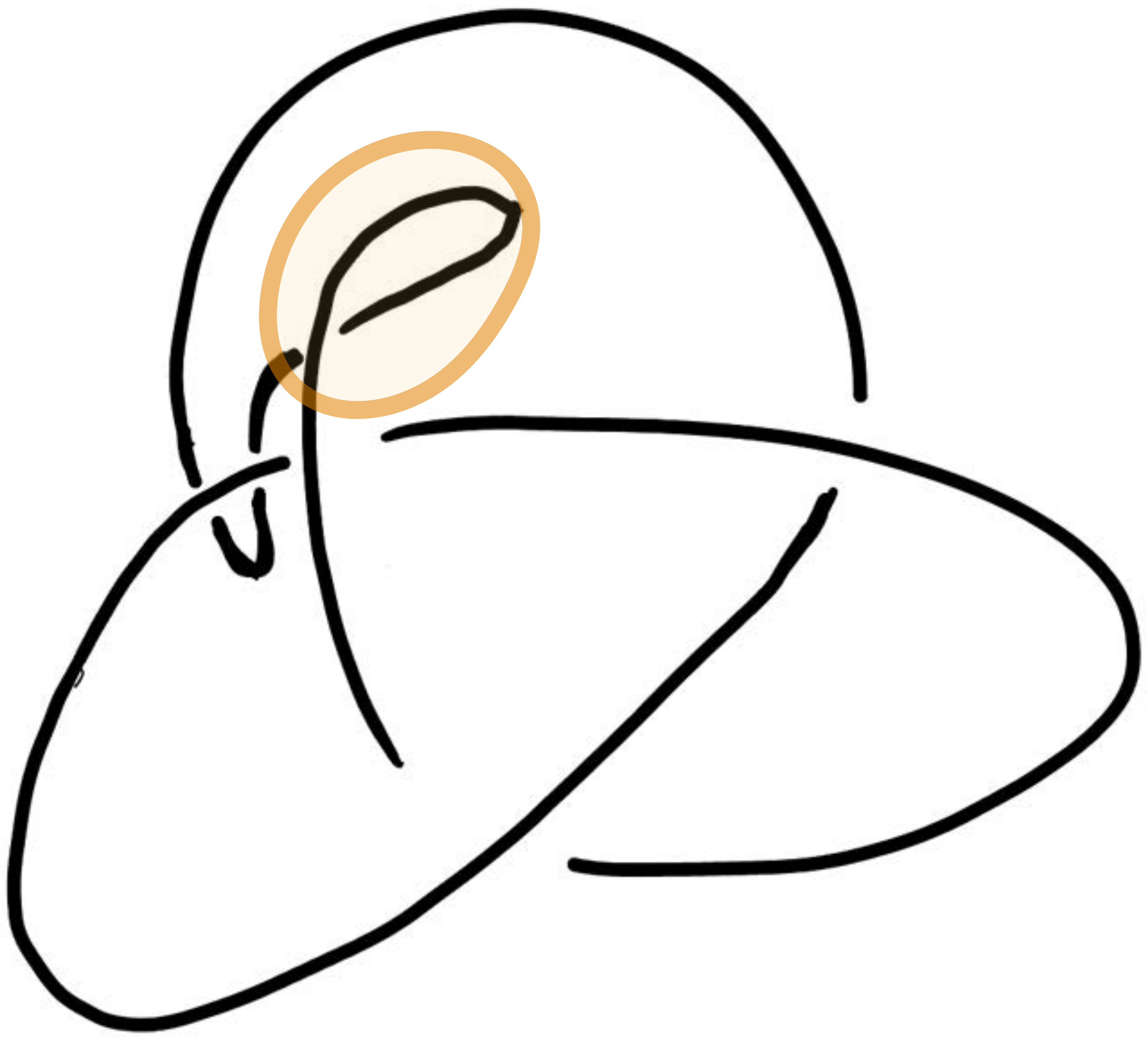

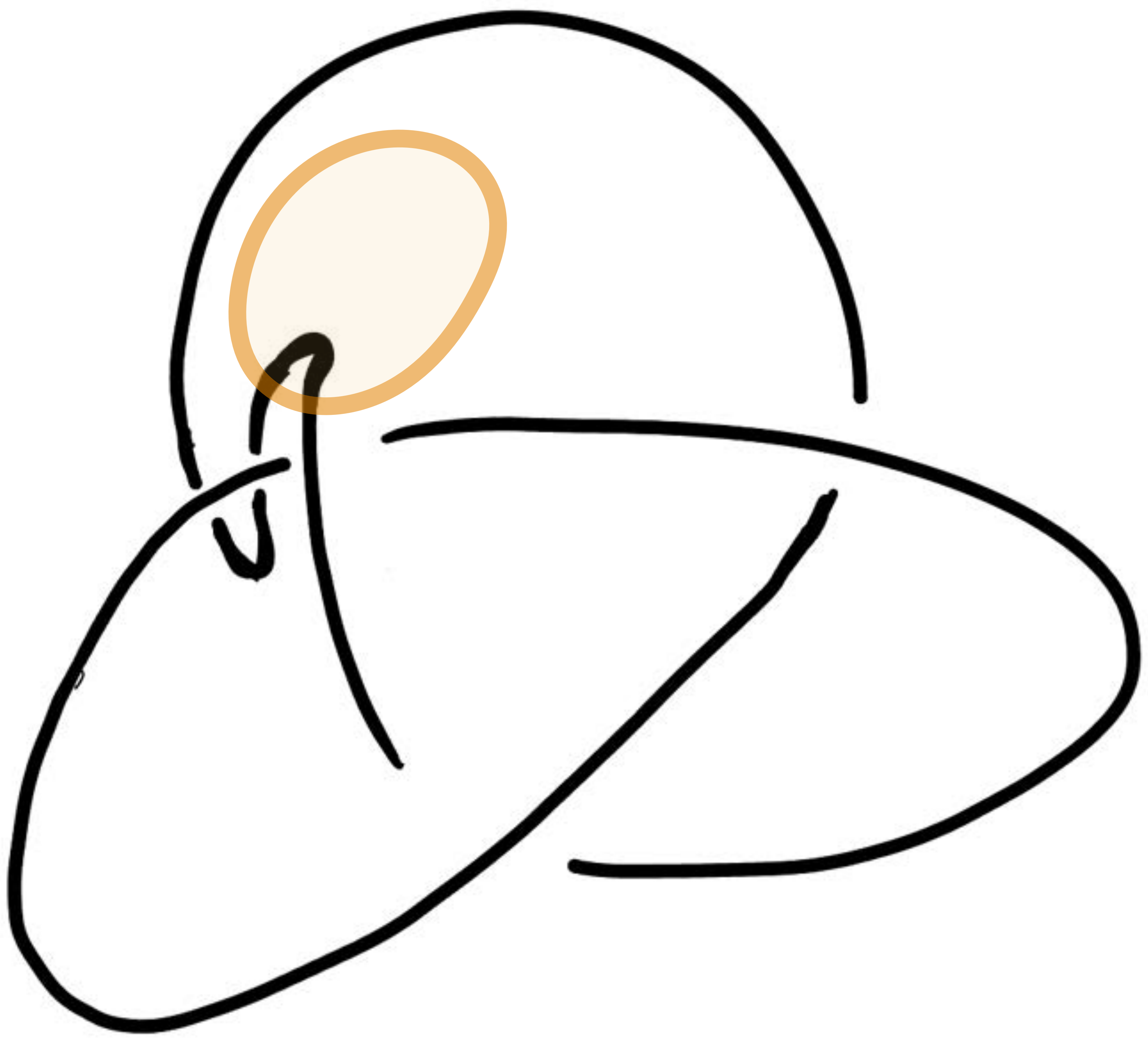

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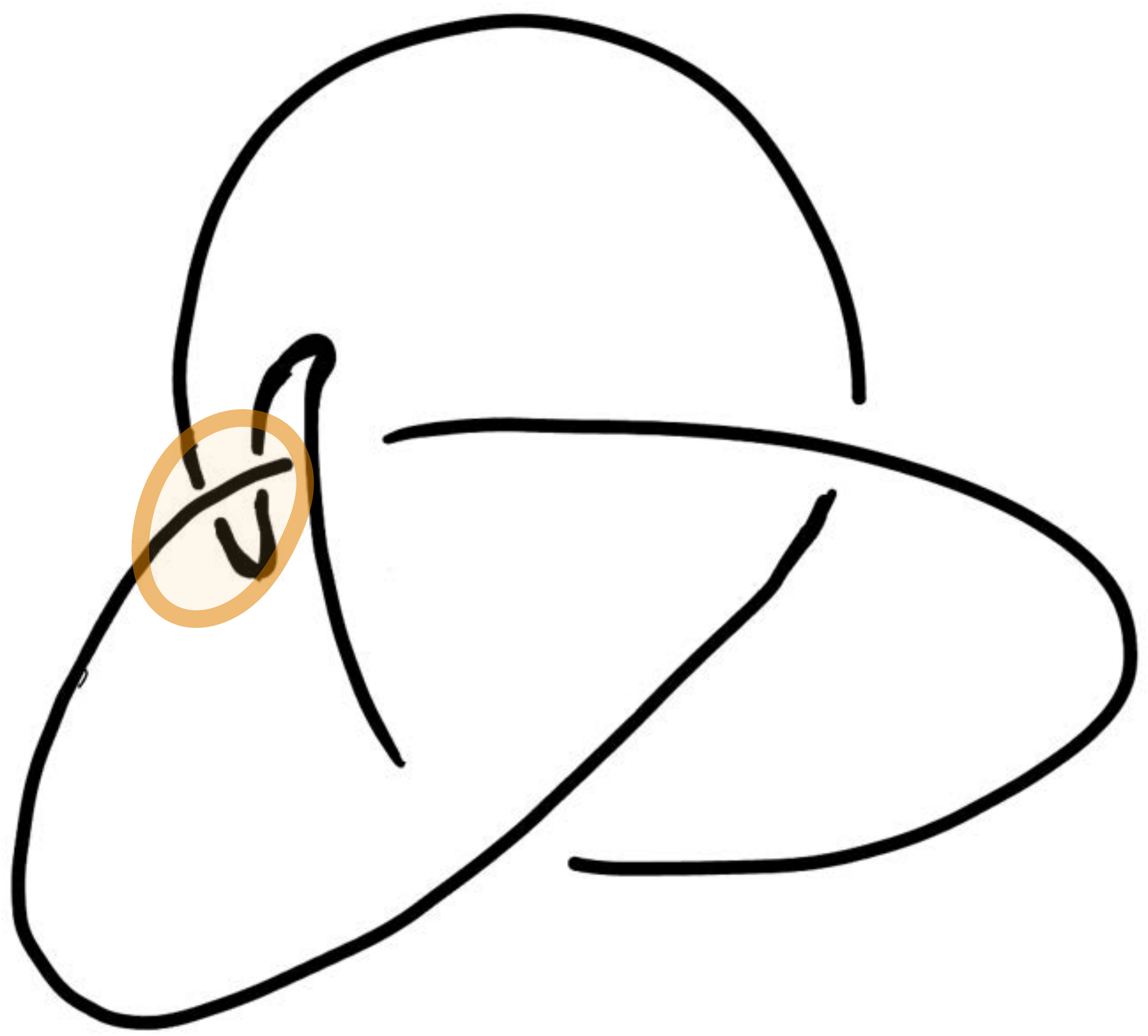

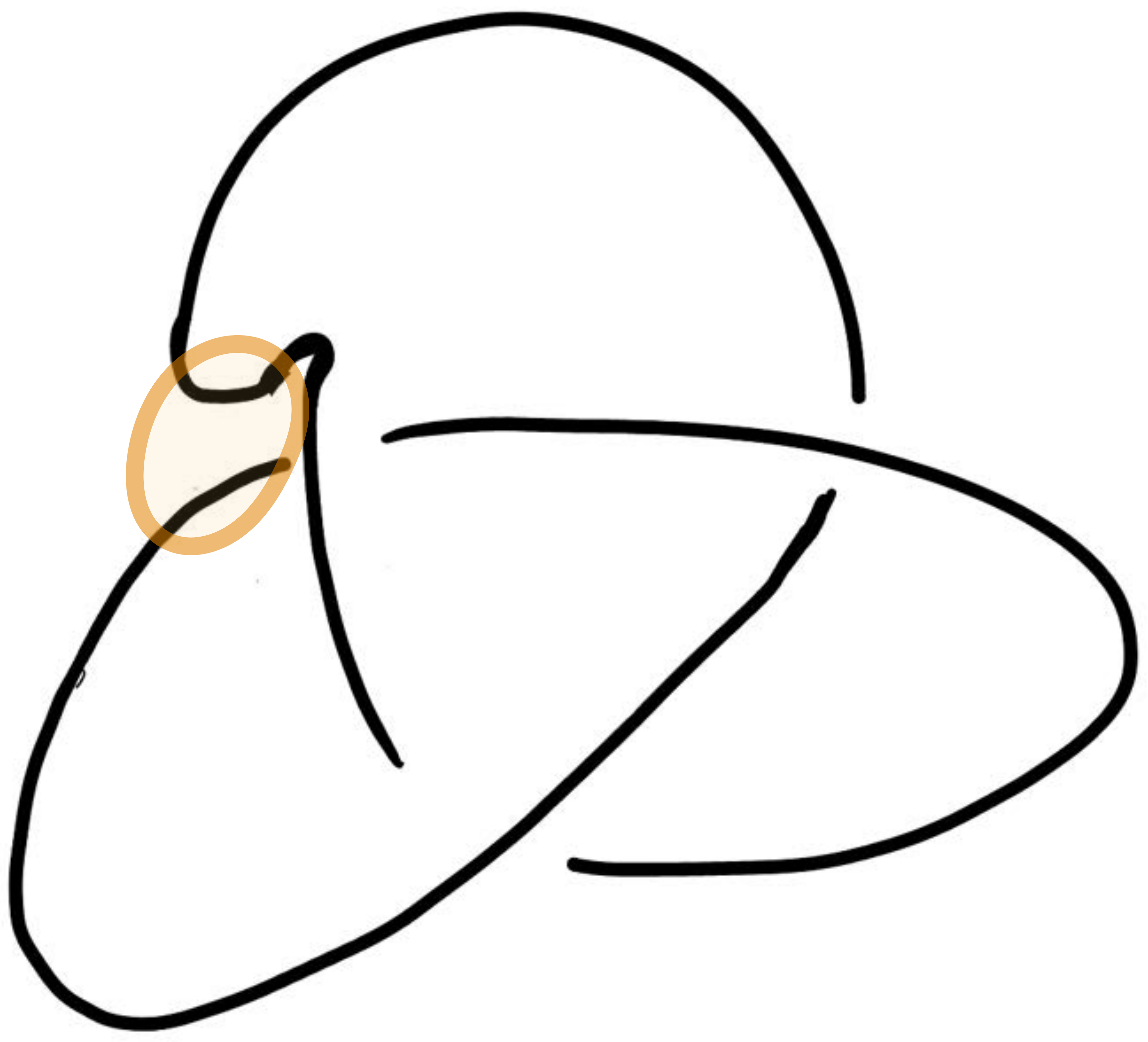

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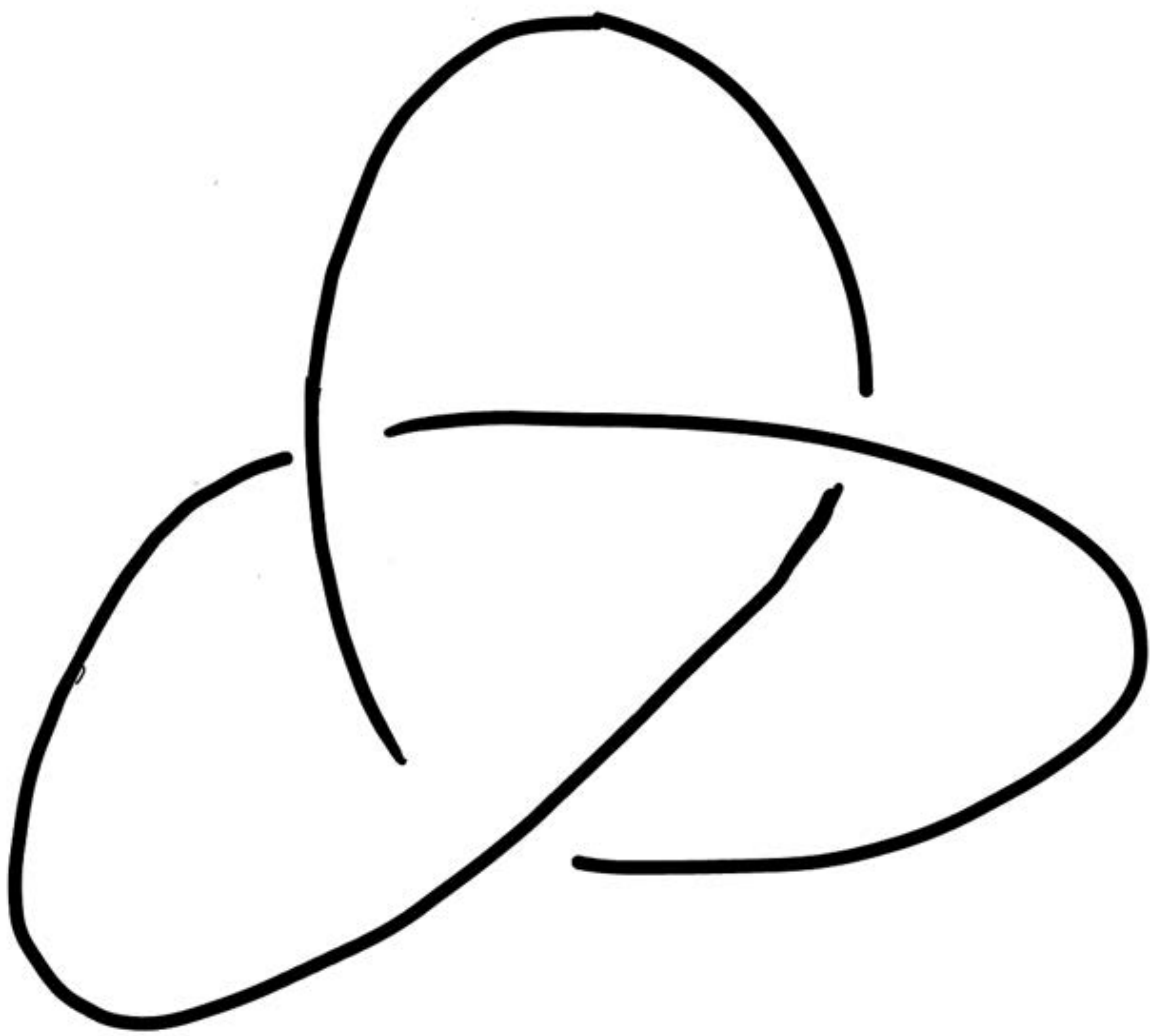
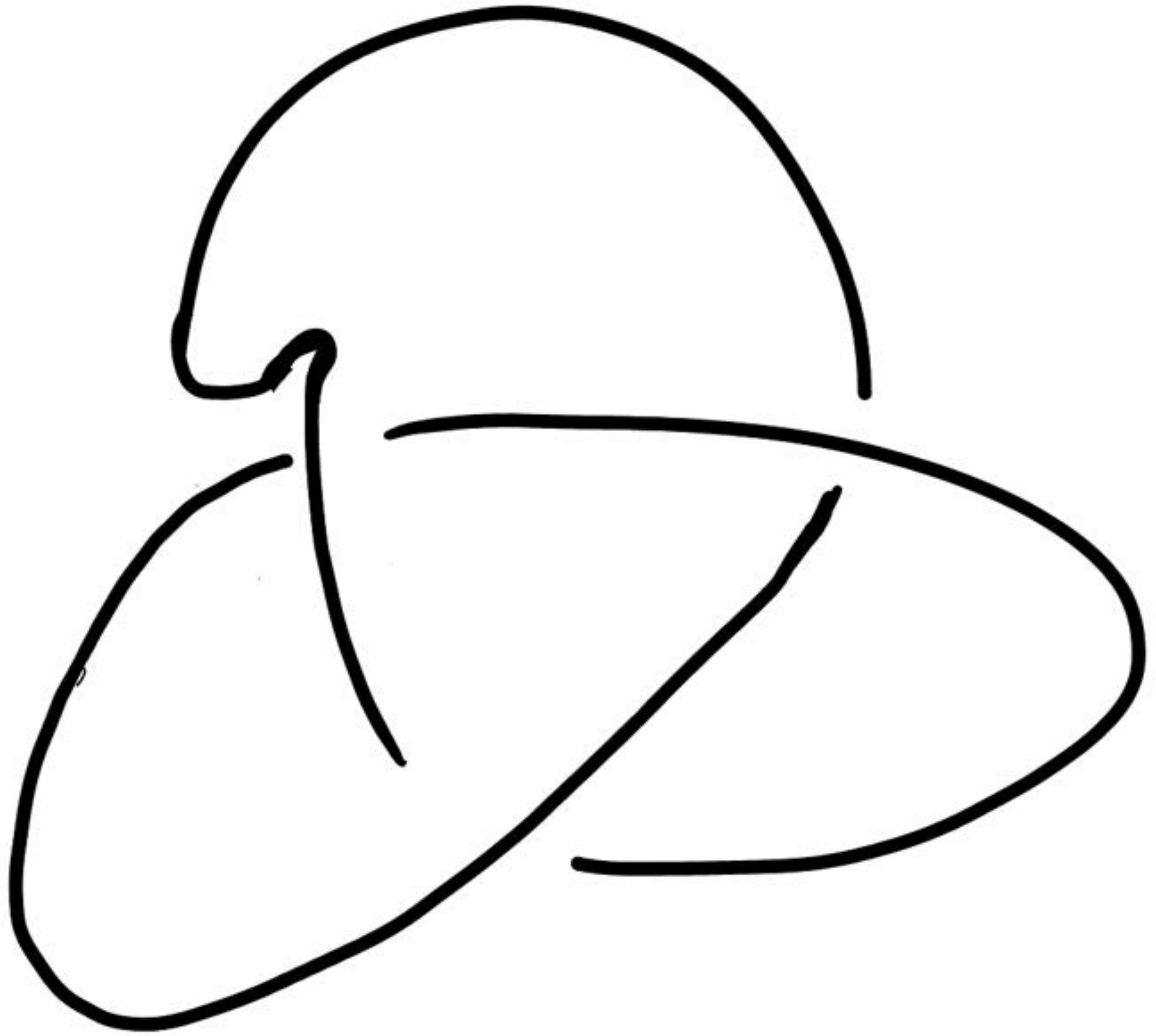
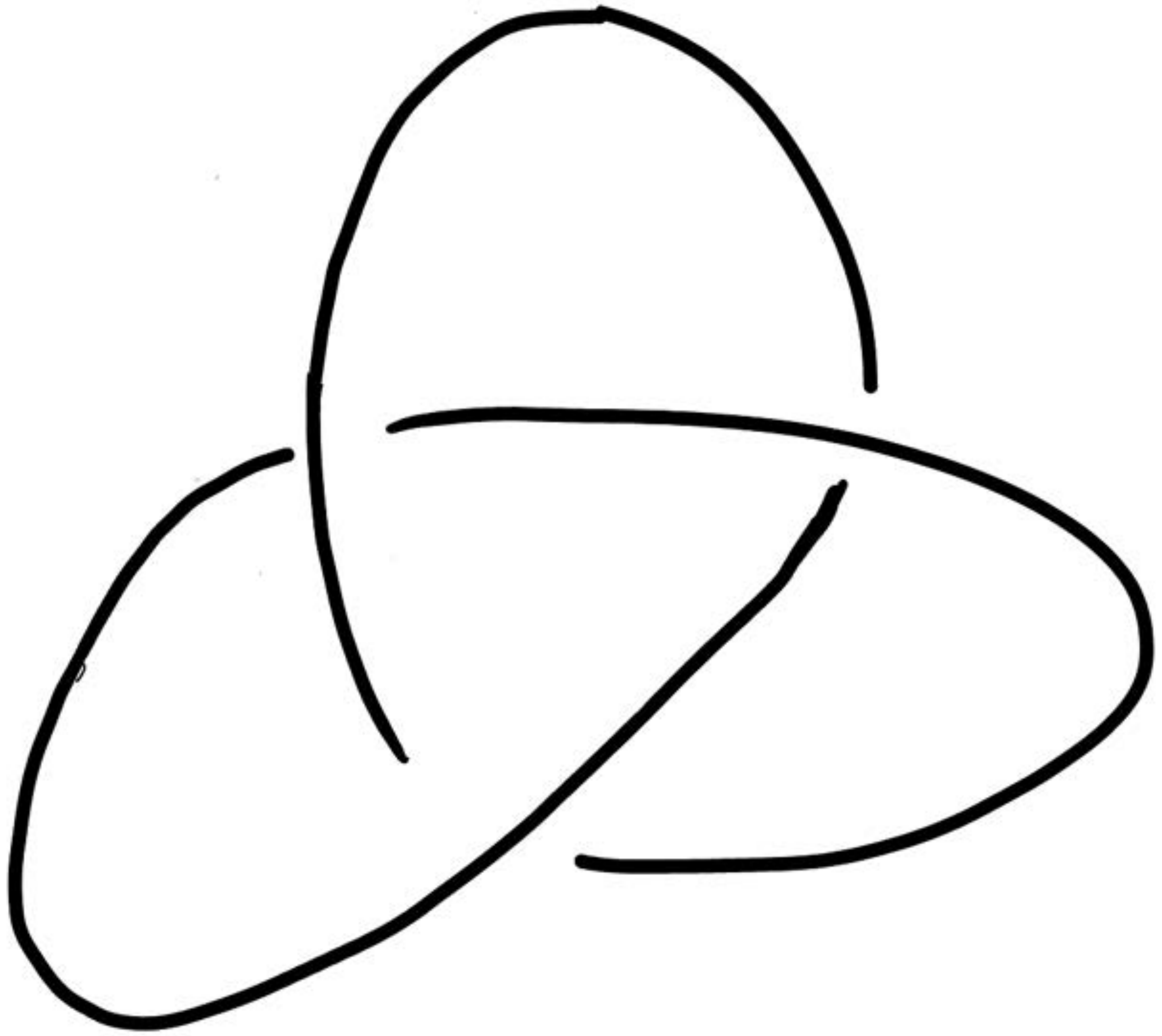




How do we show the trefoil and the circle are *different*?



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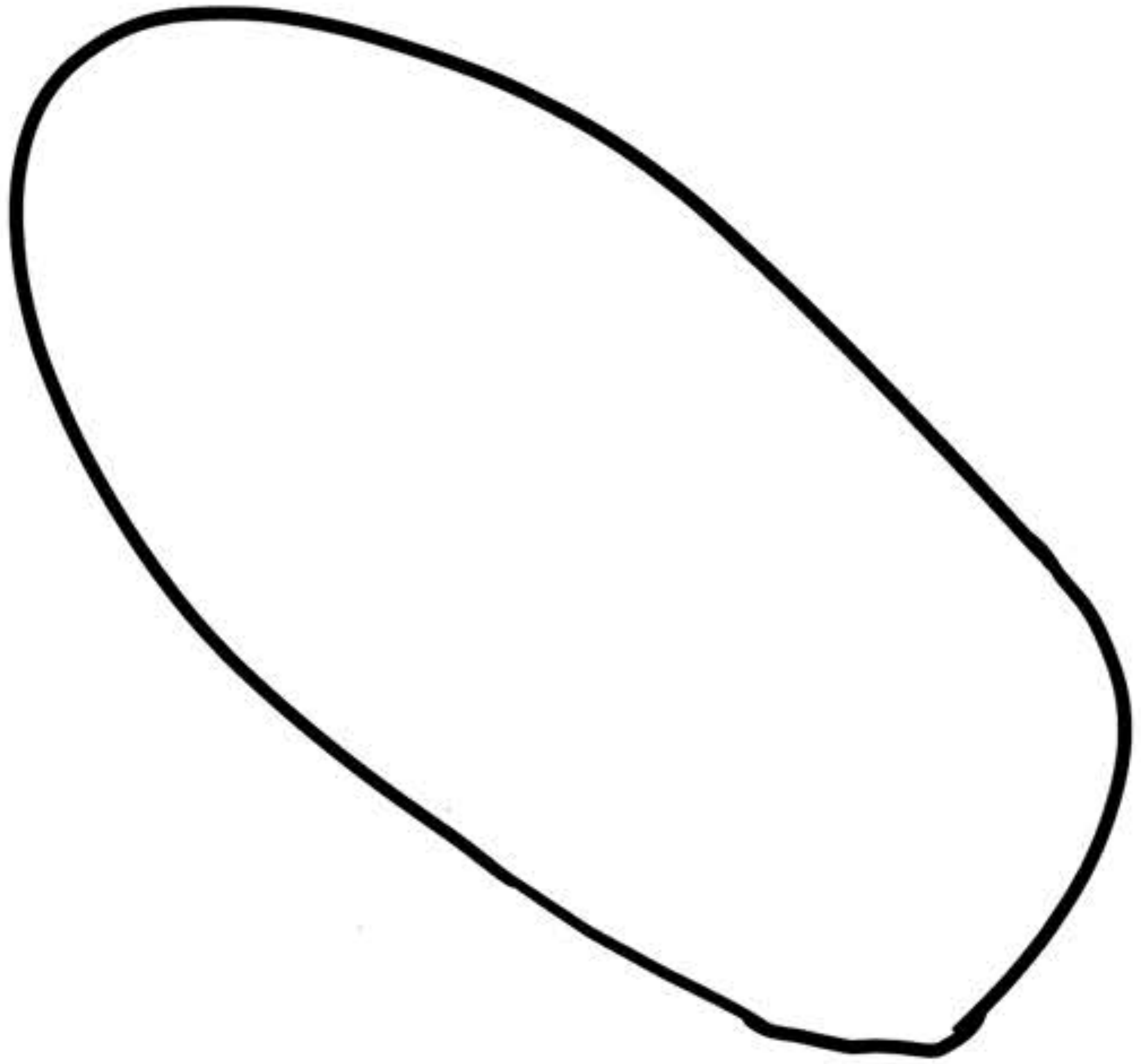
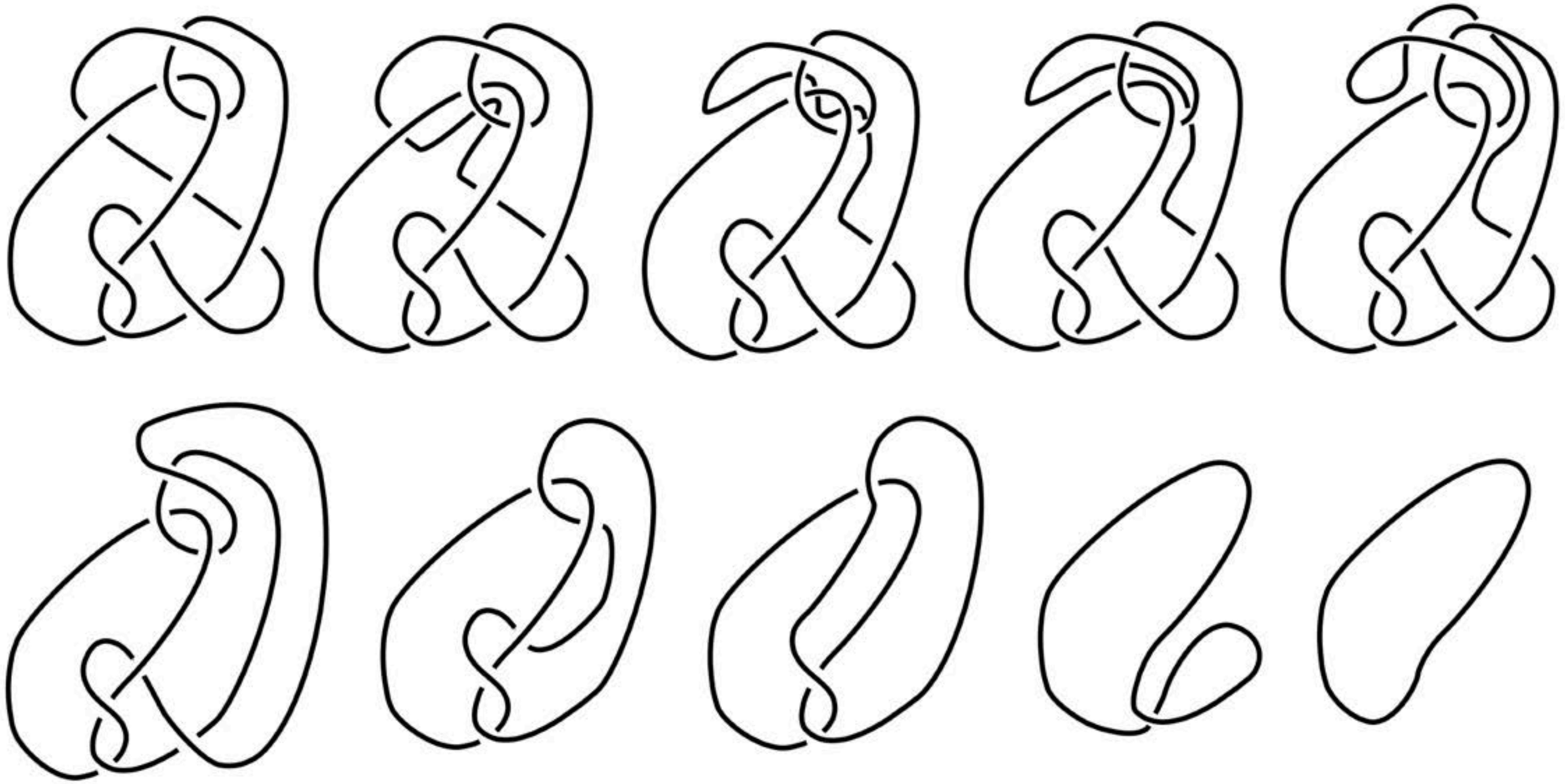


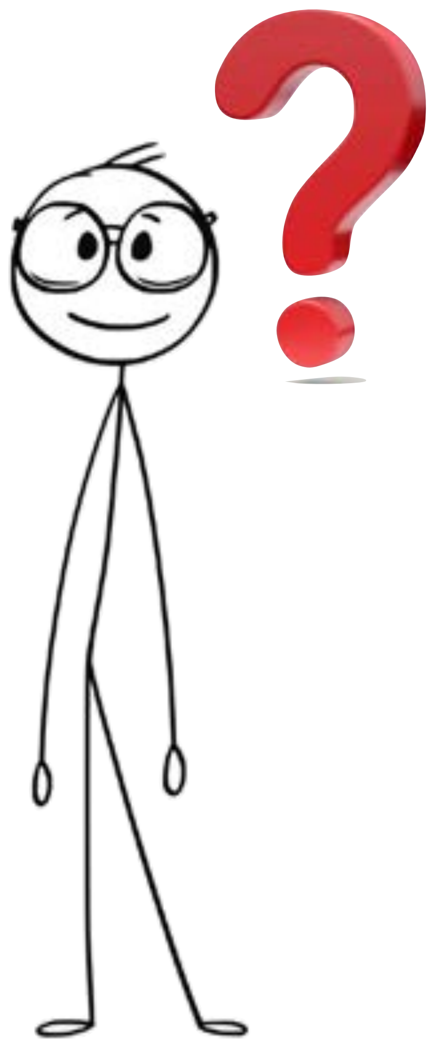
Idea: There is no Reidemeister move that can simplify the trefoil...

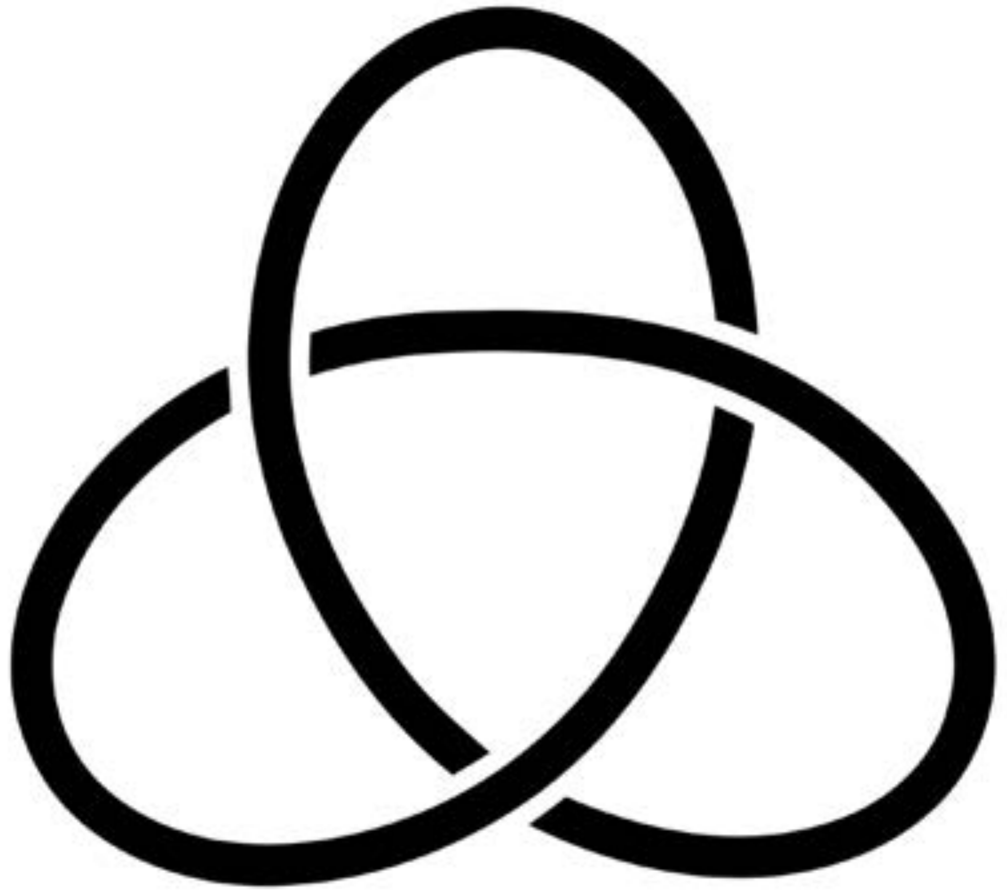
Nope - that doesn’t work

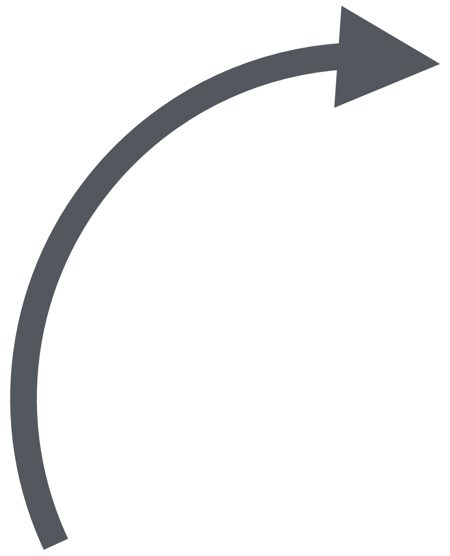
Unknotting
“The Culprit”

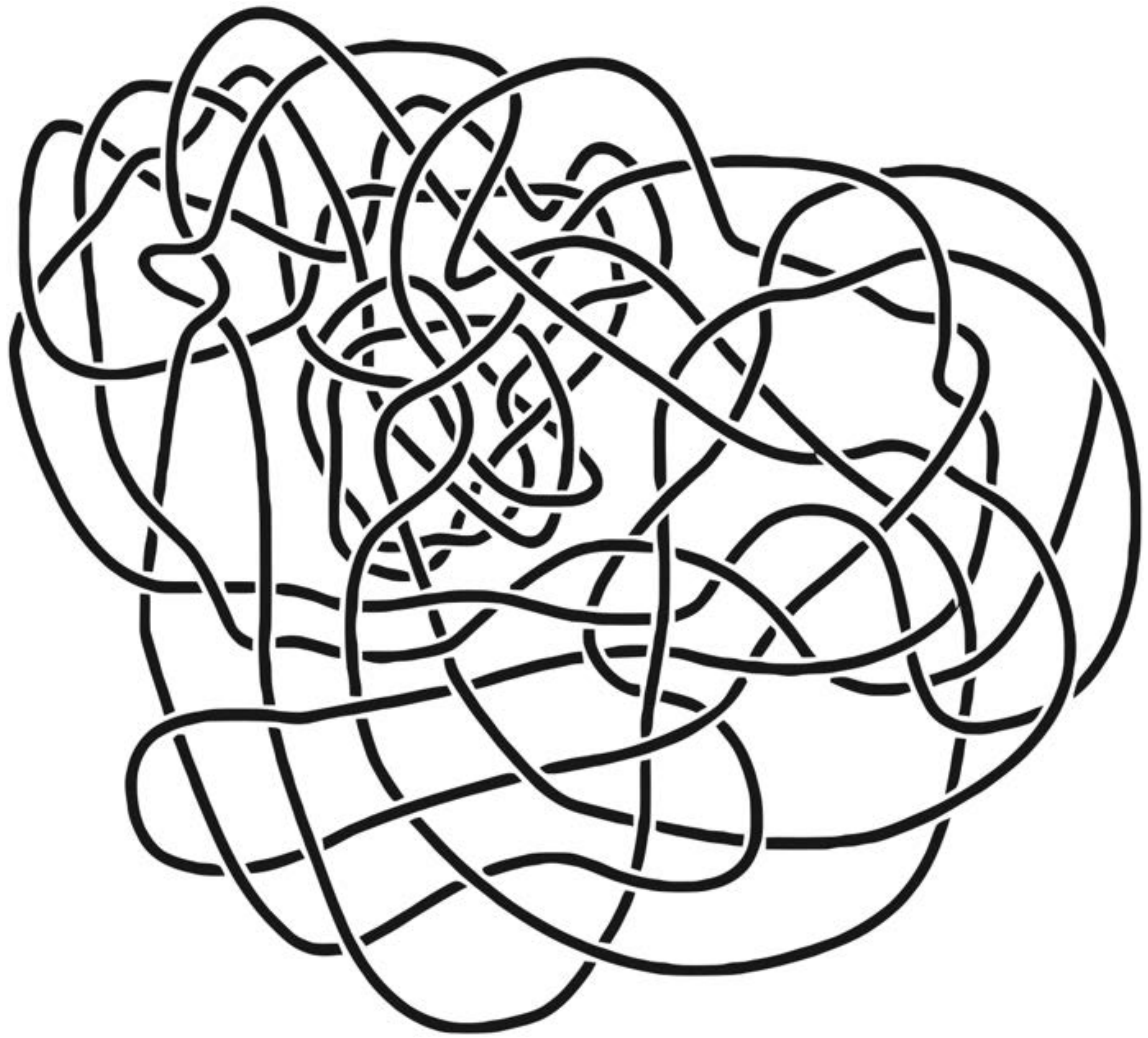
A minimal diagram, that needs to get more complicated before it simplifies



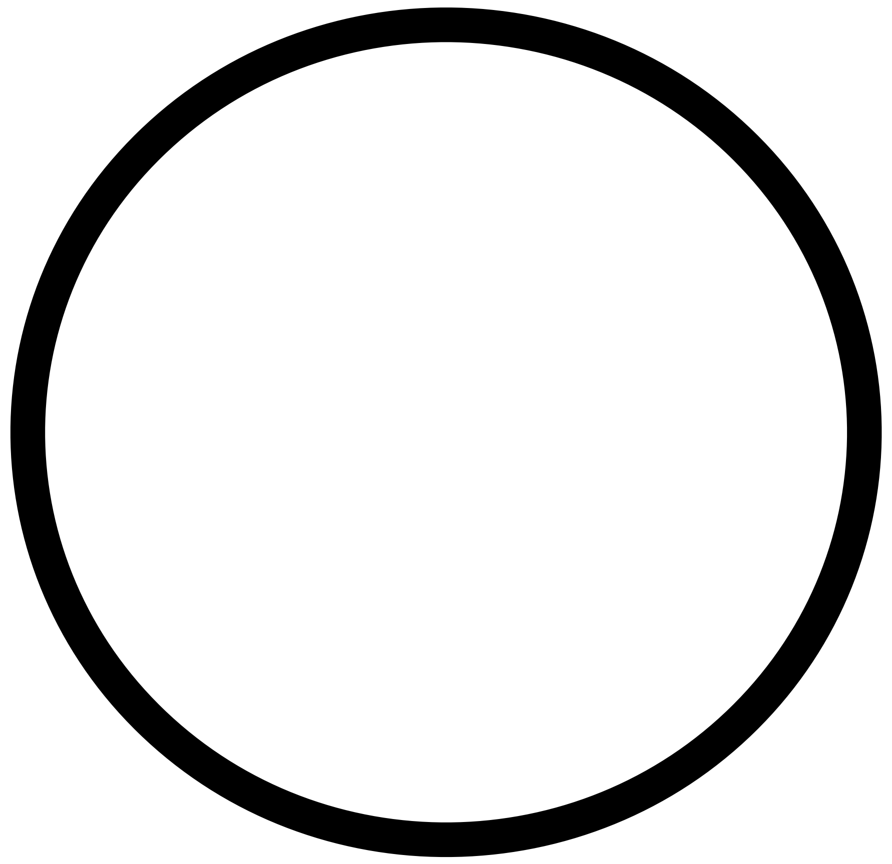


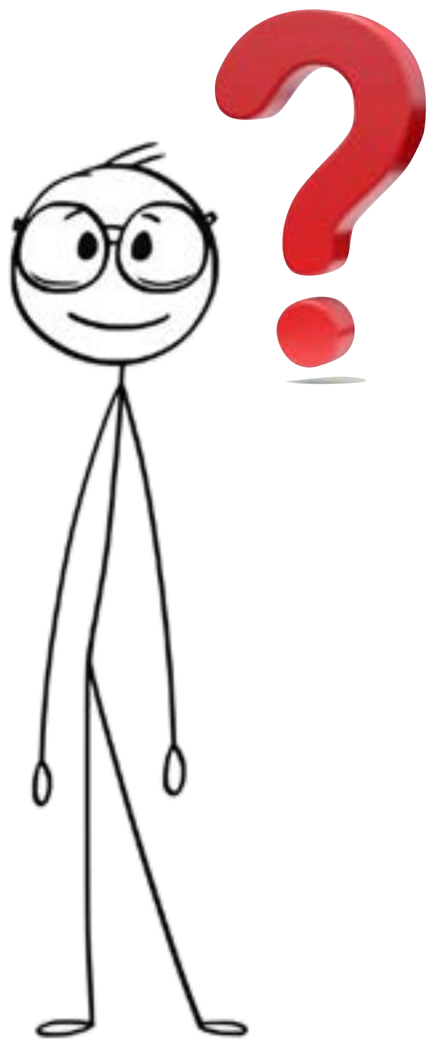


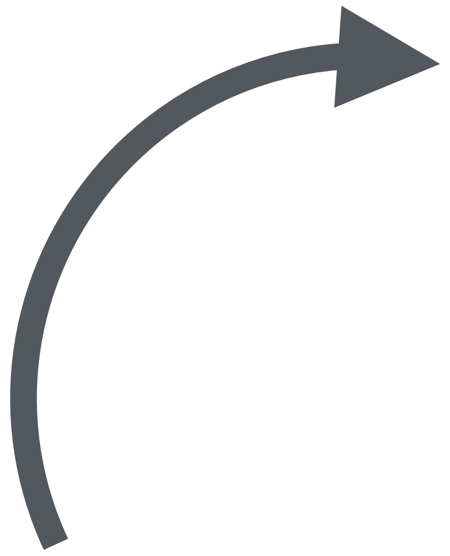


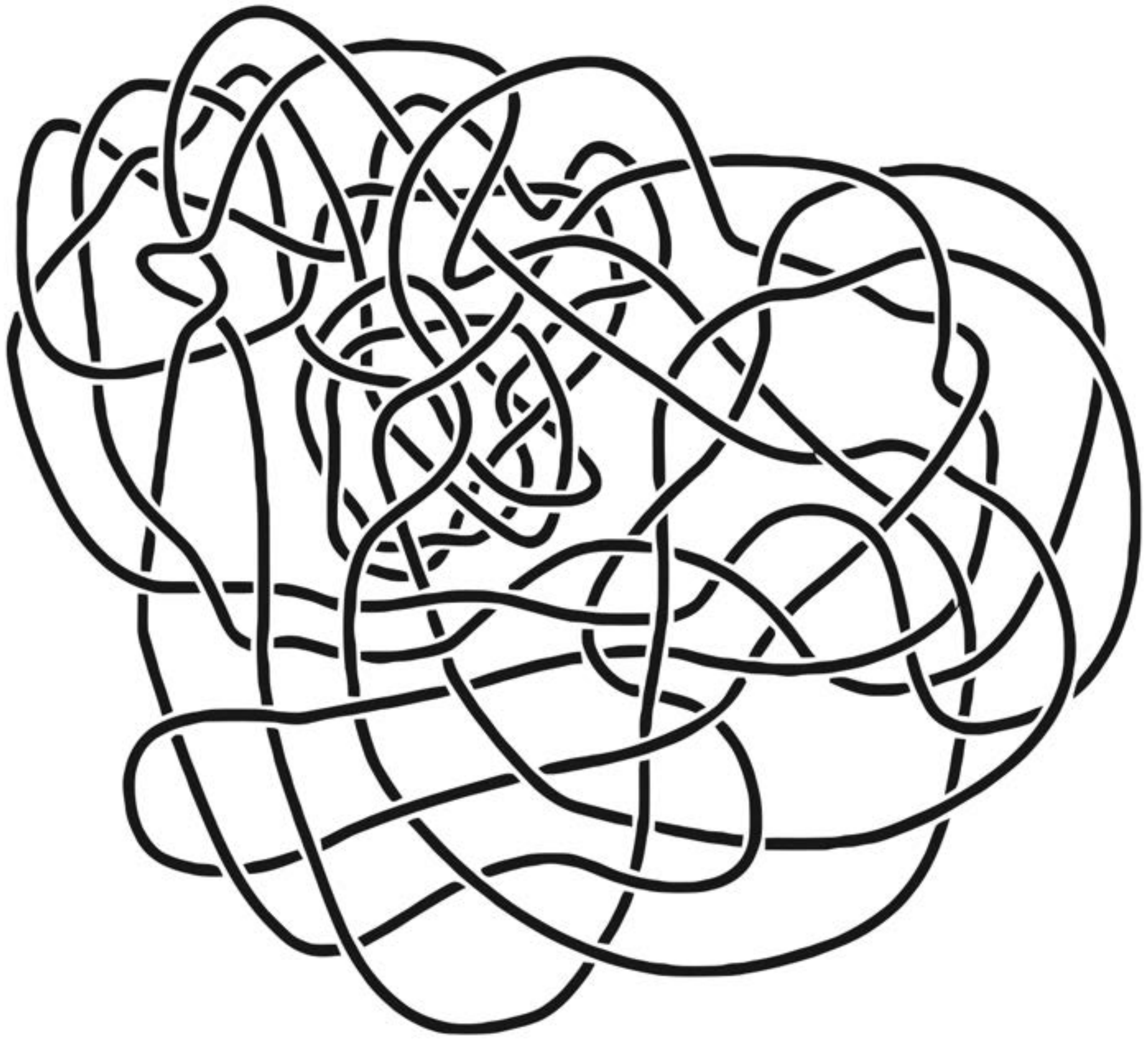


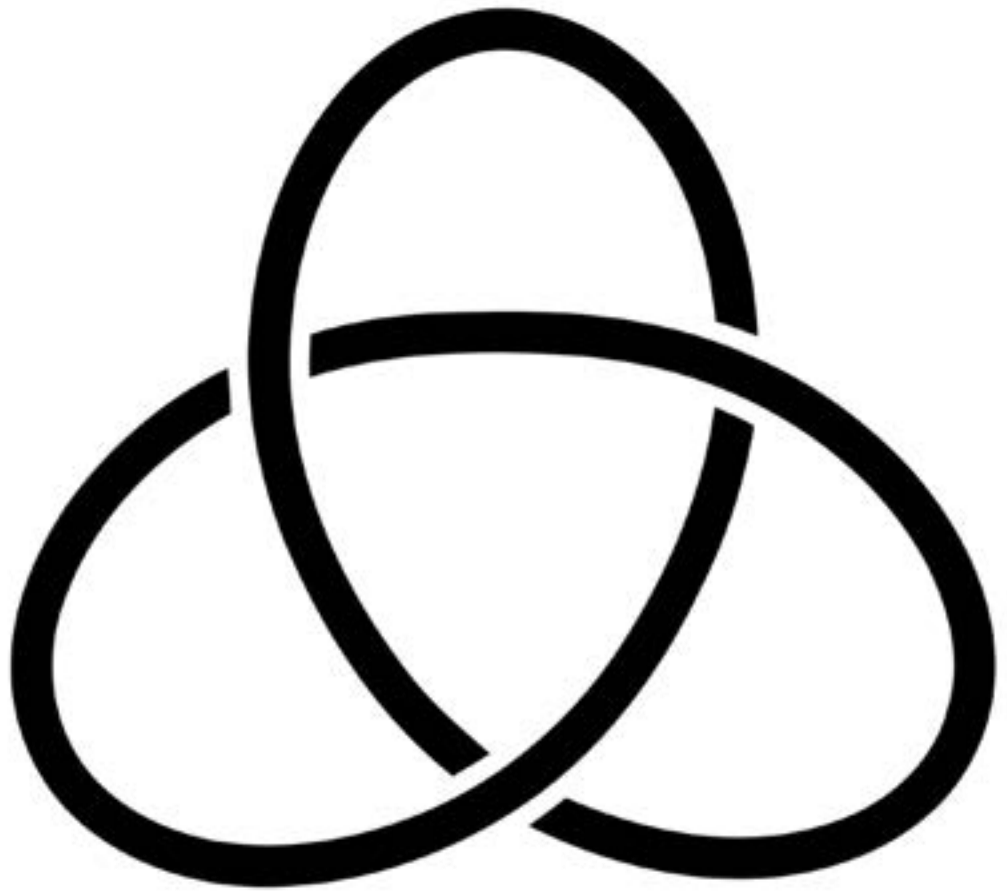
Who said we can’t unknot it by making it more complicated first?



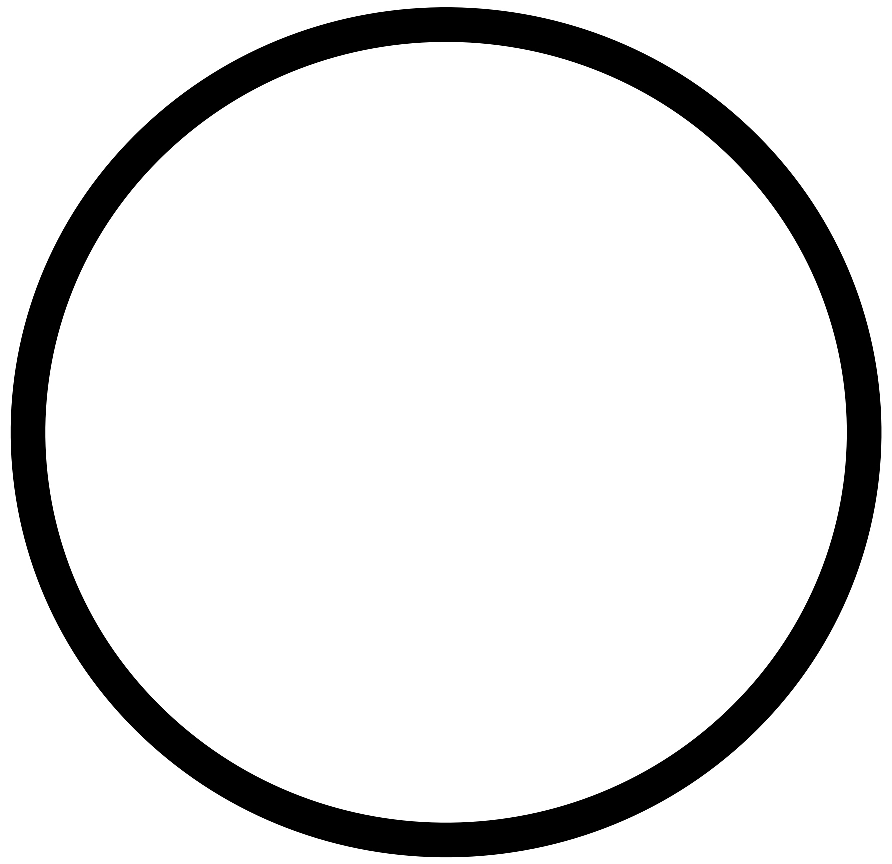








Proving the existence of a knot means proving the **nonexistence of any such complicated thing!**



Knot Invariants

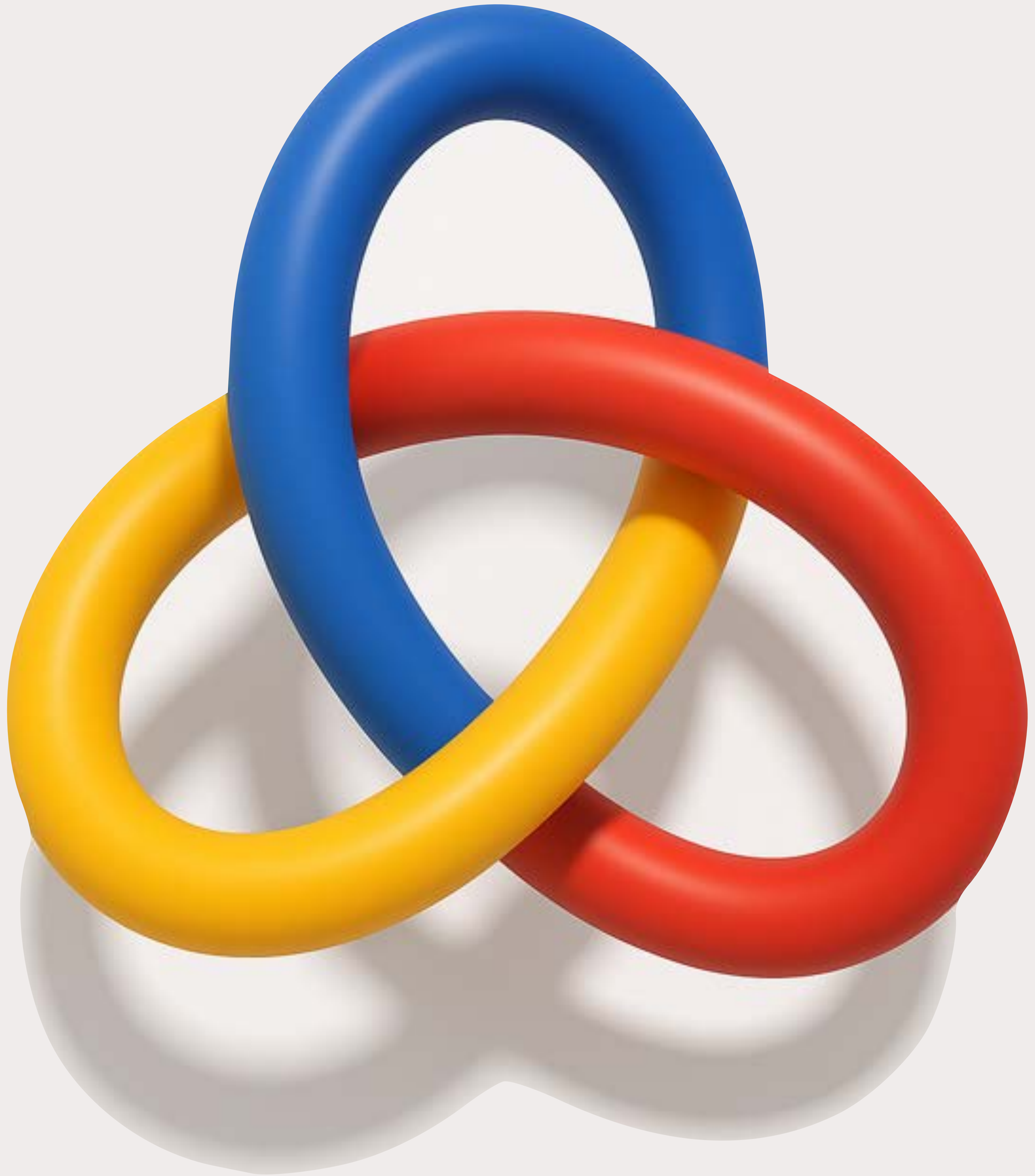
Assign a quantity to each knot (diagram) that is invariant under ambient isotope.

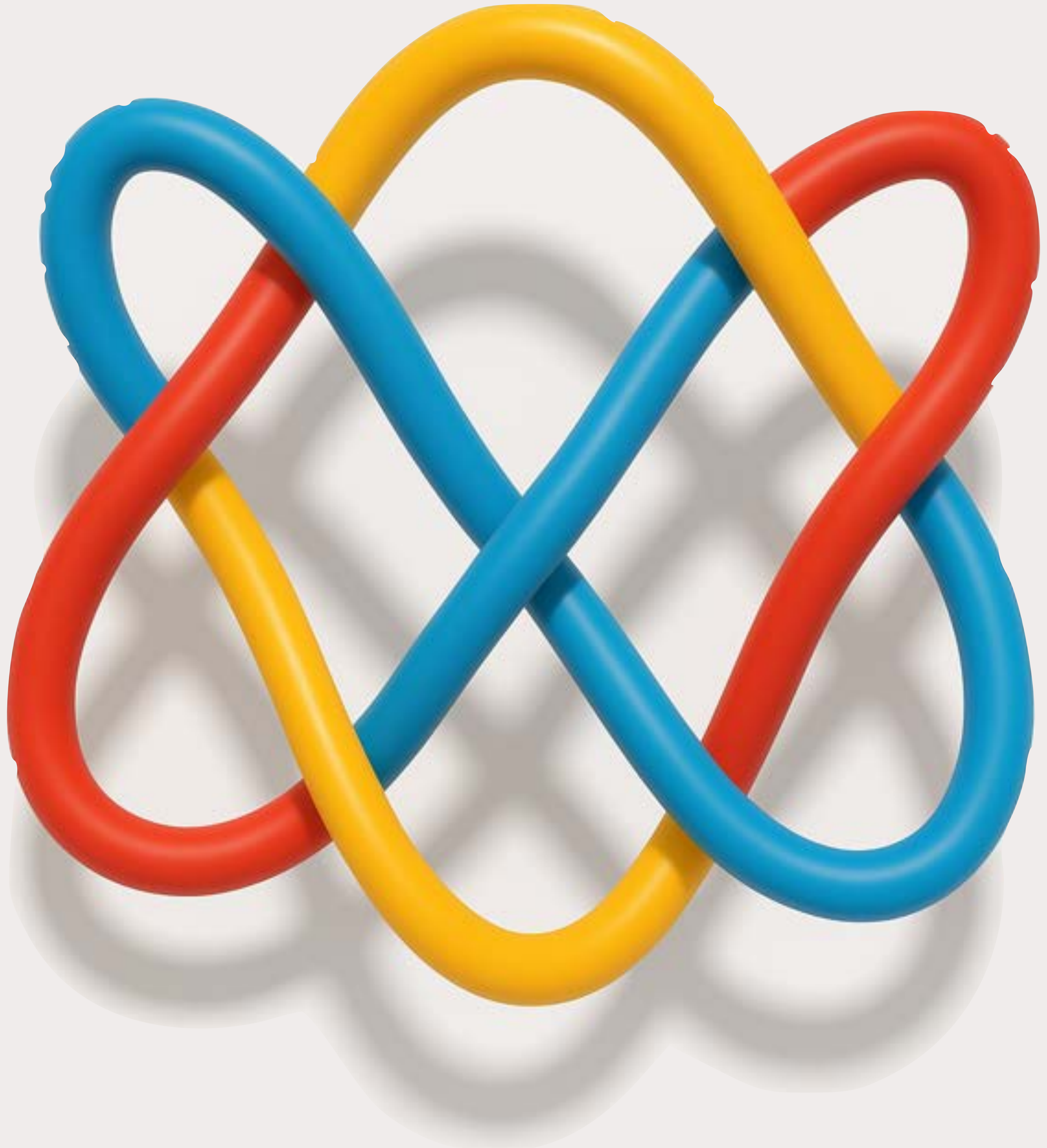
If two things are assigned different quantities, then there is *no* isotopy connecting them: they are different!

Its enough to check **invariance under the Reidemiester moves!**

Example: Tricolorability

Color a knot diagram using exactly 3 colors for strands . Each crossing has all equal, or all distinct





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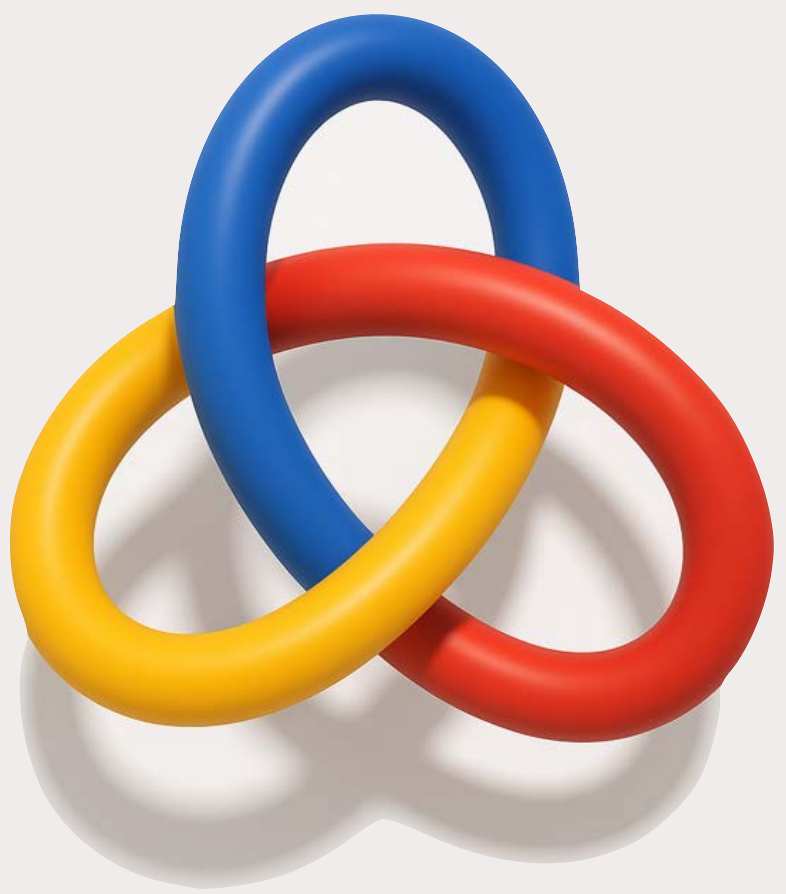
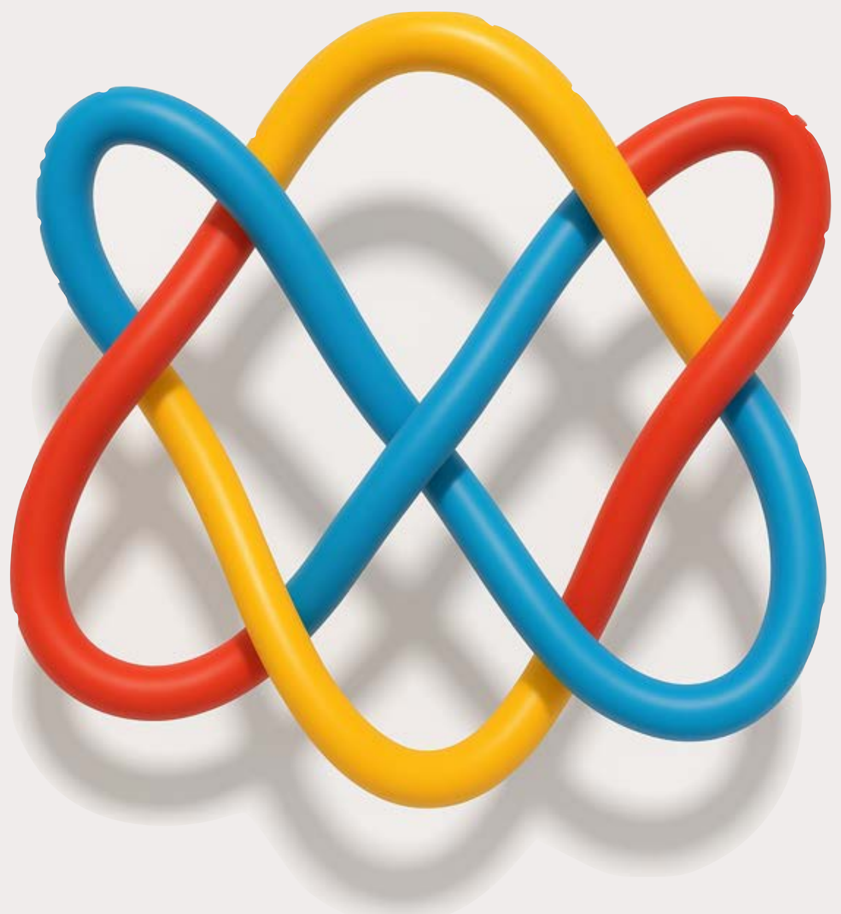
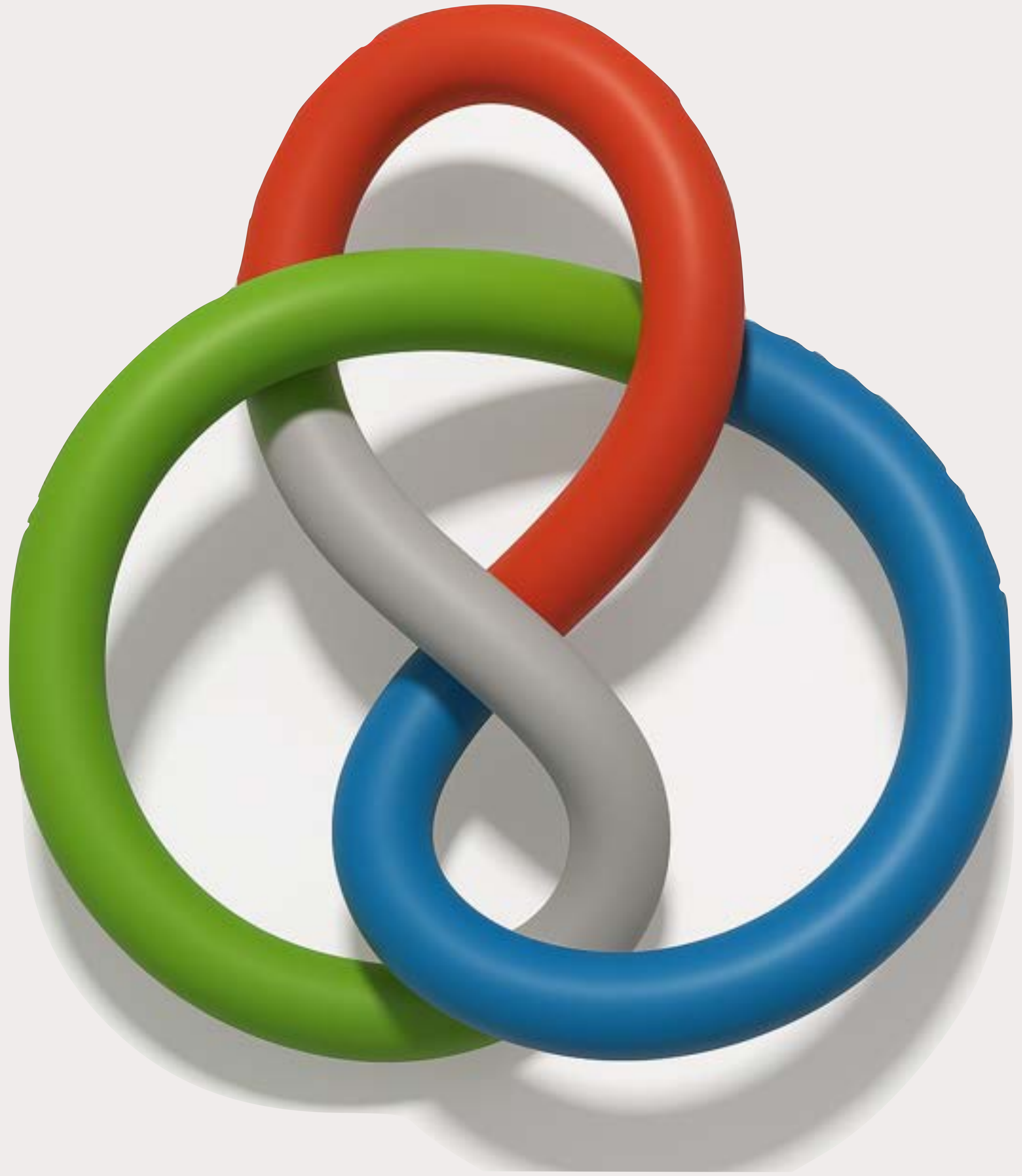




Nonexamples

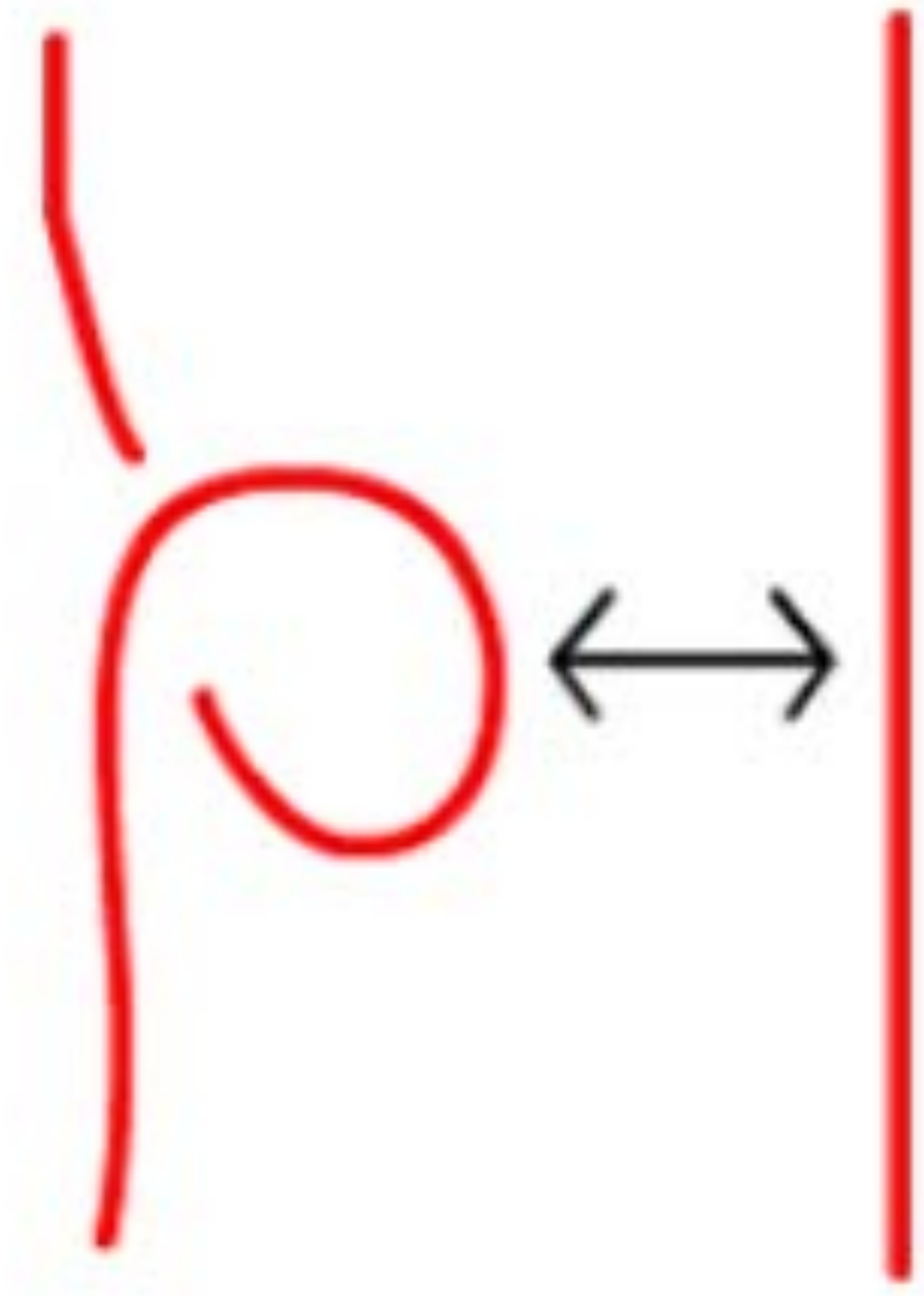


Theorem: Tricolorability

Being tricolorable is an invariant of knots.

Proof

Assume you have a 3 colored knot. Check it remains tricolored after doing a Reidemeister move locally, to any portion of it.



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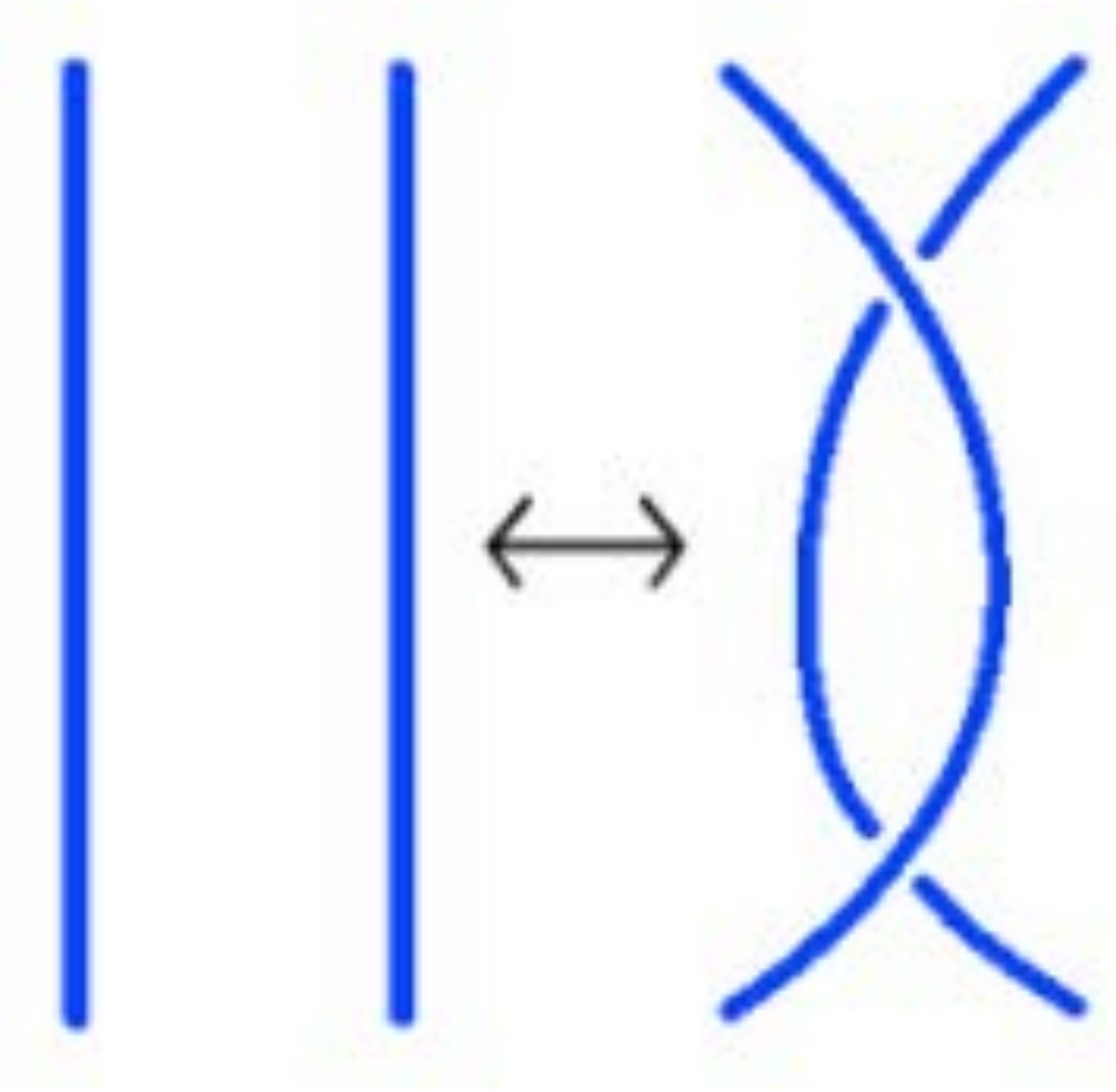

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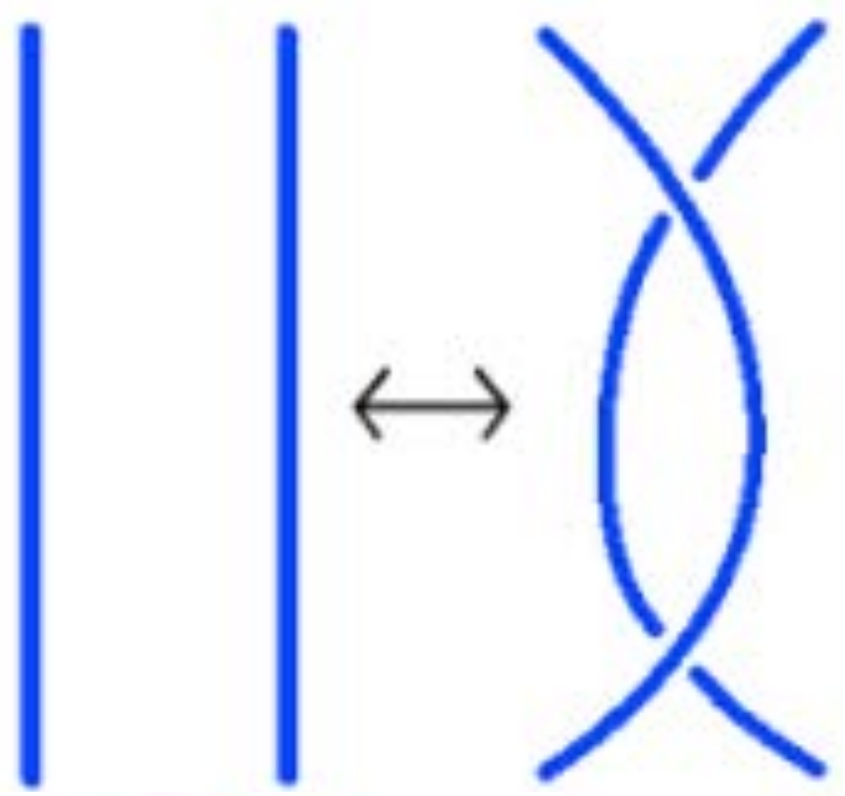

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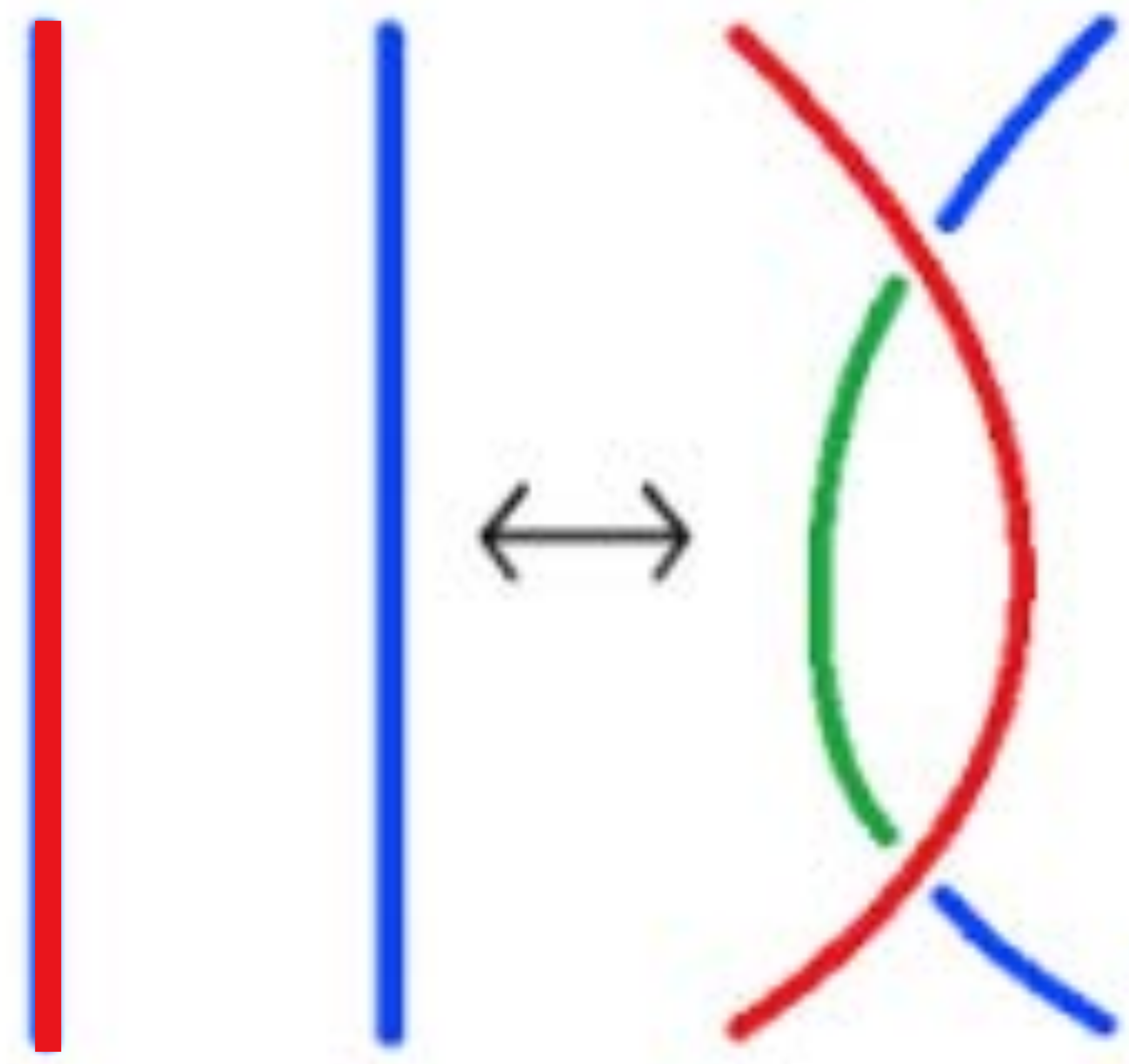

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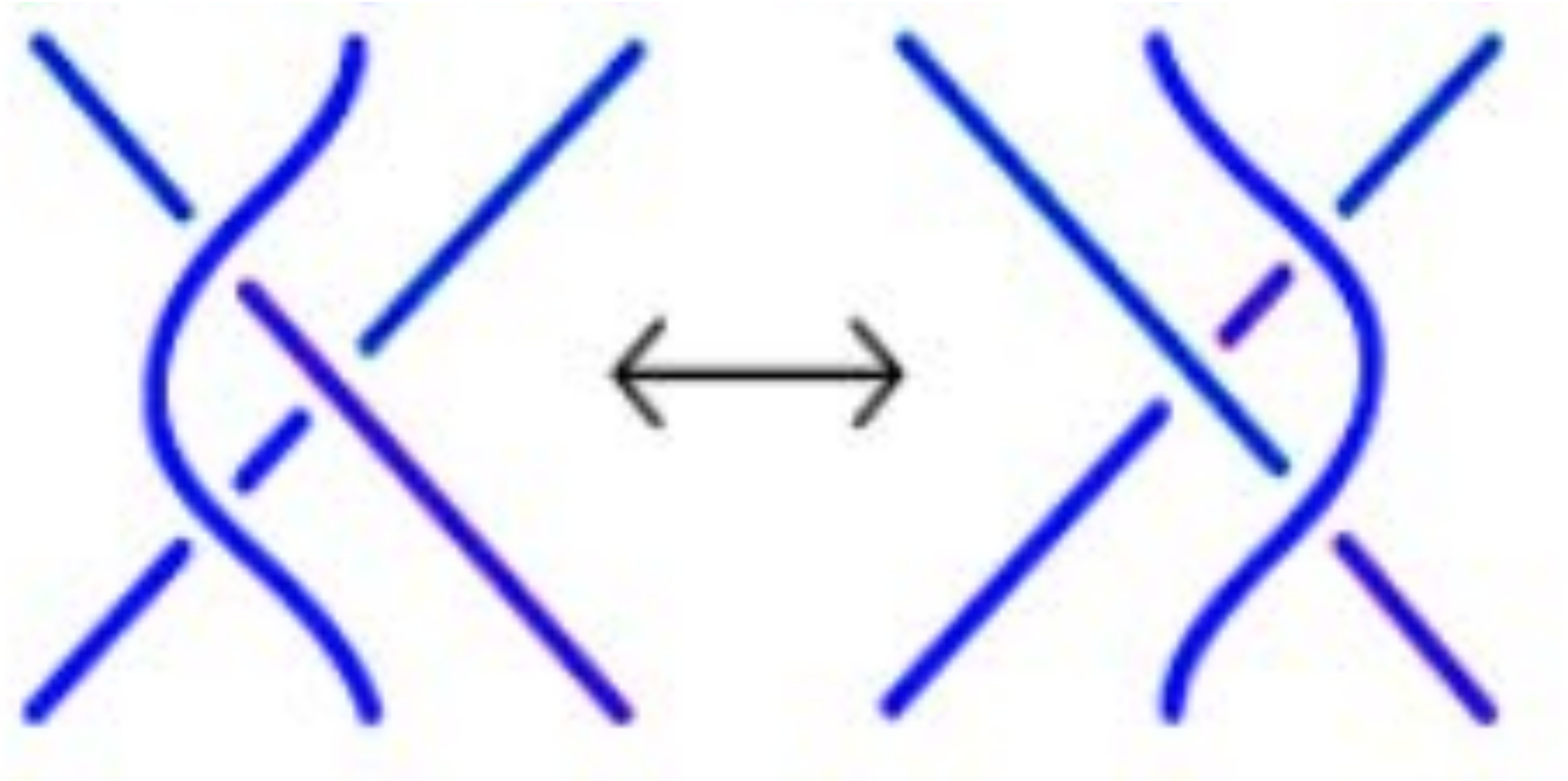

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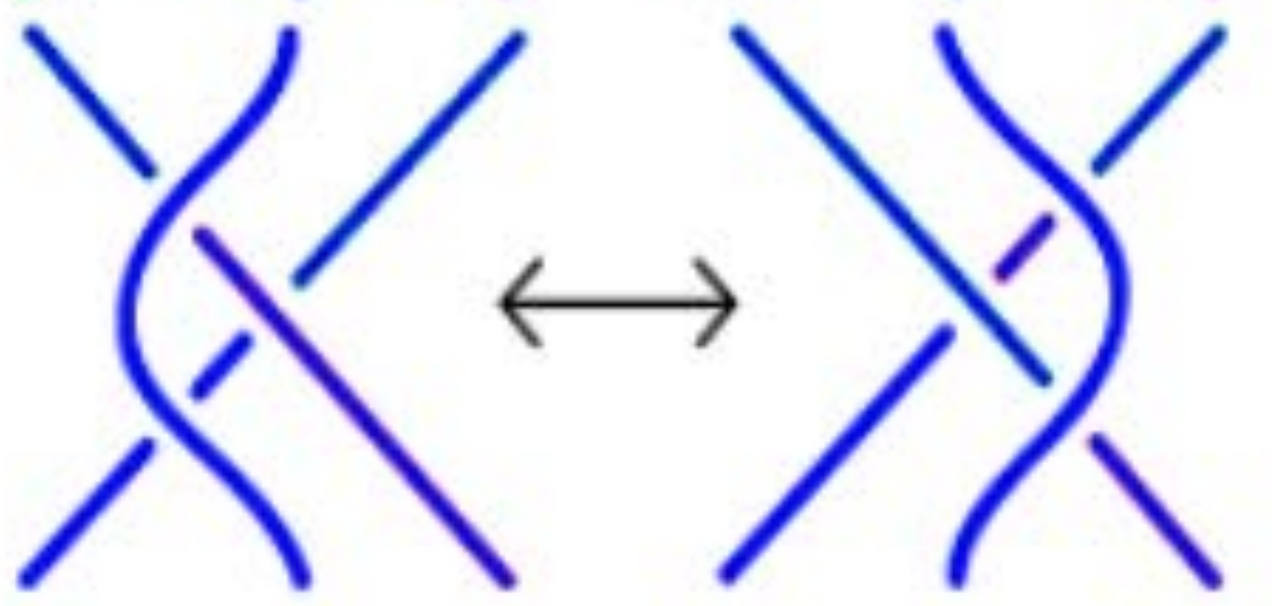

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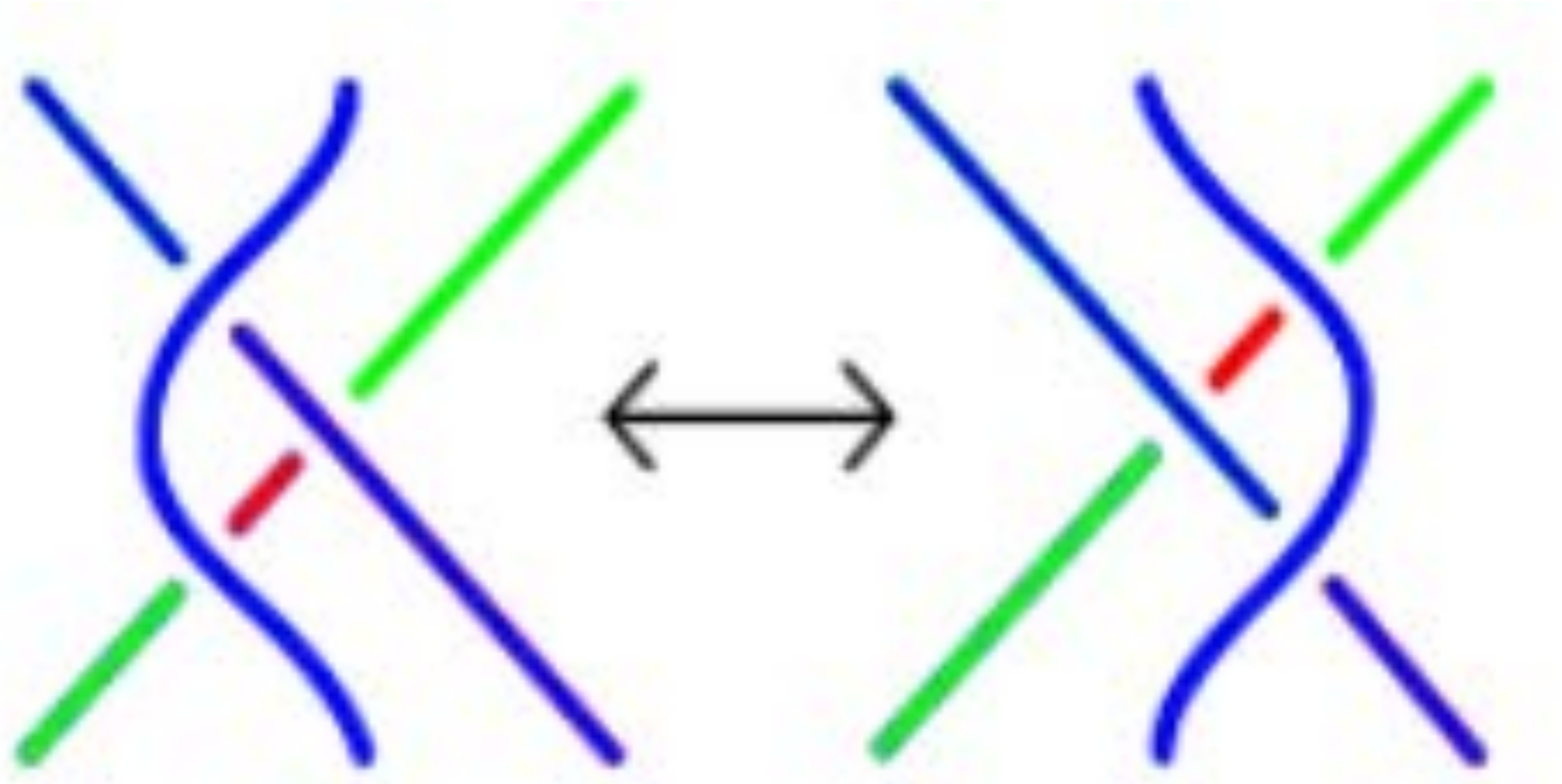
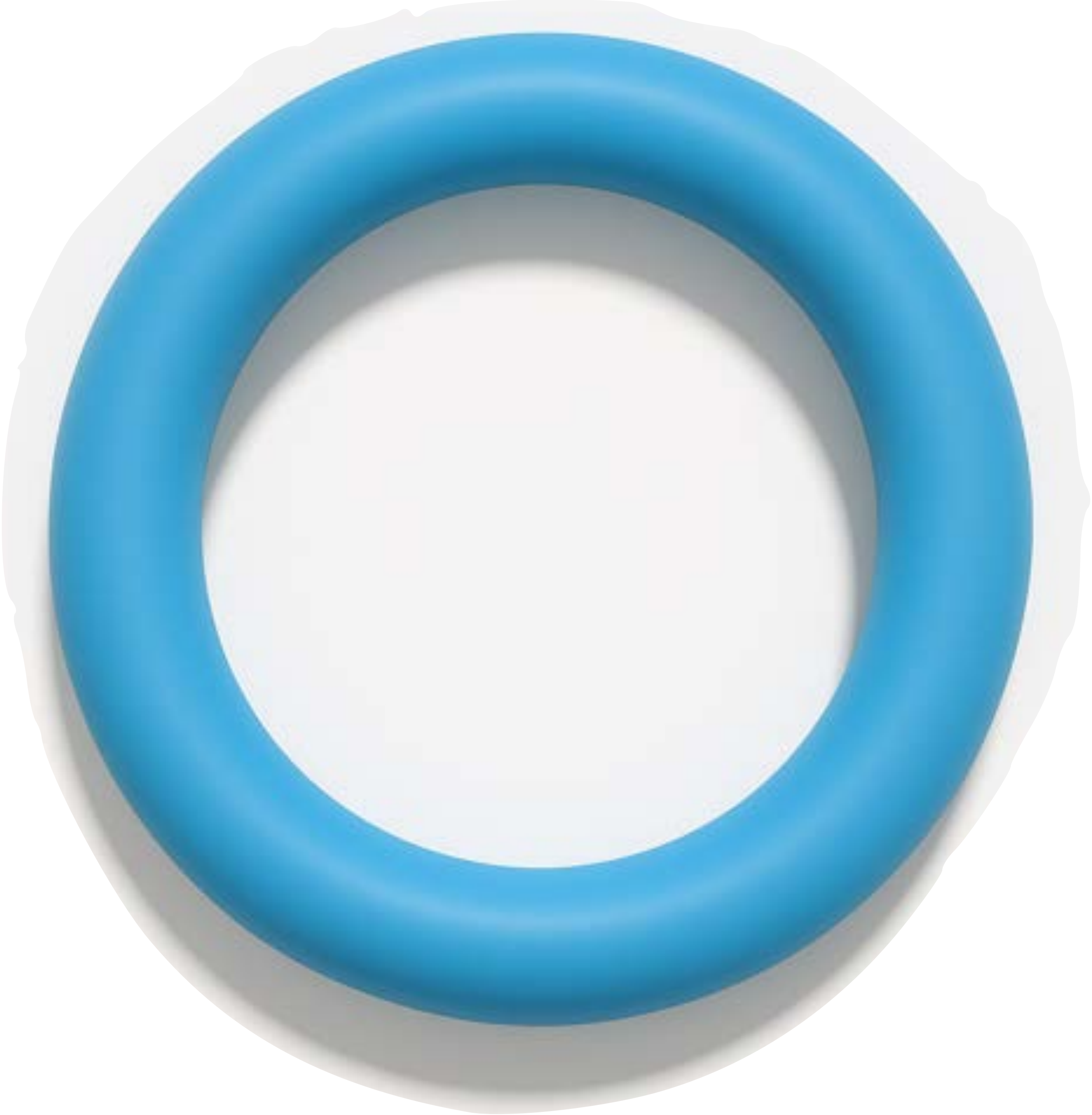


Exercise: find all other cases

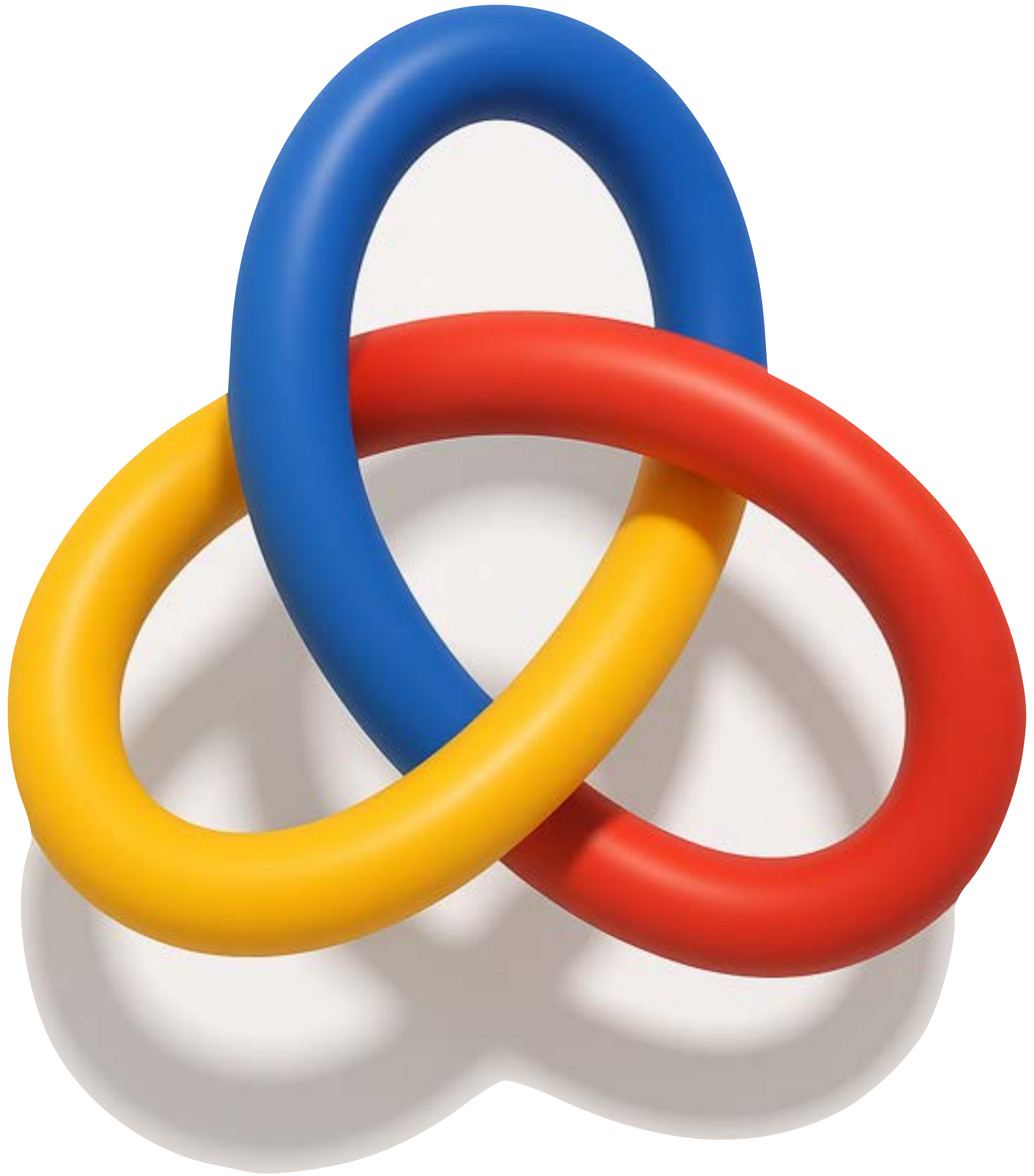
Theorem: Tricolorability

Being tricolorable is an invariant of knots.



Corollary:

Knots exist! The trefoil is different from the unknot.

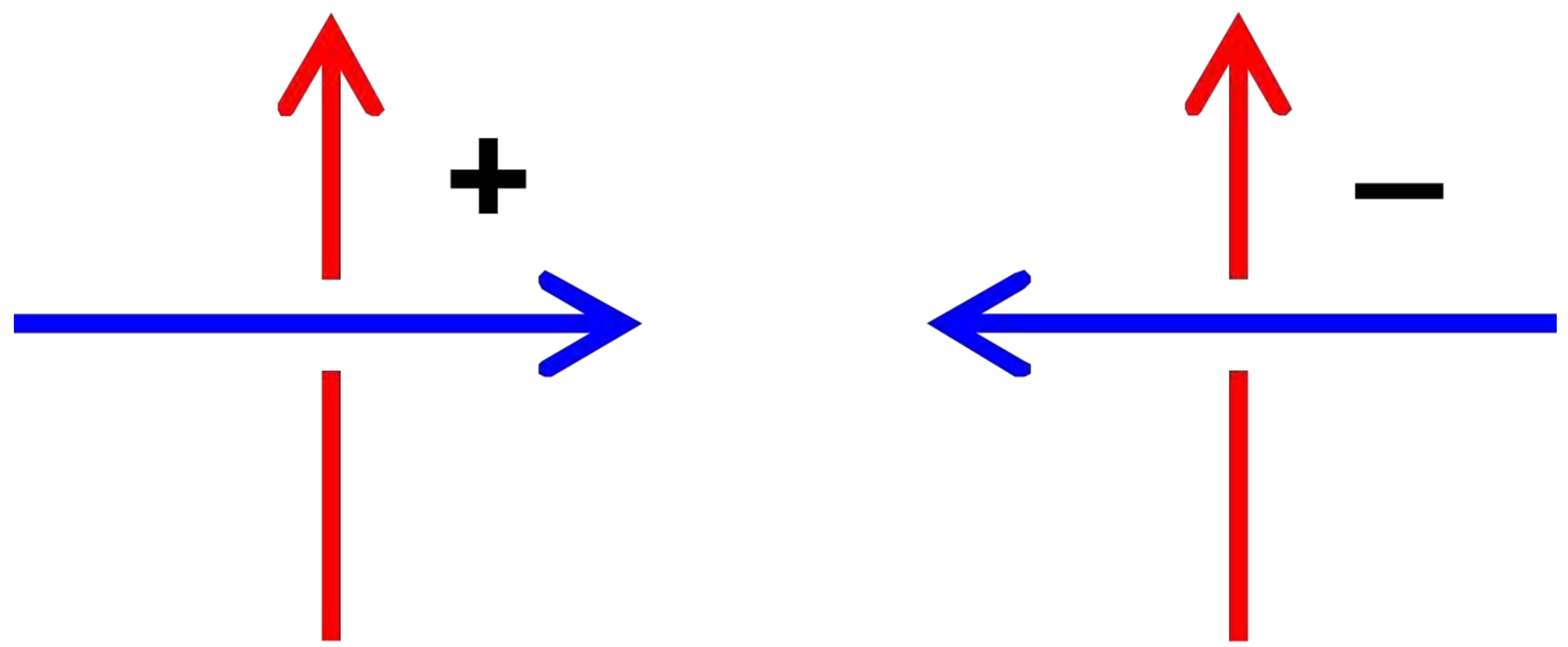


Building More Invariants:

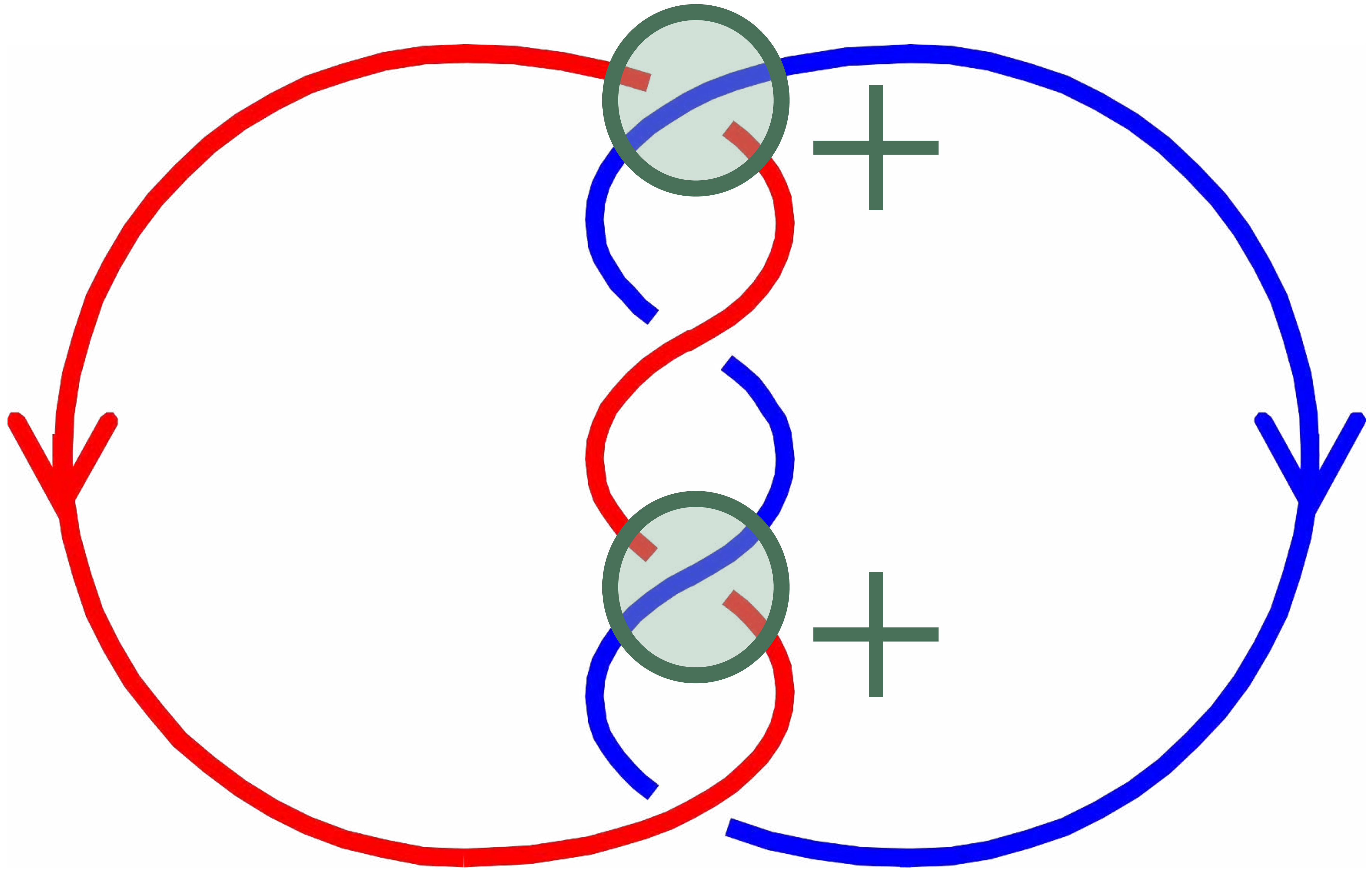
Linking Number

Given two oriented curves:

(1) Assign ± 1 to each overcrossing of the first strand



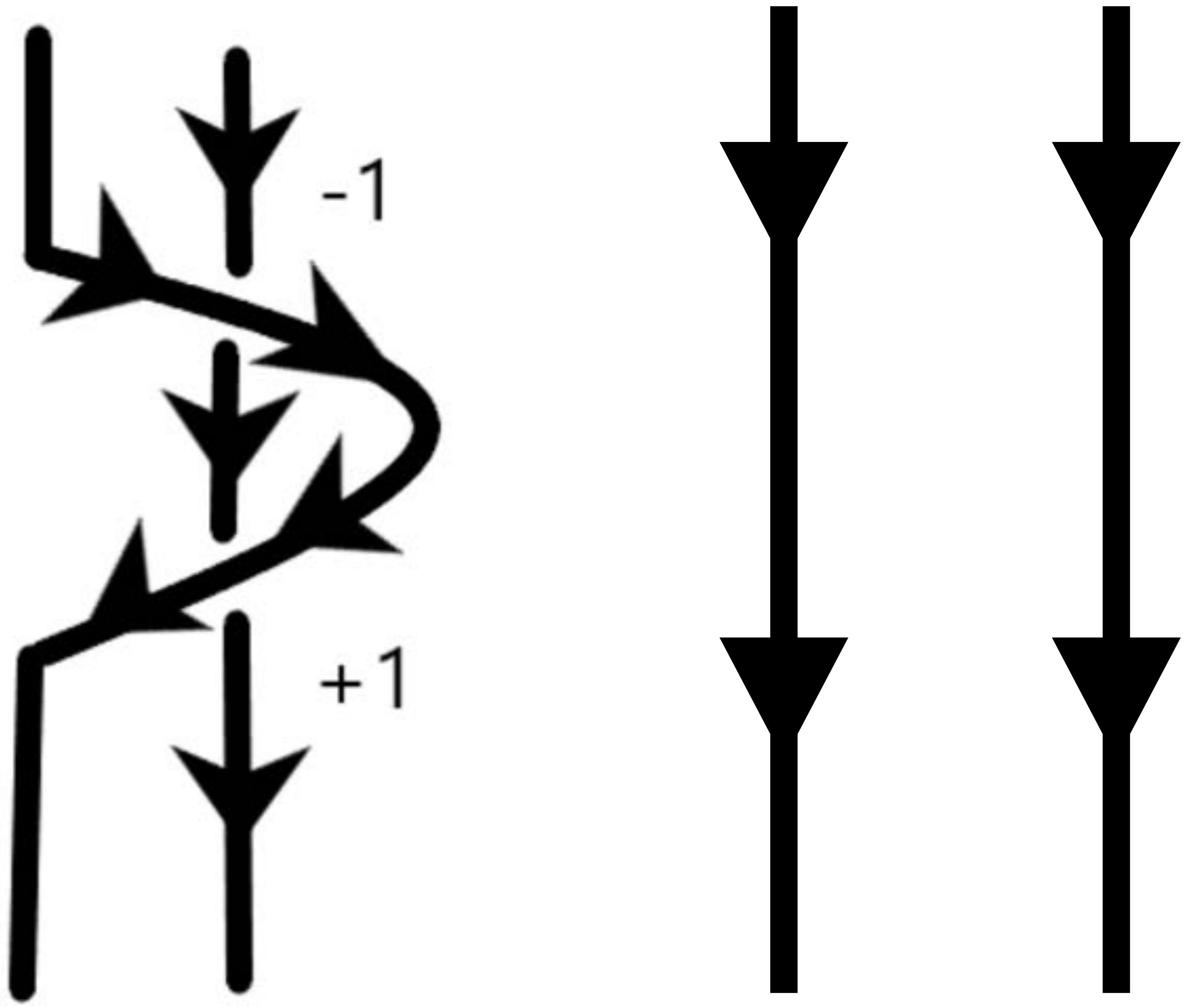
(2) Add up the net result

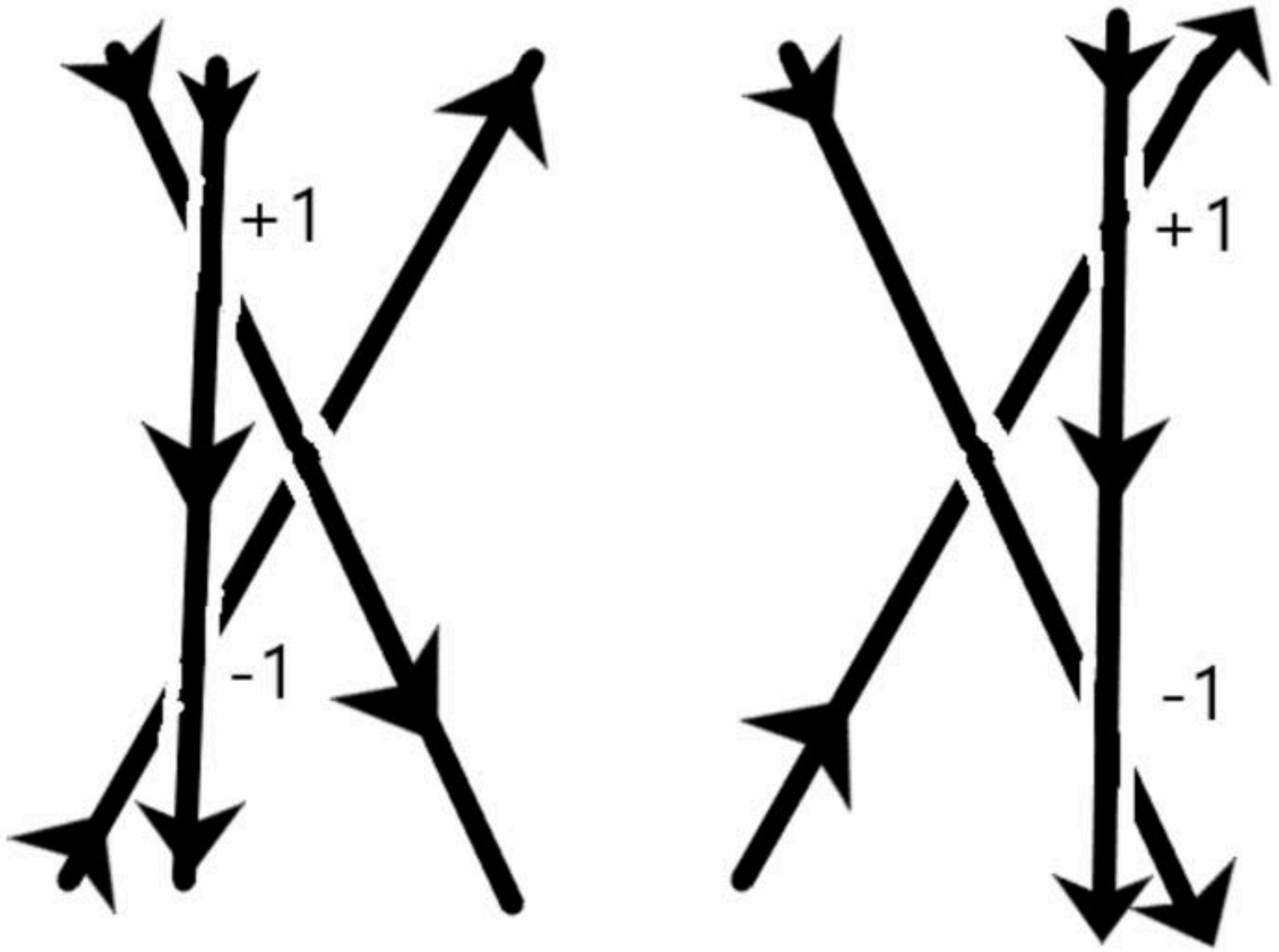


Building More Invariants:

Linking Number

Exercise: this is invariant under the Reidemeister moves





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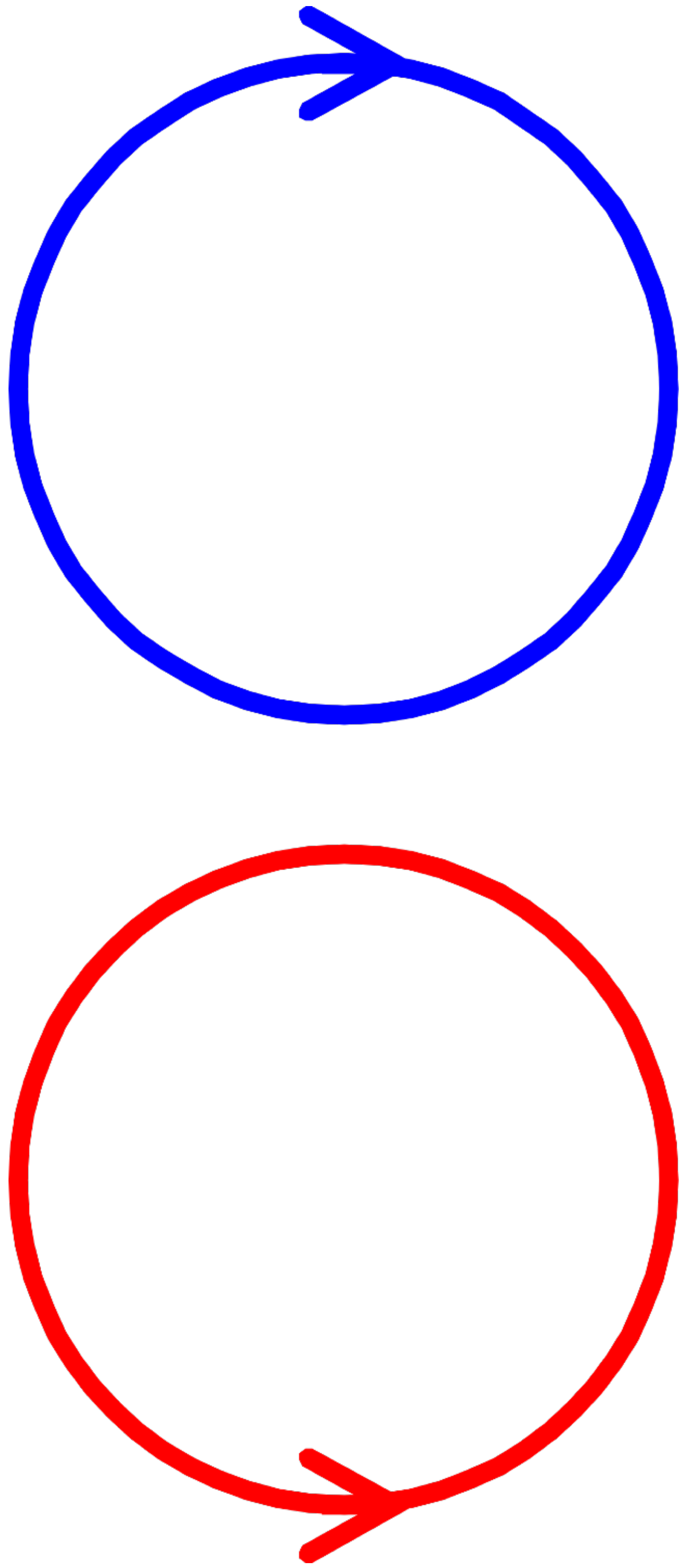
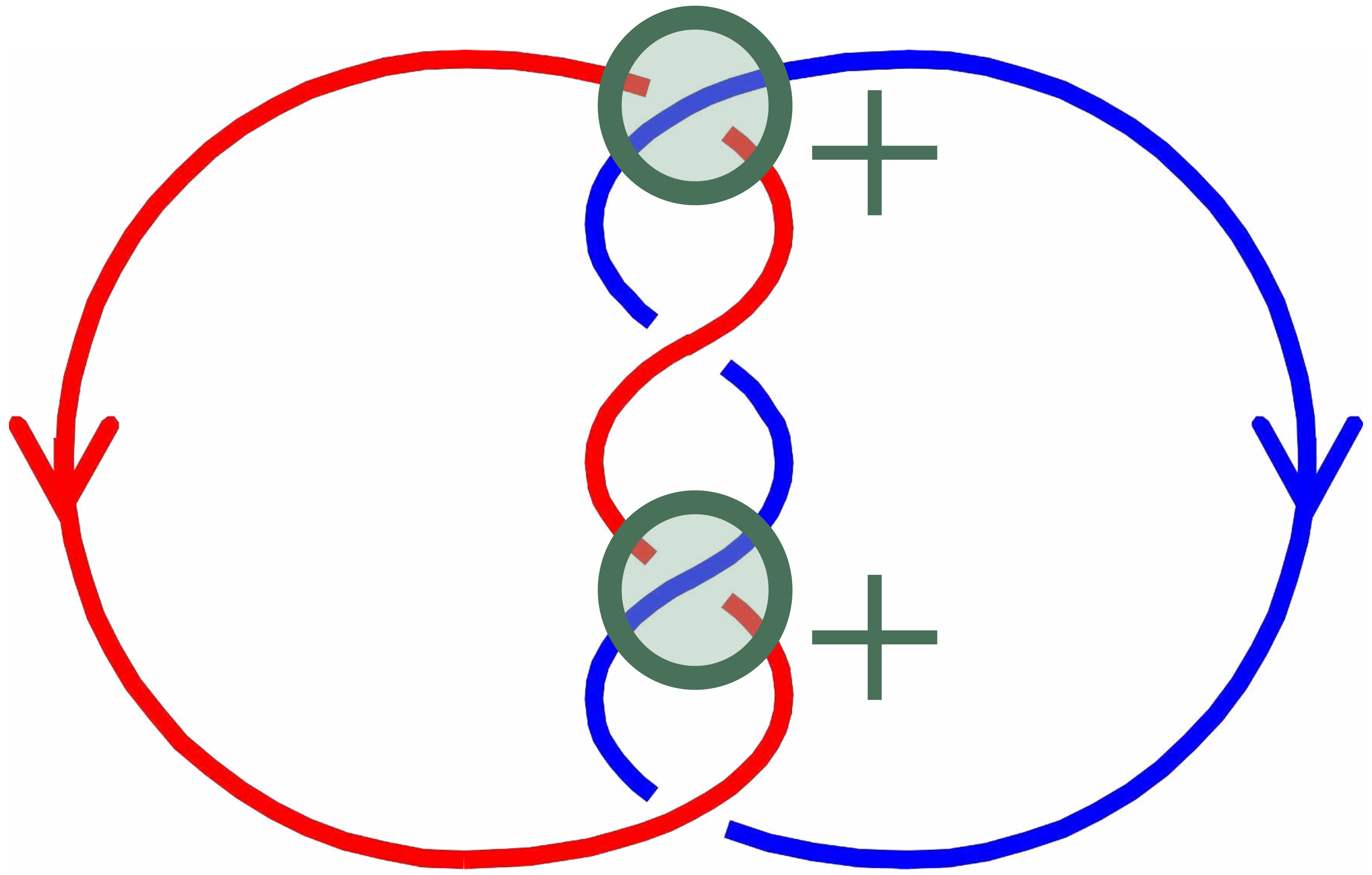
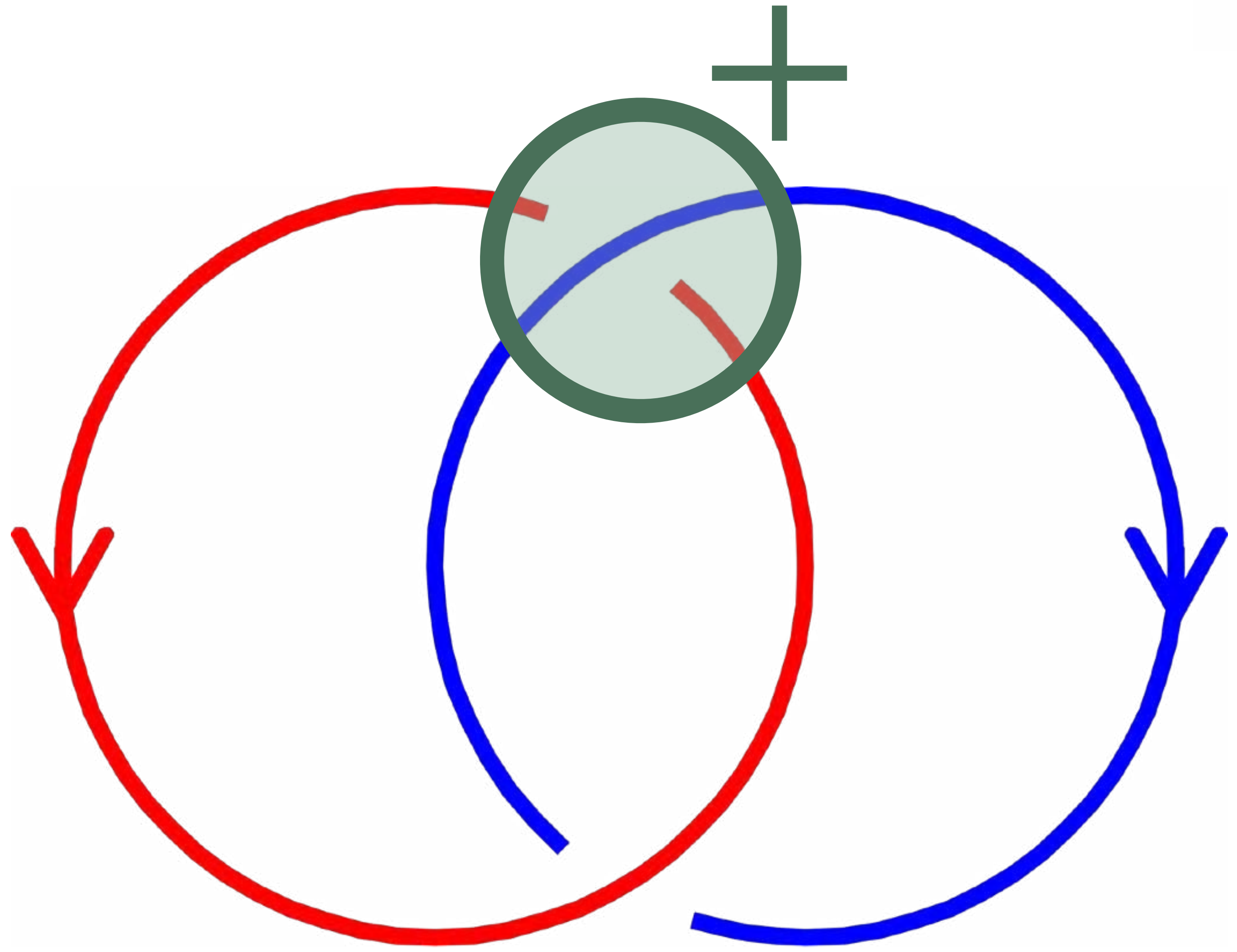


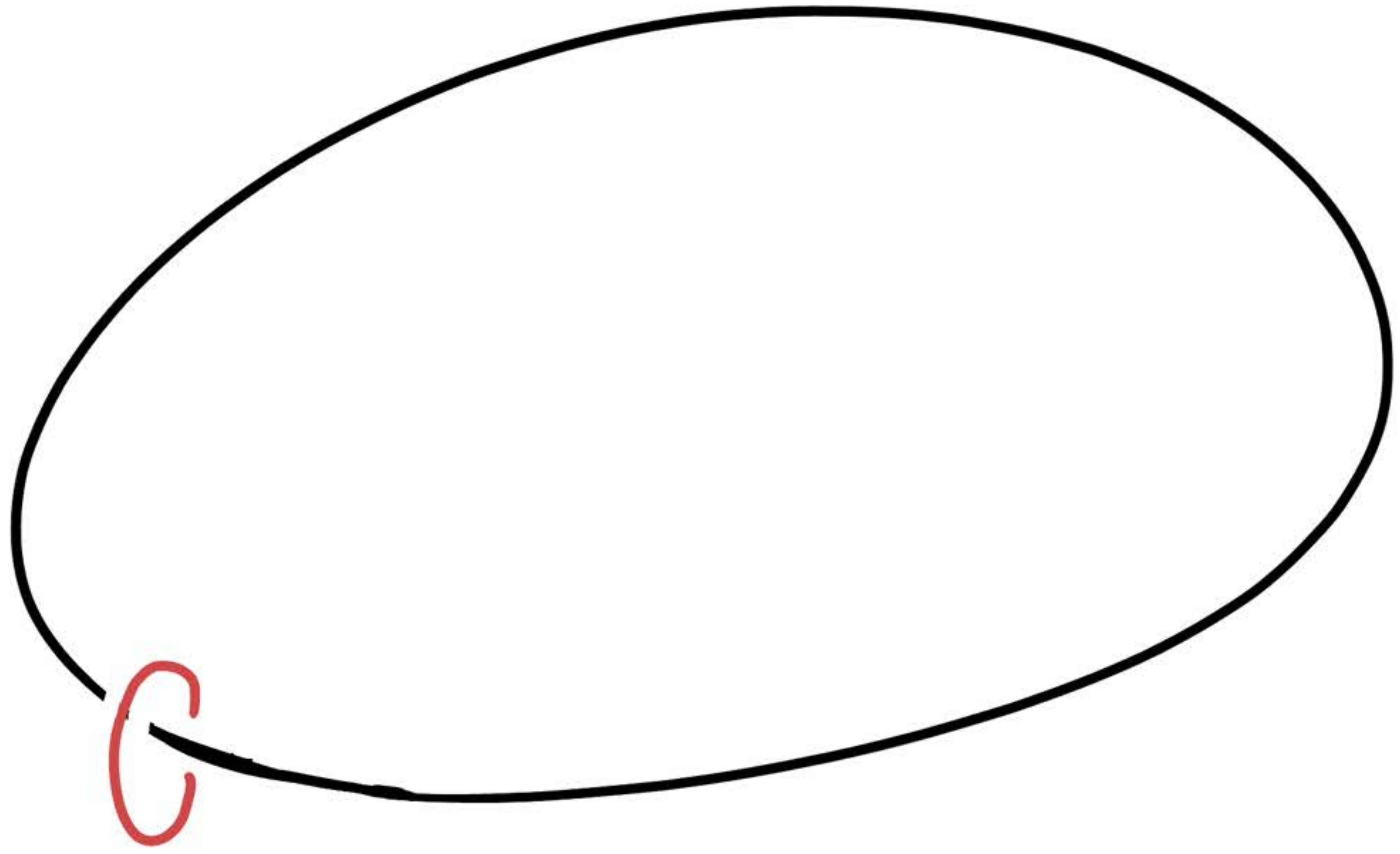
Corollary

Links Exist!

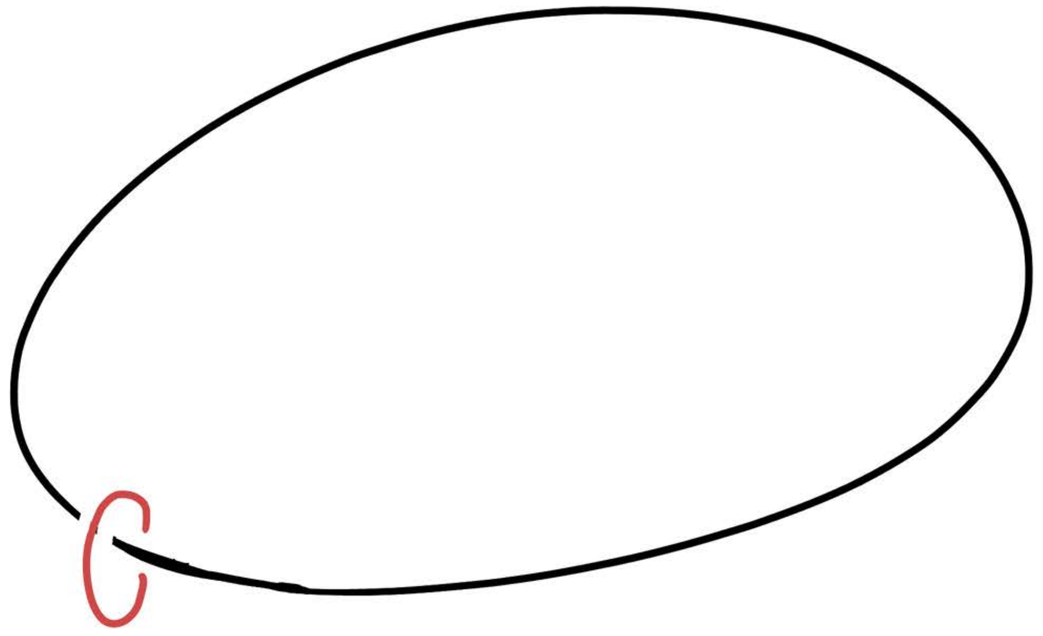


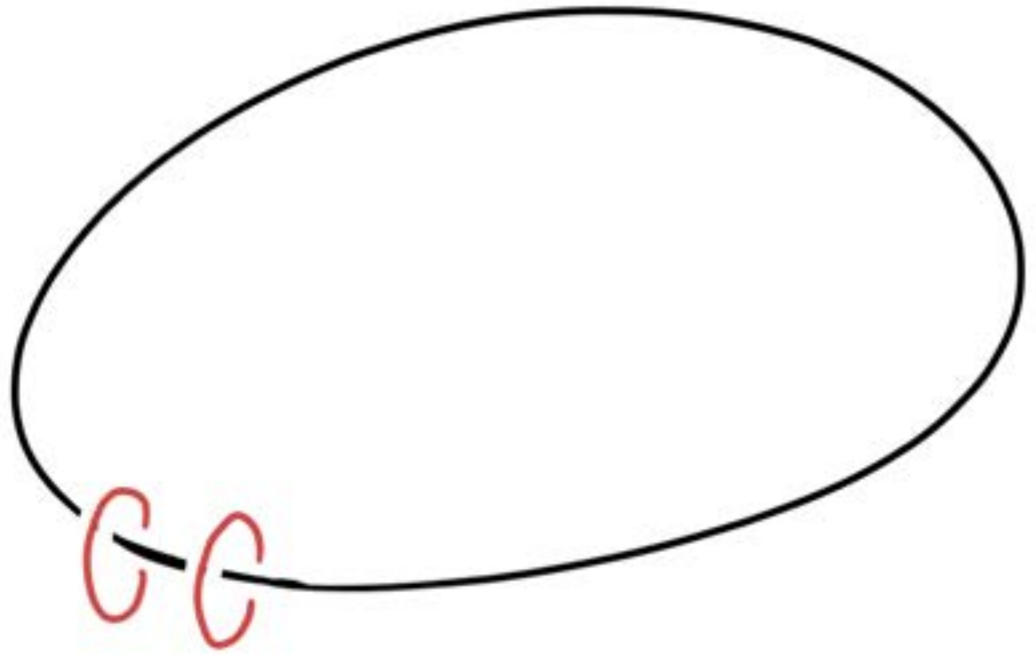


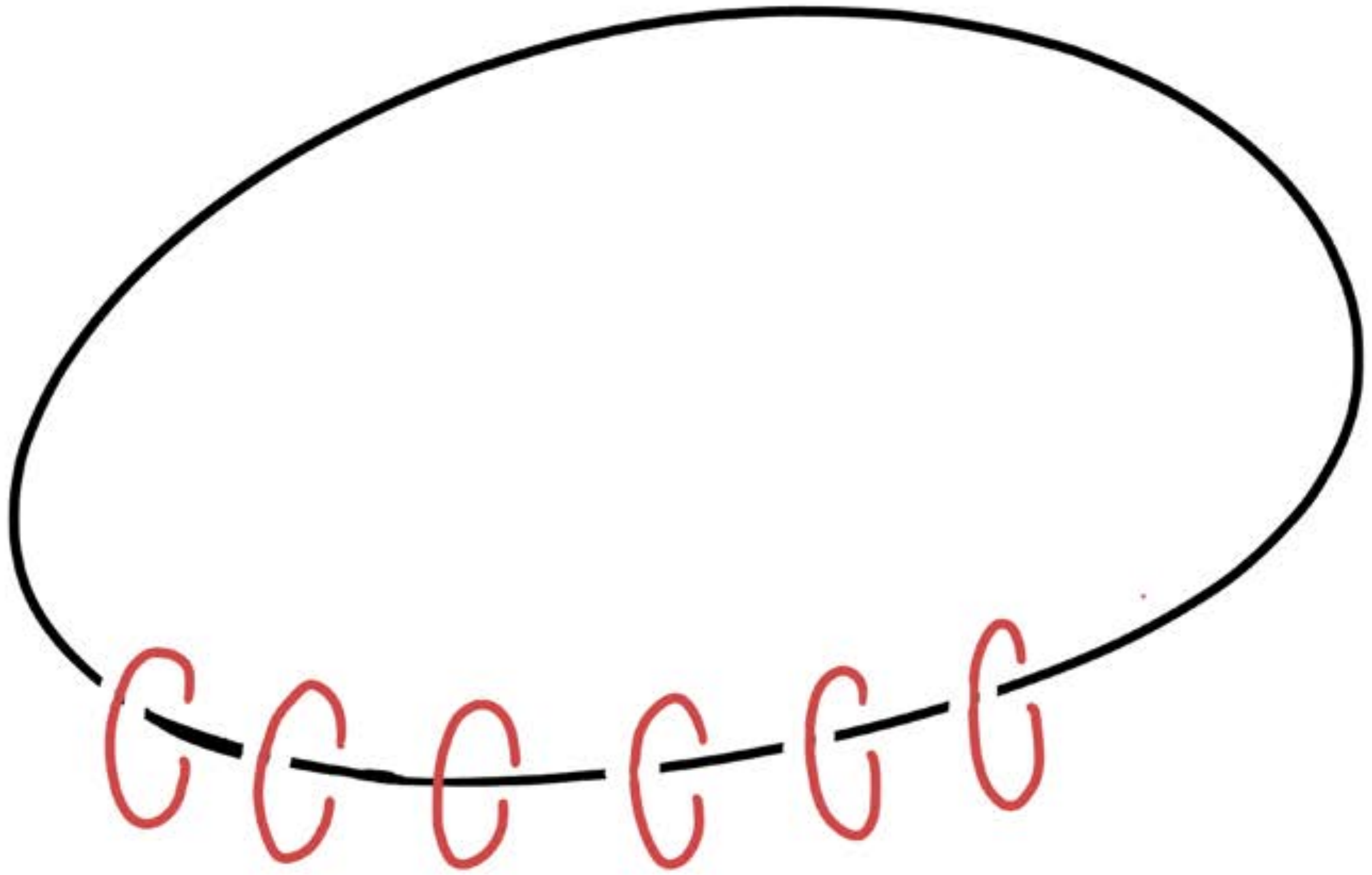
Now that we know they're linked....



Now that we know they're linked....
continue stringing along.





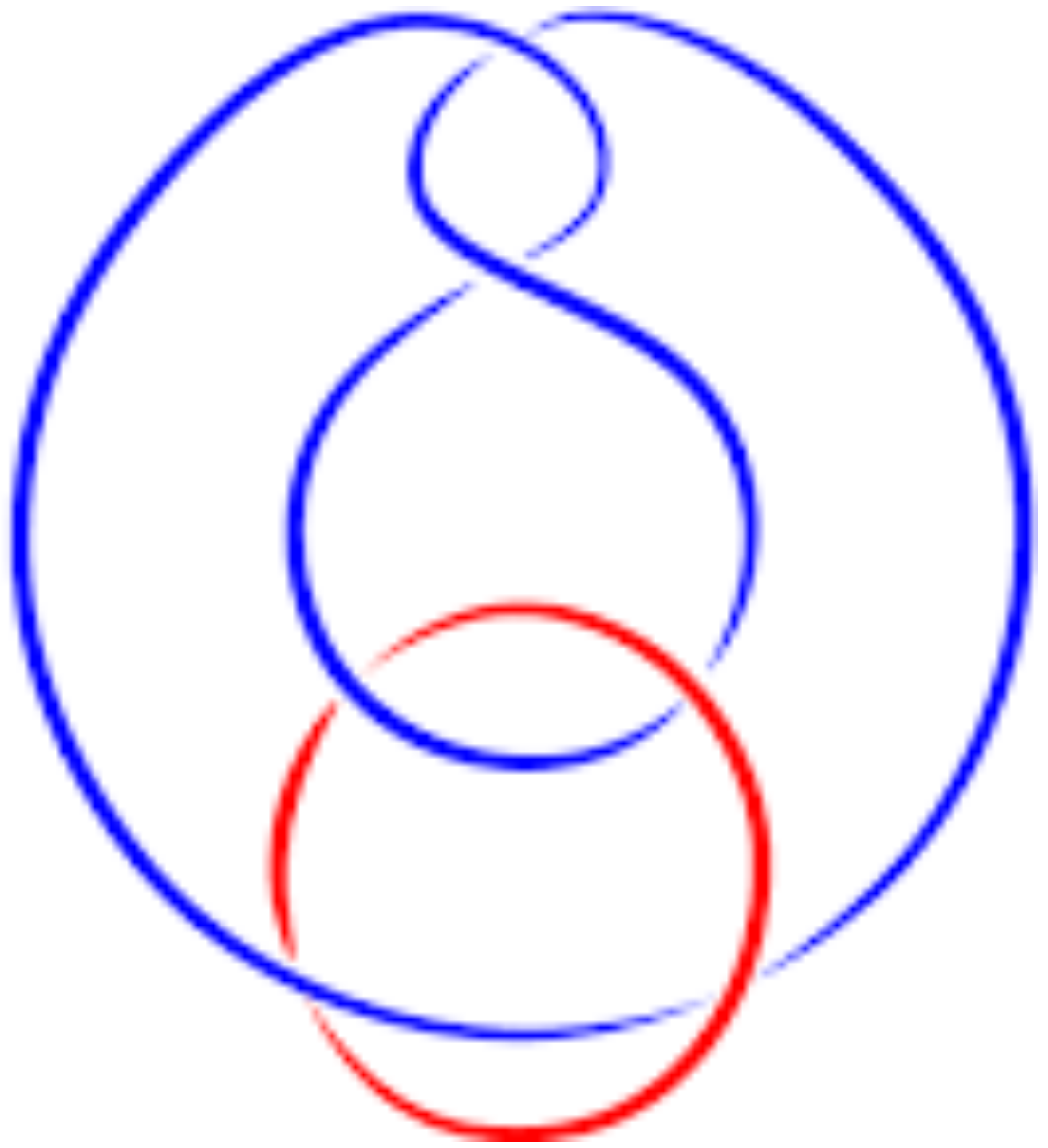


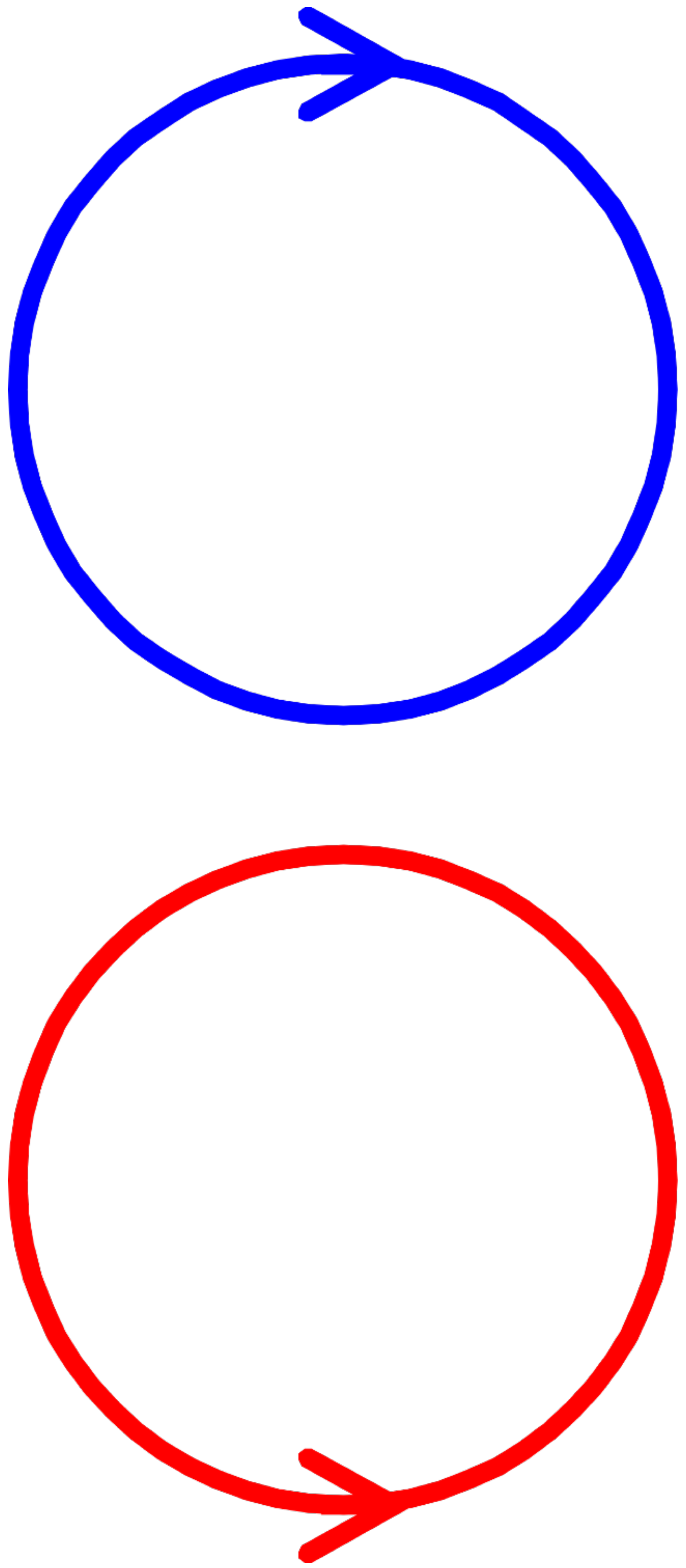
Corollary
Keychains and
Necklaces exist

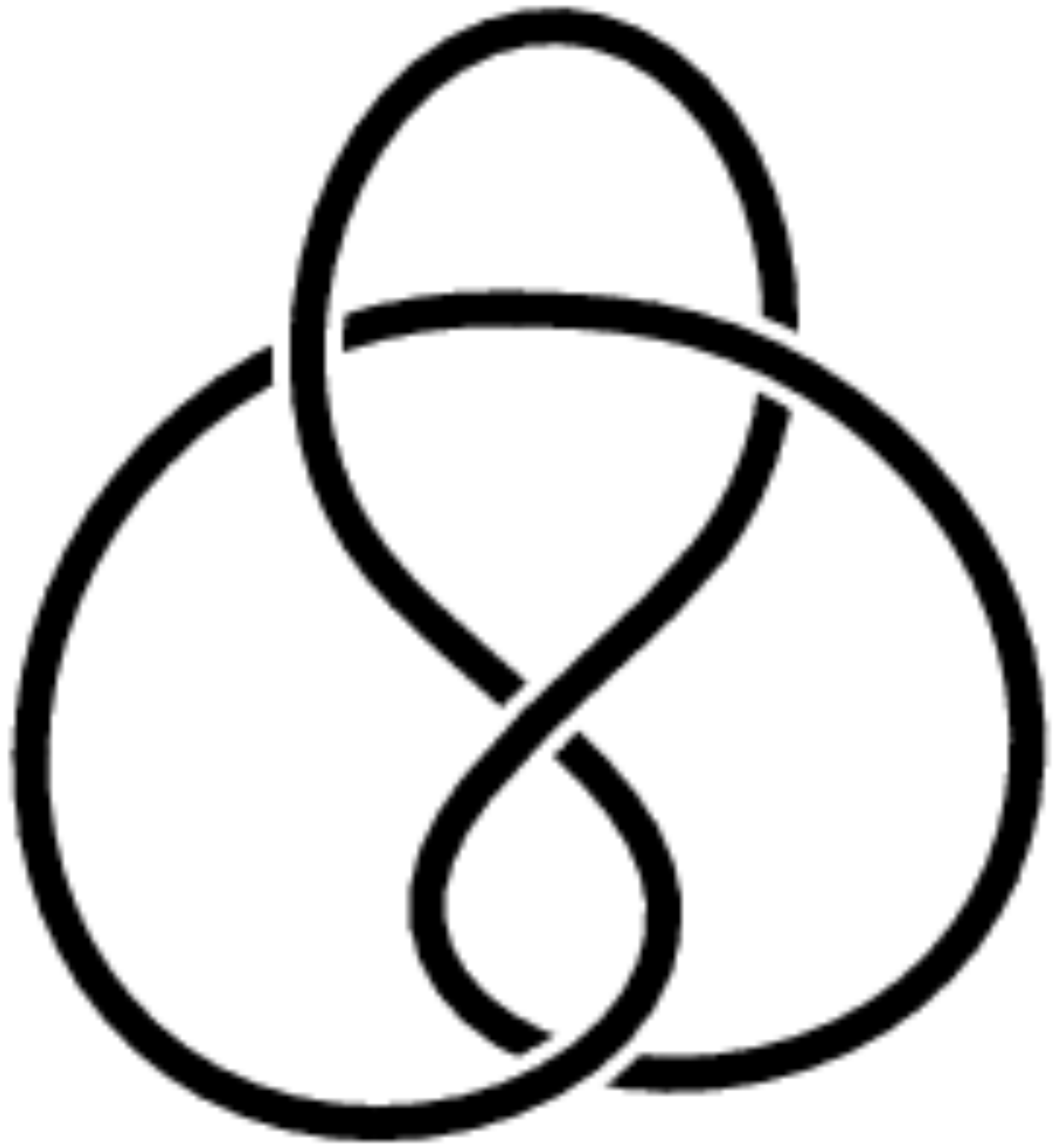
Knot Classification

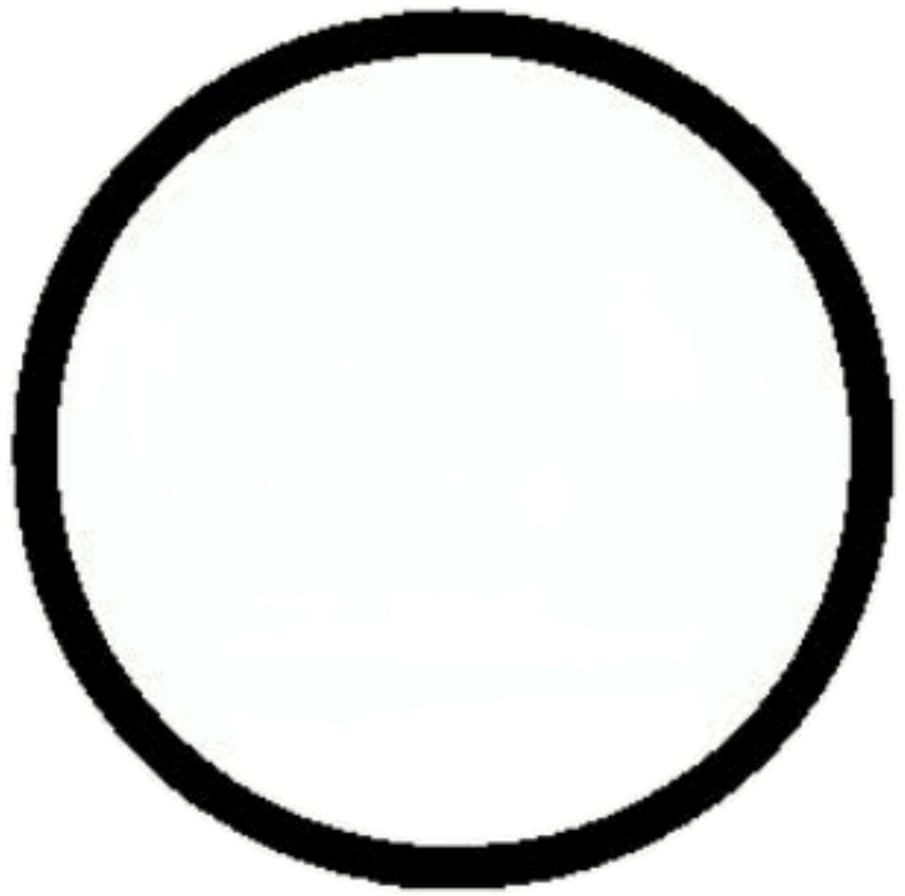
Requires many invariants!

The whitehead link has linking number zero....

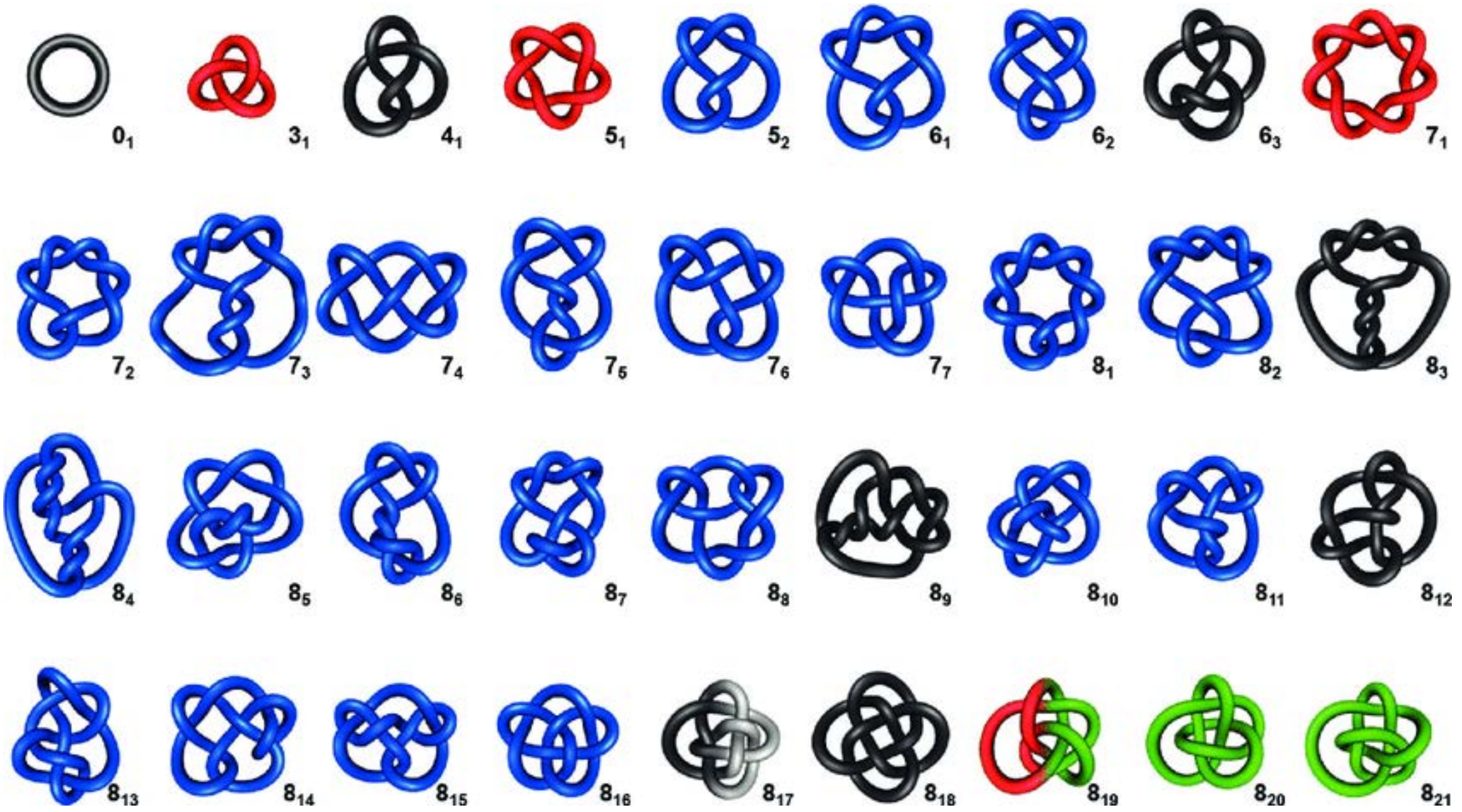


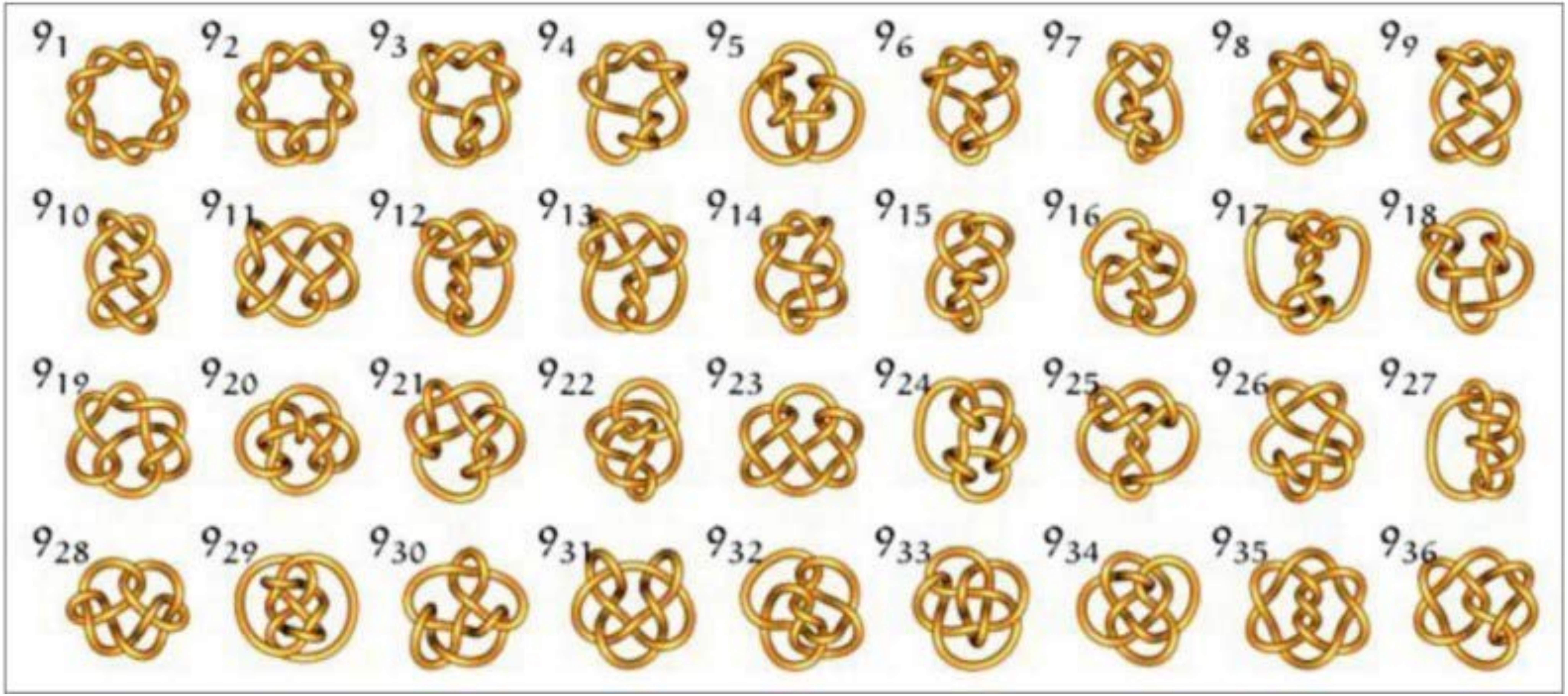






The figure 8 knot is not tricolorable...


0_1
3_1
4_1
5_1
5_2
6_1
6_2
6_3
7_1
7_2
7_3
7_4
7_5
7_6
7_7
8_1
8_2
8_3
8_4
8_5
8_6
8_7
8_8
8_9
8_{10}
8_{11}
8_{12}
8_{13}
8_{14}
8_{15}
8_{16}
8_{17}
8_{18}
8_{19}
8_{20}
8_{21}


9_1
9_2
9_3
9_4
9_5
9_6
9_7
9_8
9_9
9_{10}
9_{11}
9_{12}
9_{13}
9_{14}
9_{15}
9_{16}
9_{17}
9_{18}
9_{19}
9_{20}
9_{21}
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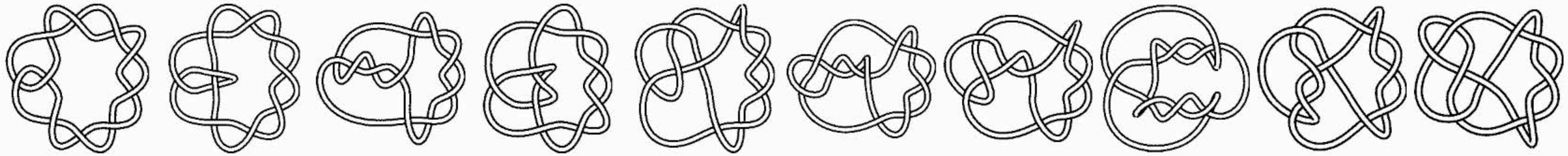
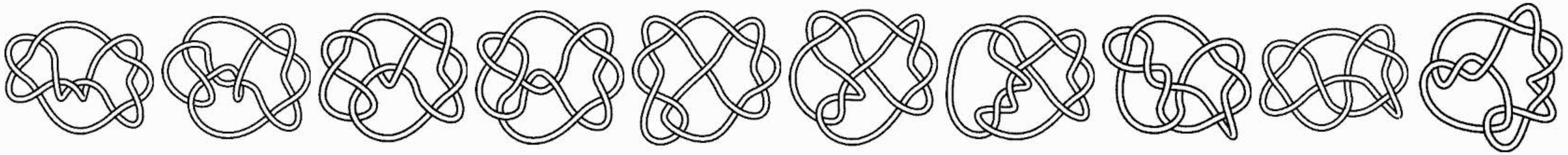
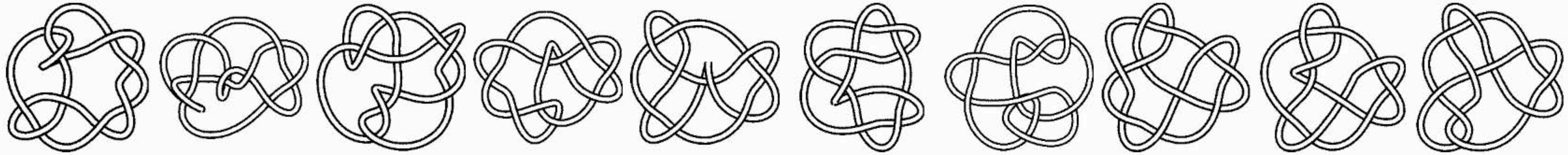
9_{31}
9_{32}
9_{33}
9_{34}
9_{35}
9_{36}



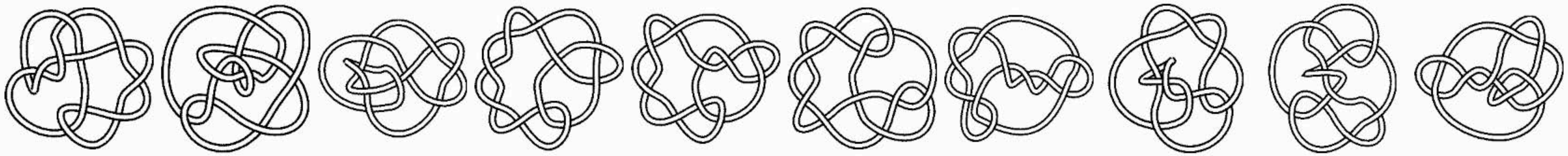
10_1 10_2 10_3 10_4 10_5 10_6 10_7 10_8 10_9 10_10



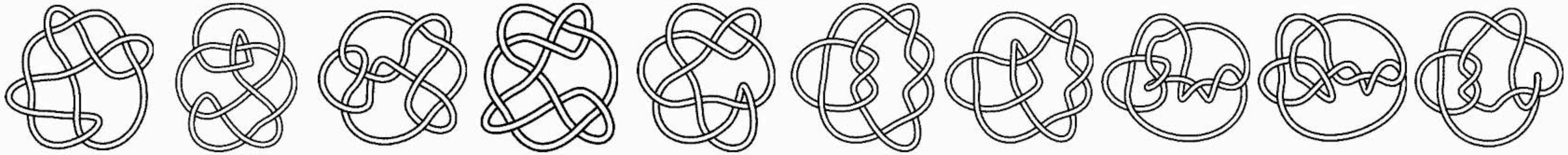
10_11 10_12 10_13 10_14 10_15 10_16 10_17 10_18 10_19 10_20



10_21 10_22 10_23 10_24 10_25 10_26 10_27 10_28 10_29 10_30



10_31 10_32 10_33 10_34 10_35 10_36 10_37 10_38 10_39 10_40

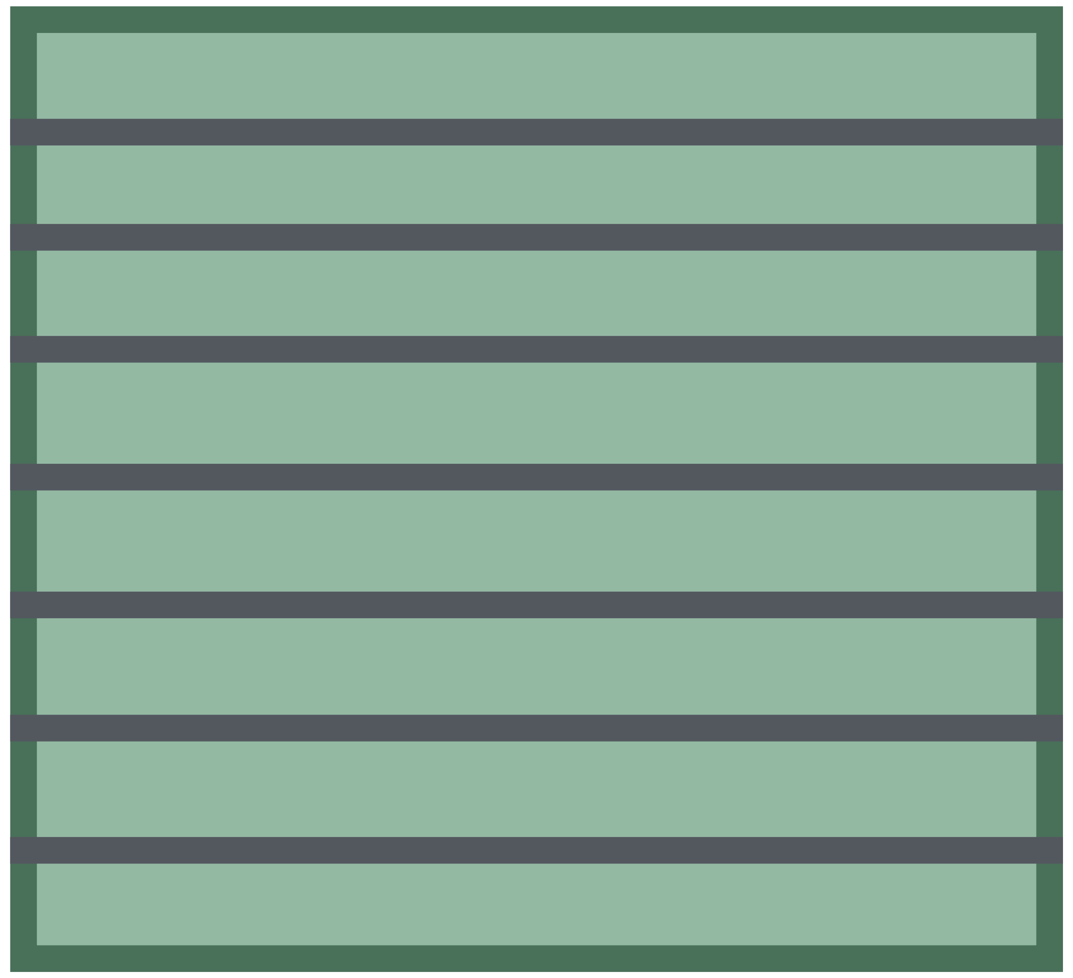


10_41 10_42 10_43 10_44 10_45 10_46 10_47 10_48 10_49 10_50

Why there are

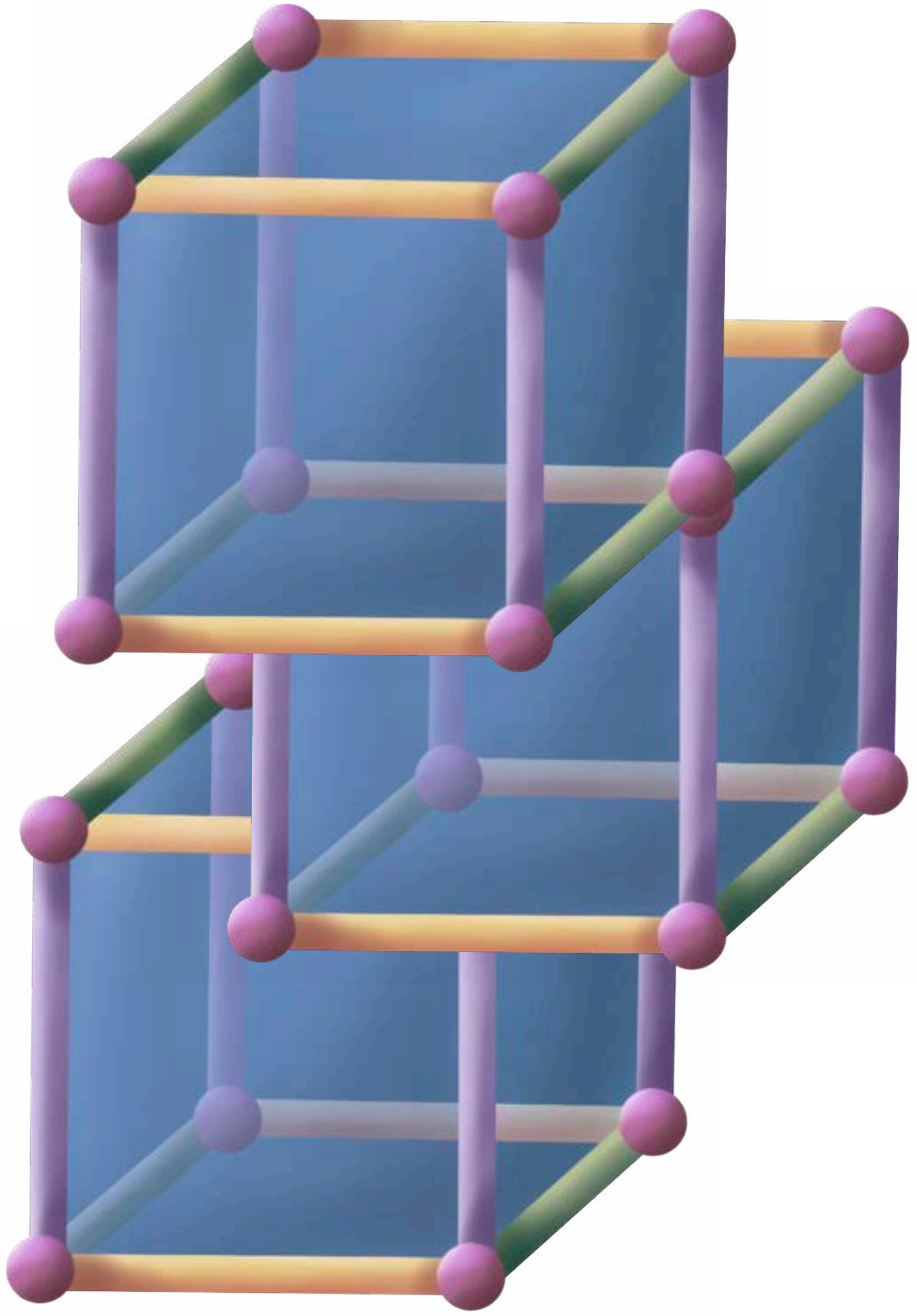
No Knots in 4 Dimensions

The 4th Dimension



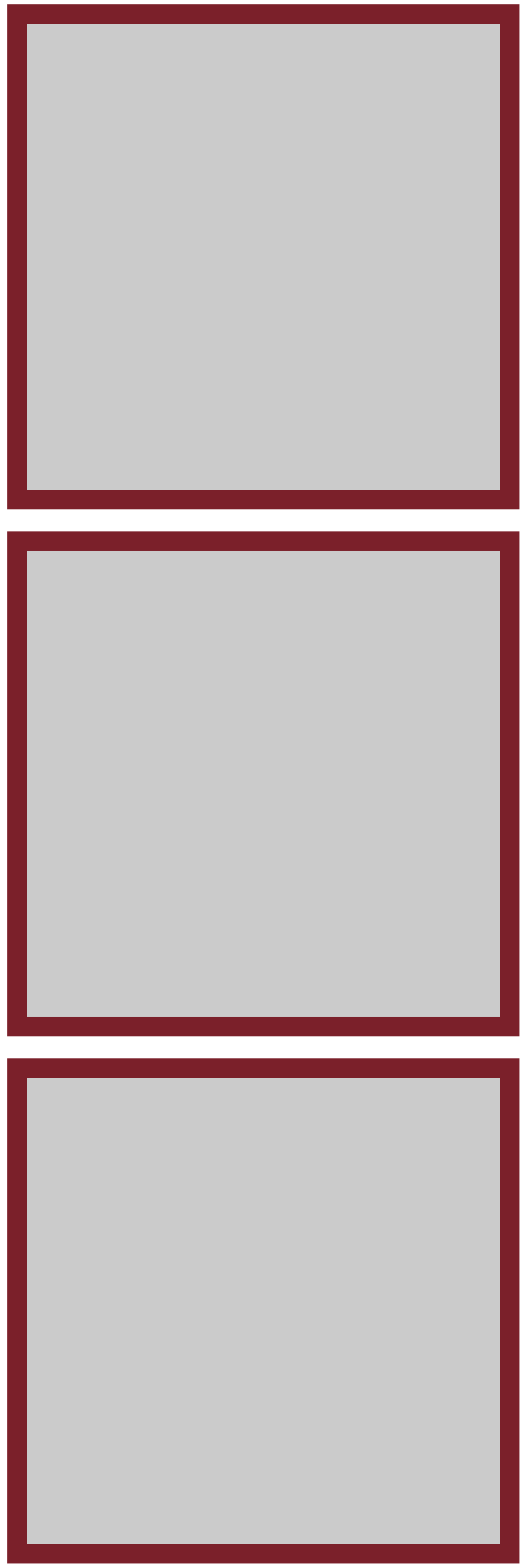
(x, y)

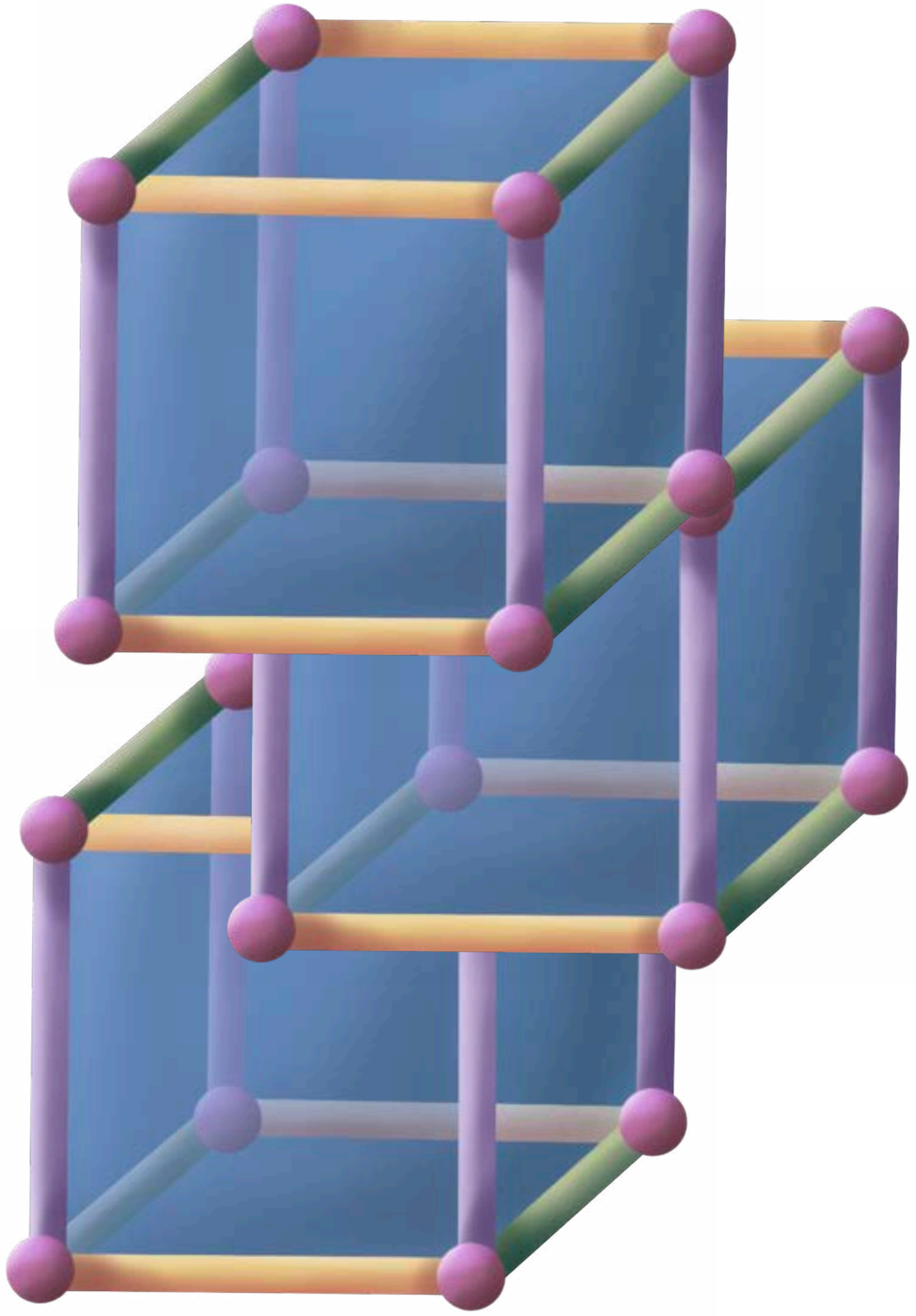
(x, y, z)



(x, y, z, w)

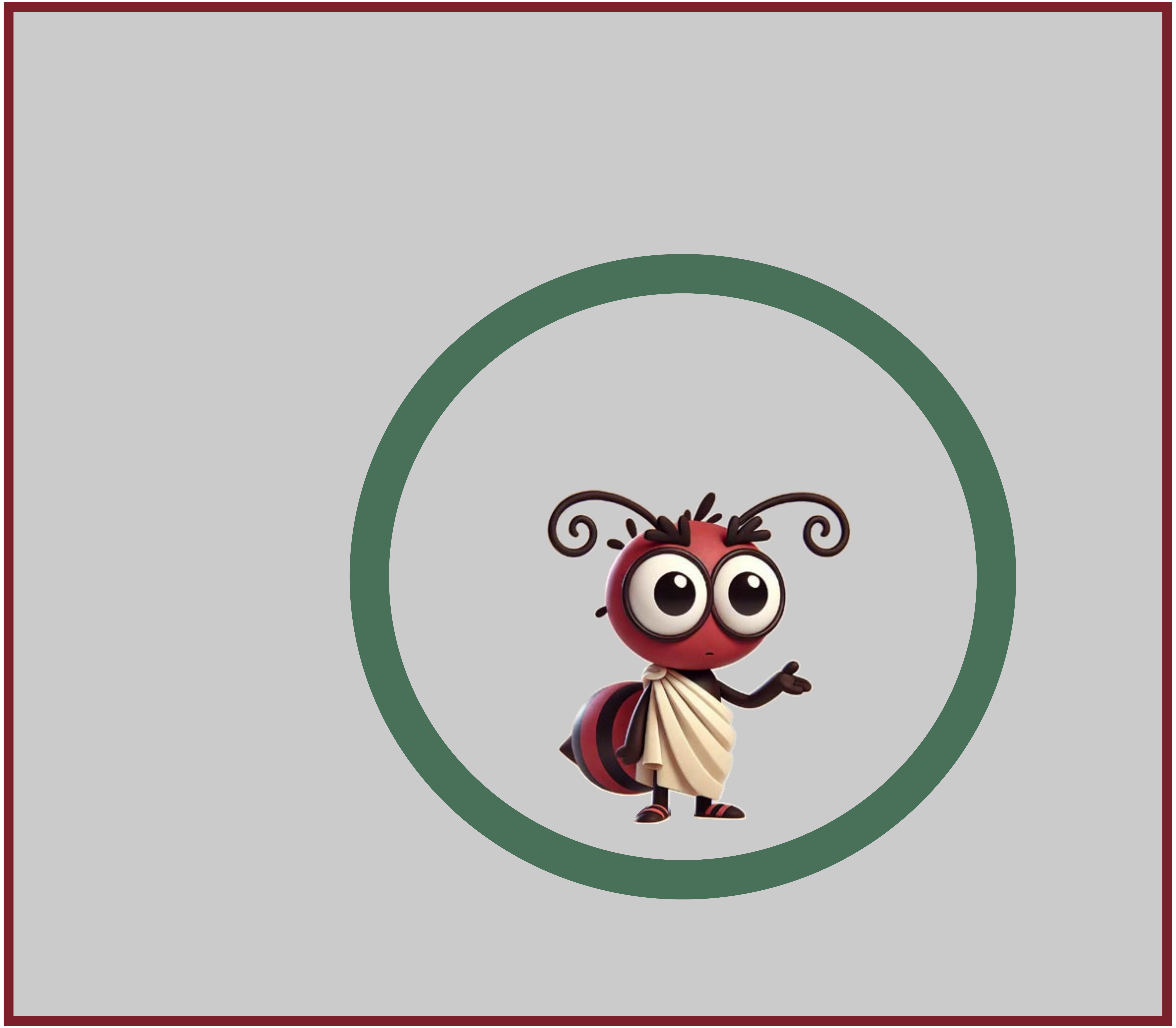
The 4th Dimension





$$(x, y, z, w)$$

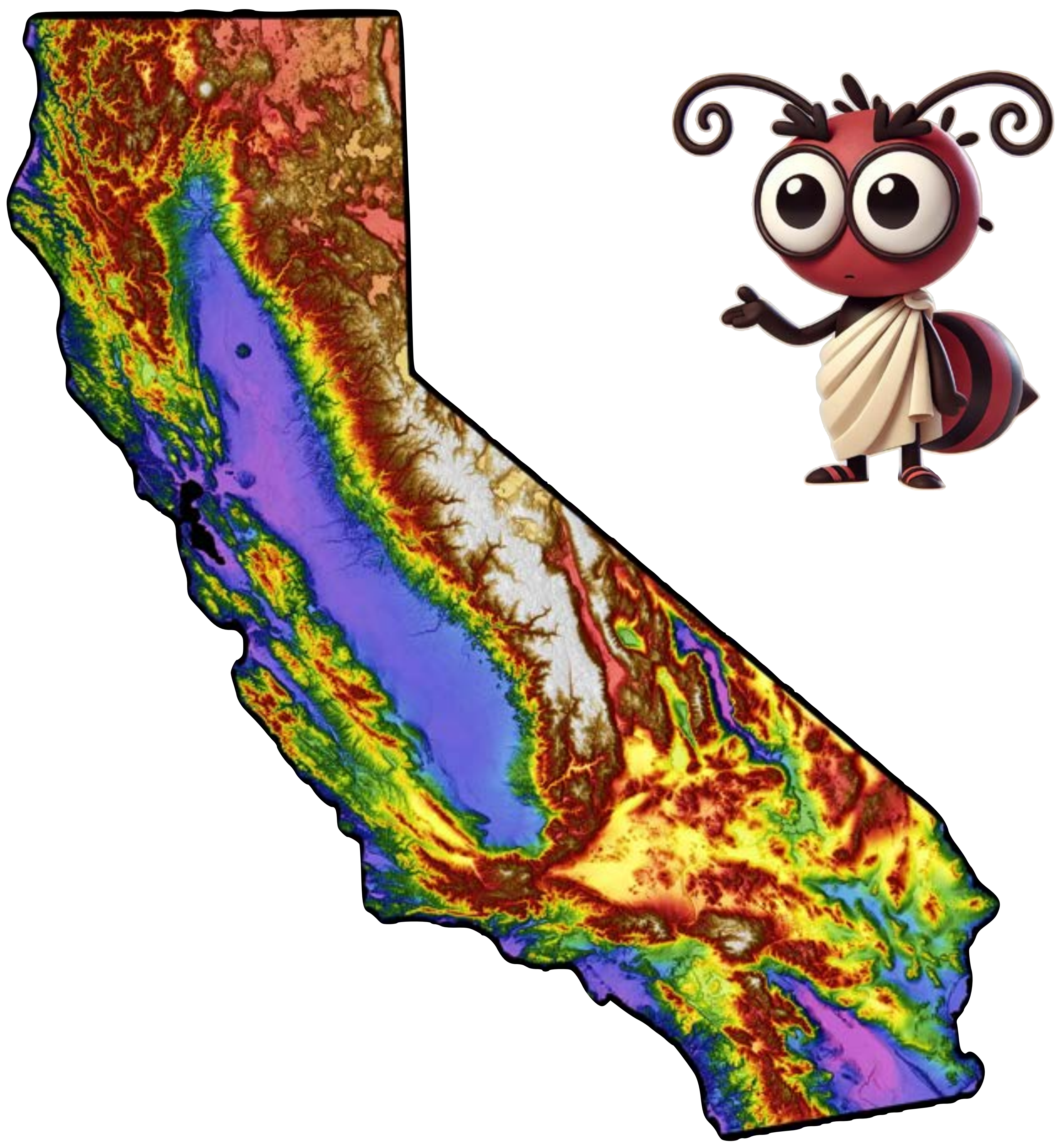
In two dimensions, a creature can be completely trapped by drawing a circle around them.

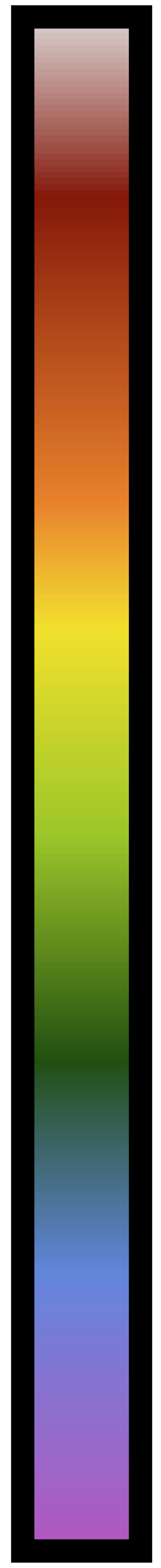


But access to the third dimension grants them an incomprehensible superpower: they can leave the circle without touching it.

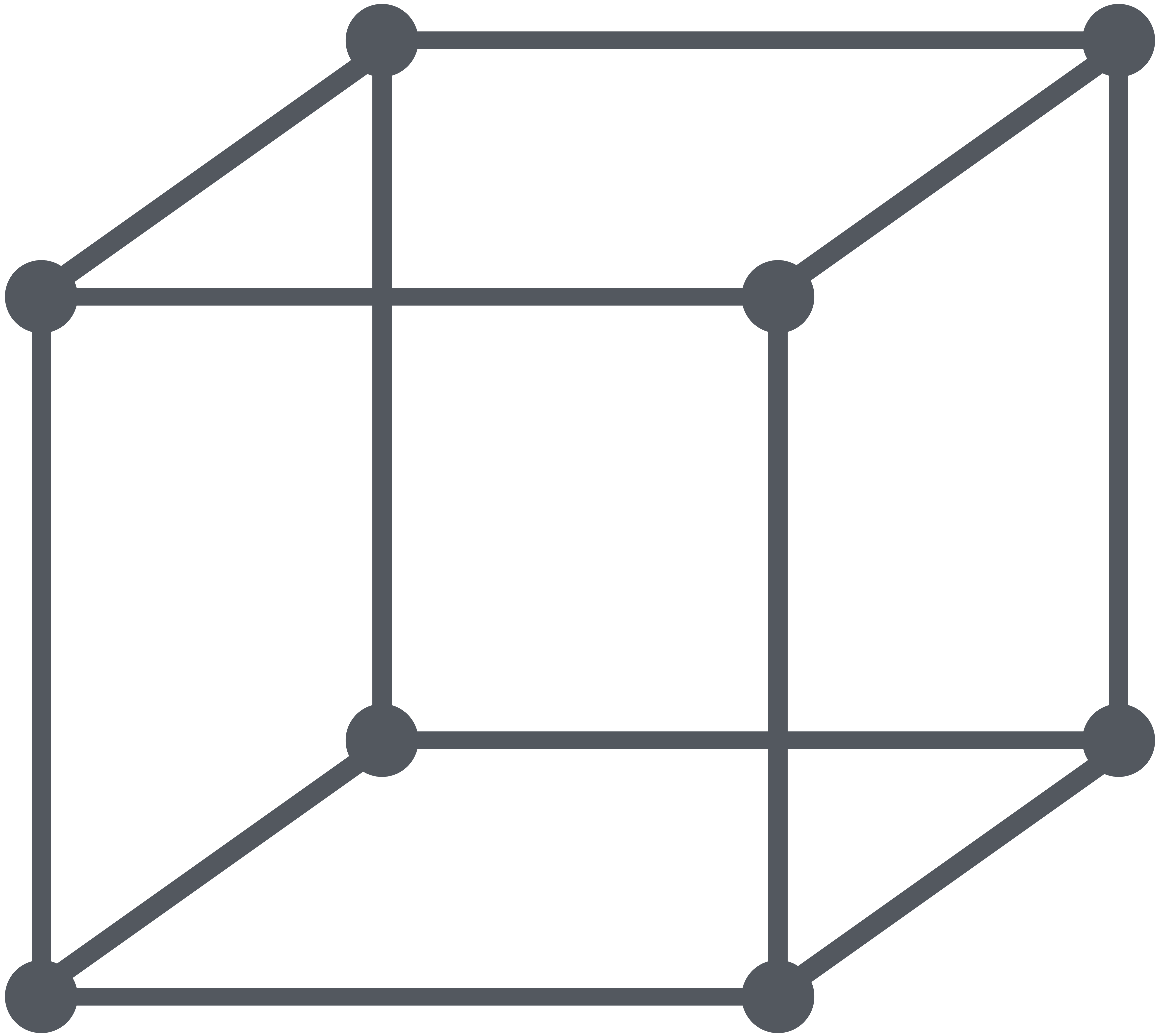
How do we explain this to our ant friend?

2D beings cannot visualize up and down: so they need to represent third dimension another way



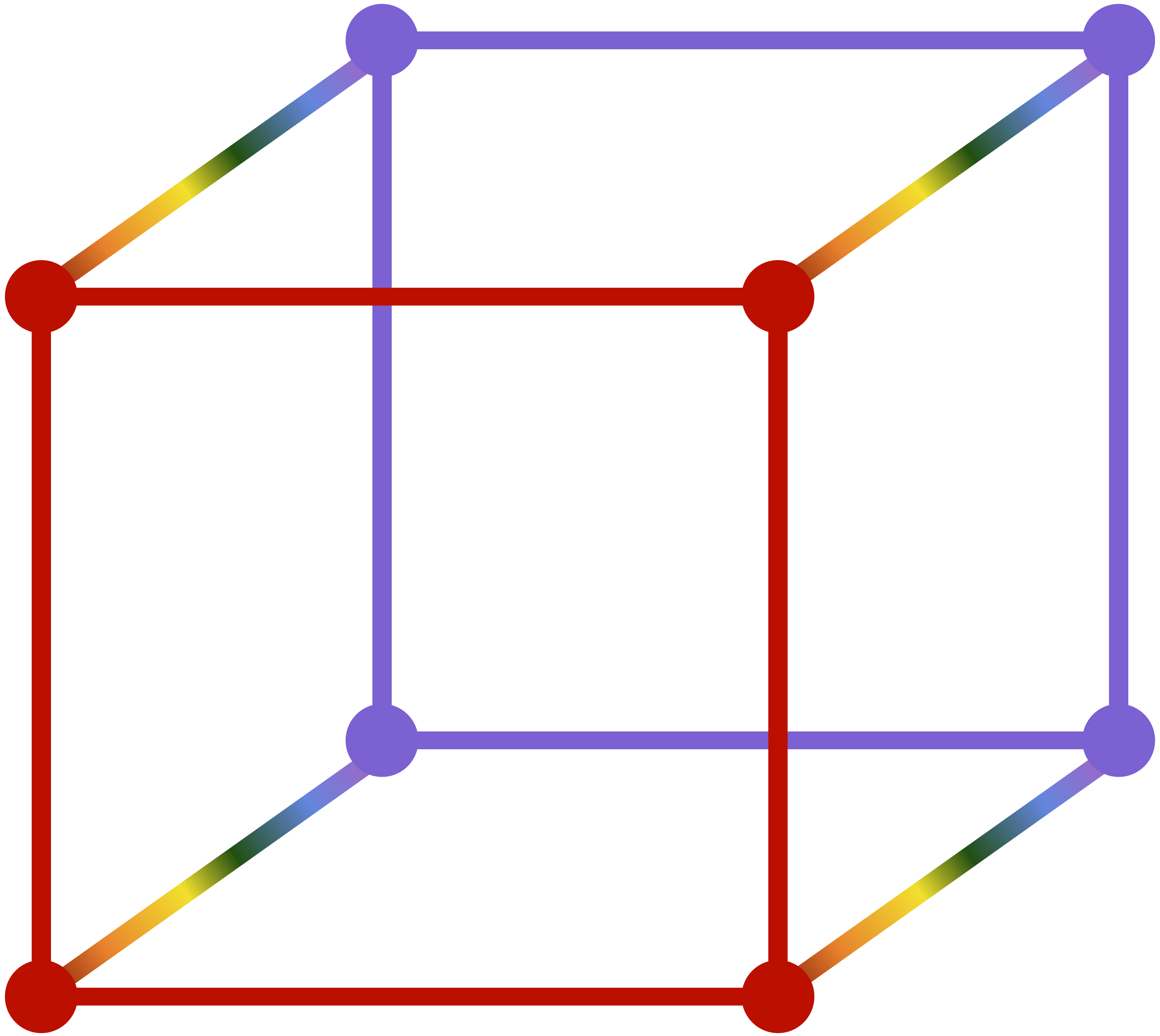


Rule: two objects of different colors are not touching

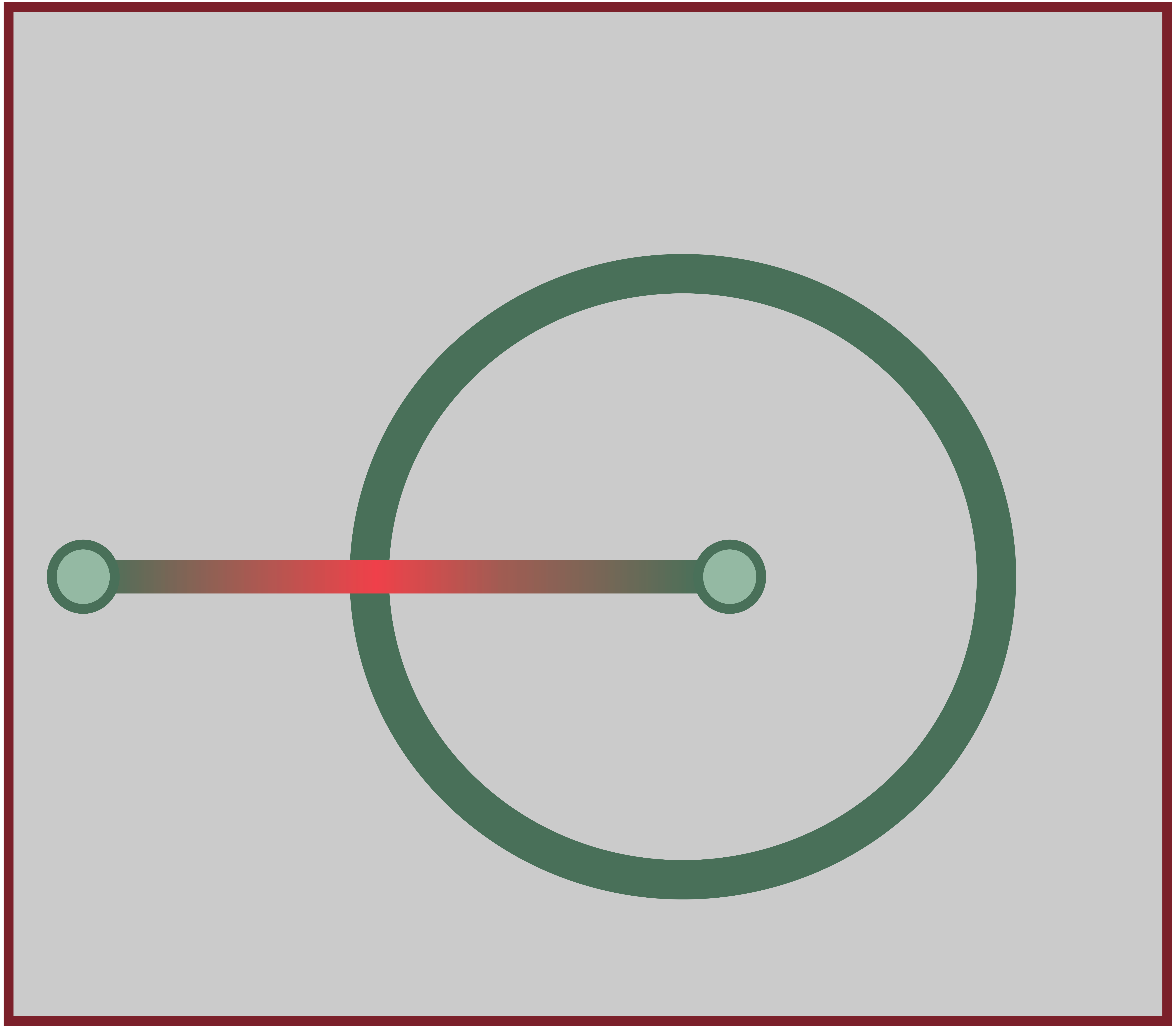


Rule: two objects of different colors are not touching

They have the same (x,y), but different z

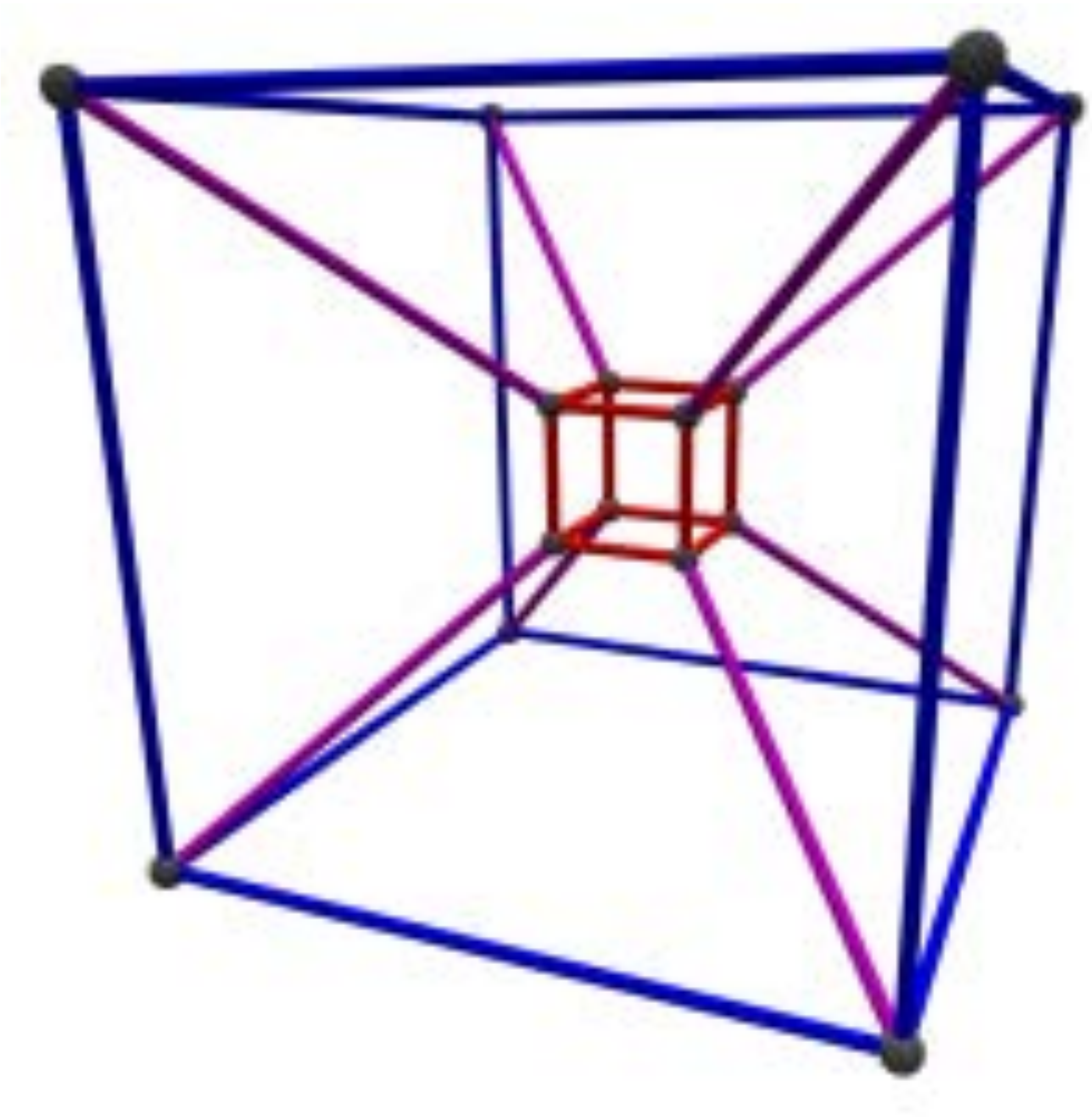


Ah! I See! Changing color, I can pass "through" the circle without touching it, because I am really "over the top" of it.



The four dimension

via color

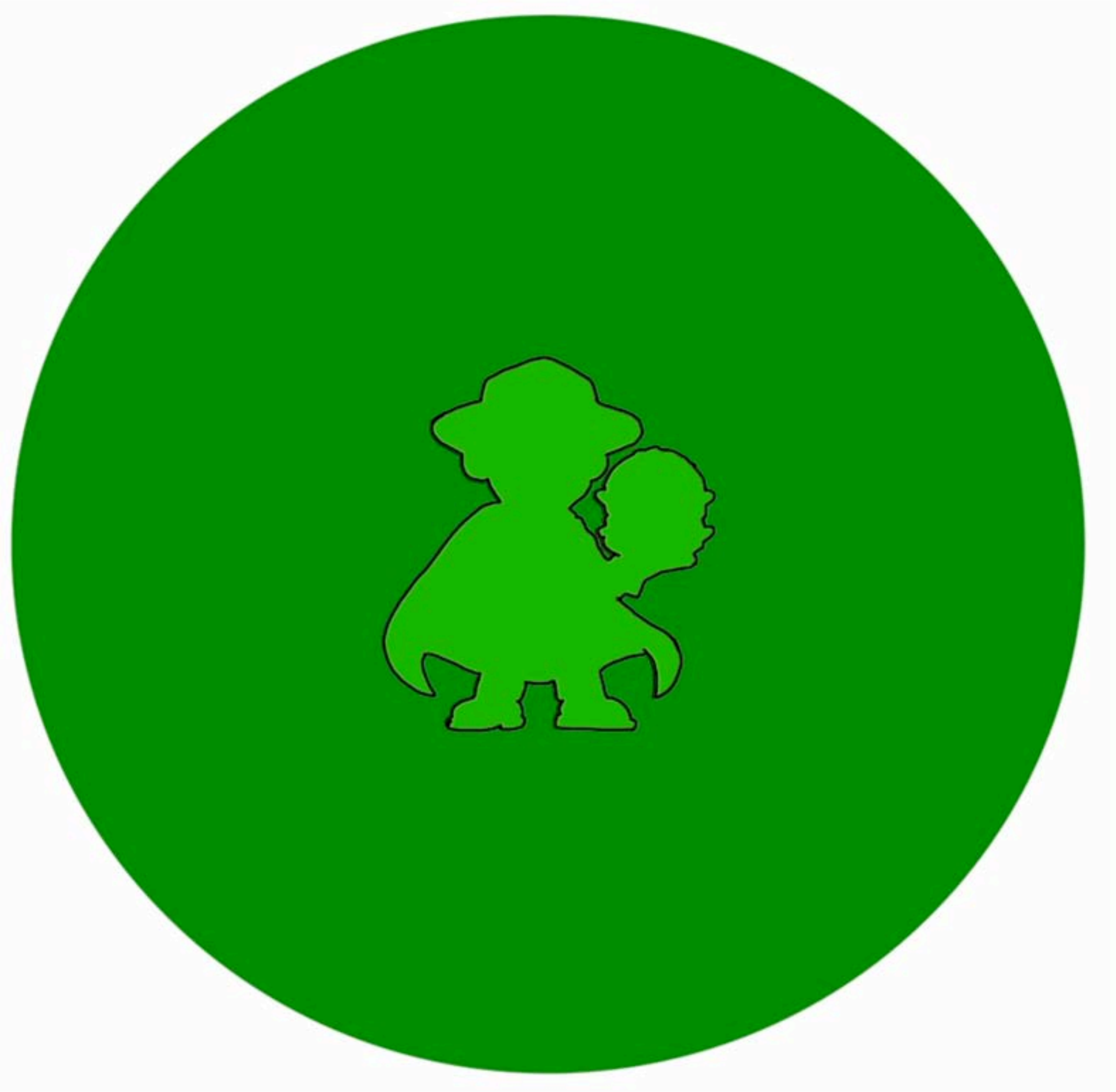


Objects in 4D can be drawn as an "elevation map": a 3D object with color at each point.

Two things occupying same space but different colors are not intersecting, but rather passing "over" one another.

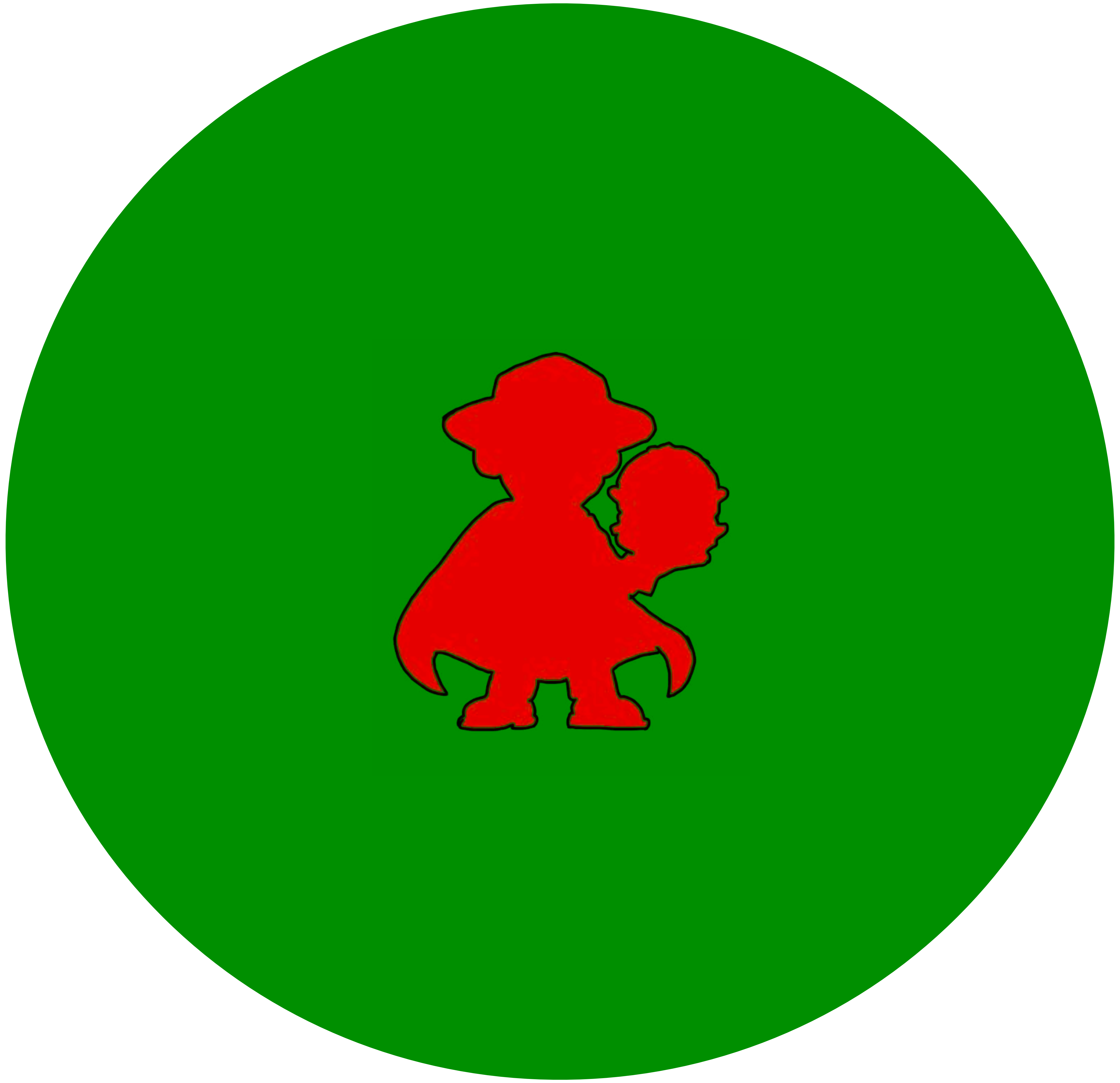
Getting out of jail

Getting out of jail



Starting in a 3D slice, both are the same color

Getting out of jail

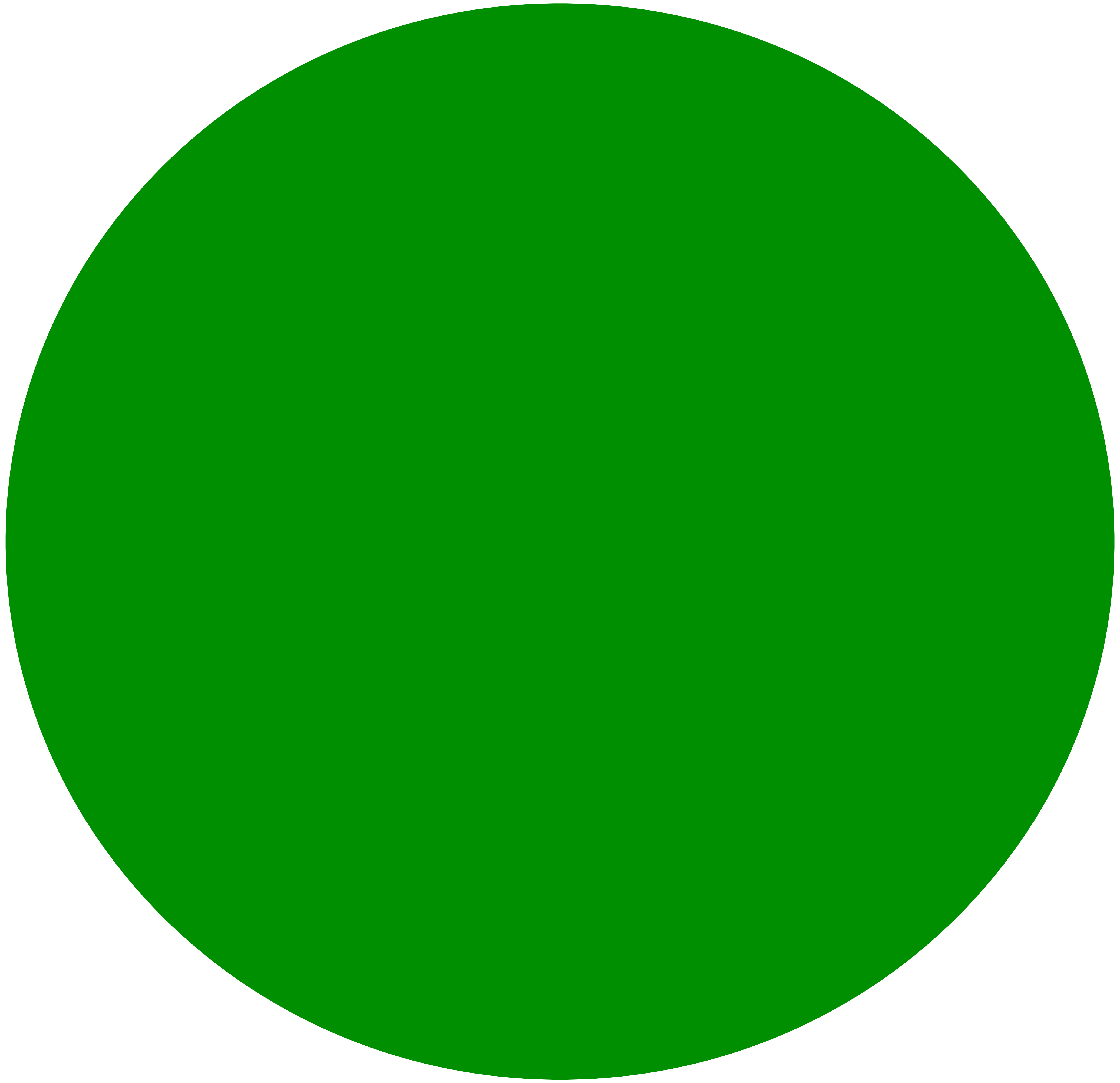


Starting in a 3D slice, both are the same color

The thief changes color, signifying movement up in the 4th direction.

This allows him to easily pass over the sphere

Getting out of jail



Starting in a 3D slice, both are the same color

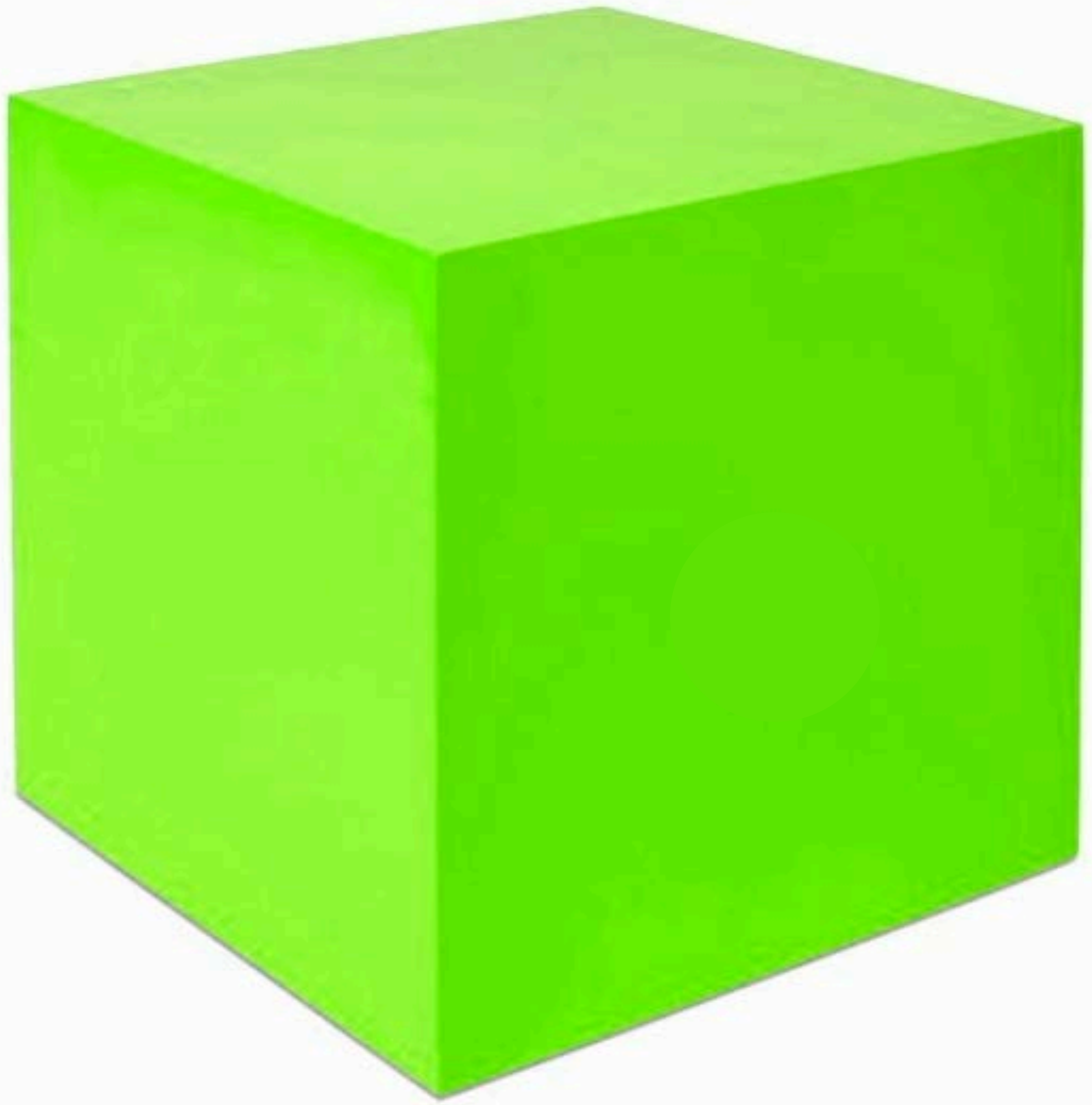
The thief changes color, signifying movement up in the 4th direction.

This allows him to easily pass over the sphere

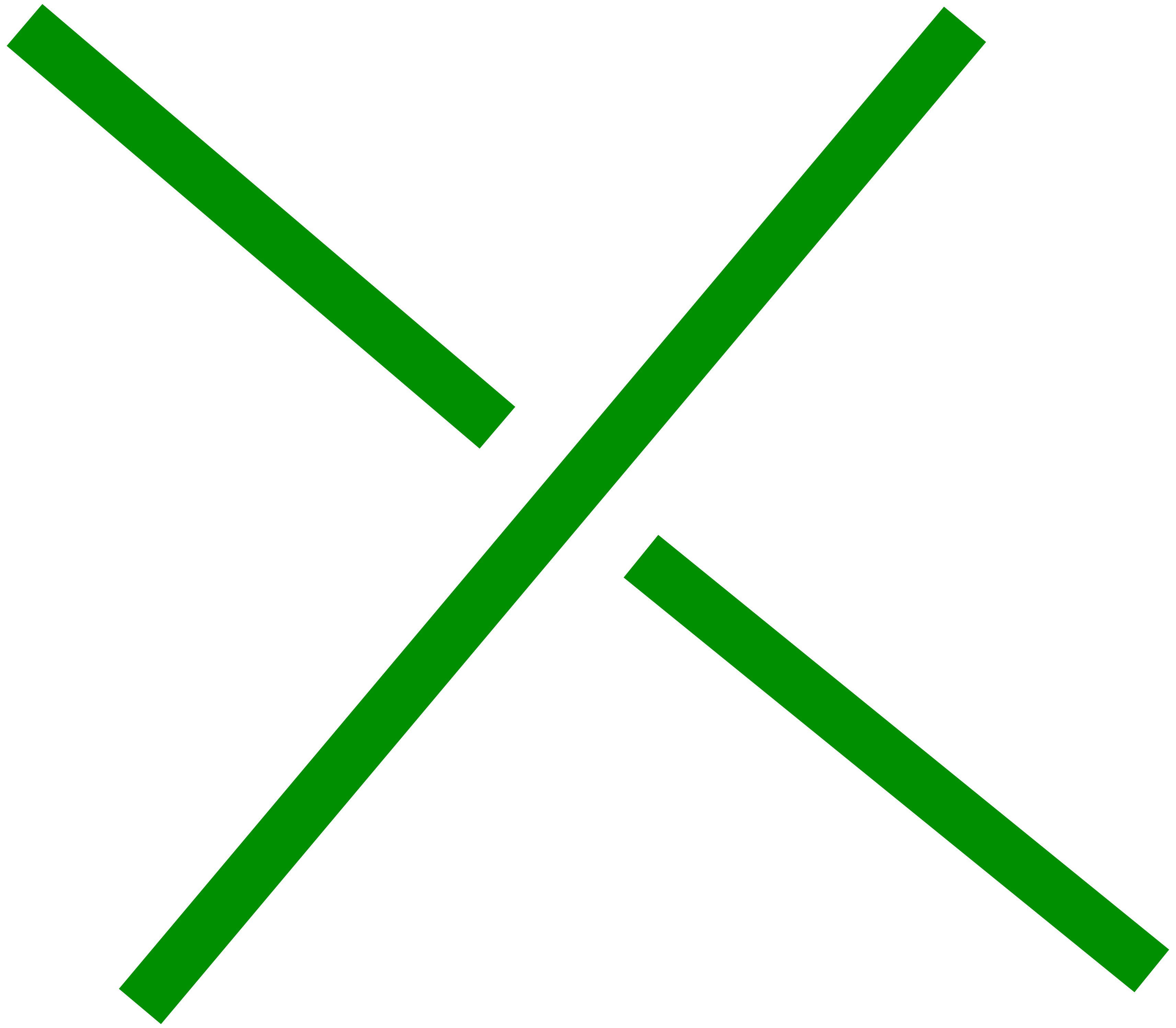
And then return to his original height.

Its easy to steal the contents of a box: just shake it!

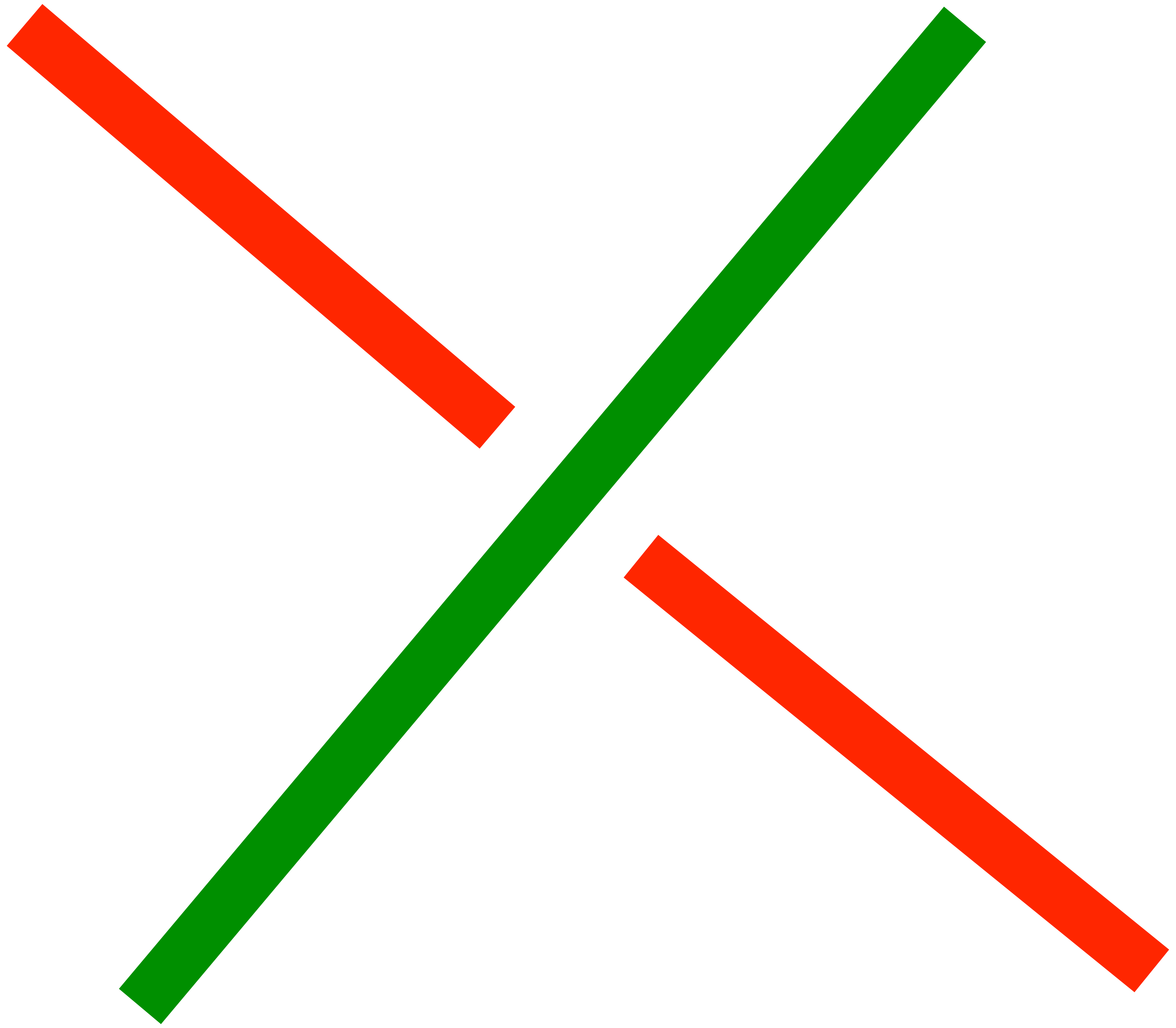
Its easy to steal the contents of a box: just shake it!



Untying Knots

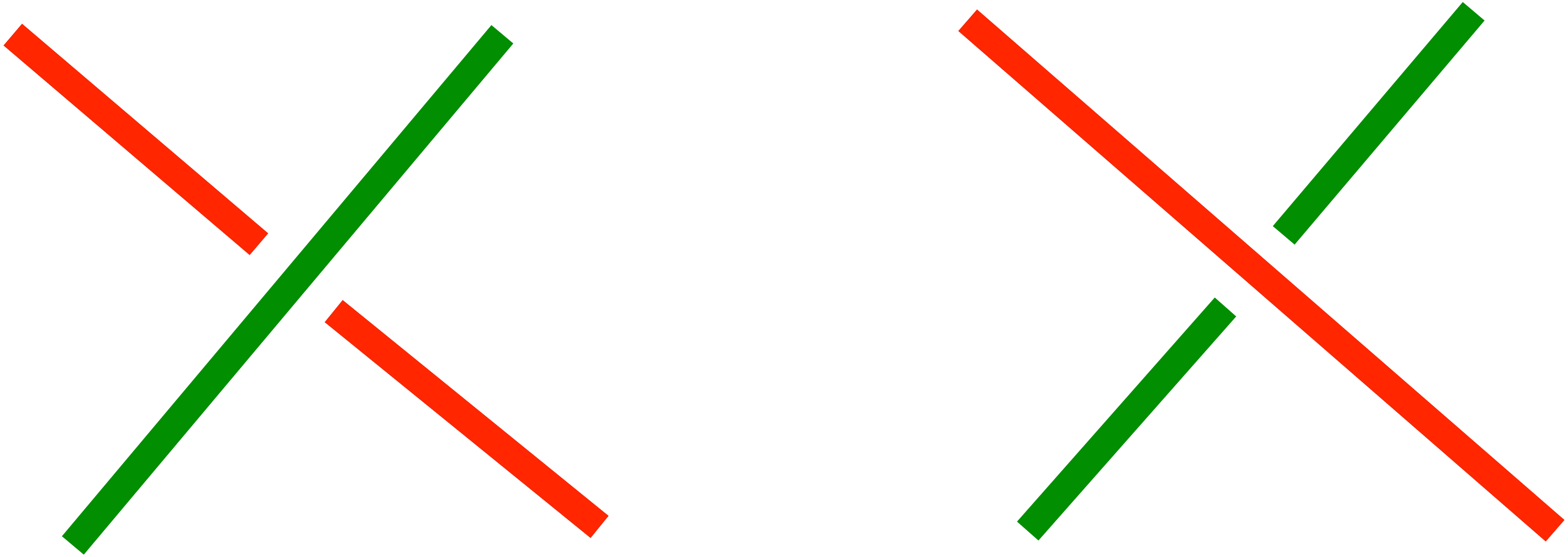


Untying Knots



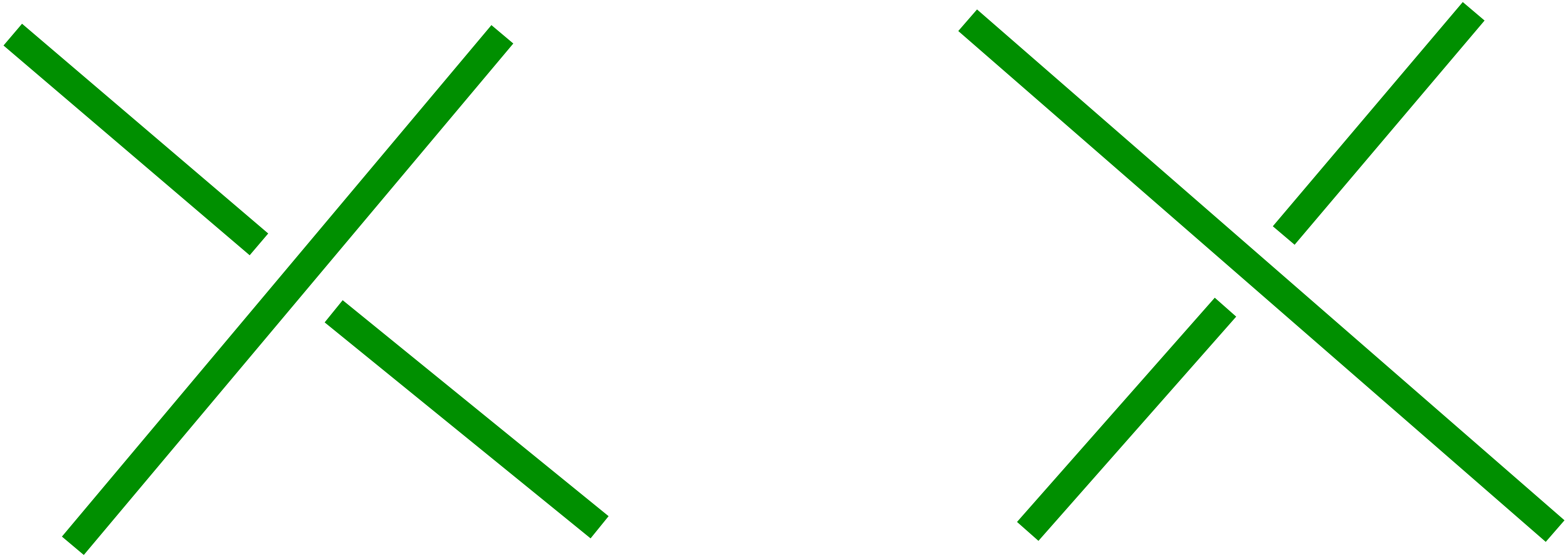
Move one of the strands in the 4th direction, leaving 3D fixed.

Untying Knots



Now pull it over the other strand in 3D

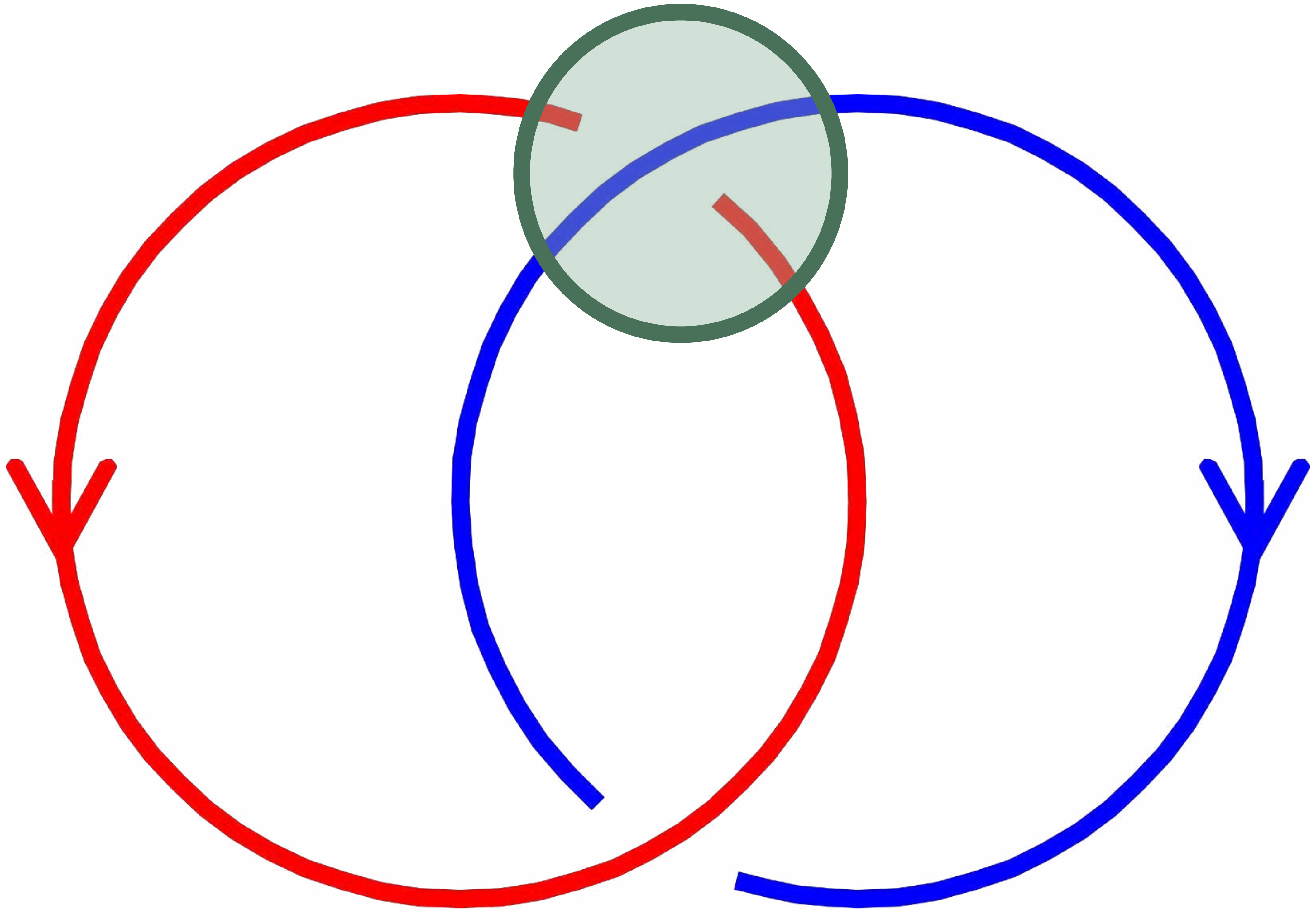
Untying Knots

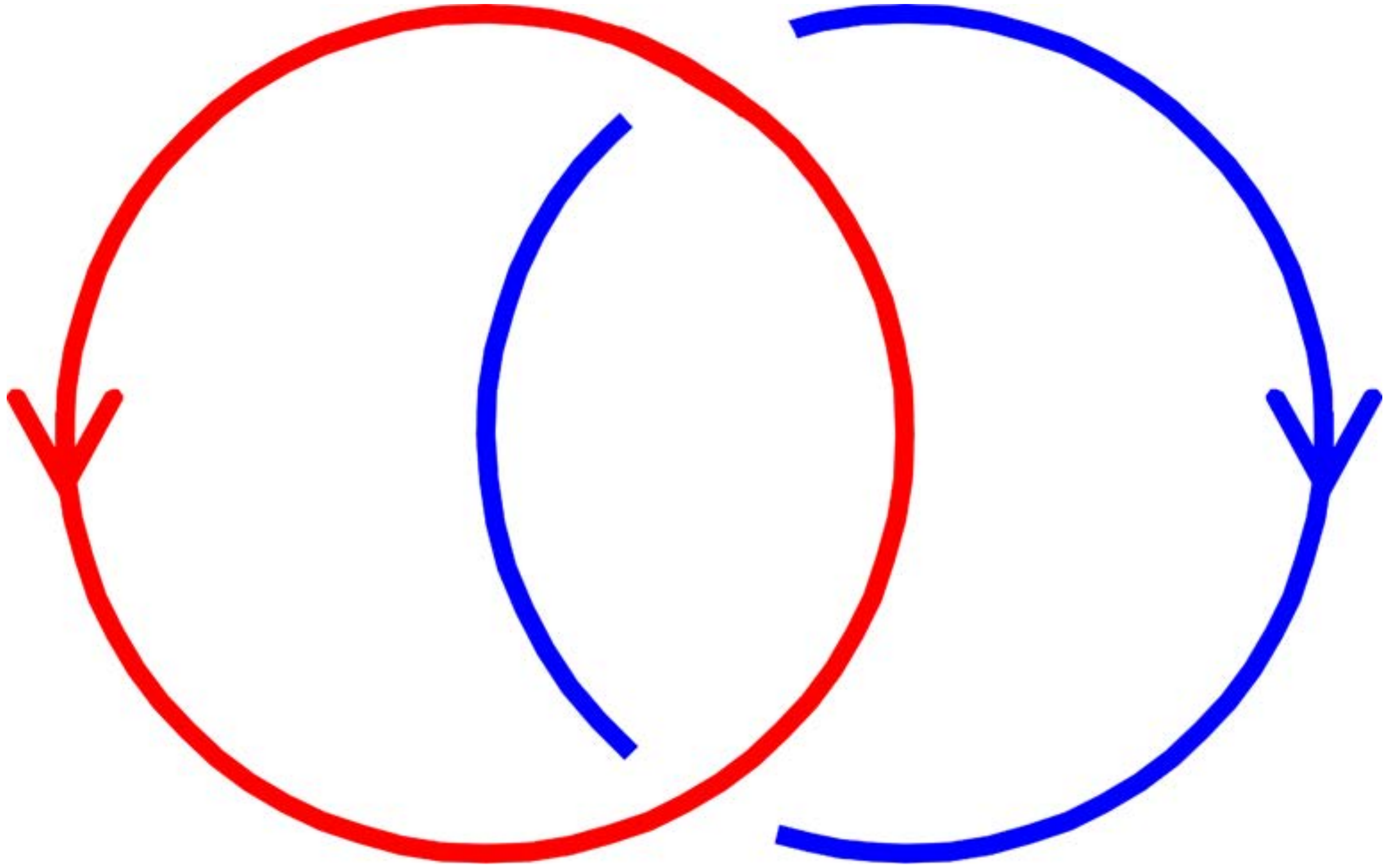


Isotopies in 4D can flip crossings!

Untying Knots

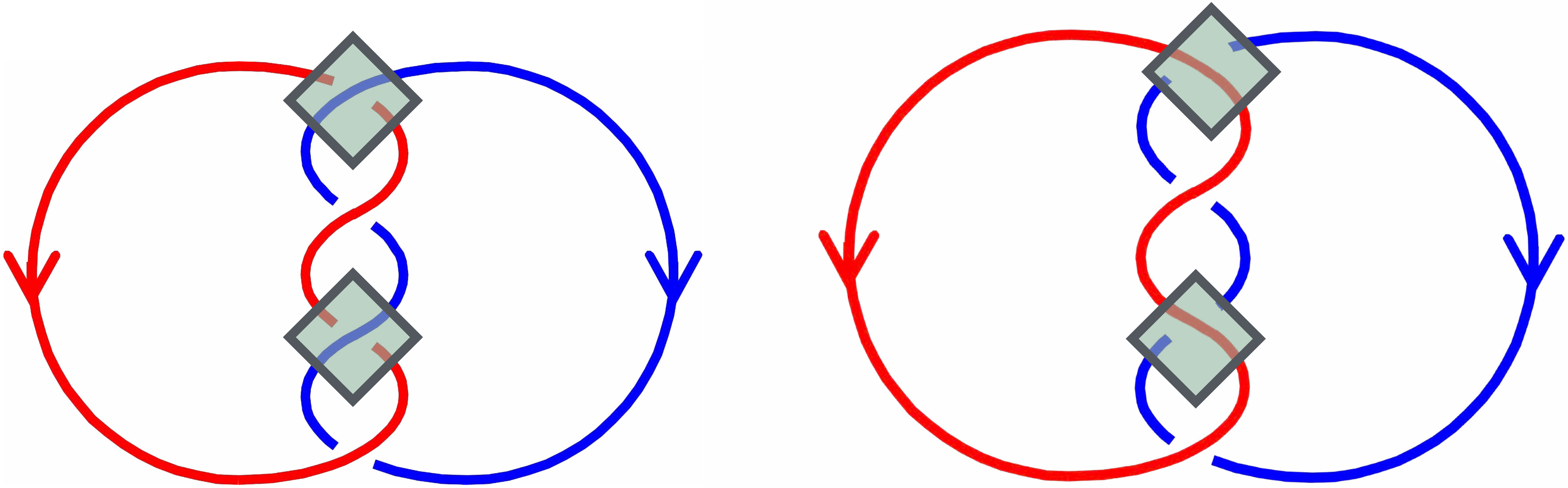
This makes the Hopf link trivial!





Untying Knots

Every link can be undone by changing some of its crossings.

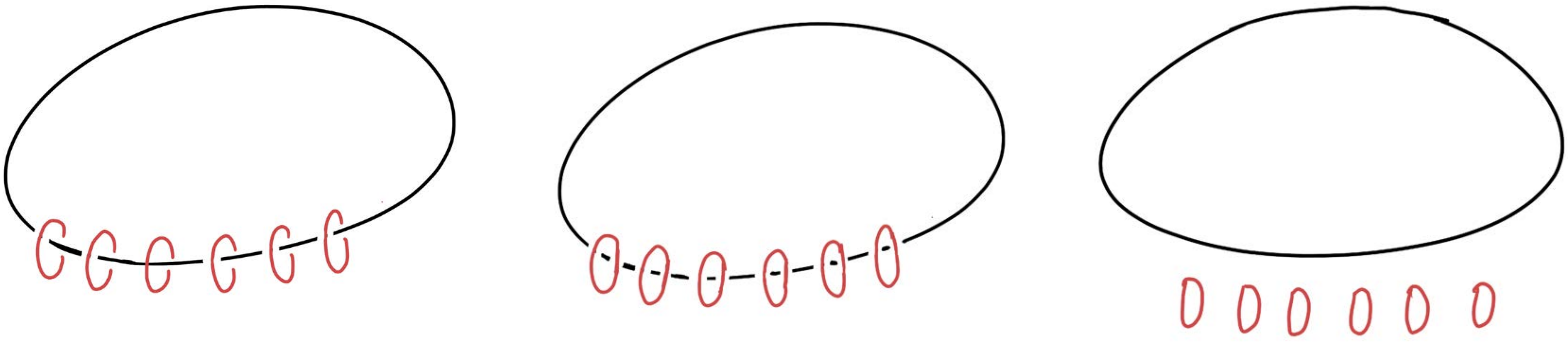


Untying Knots

Every link can be undone by changing some of its crossings.

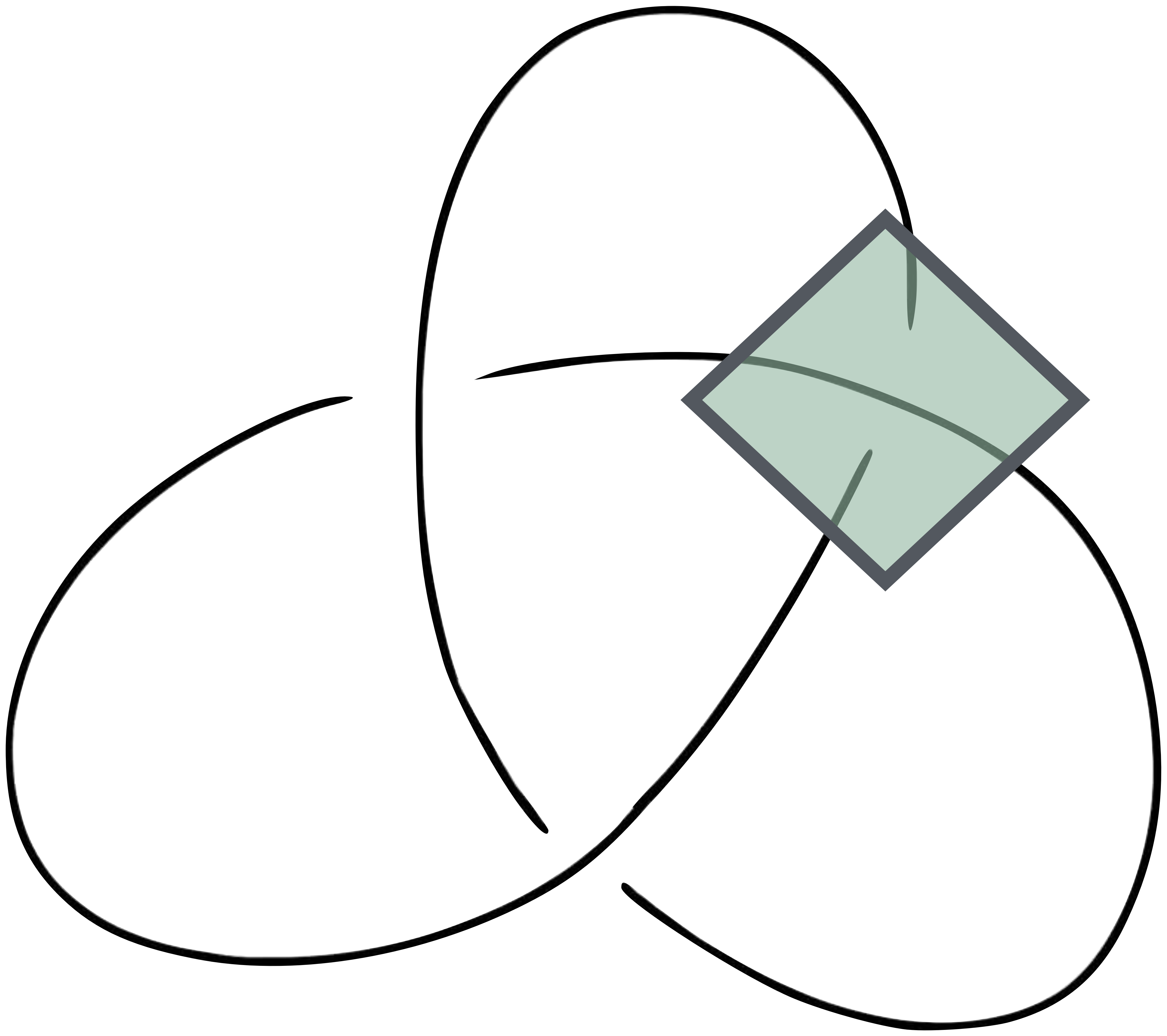
Corollary

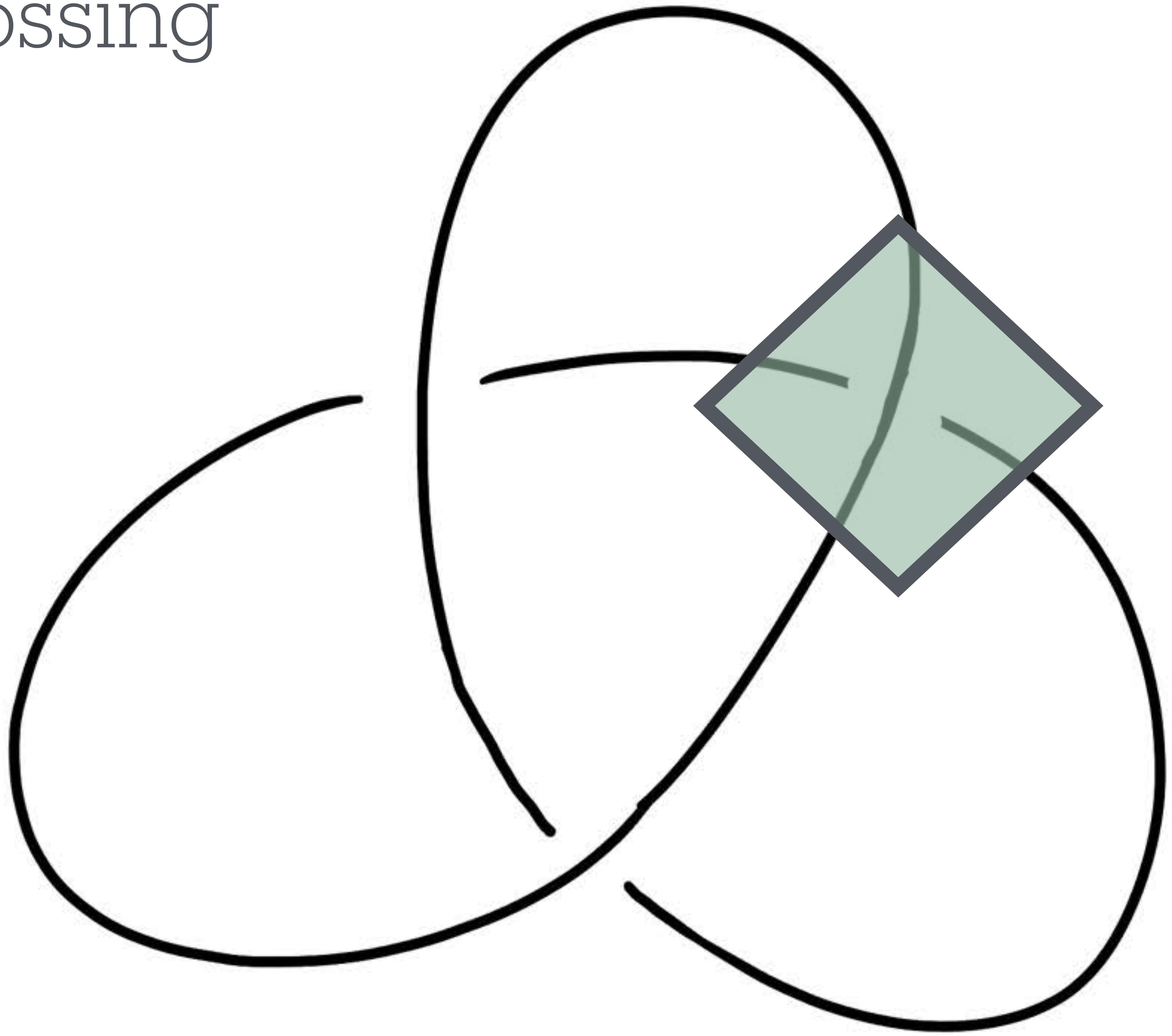
Necklaces, chains, and keyrings come apart in the 4th dimension!



Untying Knots

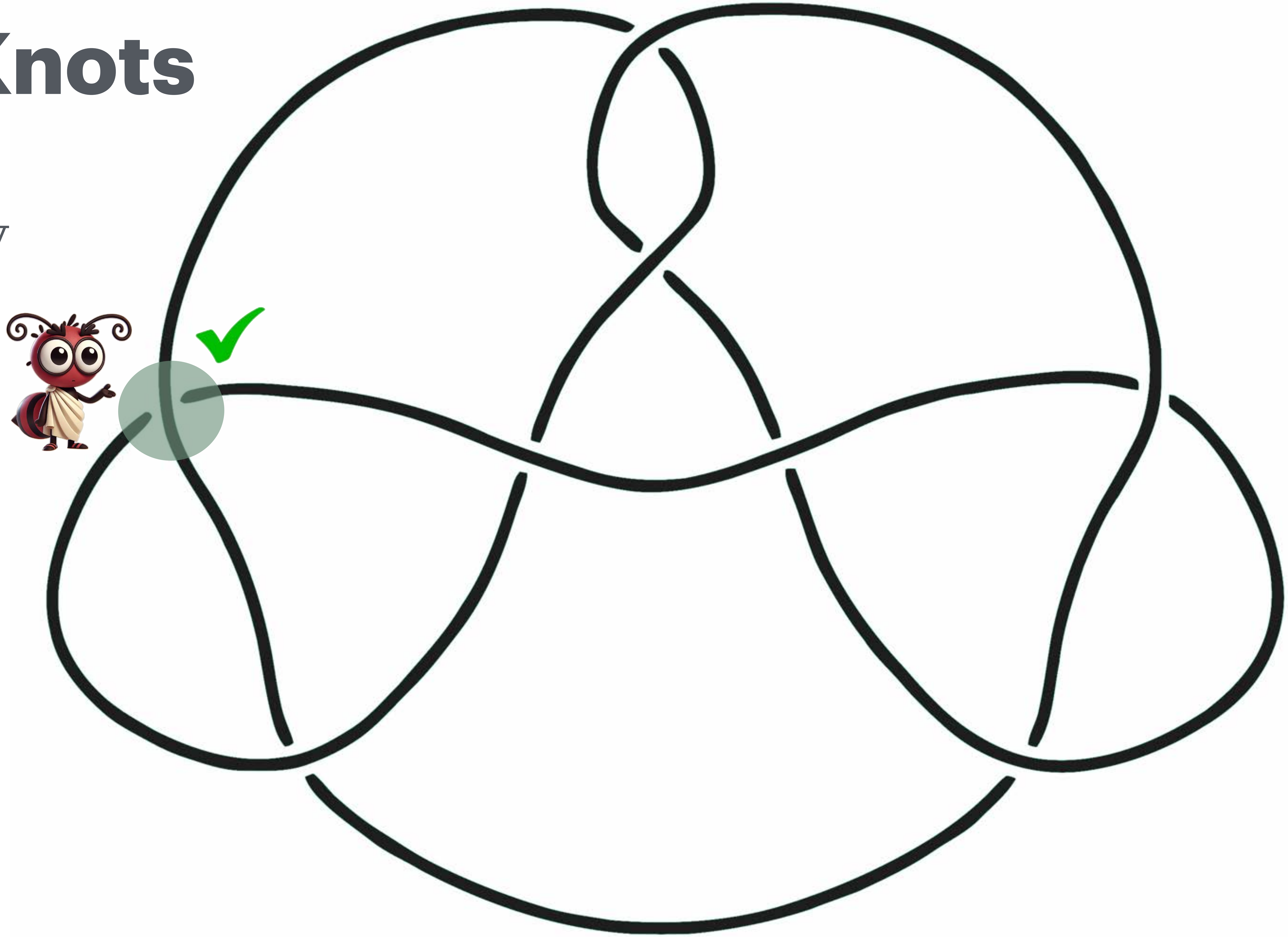
The trefoil is undone by a single crossing





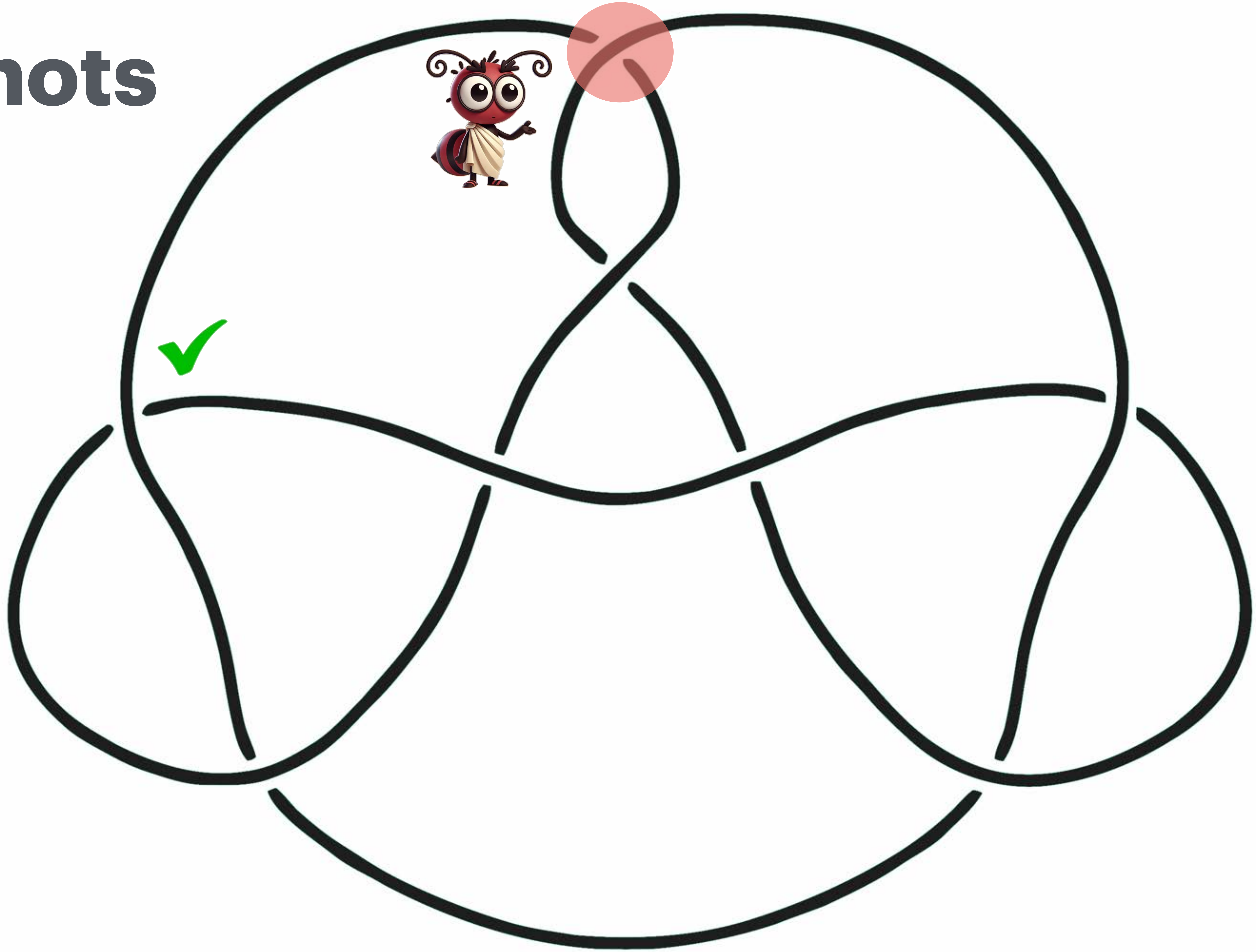
Untying Knots

All knots come undone this way



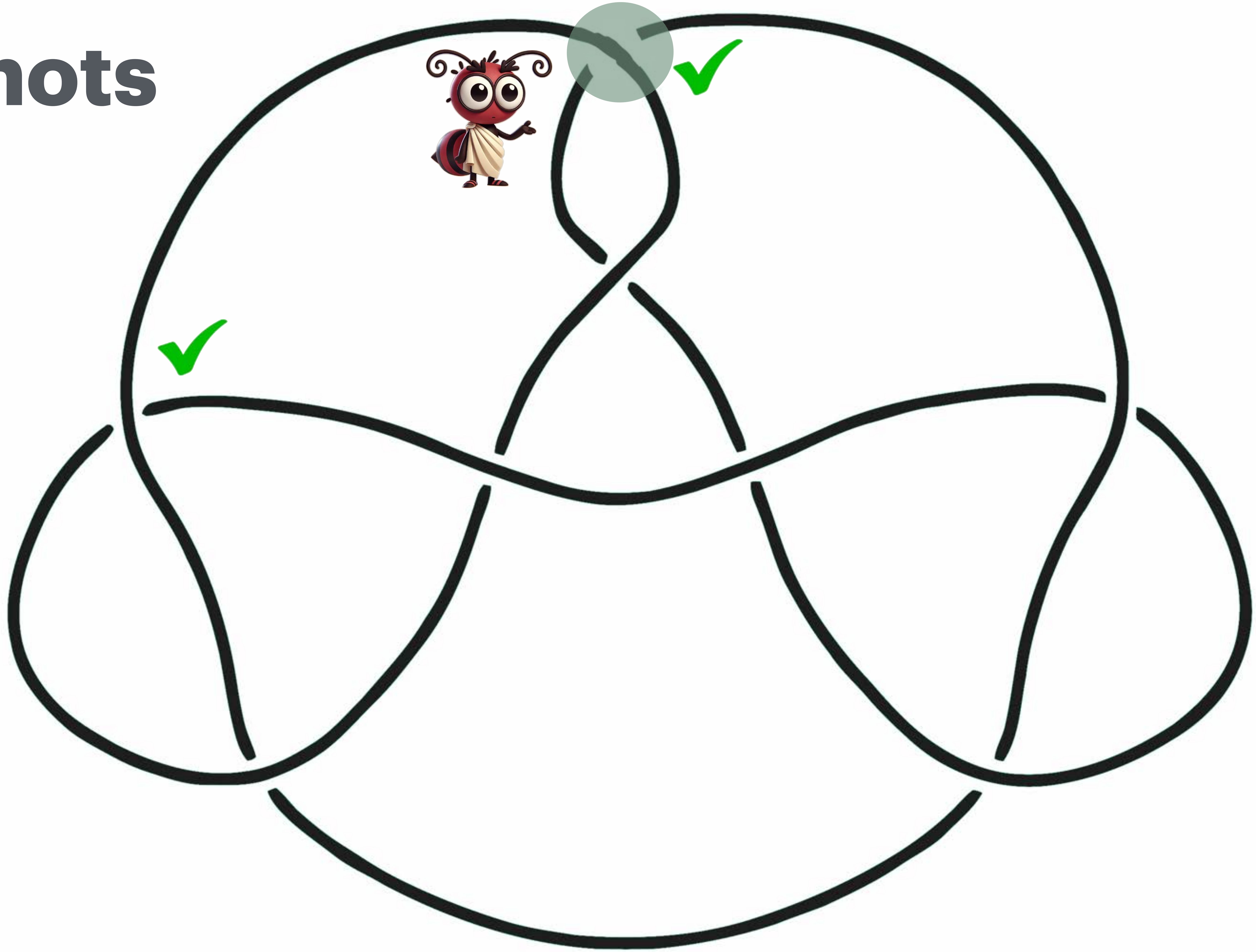
Untying Knots

All knots come undone this way



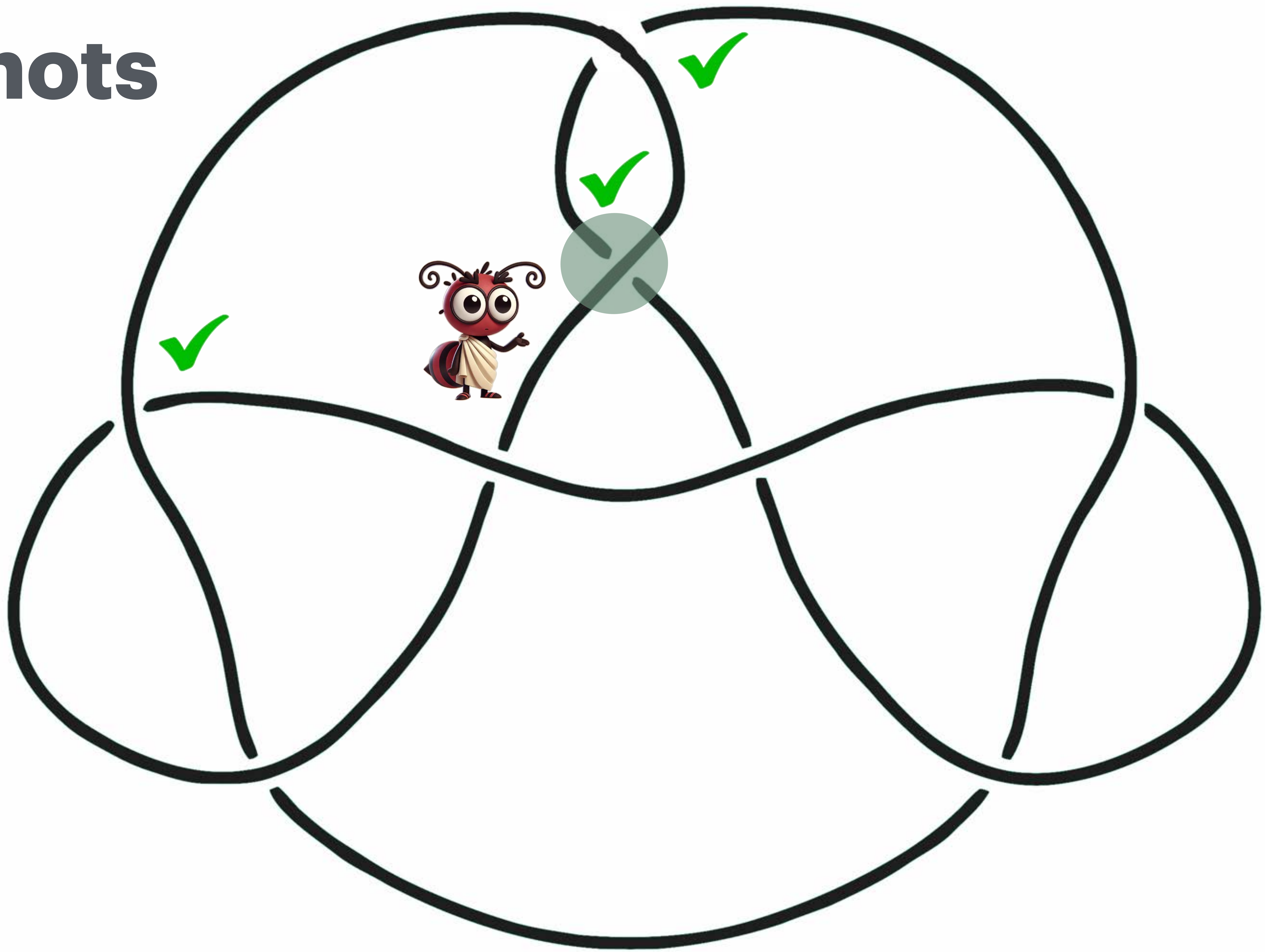
Untying Knots

All knots come
undone this way



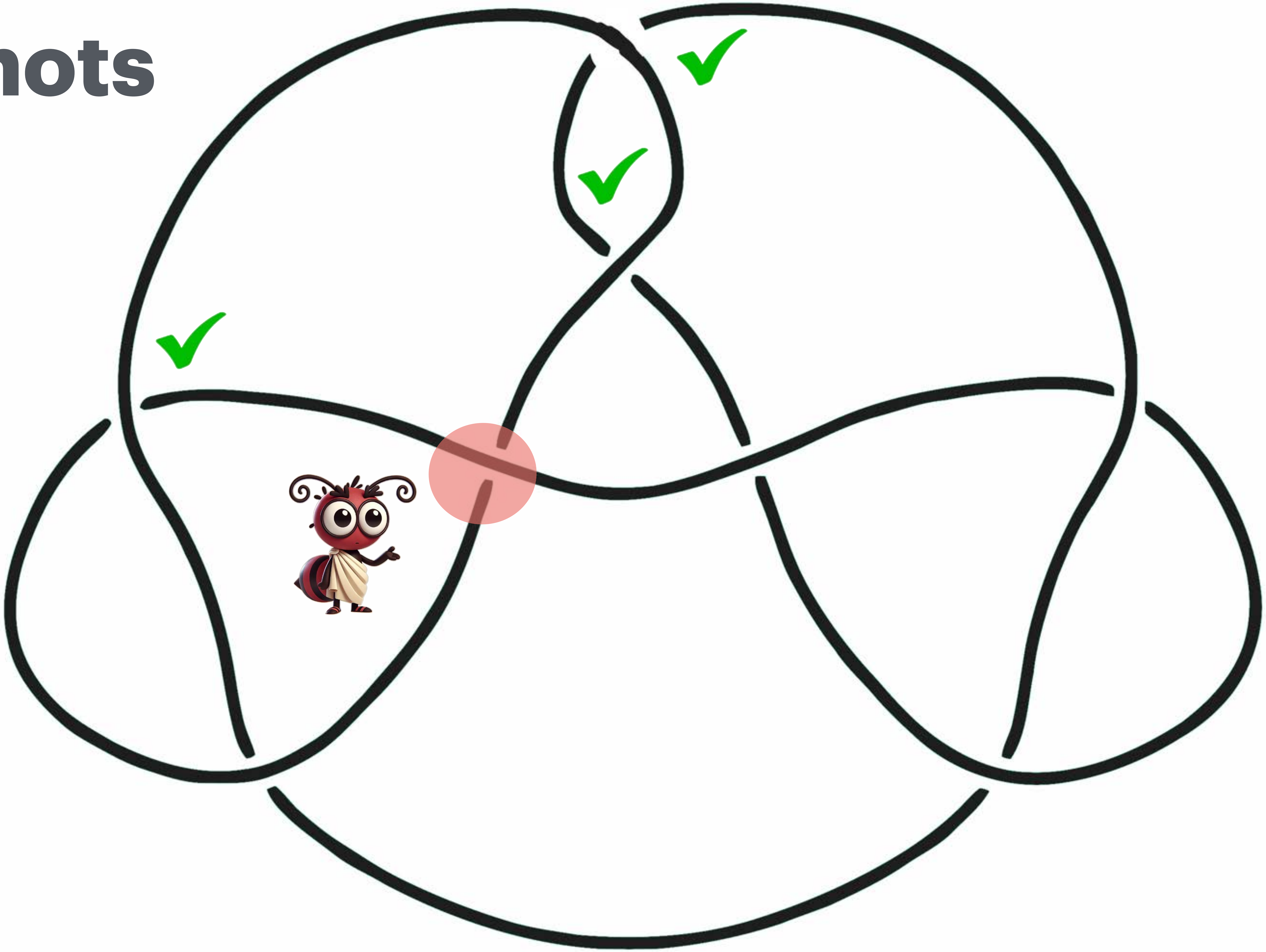
Untying Knots

All knots come undone this way



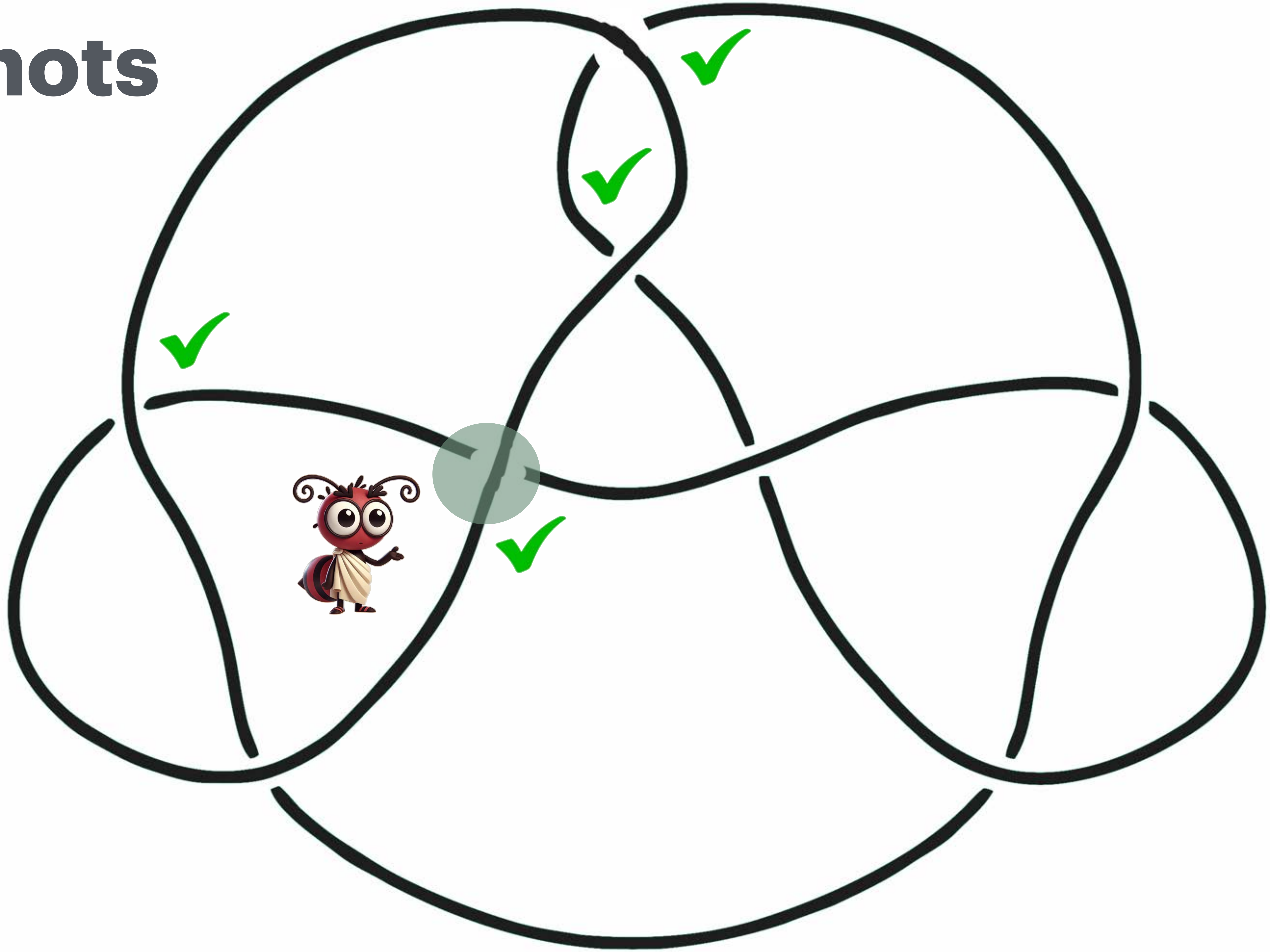
Untying Knots

All knots come undone this way



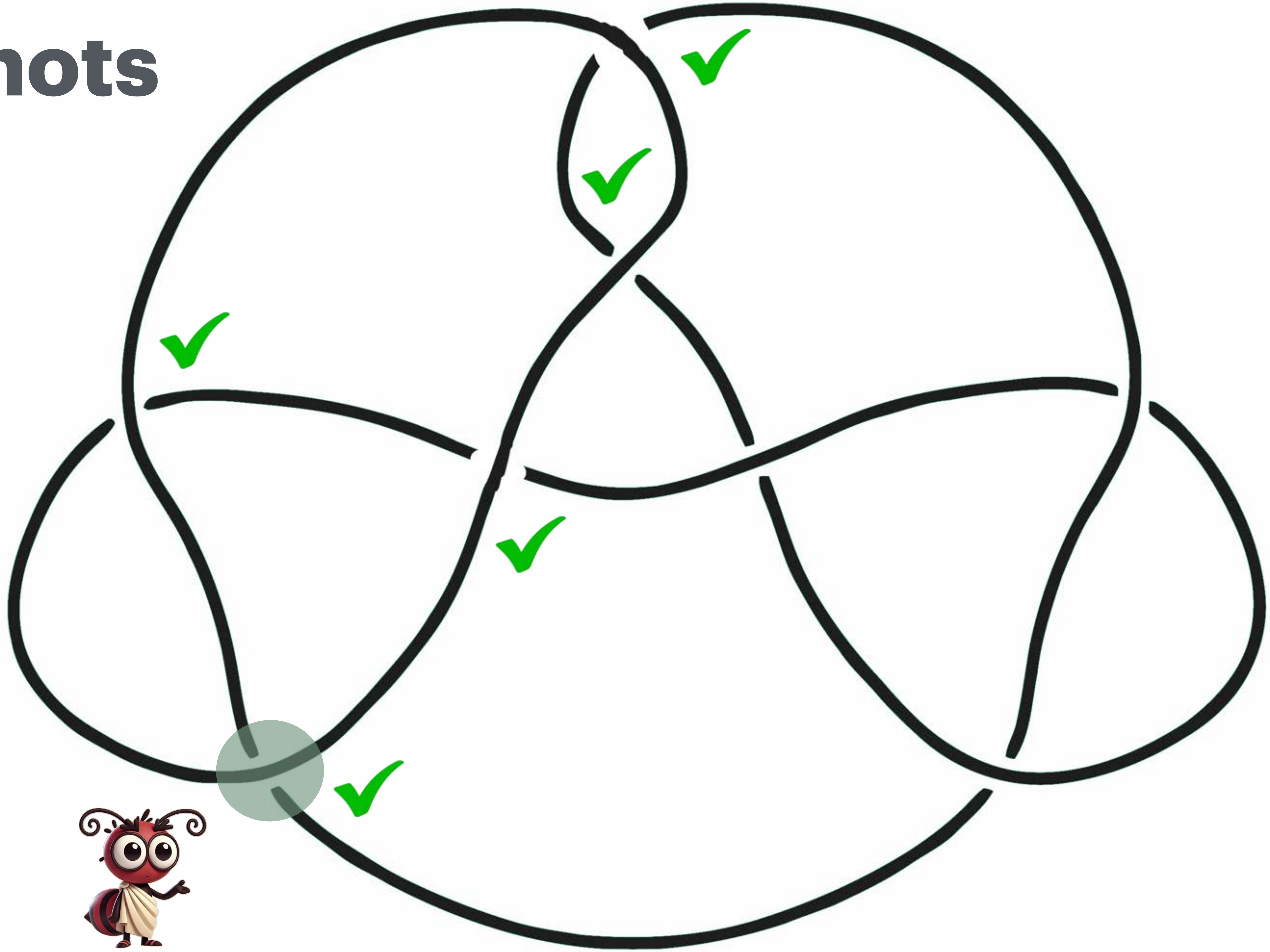
Untying Knots

All knots come undone this way



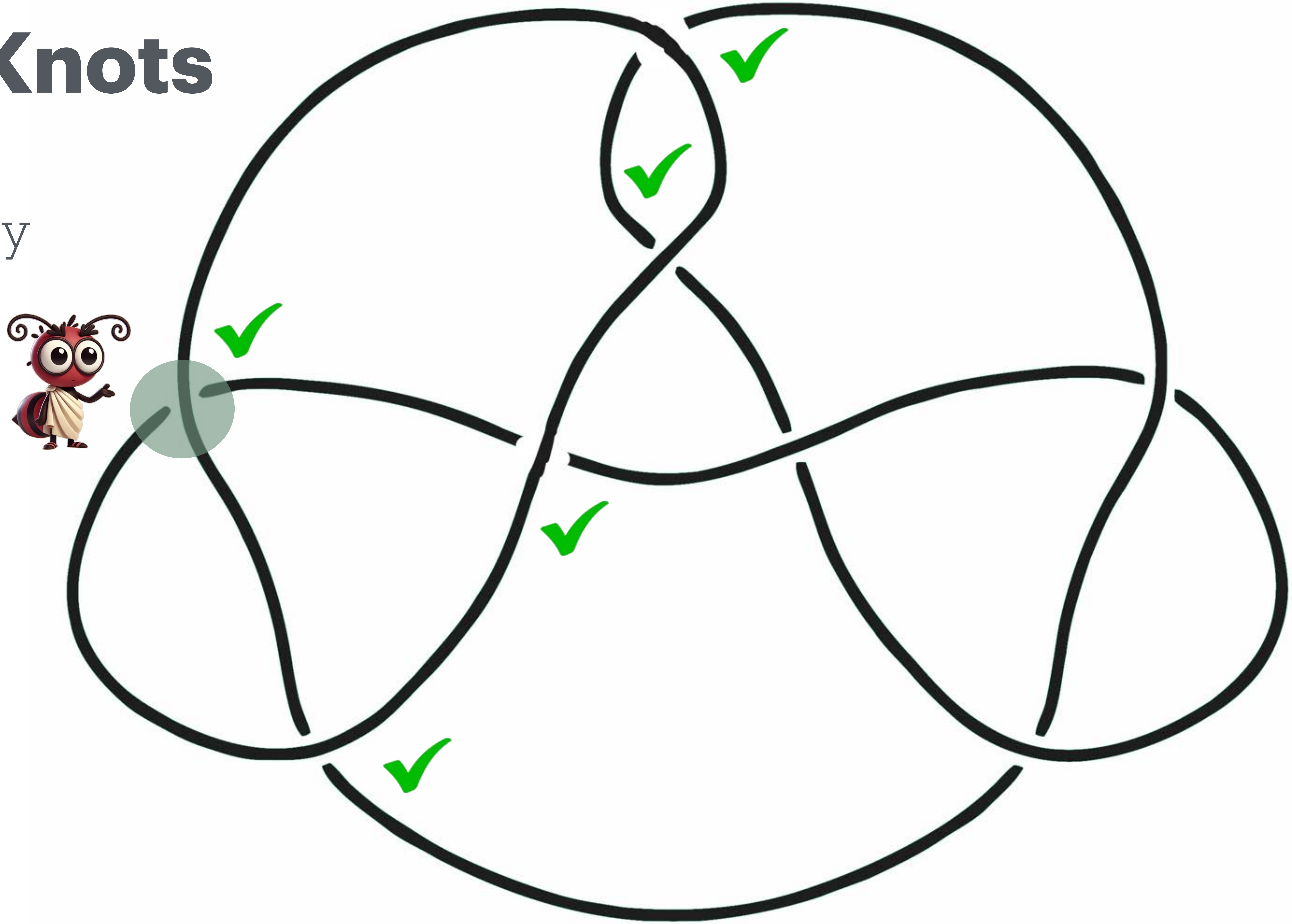
Untying Knots

All knots come
undone this way



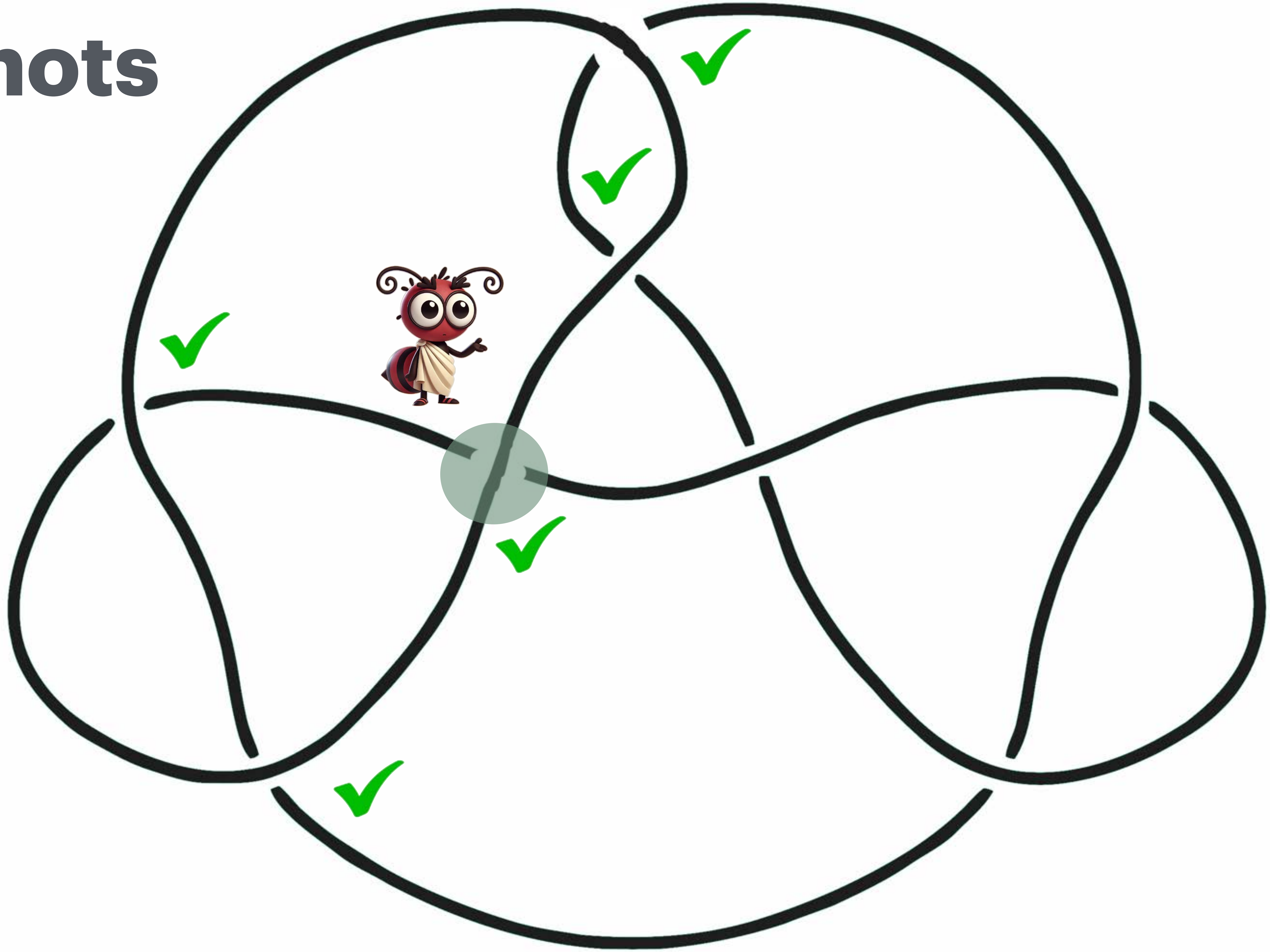
Untying Knots

All knots come undone this way



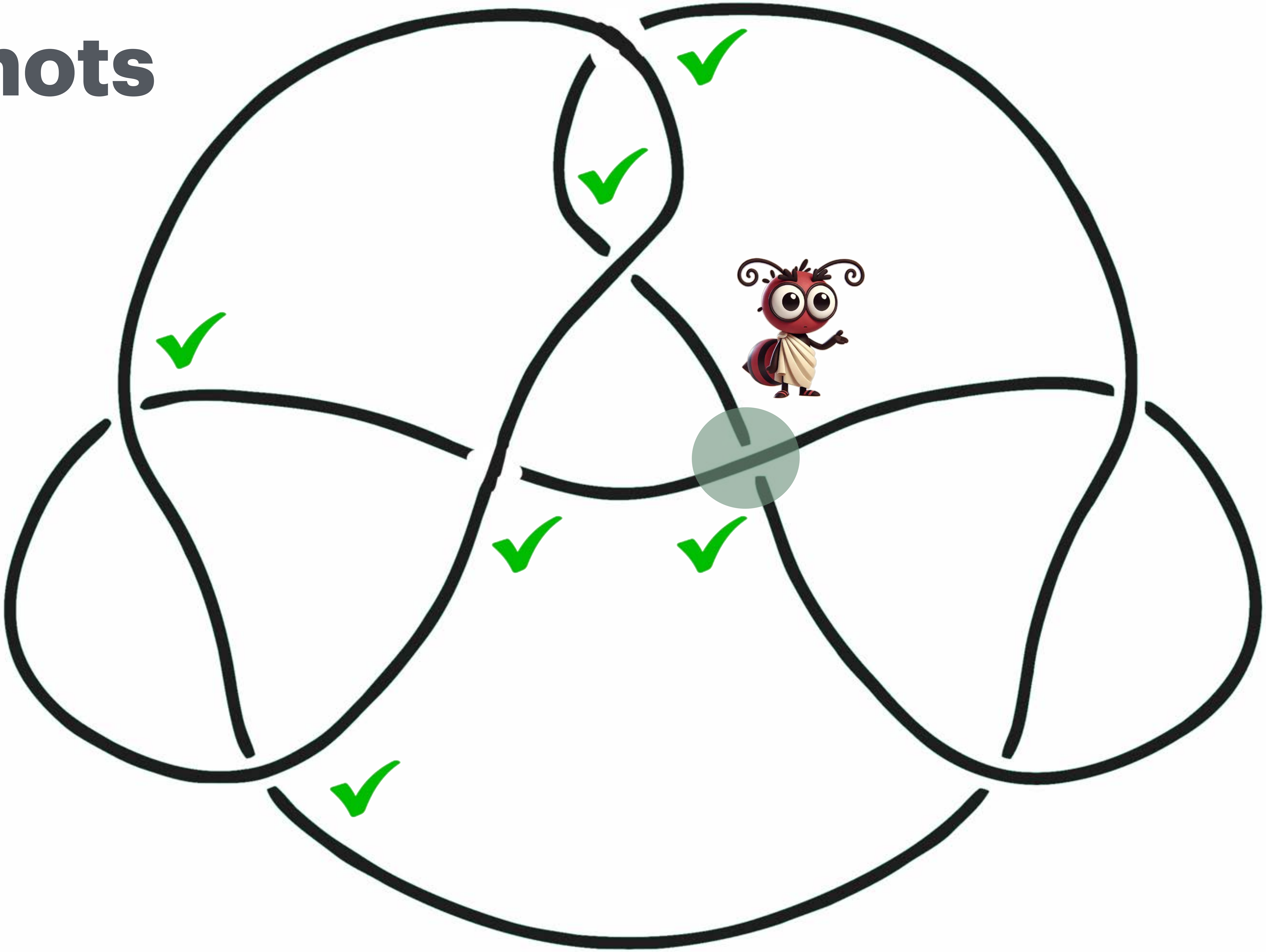
Untying Knots

All knots come undone this way



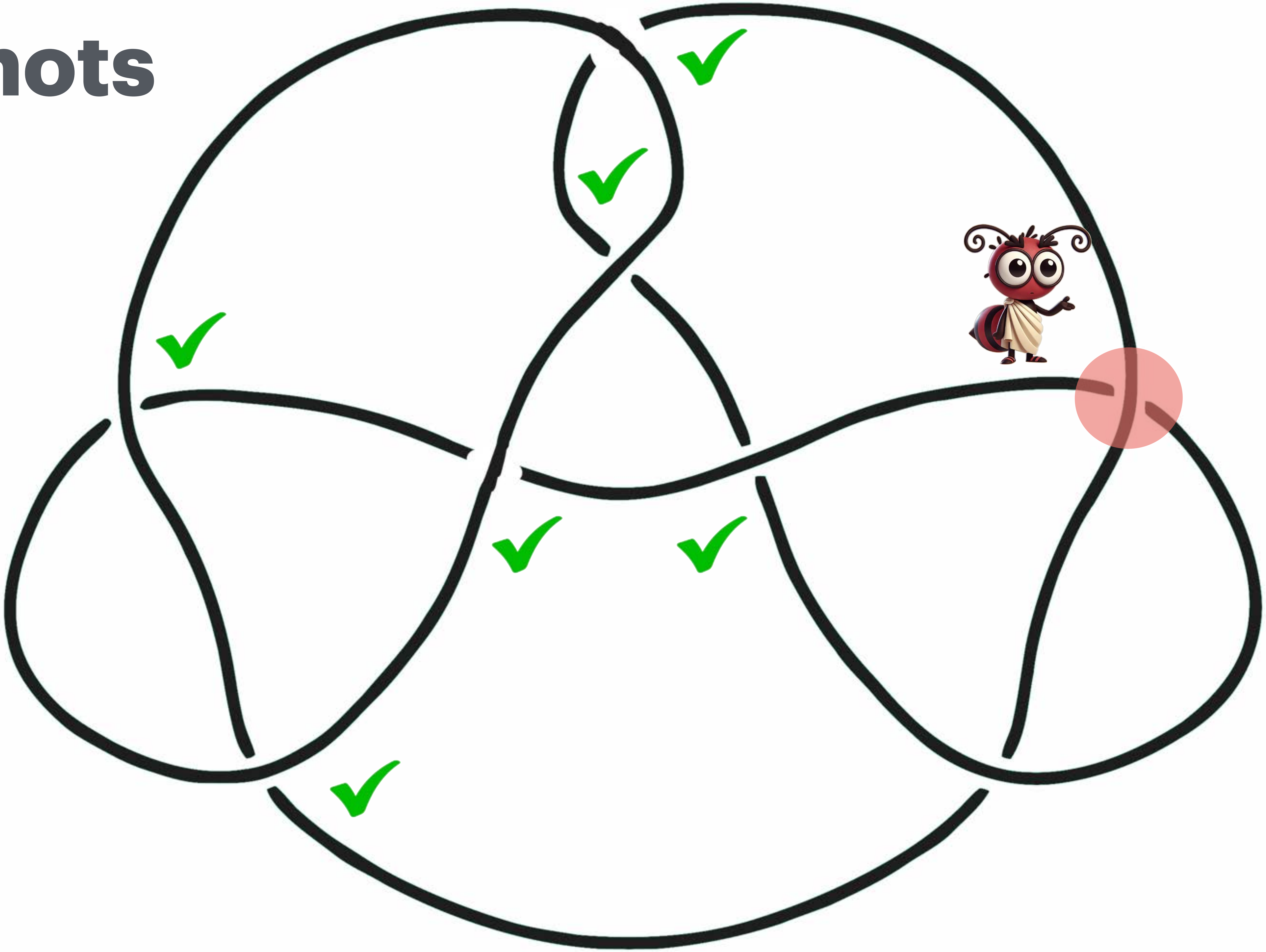
Untying Knots

All knots come
undone this way



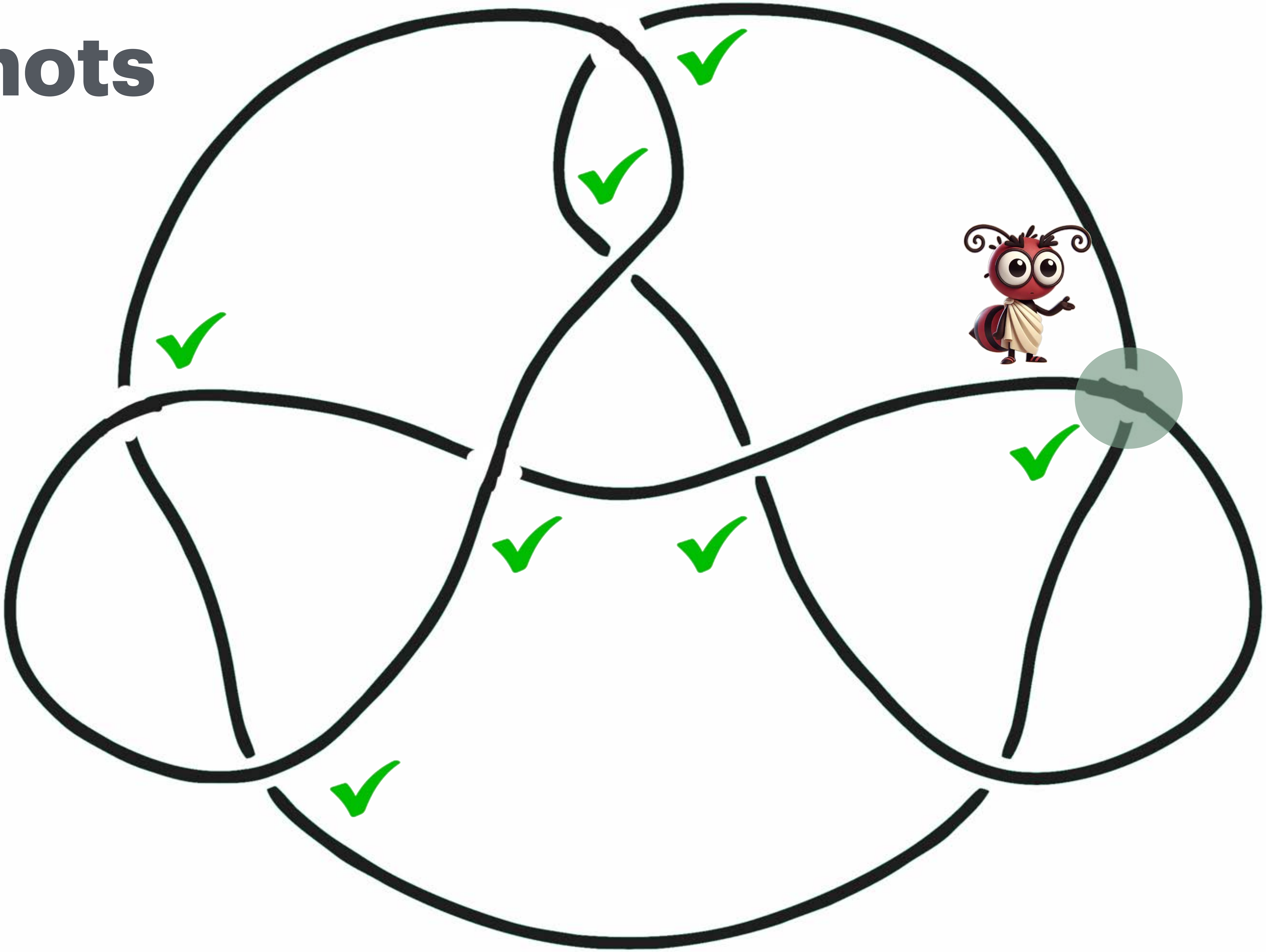
Untying Knots

All knots come undone this way



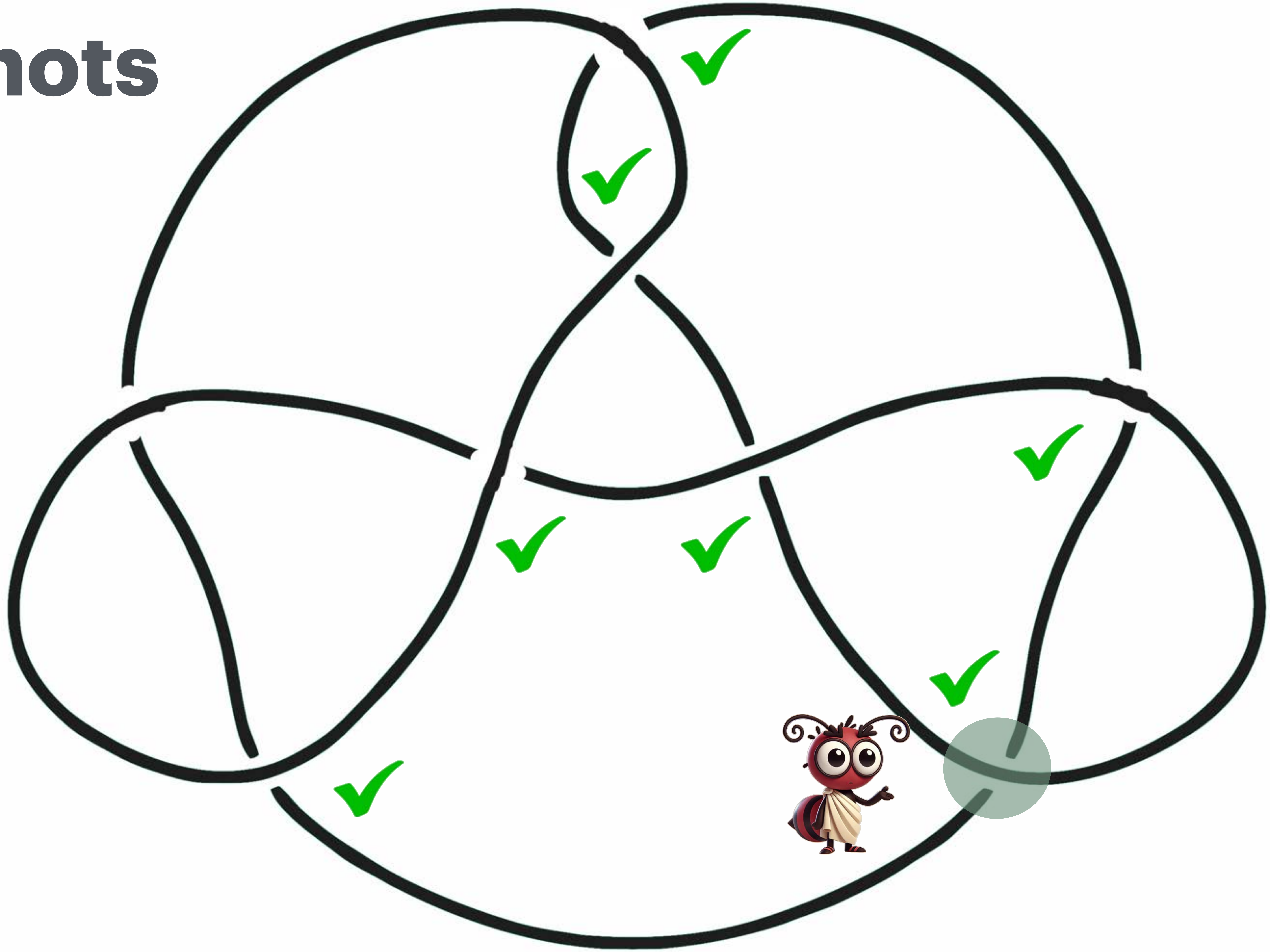
Untying Knots

All knots come
undone this way



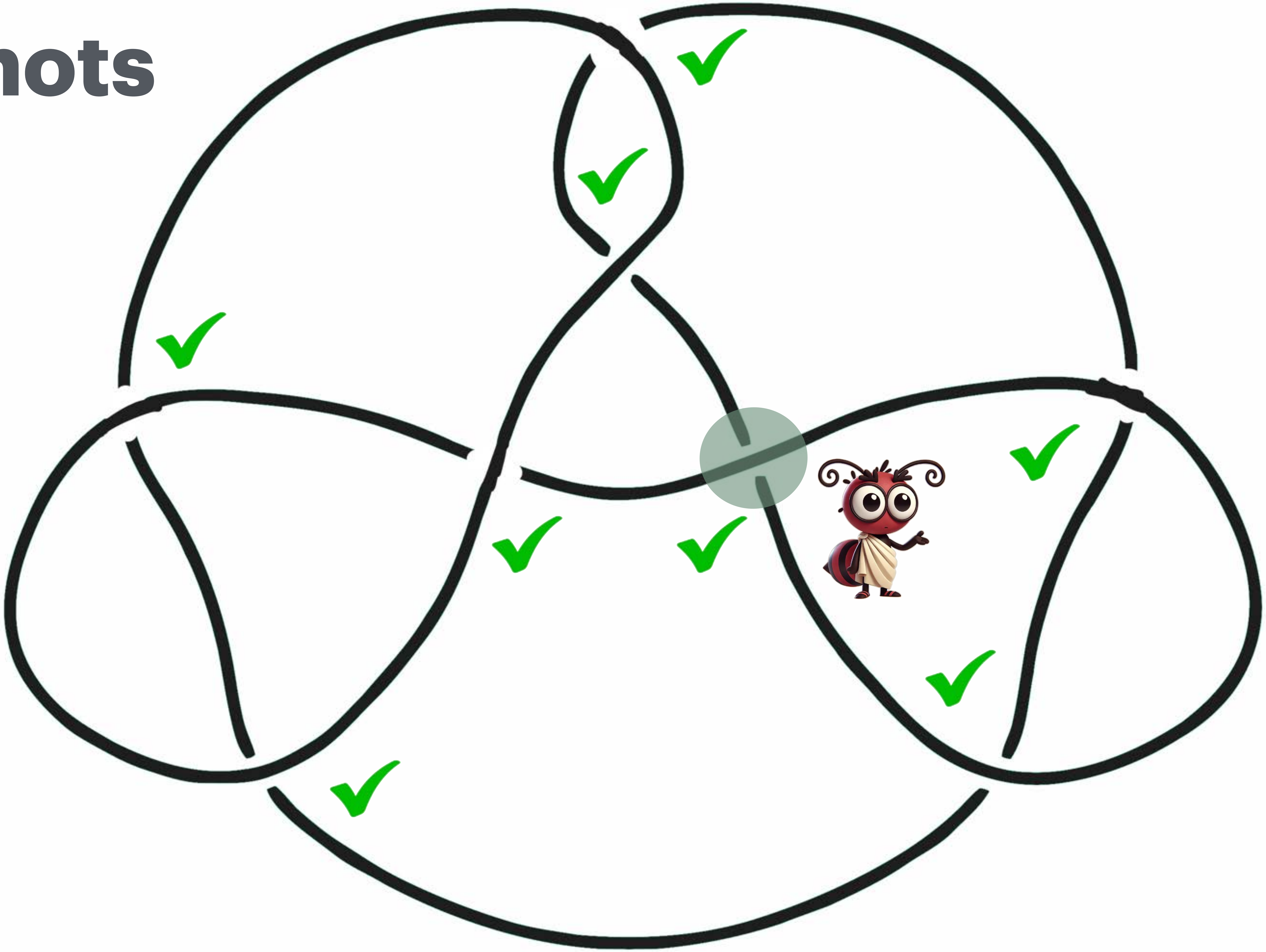
Untying Knots

All knots come
undone this way



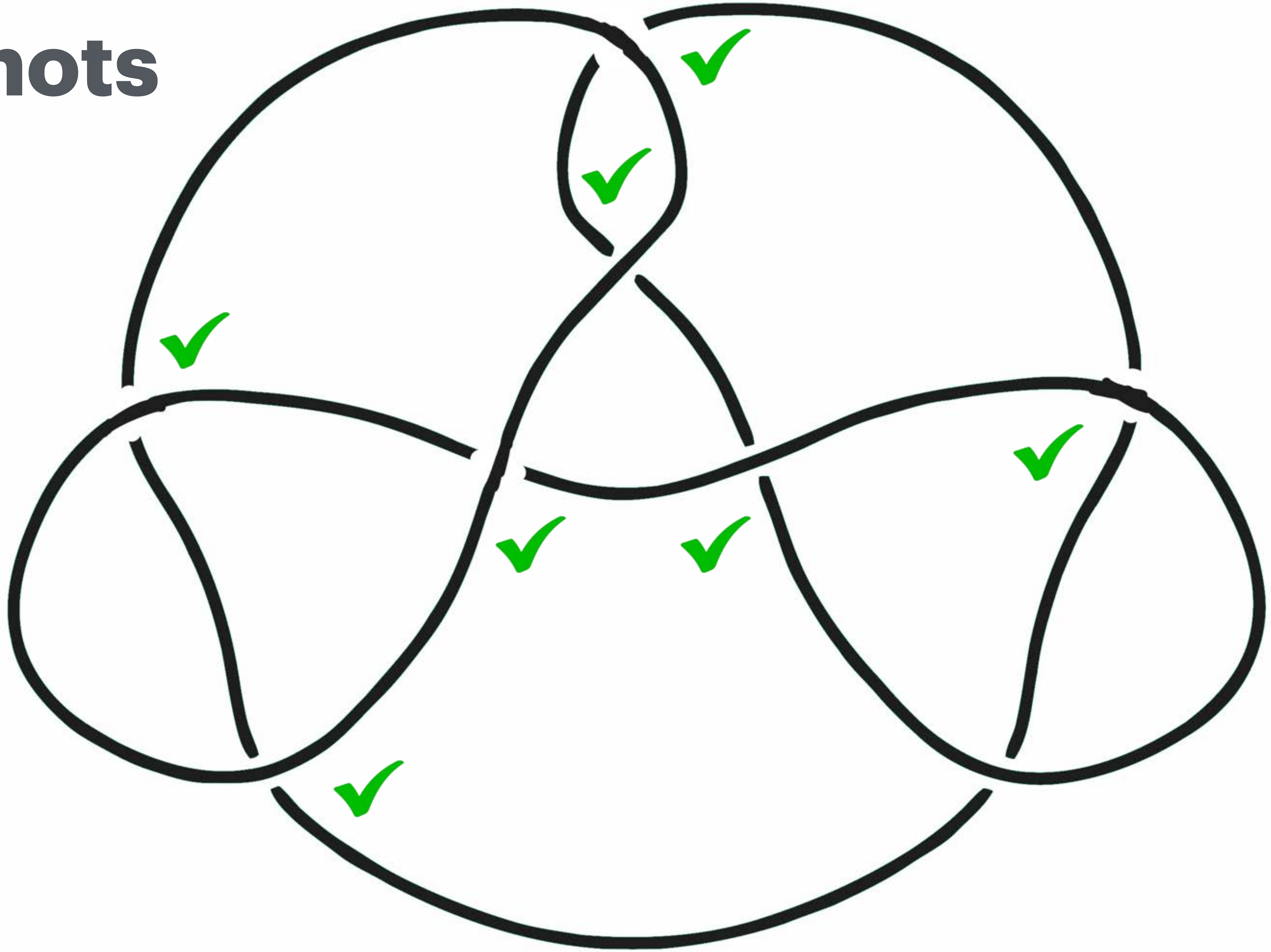
Untying Knots

All knots come
undone this way



Untying Knots

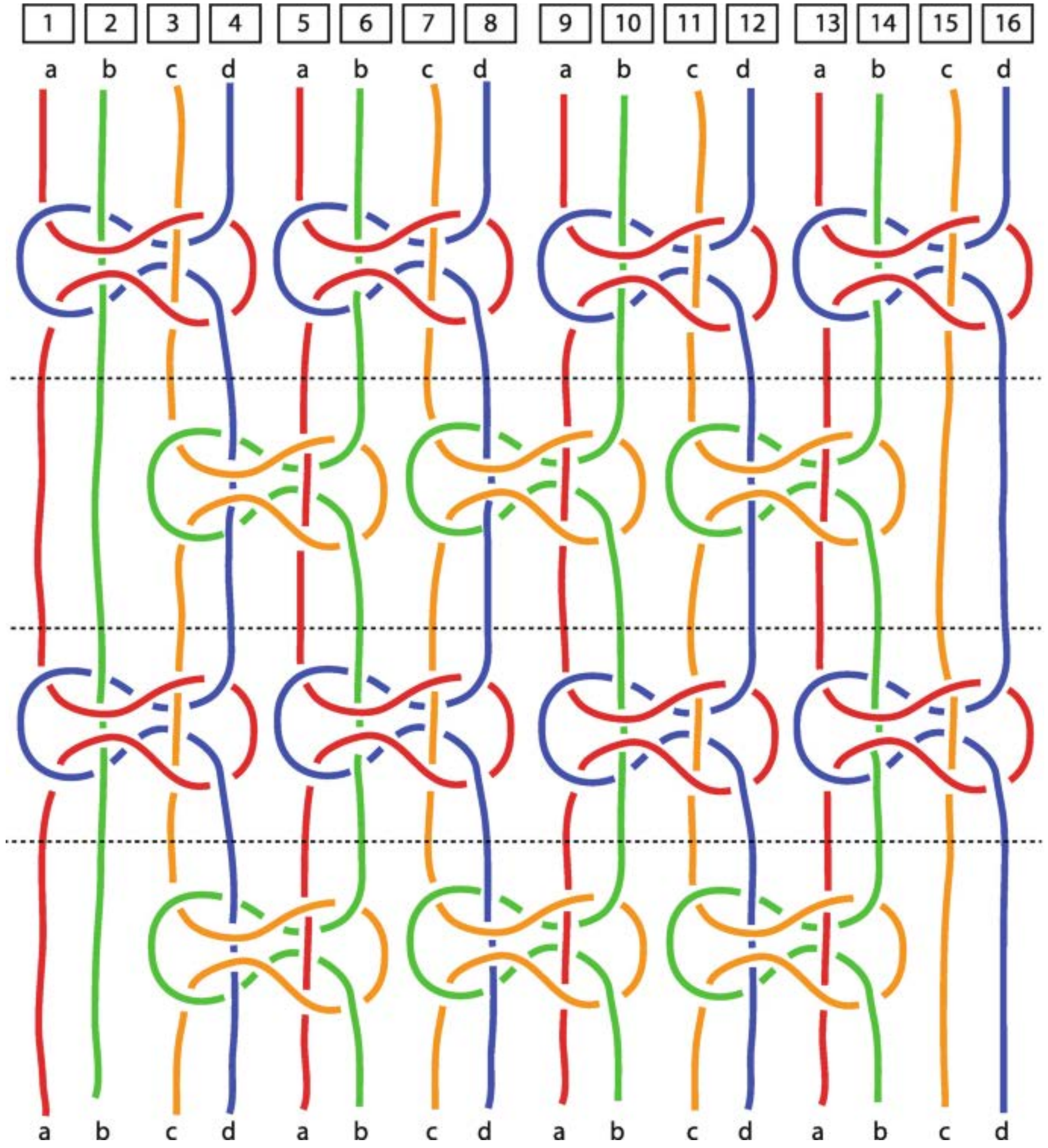
All knots come undone this way



Between the start and end, the string is now *monotonically decreasing in height!*

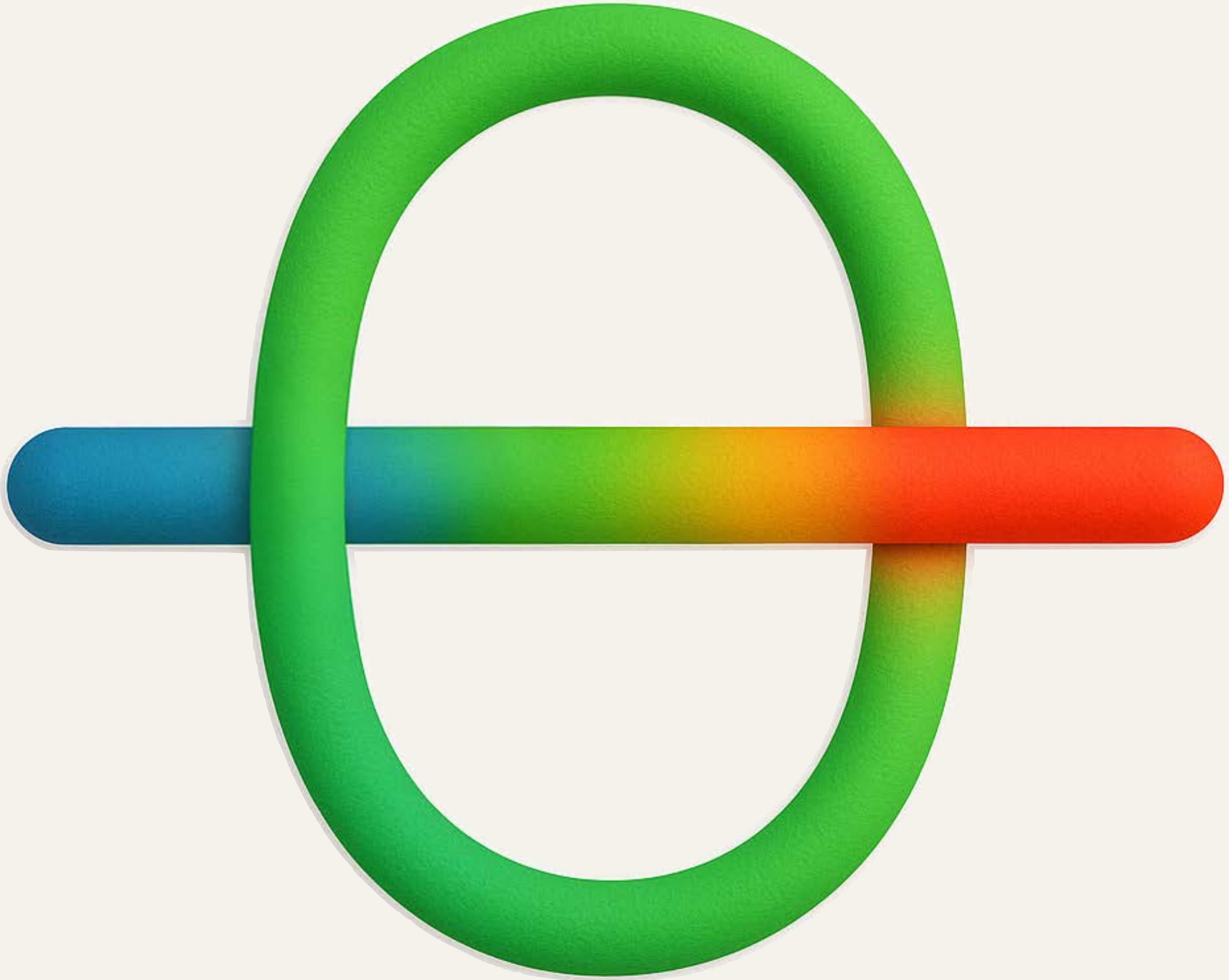
Untying Knots

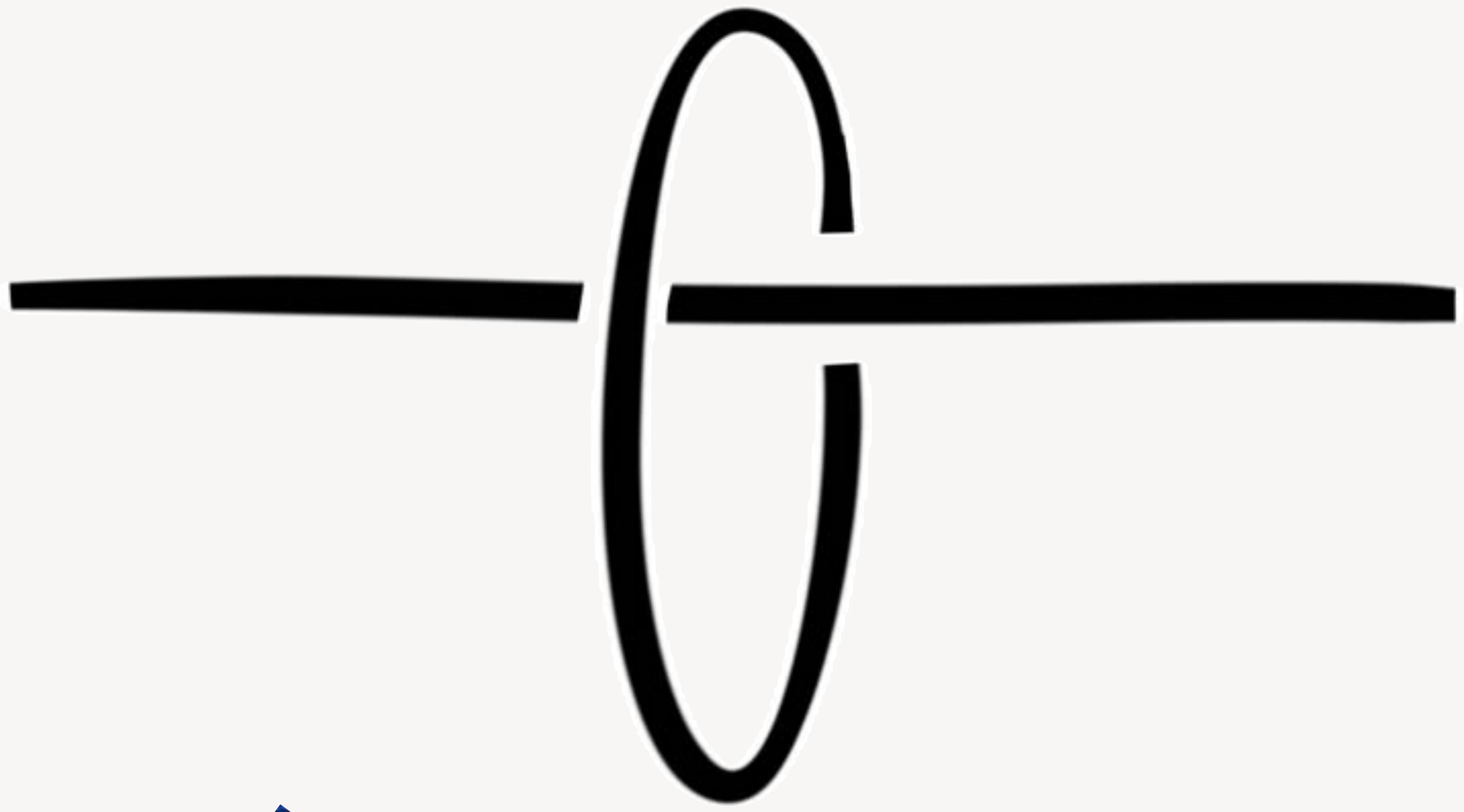
Knitted fabric all falls apart!

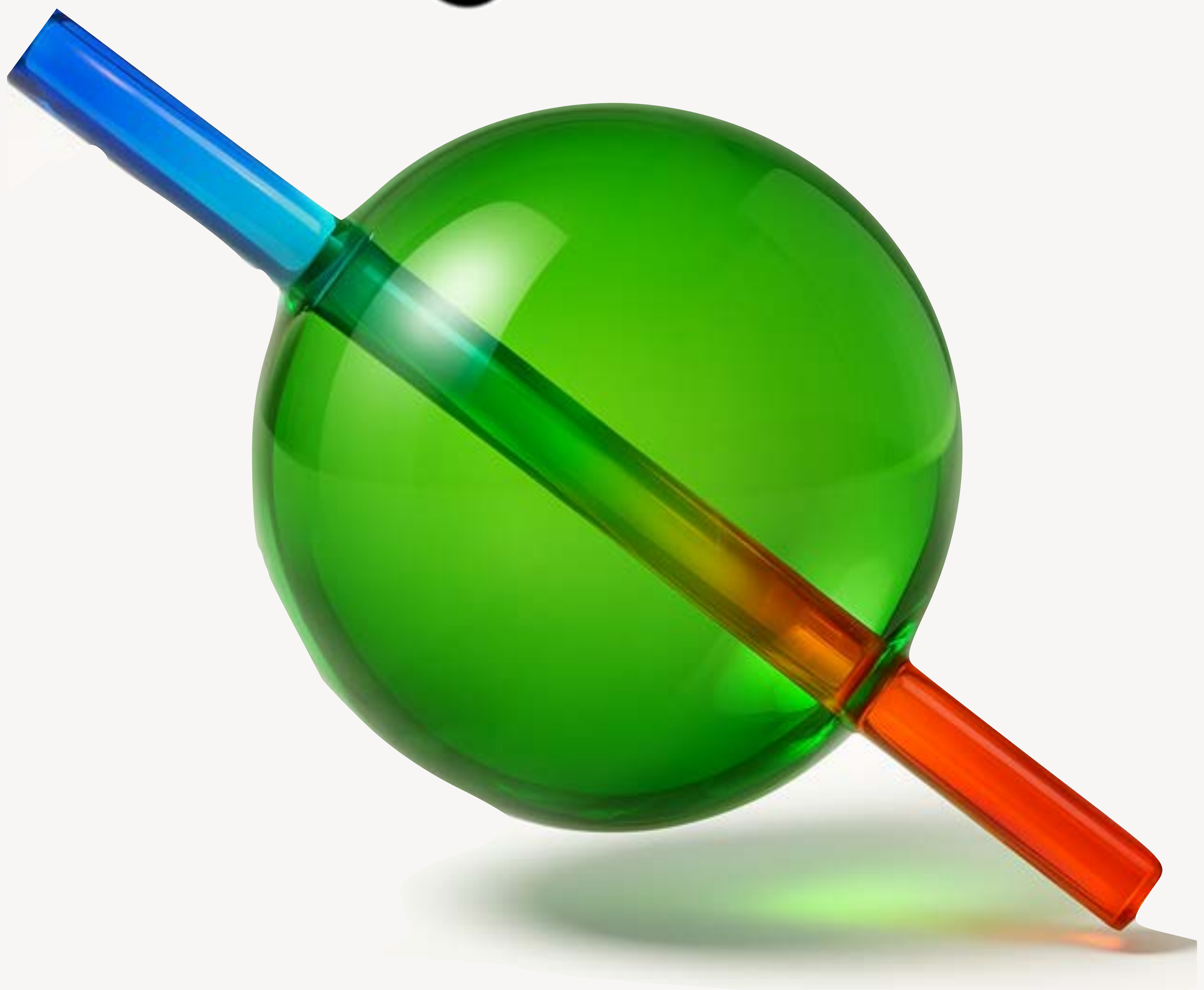


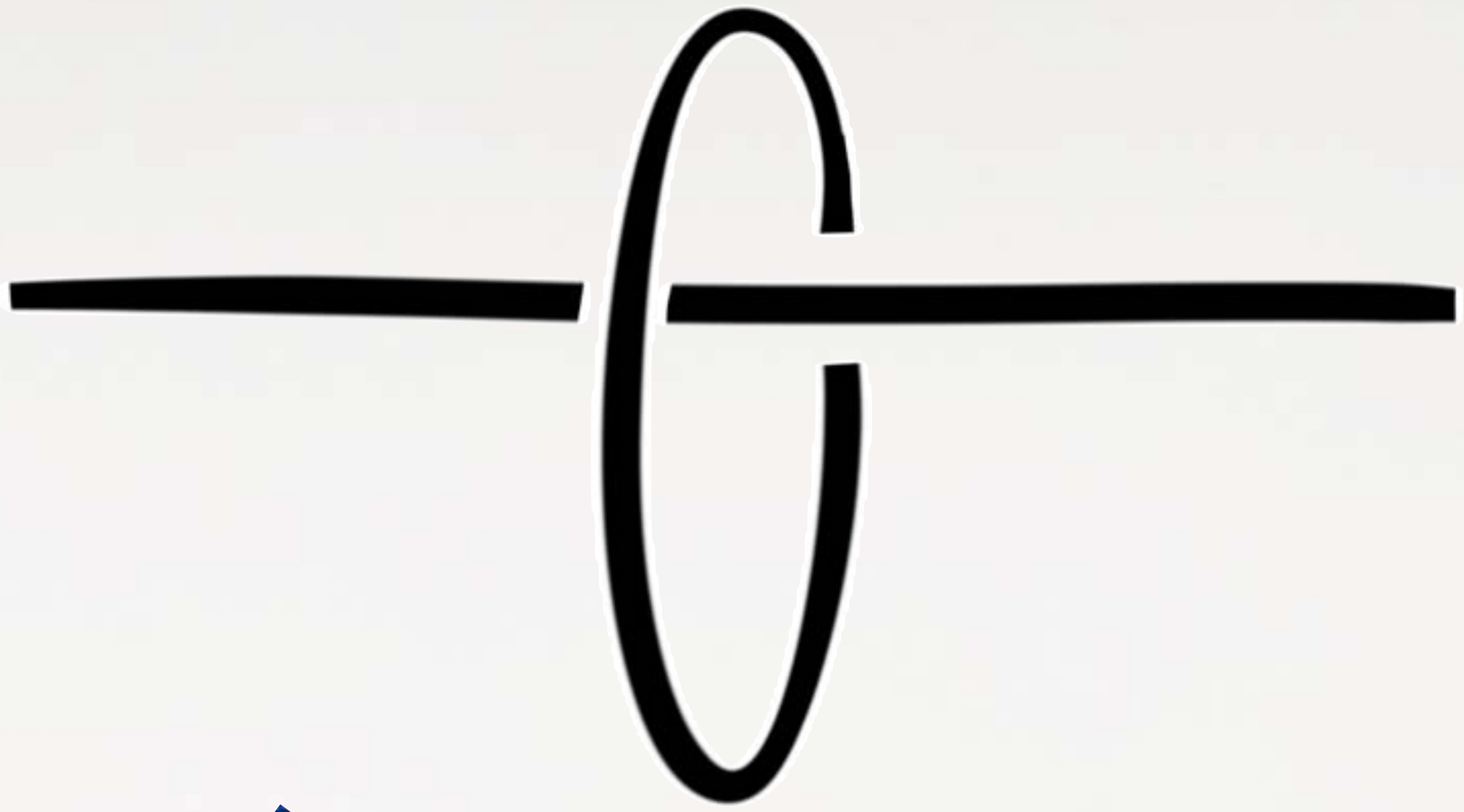
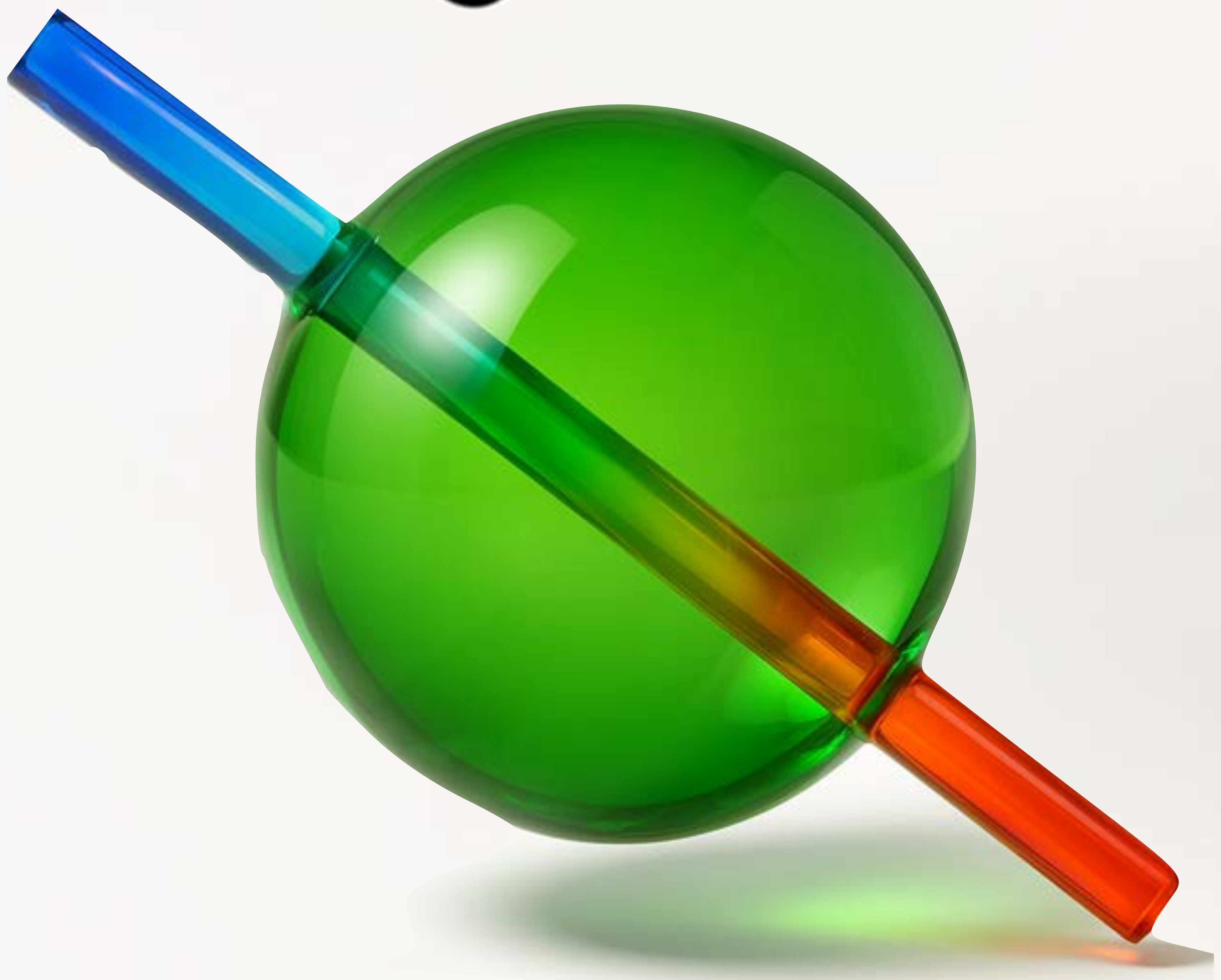
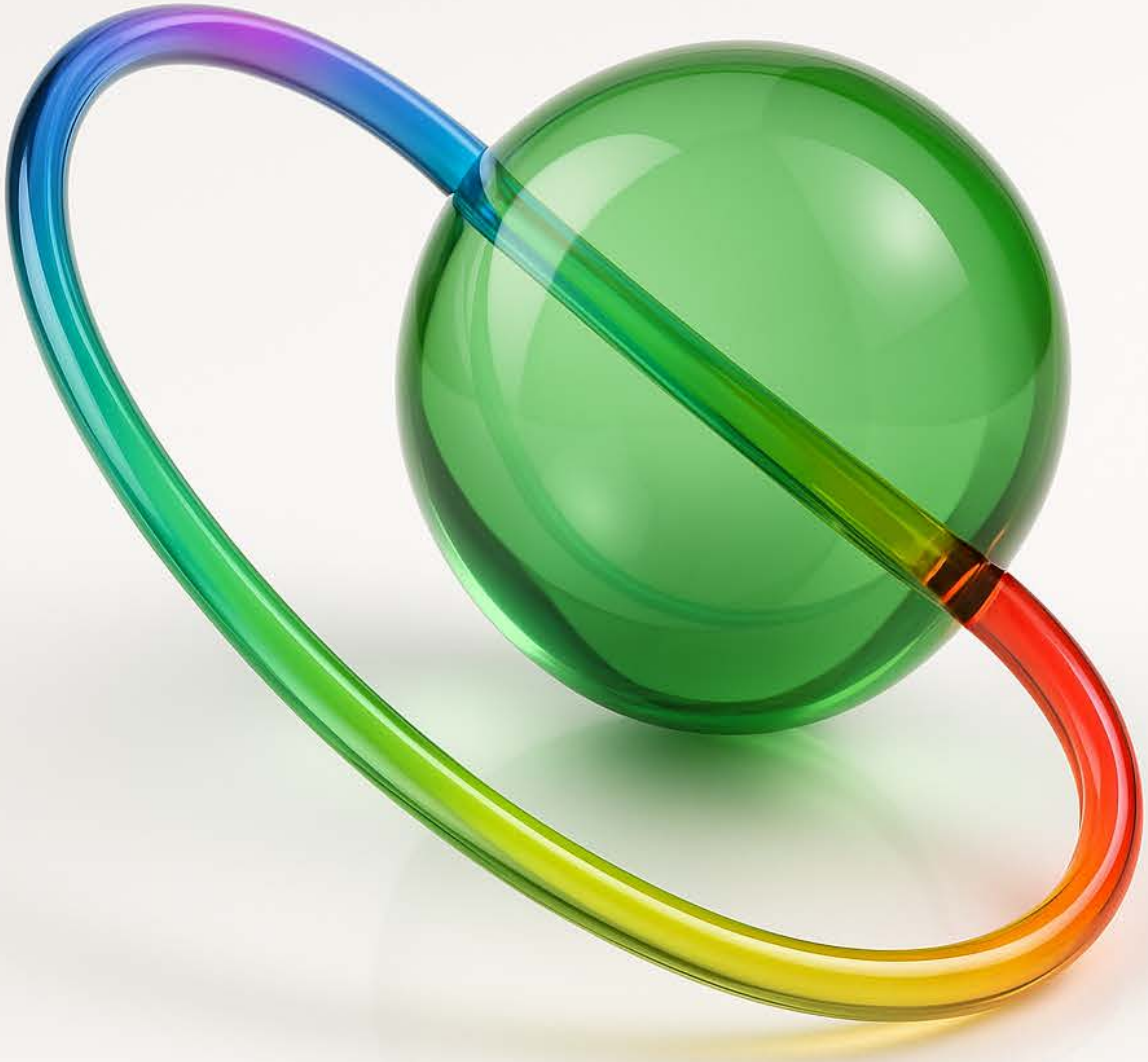
Weird and Wonderful

Creations in the 4th dimension



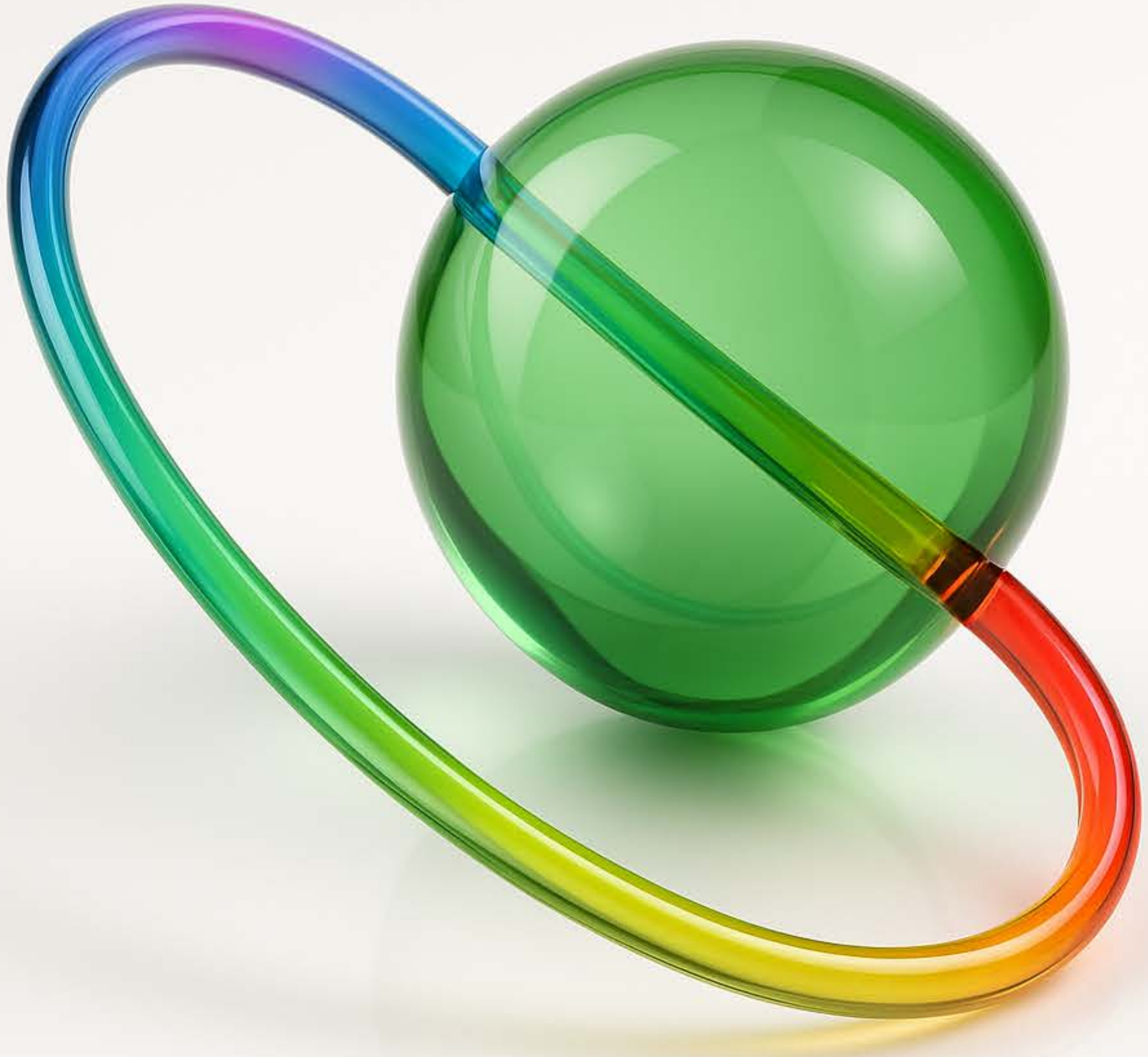




This is a nontrivial closed loop in the complement of a 2 sphere in $\mathbb{R}^4$

In fact, it generates $\pi_1\left(\mathbb{R}^4 \setminus \mathbb{S}^2\right)$!



There are *knotted circles in* $\mathbb{R}^3$....

Are there ***knotted spheres in*** $\mathbb{R}^4$?

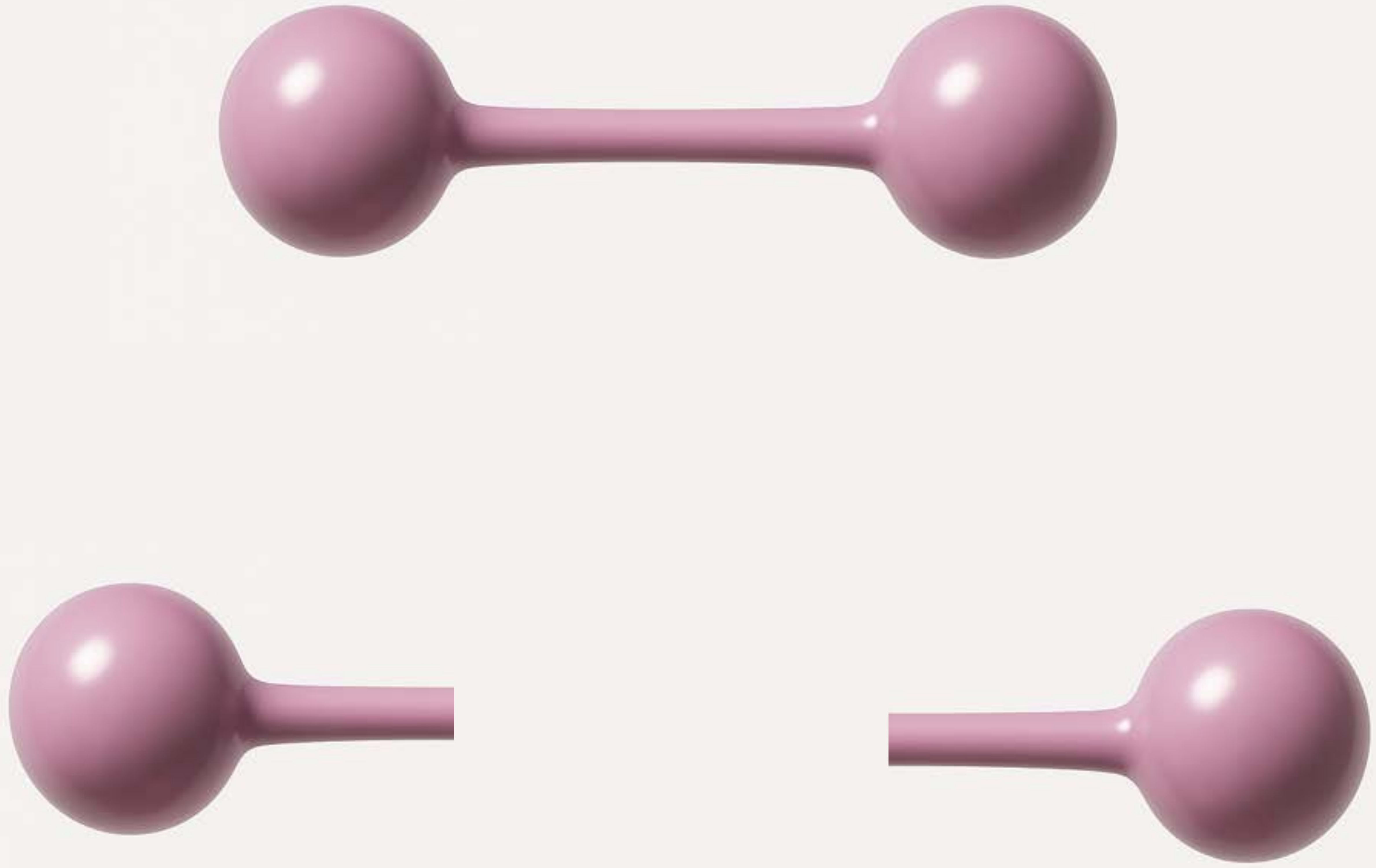
We can distort the sphere topologically if it helps.

We can even start by knotting a 'dumbbell' and then '*inflating*'.

There are *knotted circles in* $\mathbb{R}^3$....

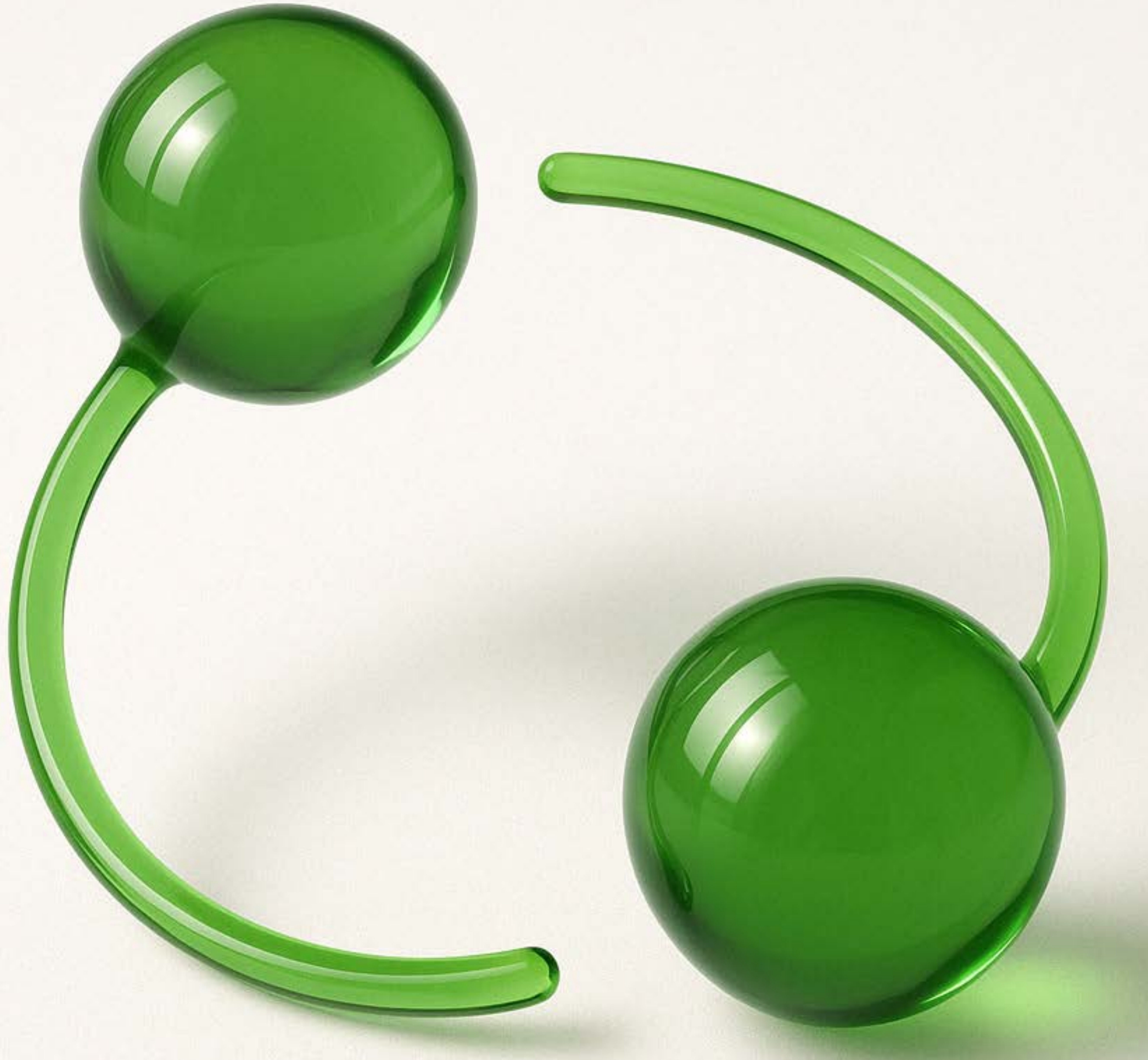
Are there *knotted spheres in* $\mathbb{R}^4$**?**

To knot a dumbell, we can start with two spheres with 'tails', and eventually fuse them.



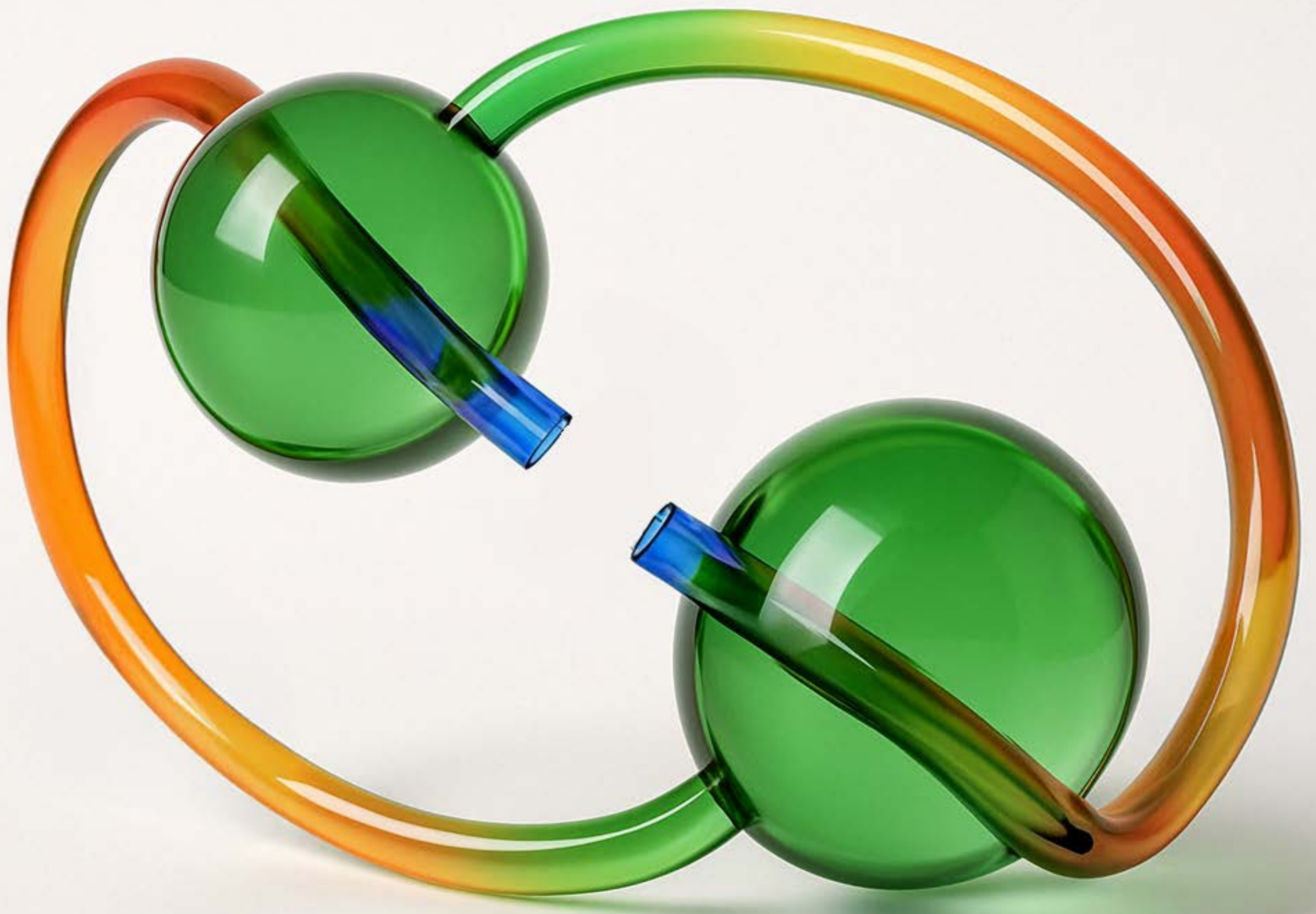
Everything starts in the same 3-dimensional slice, so is uniformly colored.

We extend the tails, remaining in the same 3-dimensional slice.

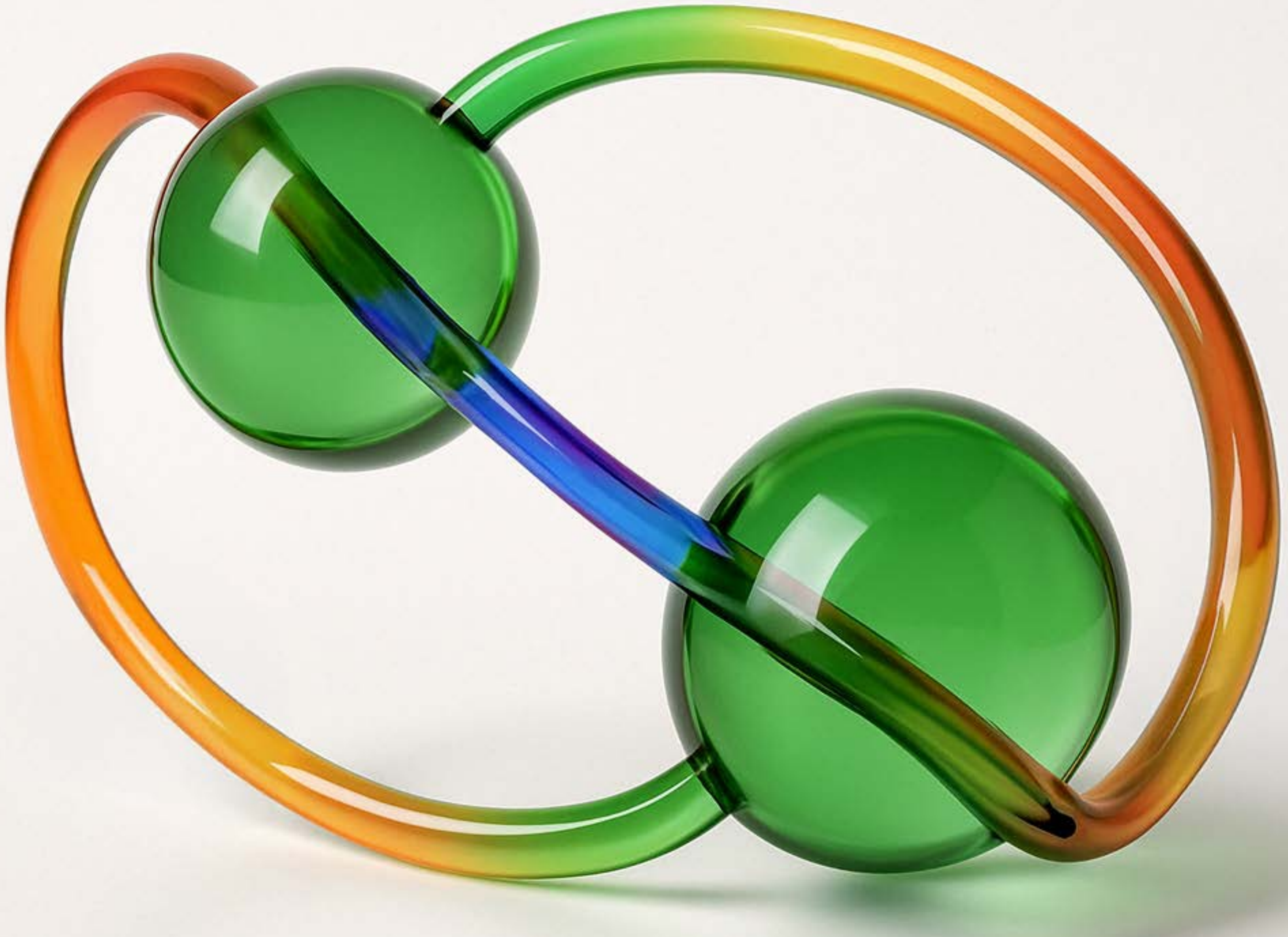


We now lift the tails up ‘over the top’ of the spheres, in the 4th direction.

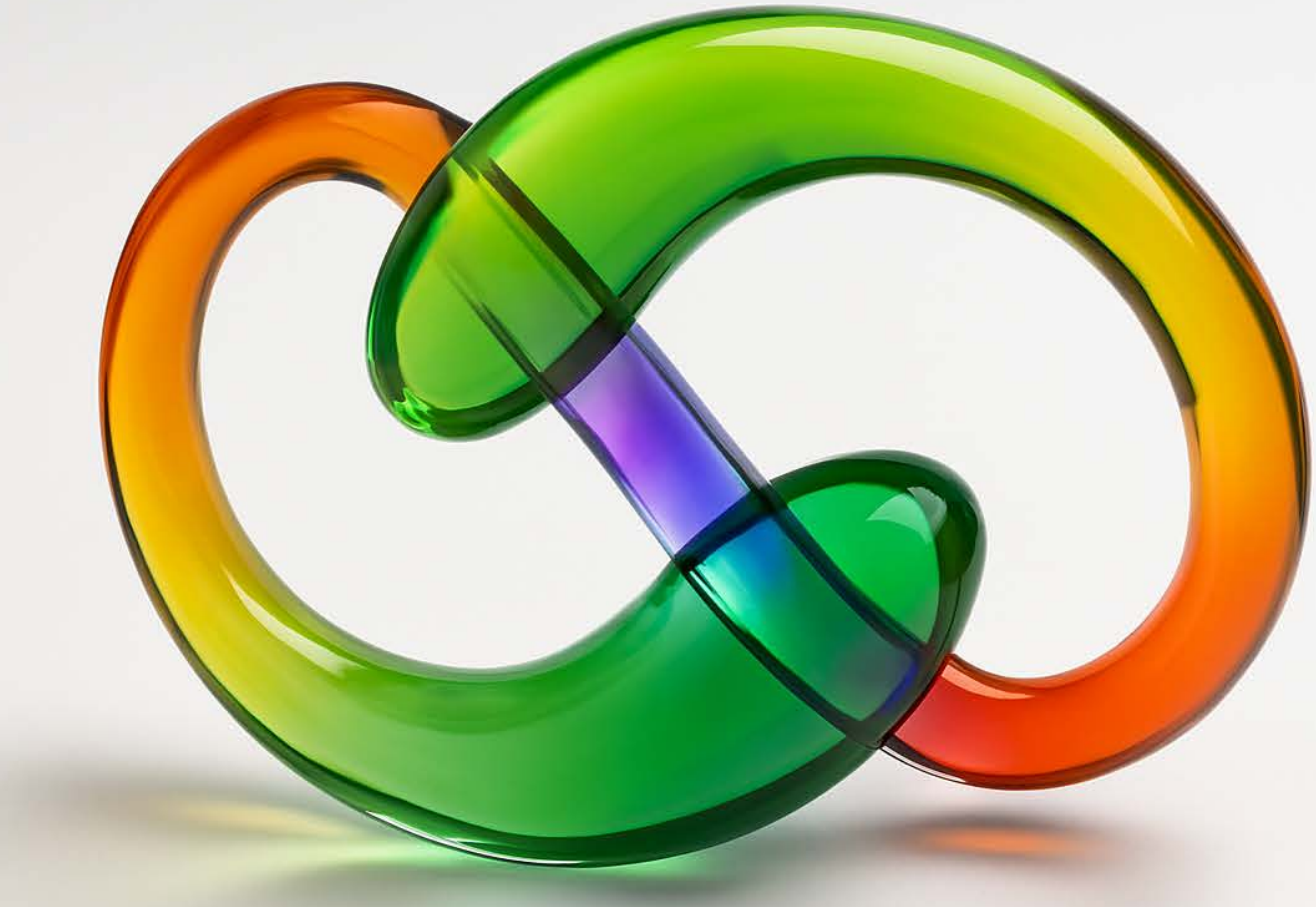
We now thread the tails through the spheres, entering above and leaving below.



The tails are now below the spheres at the same level, so we can ‘fuse them’



Now: inflate from a dumbbell to a sphere!



This sphere is knotted: there is no ambient isotopy of $\mathbb{R}^4$ taking it to the standard embedding.

This sphere is knotted: there is no ambient isotopy of $\mathbb{R}^4$ taking it to the standard embedding.

Idea; compute the fundamental group of its complement.

Thanks!