Geometry, Topology *and the* **Poincaré Conjecture**

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University of San Francisco

Math is a conversation
Spanning Centuries

2100 years

Euclid, 300BCE

Bolyai, Lobachevsky, Gauss, 1800CE

“Is my 5th axiom actually a consequence of the other 4?”

“No! There is more to geometry than you could have imagined”

Math is a conversation
Spanning Centuries

Archimedes 250 BCE

“Is to π irrational?”

2010 years

Johann Lambert, 1760CE

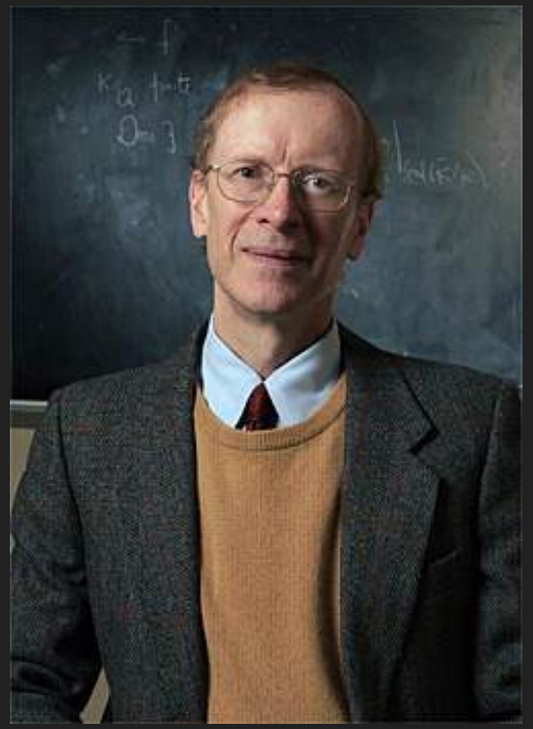
“Yes!”

Math is a conversation **Spanning Centuries**

Fermat, 1637

"Are there nontrivial solutions to $x^n + y^n = z^n$?"

358 years



Andrew Wiles, 1994

"There are not!"

Math is a conversation
Spanning Centuries

Modern math holds a conference every century to lay out its biggest problems...

100 years

CMI

1900: Hilbert's List

2000: Millennium Problems

- **Birch and Swinnerton-Dyer conjecture:**
  How well do we understand geometry with modular arithmetic?
- **Hodge Conjecture:**
  How do topology and algebra interact when solving equations?
- **Navier Stokes:**
  How well do we understand difficult differential equations?
- **P vs NP**
  How well do we understand computation?
- **The Riemann Hypothesis**
  How much does calculus know about prime numbers?
- **Yang Mills Mass Gap**
  Can we rigorously understand modern quantum theory?
- **The Poincare Conjecture**
  How well do we understand shapes (manifolds)?

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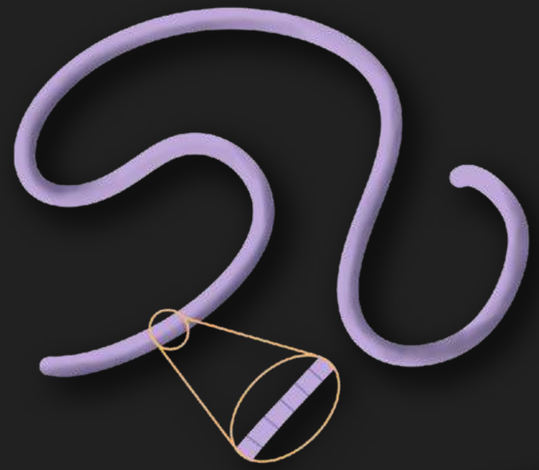
  How well do we understand shapes (manifolds)?

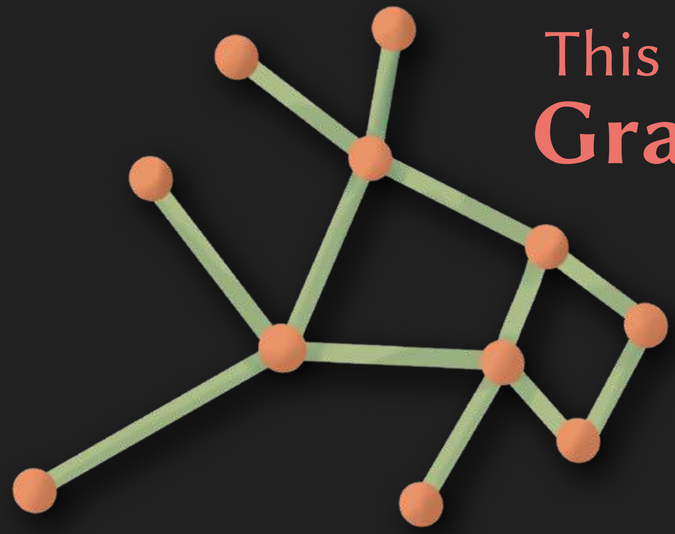
Manifolds

Spaces that look like $\mathbb{R}^n$ everywhere

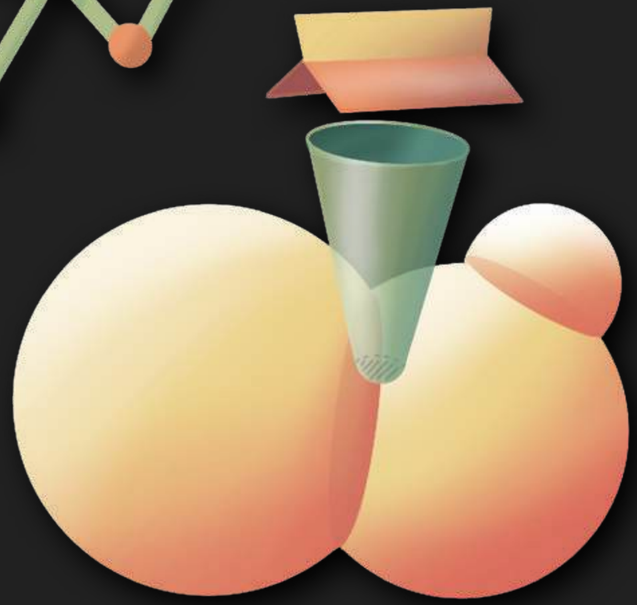


This **Curve** is a **Manifold**

This **Surface** is a **Manifold**



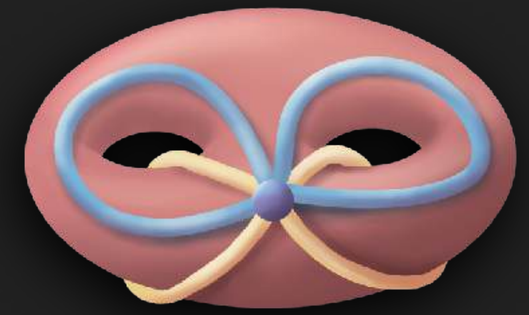
This **Graph** is **Not**



Soap **Bubbles** are **Not**

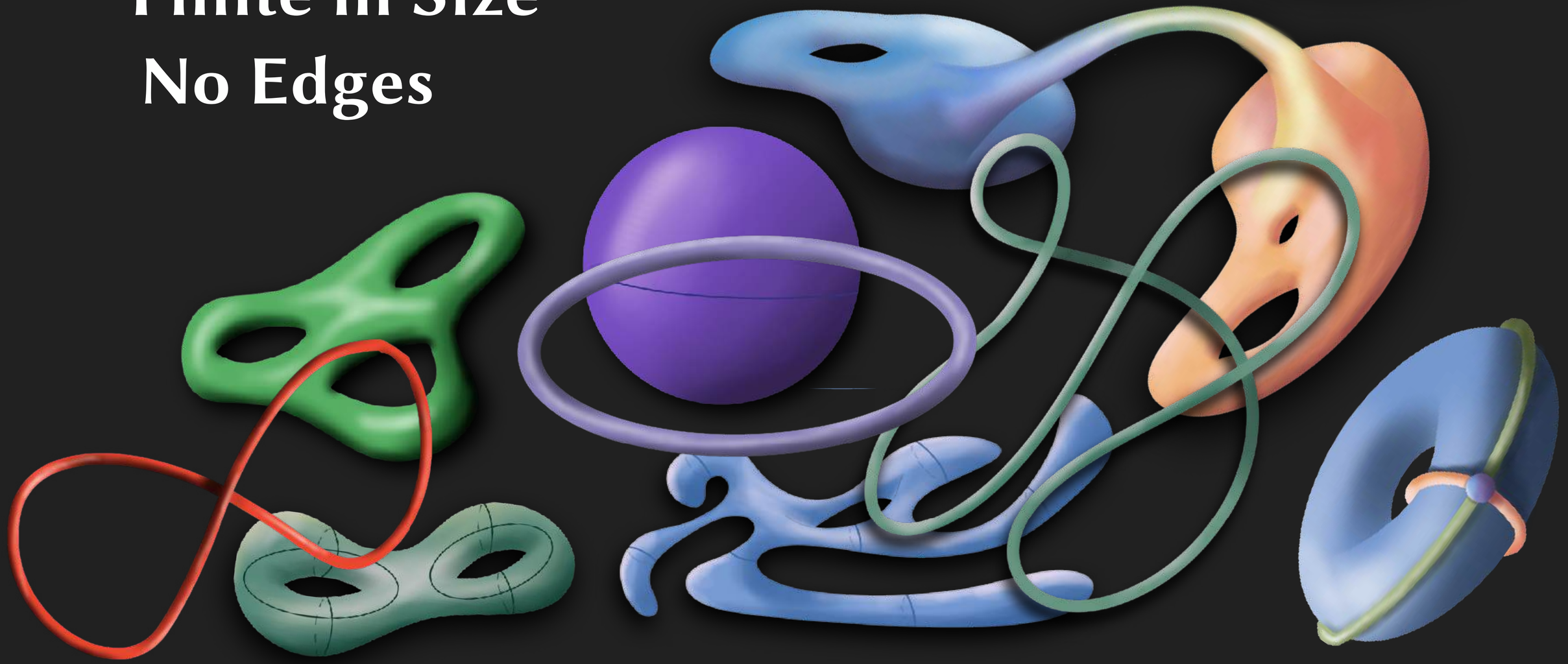
Manifolds

Spaces that look like $\mathbb{R}^n$ everywhere

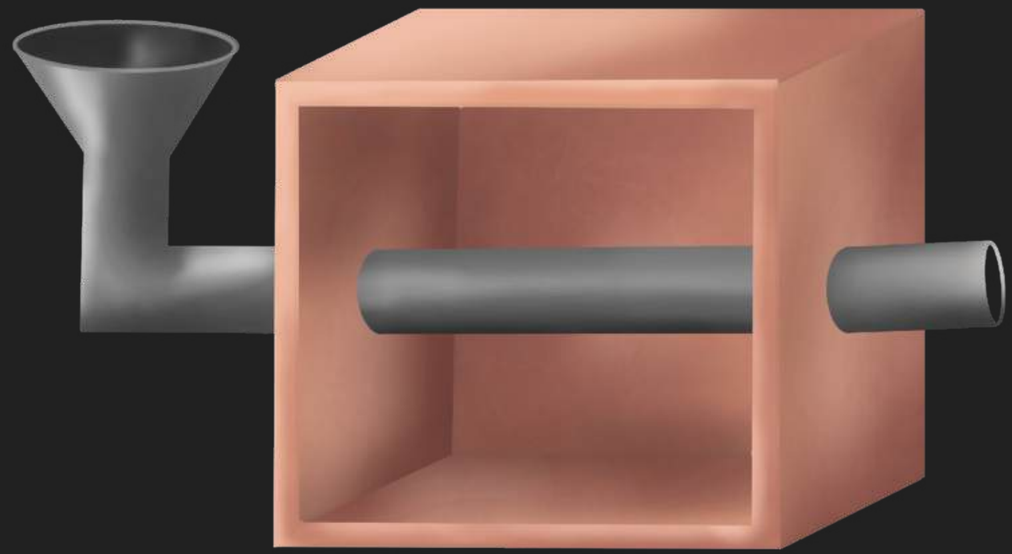


Finite in Size

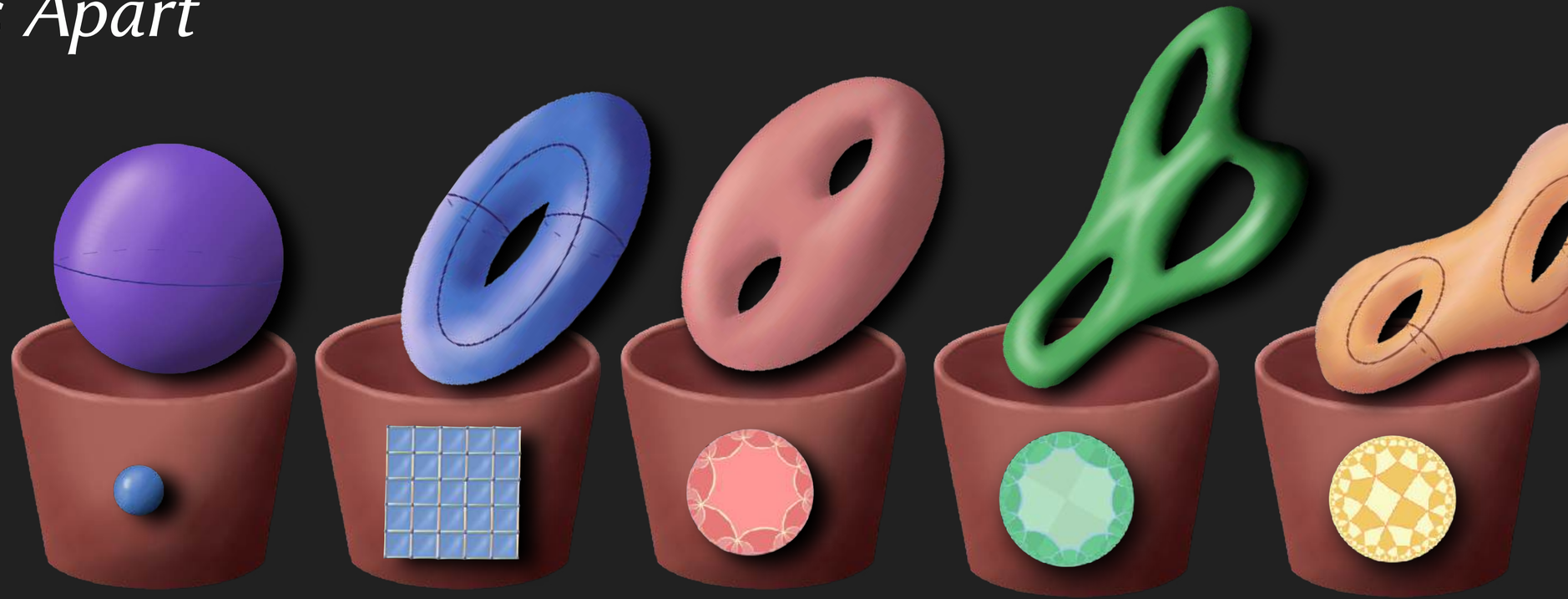
No Edges



Part I



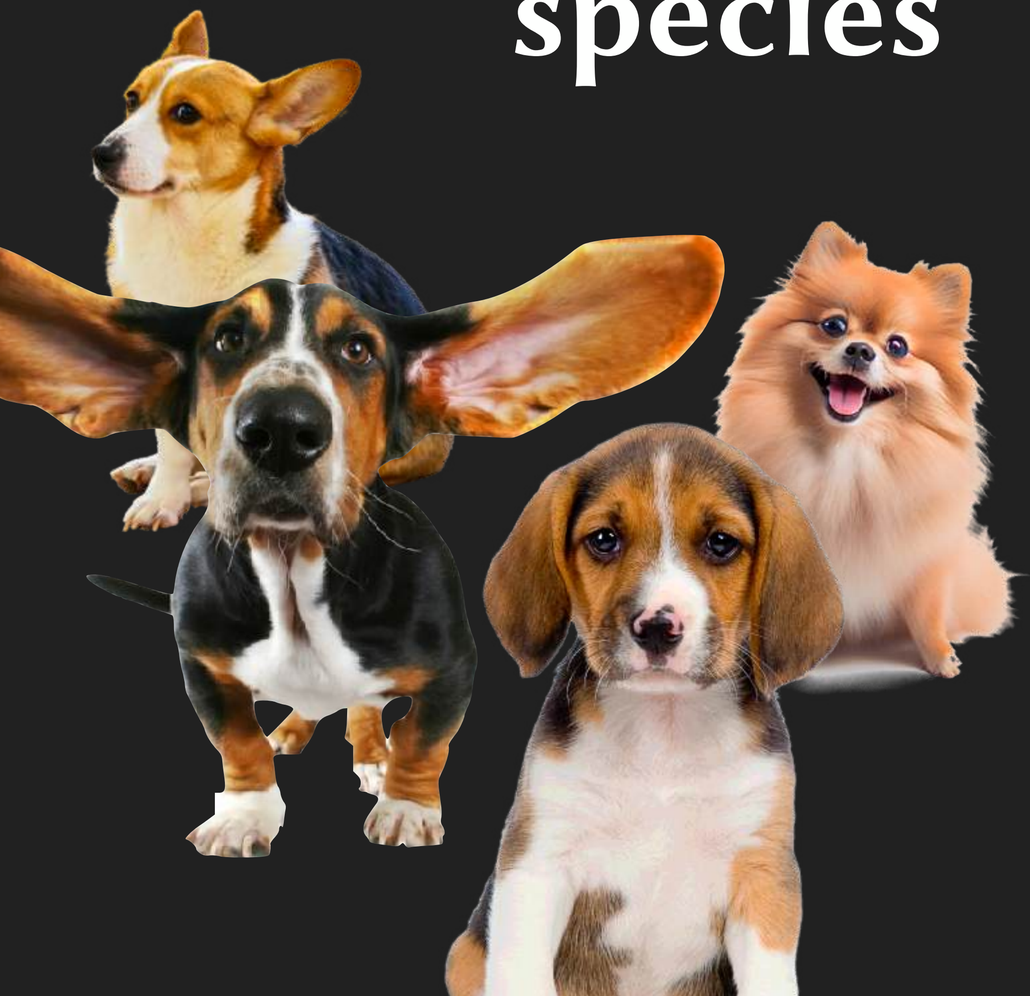
Telling Shapes Apart



When are Two Things the Same?

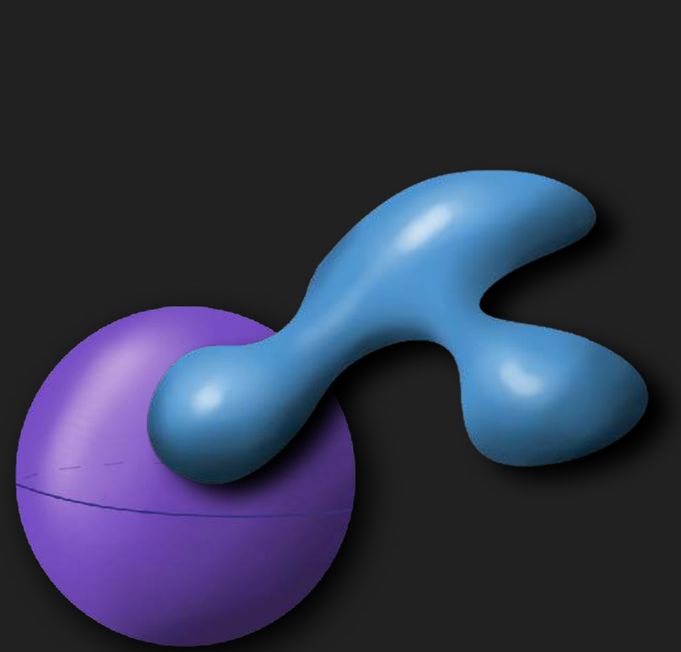
For **animals,** individuals are classified by **species**

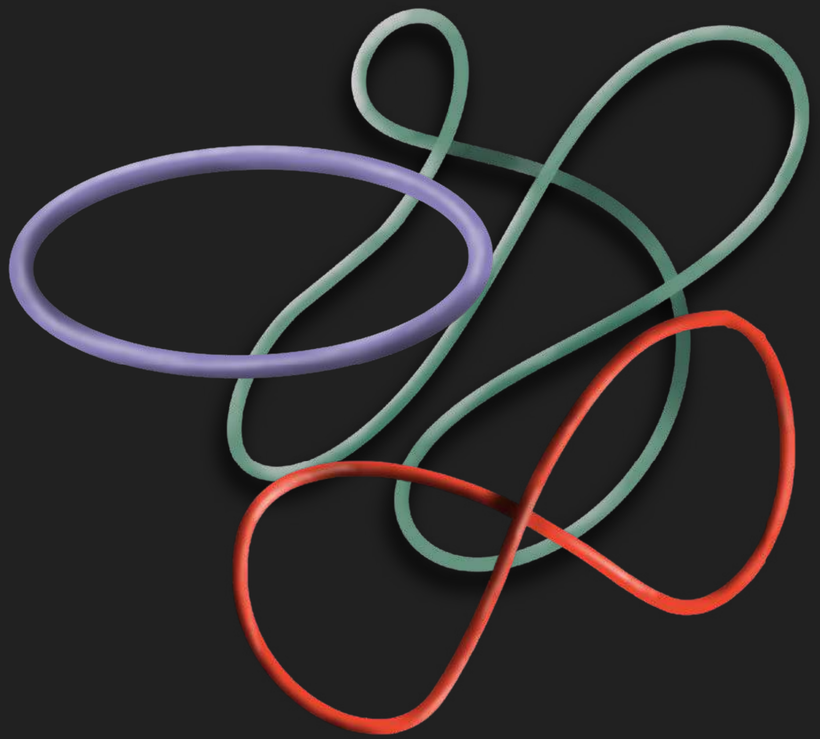
When are **Two Things** the **Same**?



For **shapes,** individuals are classified by **homeomorphism**

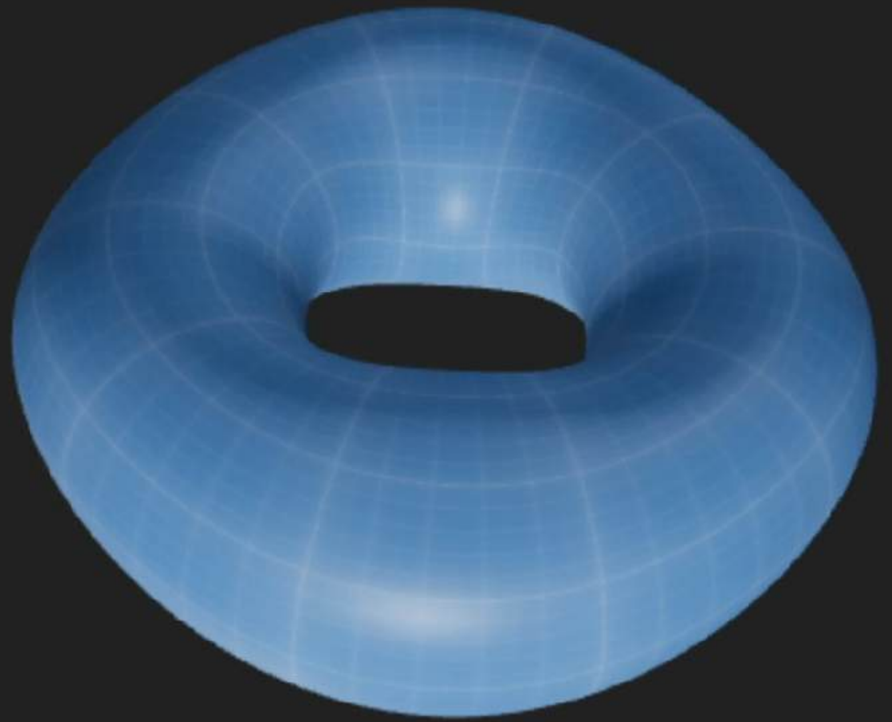
When are **Two Things** the **Same**?





When are
Two Things
the **Same**
?

Can be
continuously deformed
into one another,
without tearing

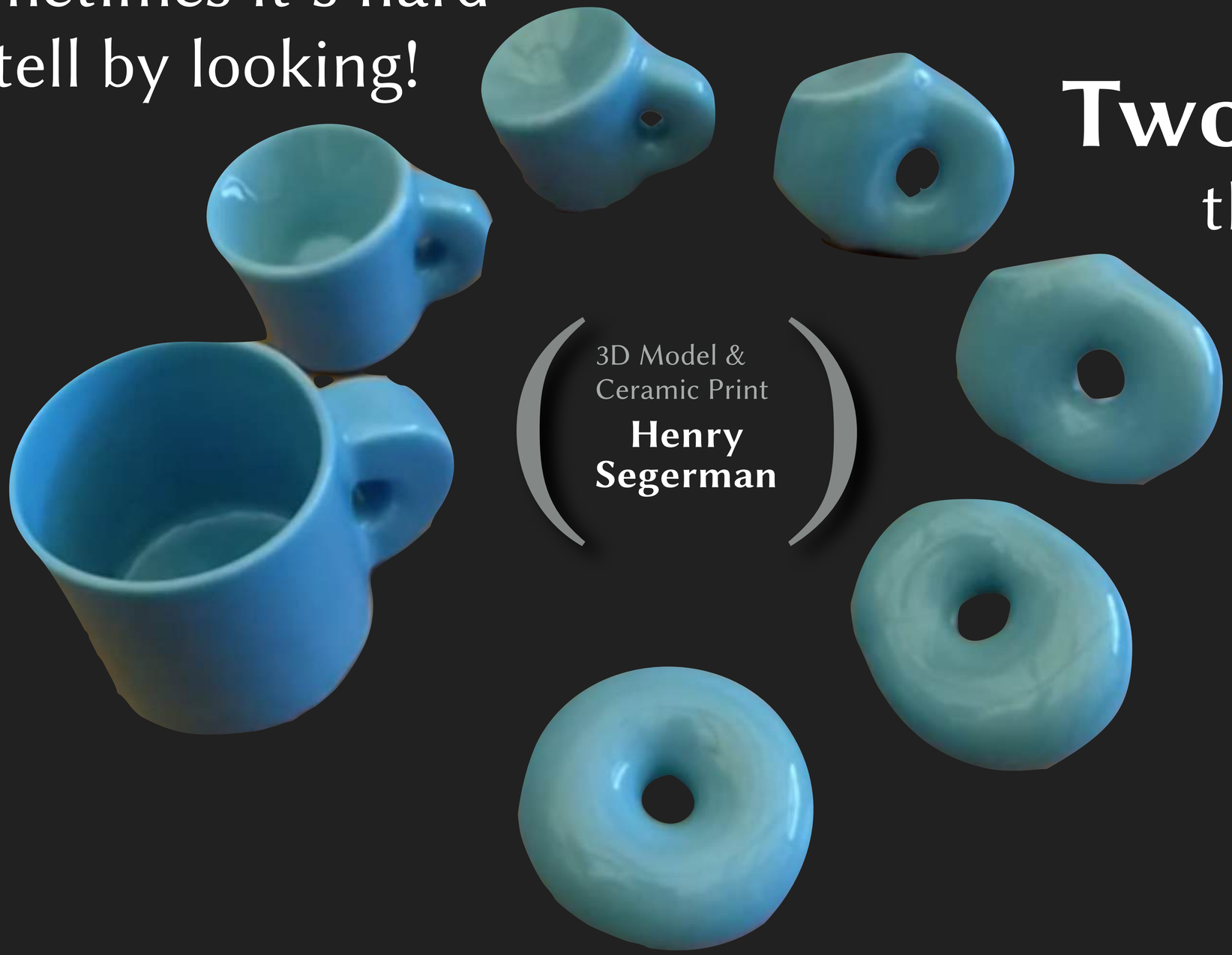


This is always a torus to topologists!

Sometimes it’s hard
to tell by looking!

When are
Two Things
the **Same** ?

Sometimes it’s hard to tell by looking!

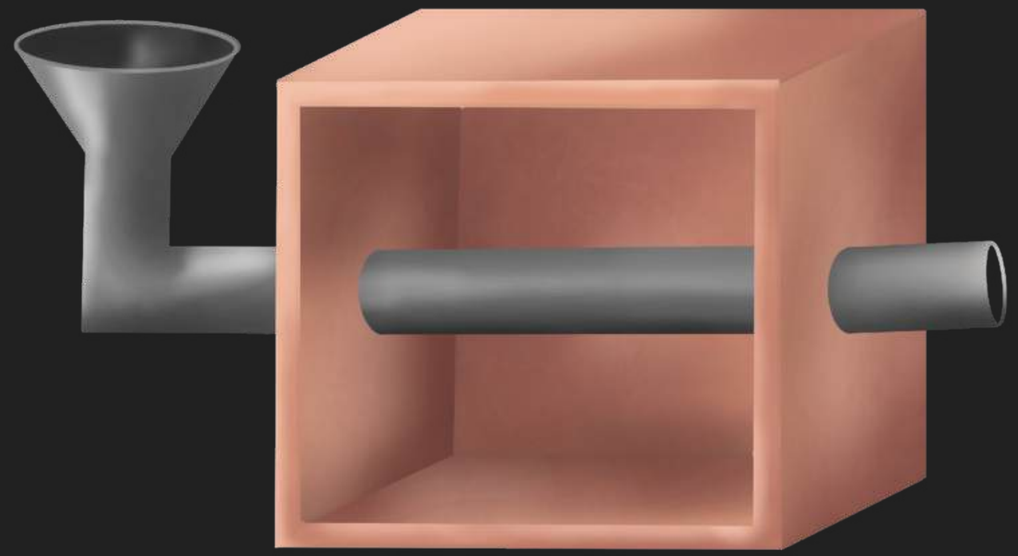


3D Model &
Ceramic Print
Henry Segerman

When are **Two Things** the **Same**?

The coffee cup & donut are the same topological ‘species’

The idea of **Invariants**



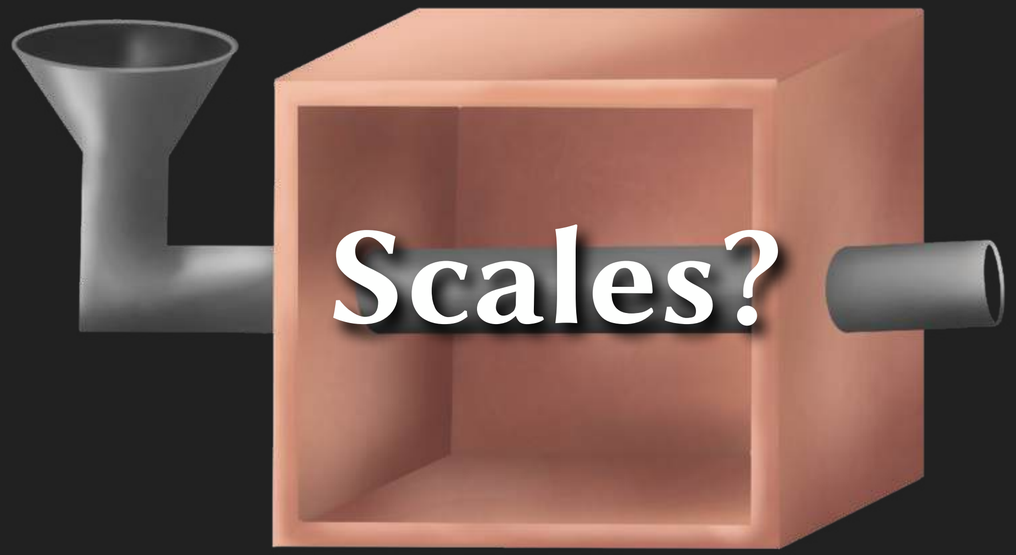
A quantity that is **constant on objects of the same type** *is called an invariant.*

When are **Two Things** the **Same** ?

The value of an invariant can be used to sort objects.

Biological **Invariants**

Biological Invariants



Biological Invariants

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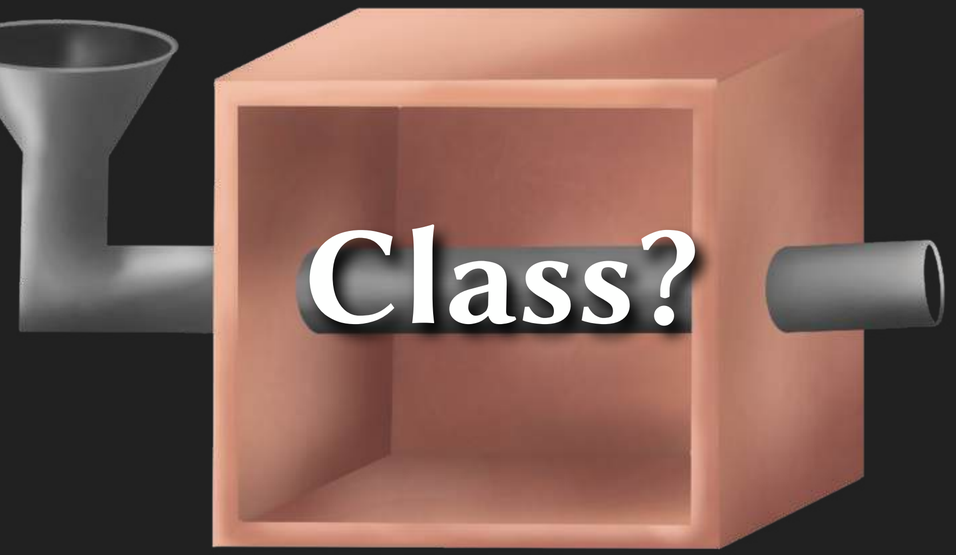
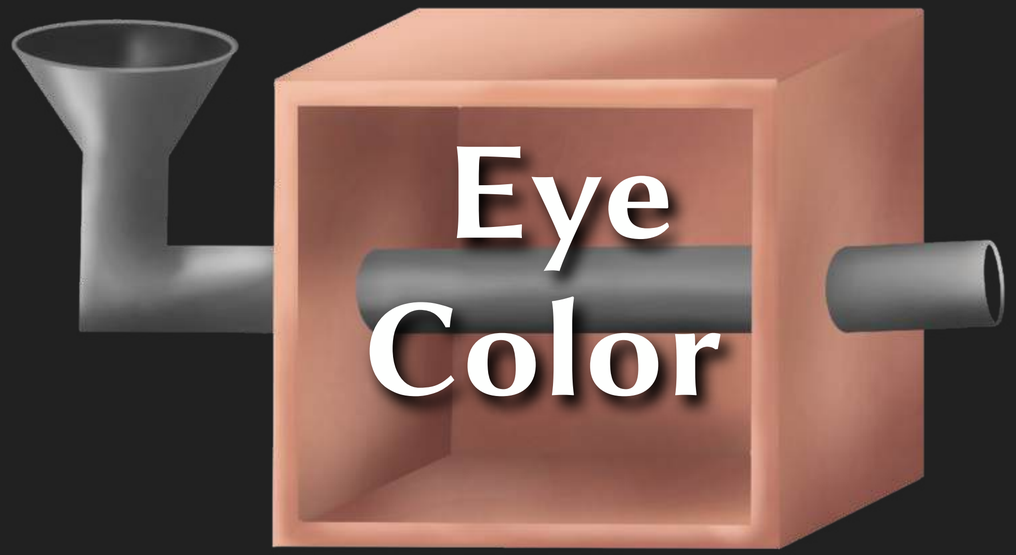


Biological Non-Invariants



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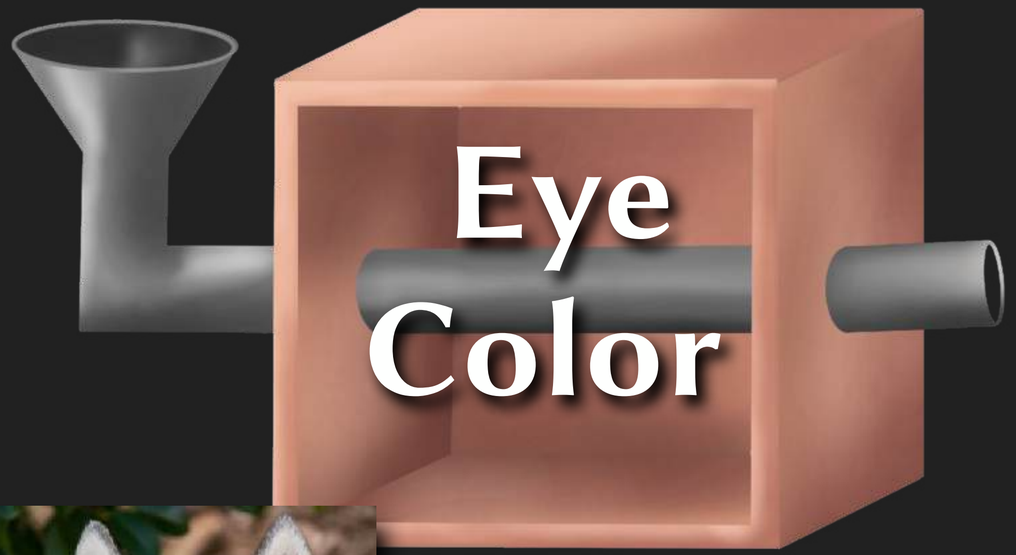

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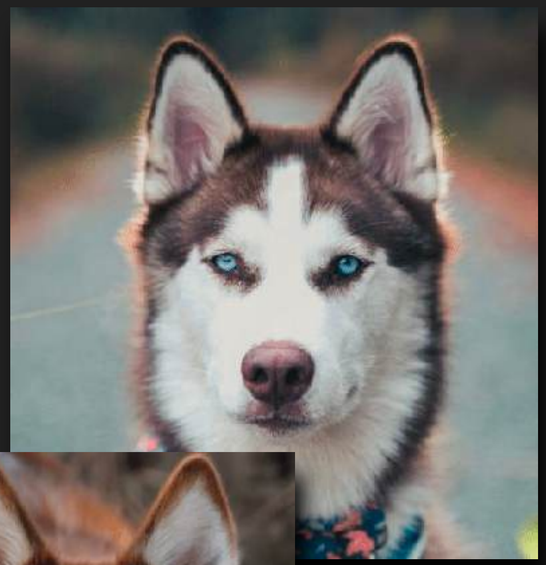
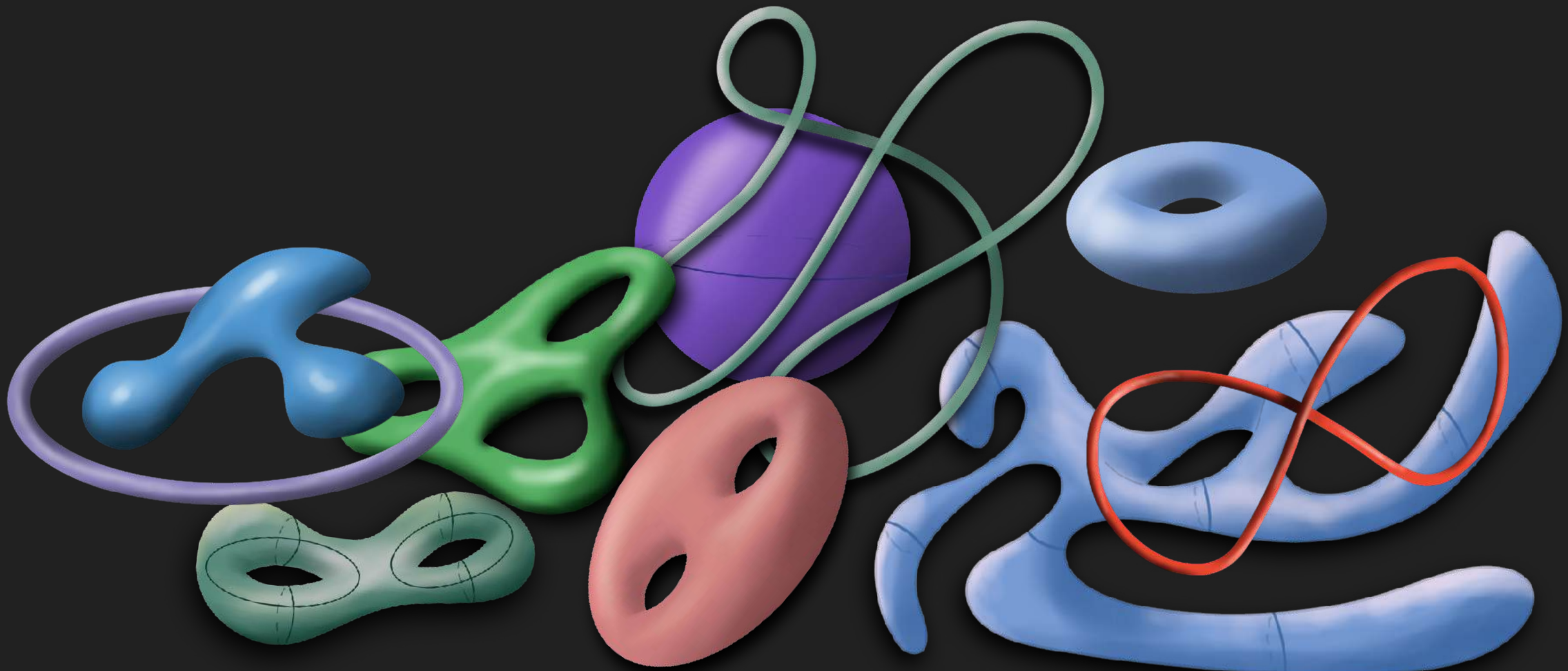
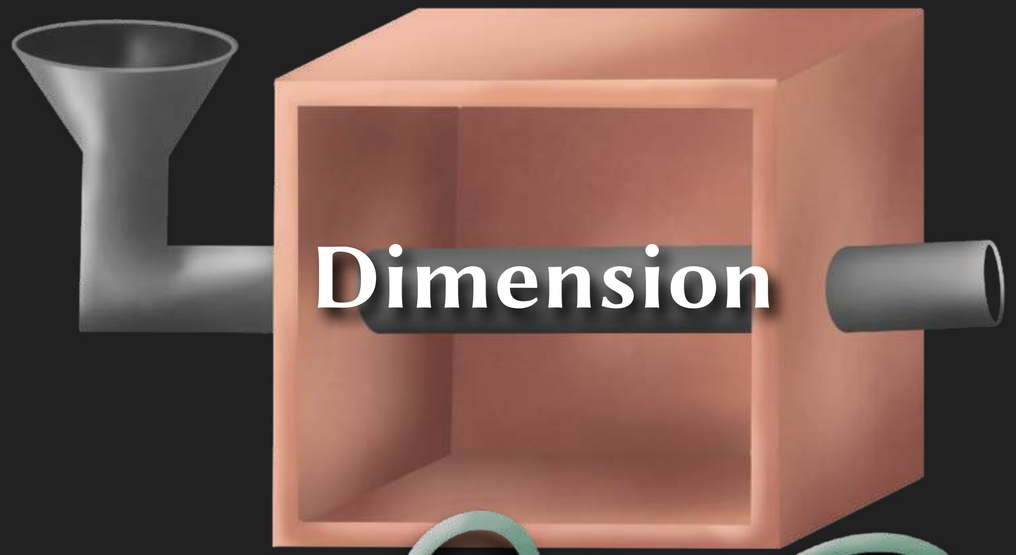
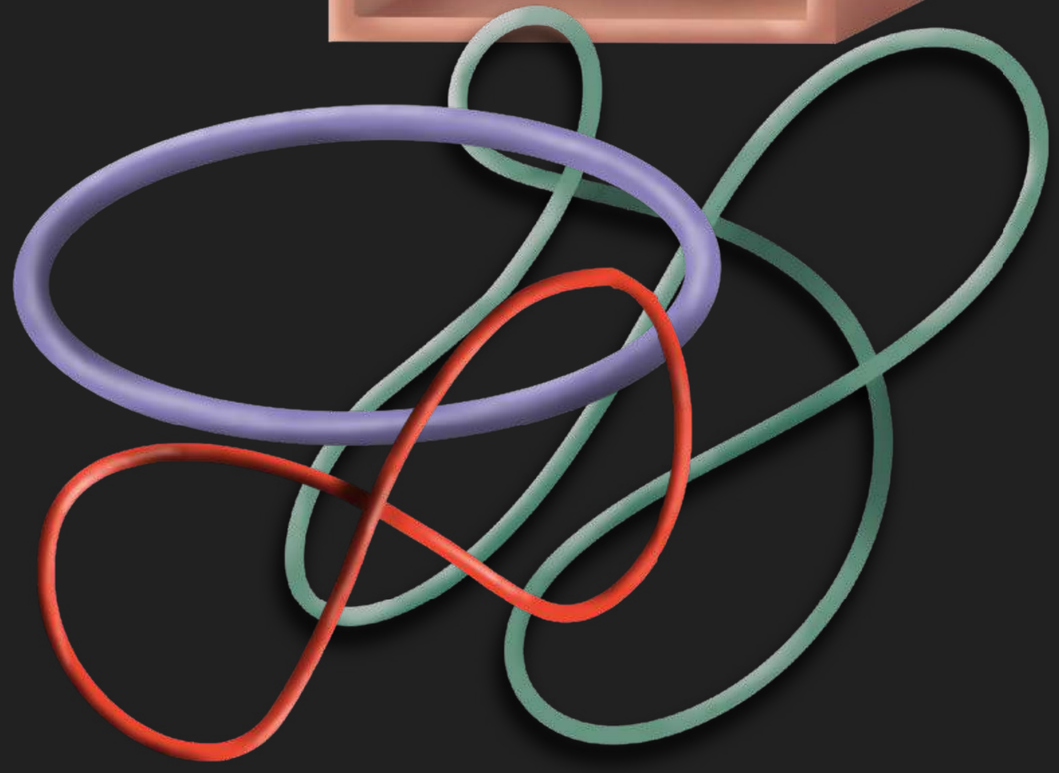


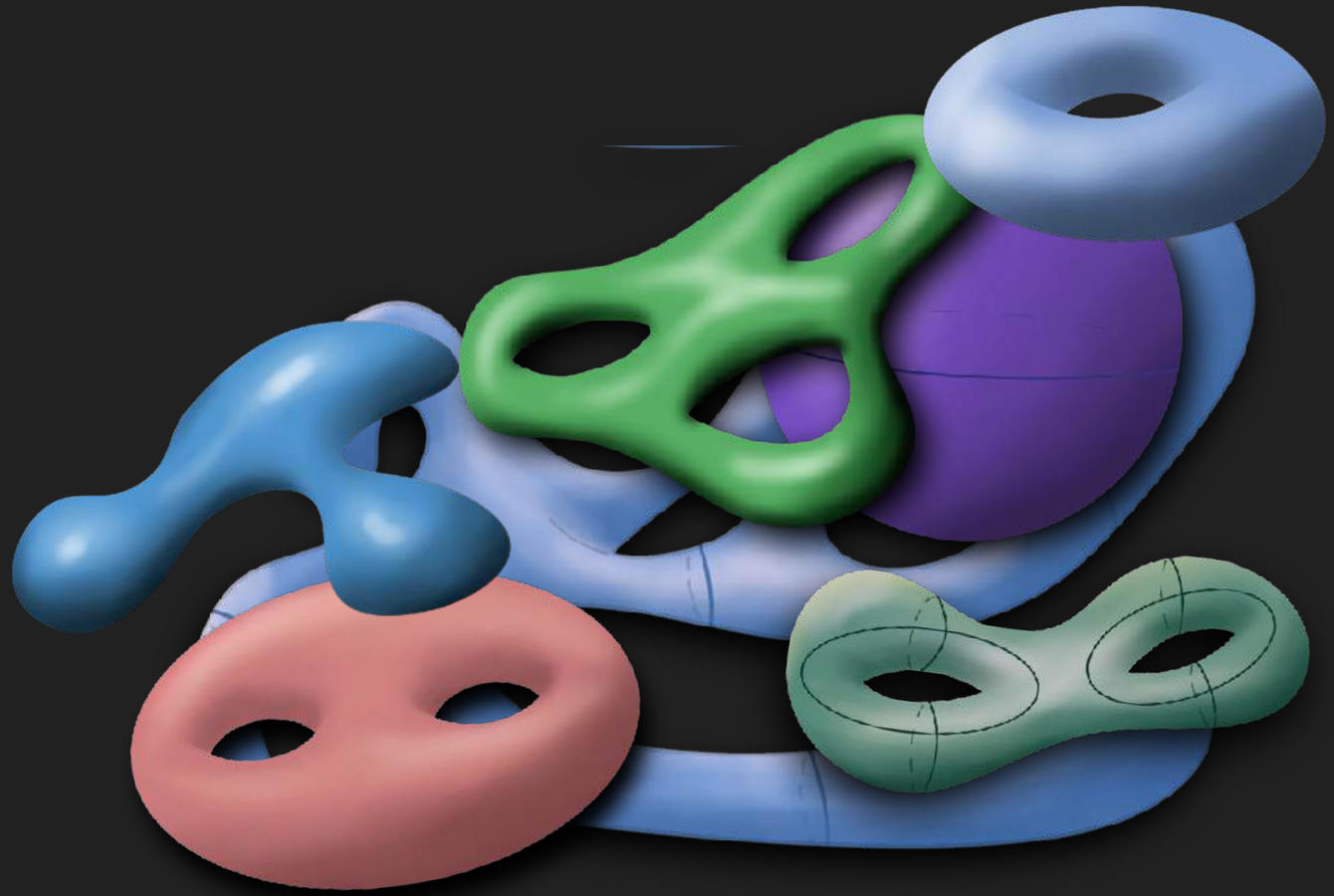
Topological **Invariants**



Topological **Invariants**

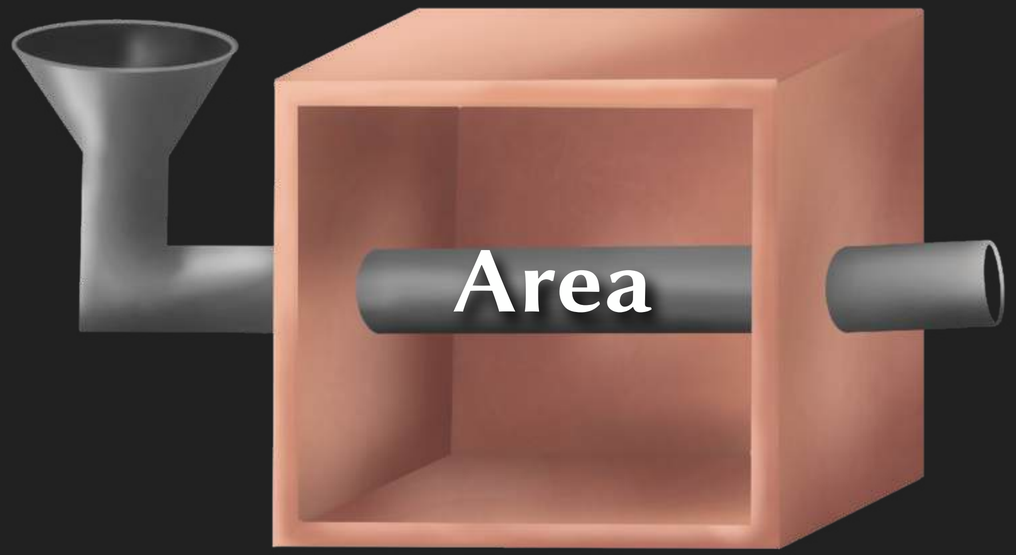


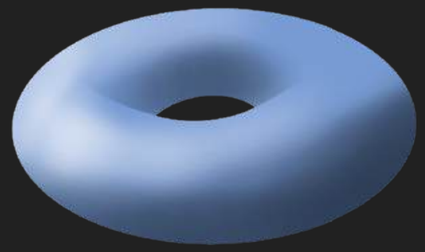


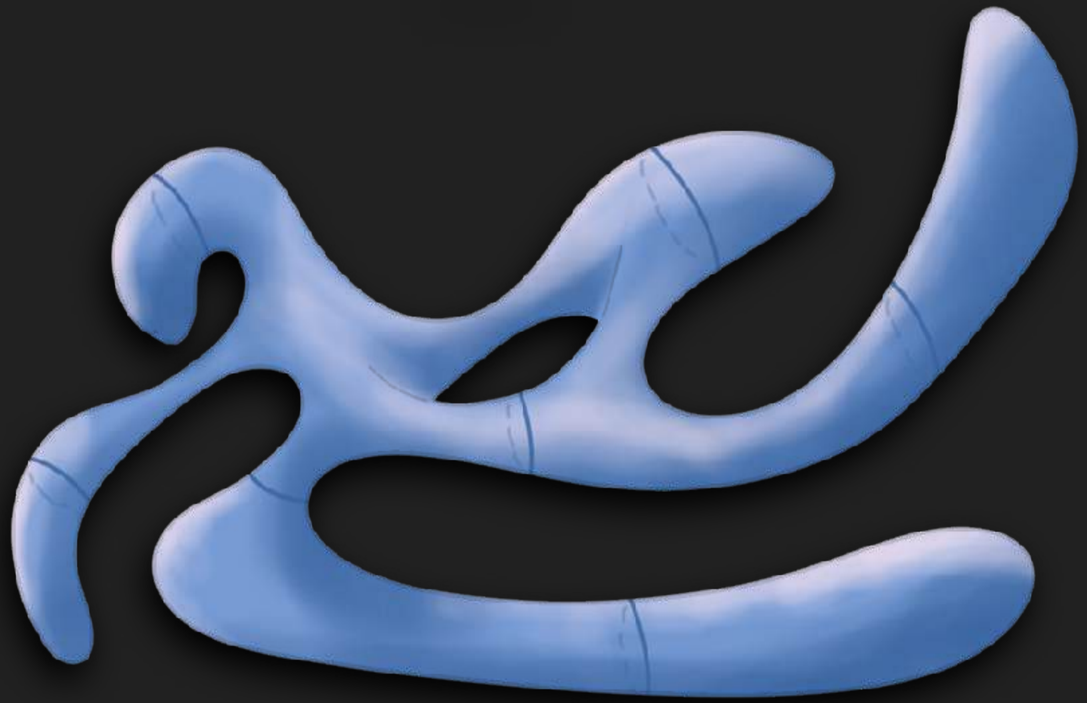


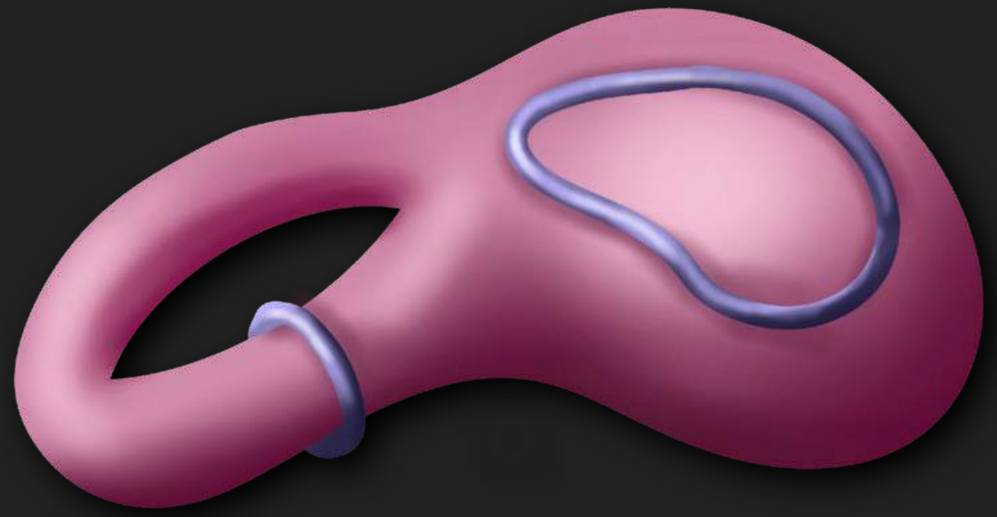
Topological

Non-Invariants









The Fundamental Group (1892)

A loop is “stuck” if you can’t shrink it to a point, or disconnect the surface with it.

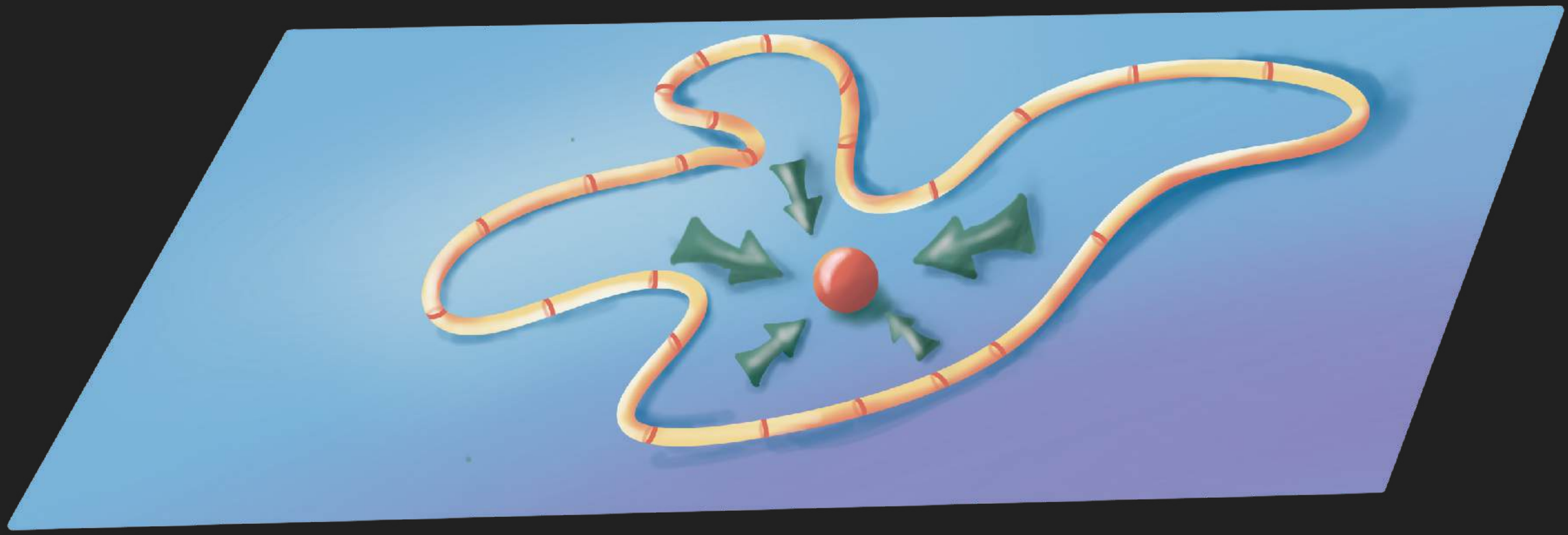
This property of a loop is unchanged by continuous deformation.

Henry Poincare

The collection of all stuck loops is a topological invariant.

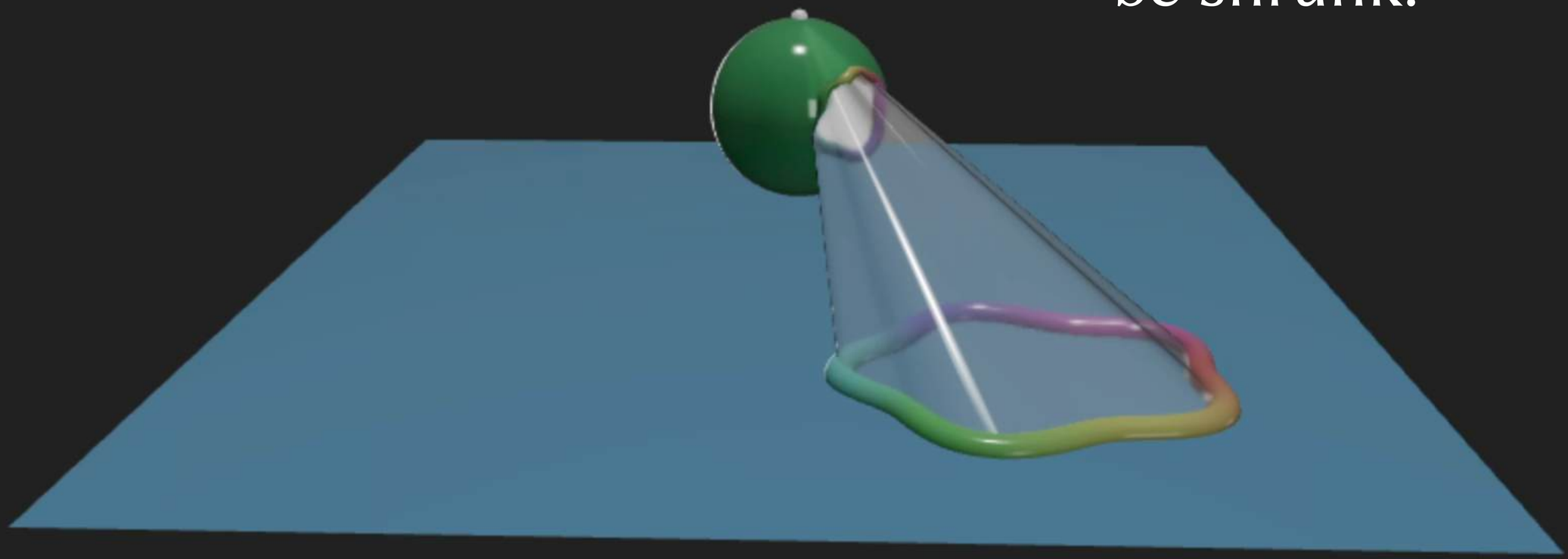
The **Plane** has **No “Stuck” Loops**

$$f(s) \mapsto t \cdot f(s)$$



The **Sphere** has **No “Stuck” Loops**

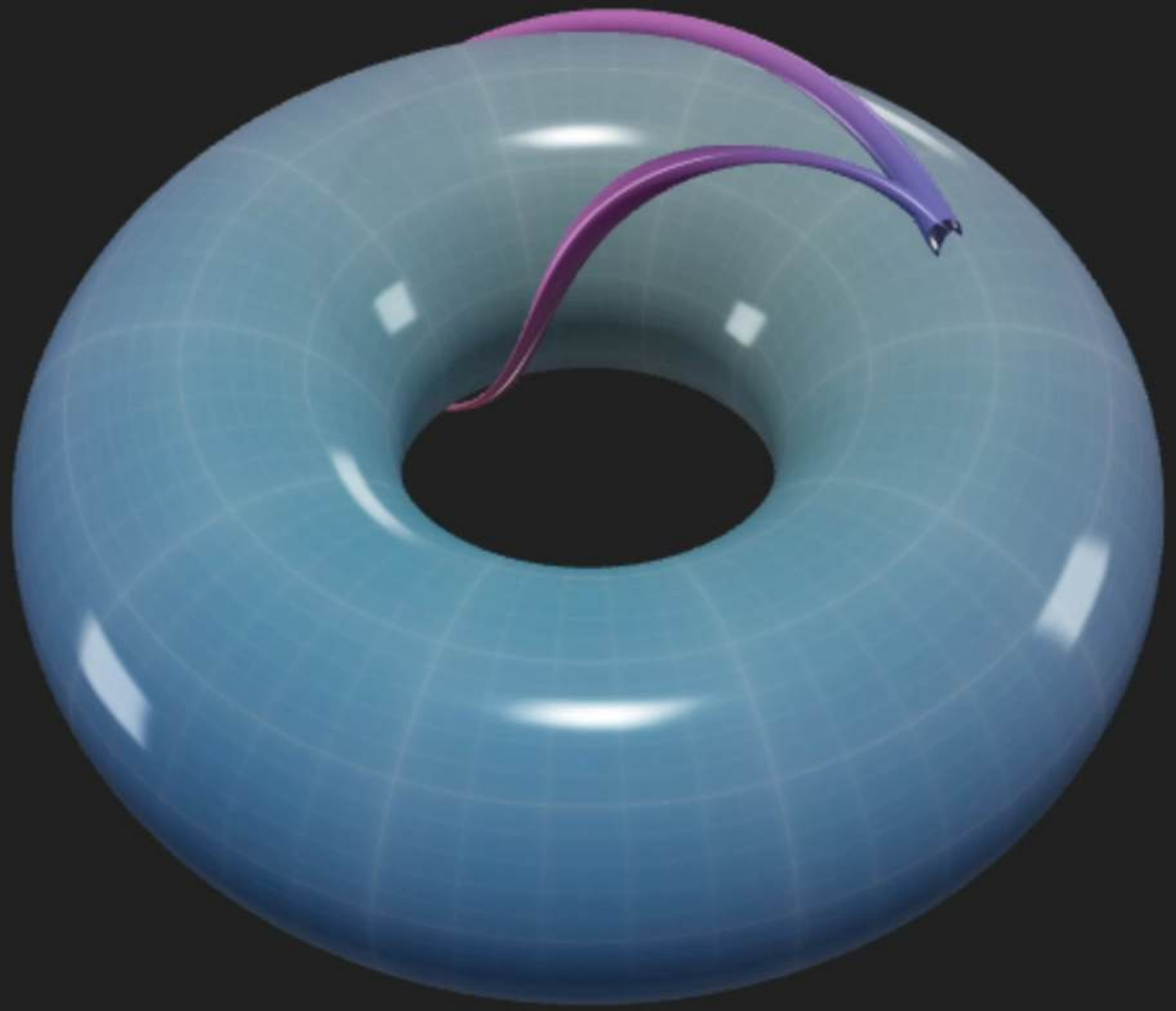
Look at planar shadow, where everything can be shrunk!

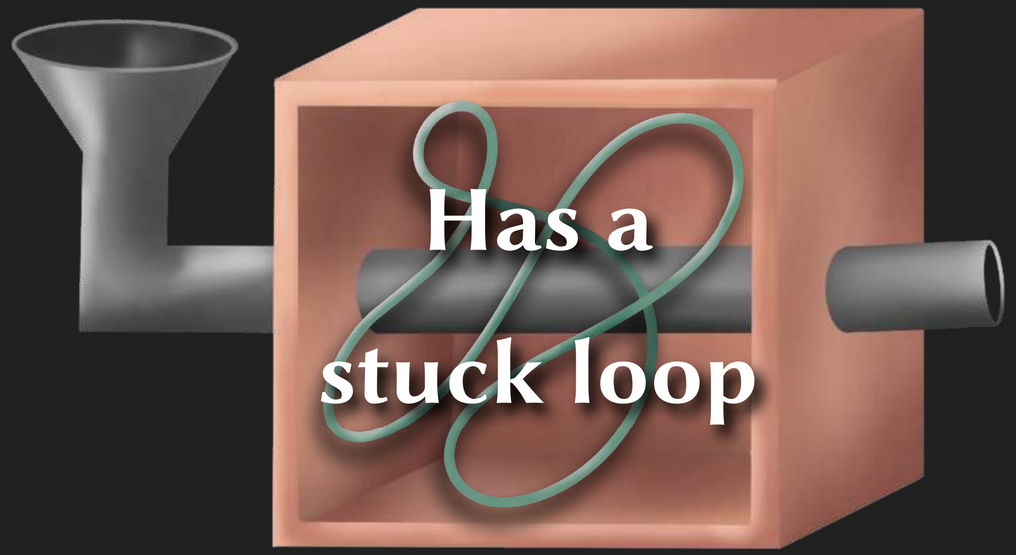


The **Torus** has a

“Stuck” Loop!

No amount of wiggling will shrink this loop to a point!



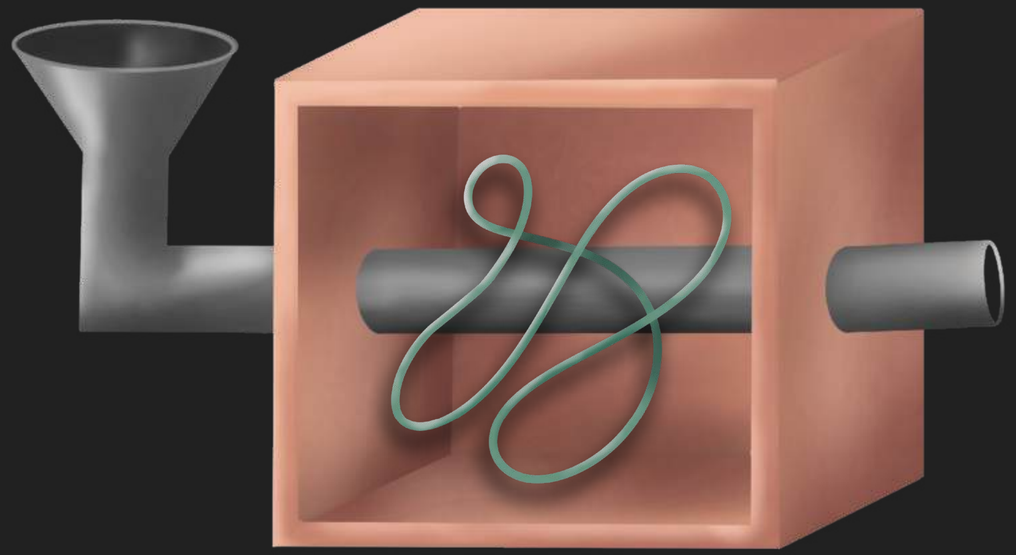


How good is

The Loop Invariant

?

Loops can tell apart the sphere and the torus.



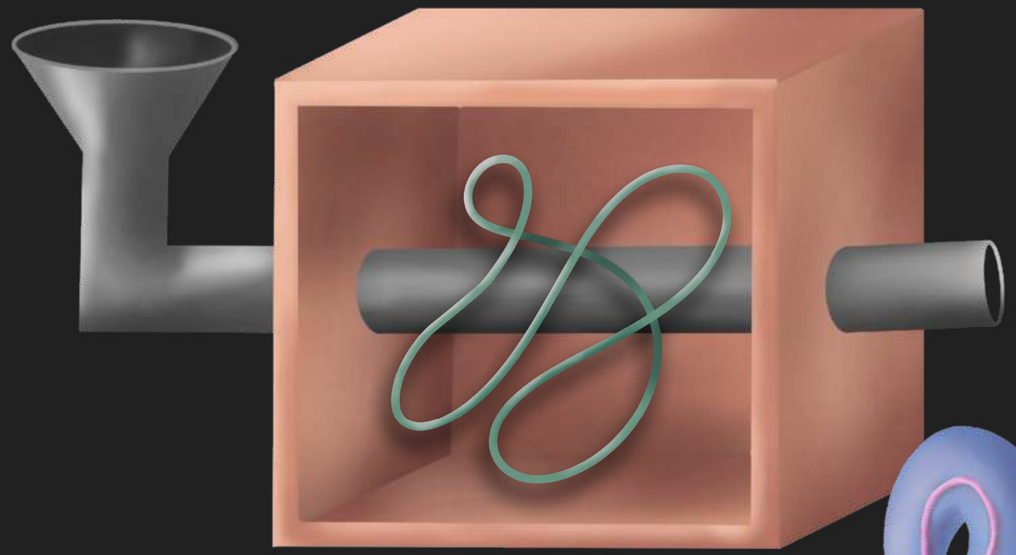
How good is

The Loop Invariant ?

Loops are a

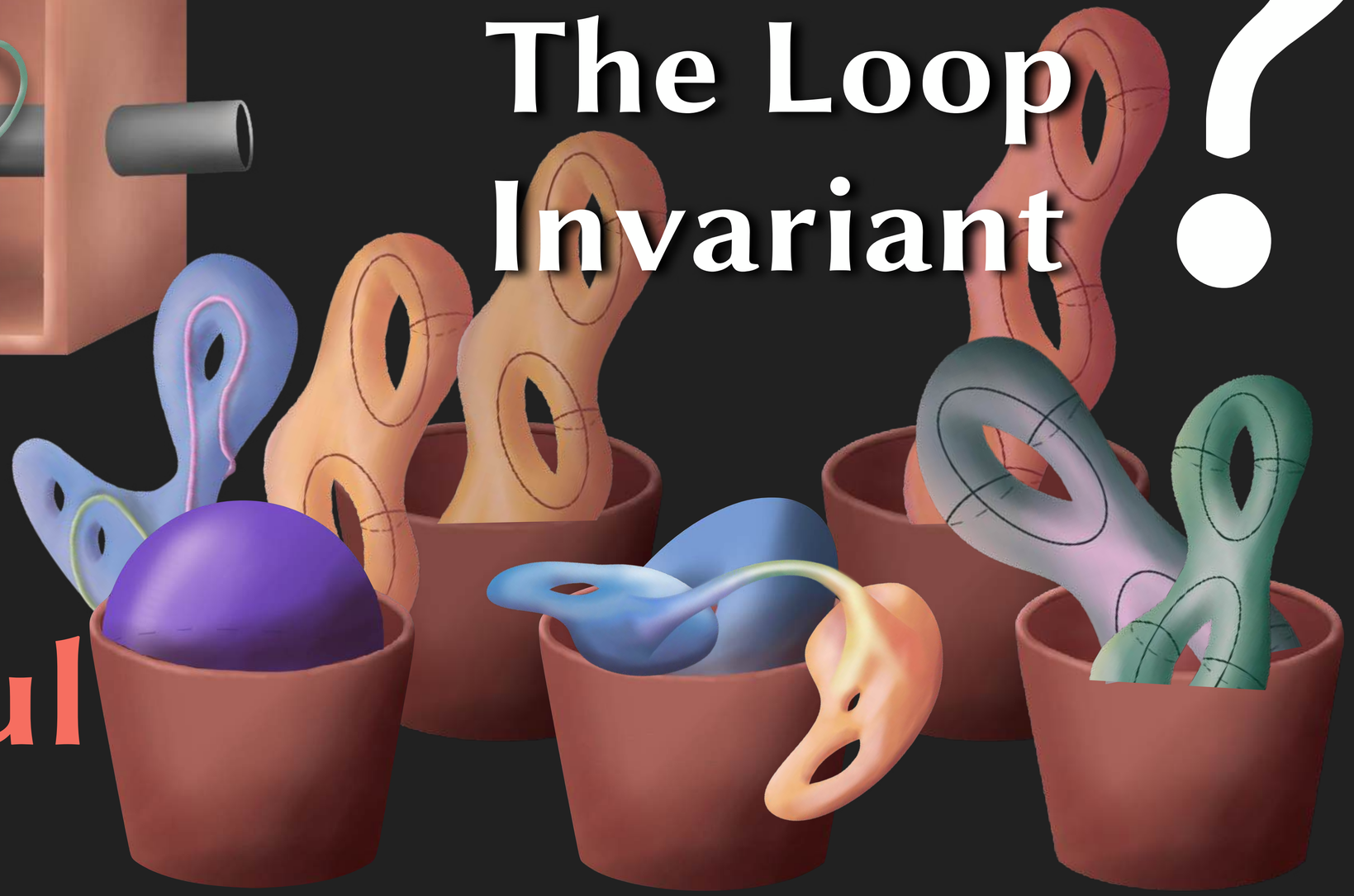
Helpful

Invariant

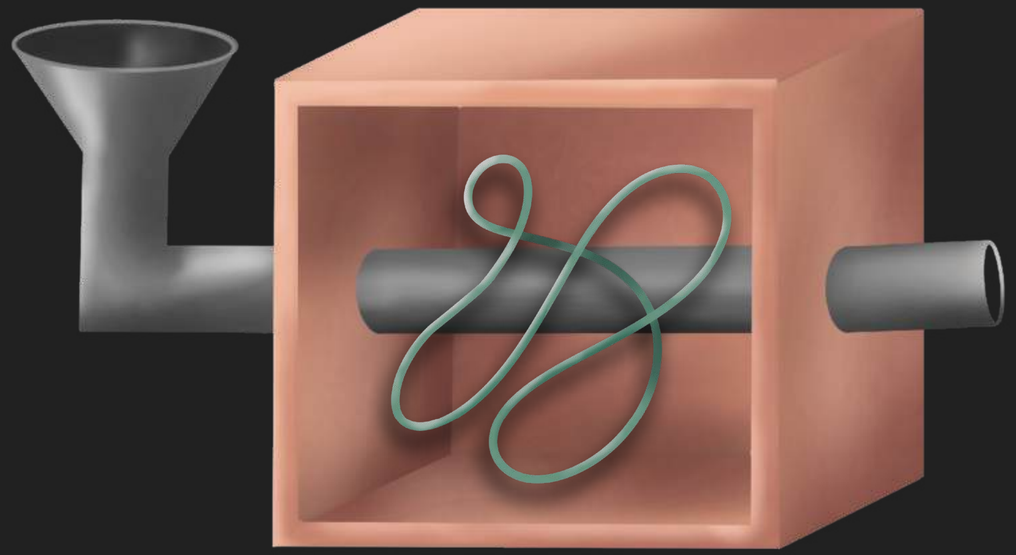


How good is

The Loop Invariant ?



Loops are an Unhelpful Invariant



How good is

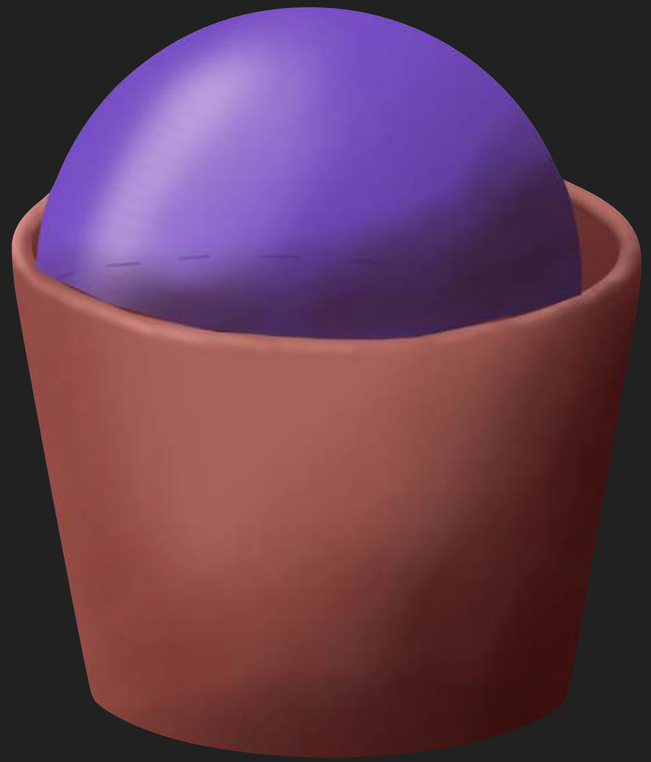
The Loop Invariant

Can loops find The Sphere?

(Generalized)
The Poincaré Conjecture

“I think loops are a good invariant. They can tell if a shape is a sphere.”

-Poincaré, 1904



The Poincare Conjecture

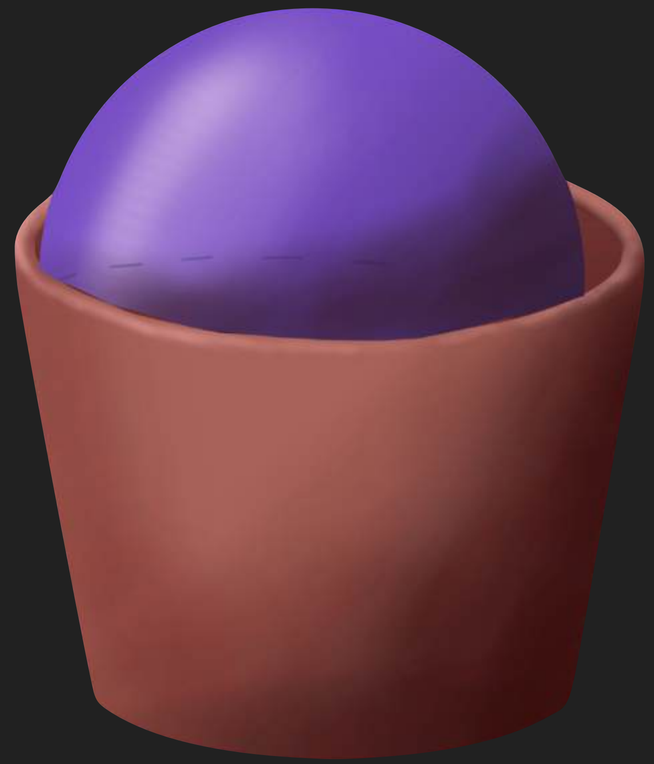
If a shape has the same

Loop Invariants

as the sphere, then

it is the sphere

-Poincaré, 1904



The Poincare Conjecture

If it

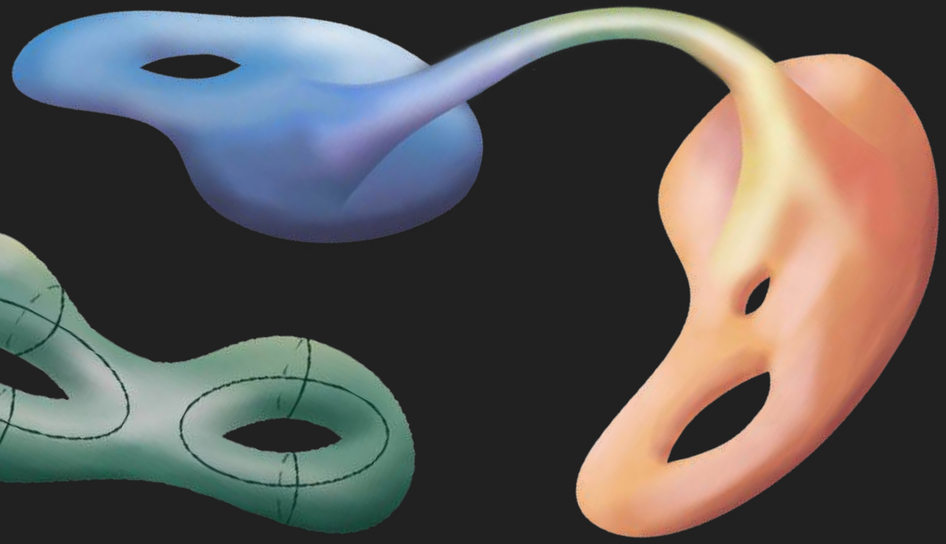
walks & quacks

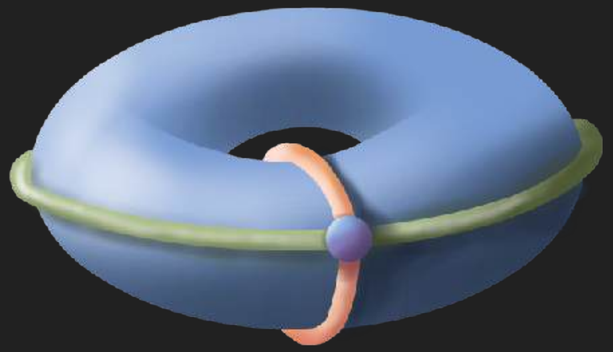
like a duck, then

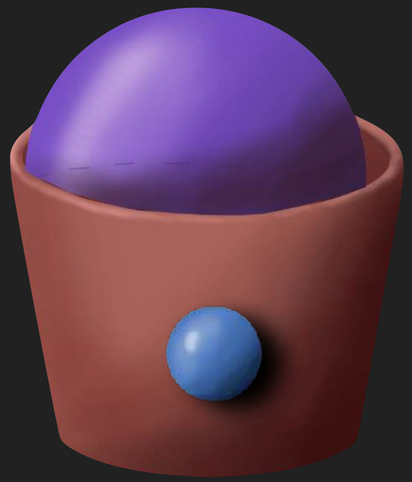
it is a duck

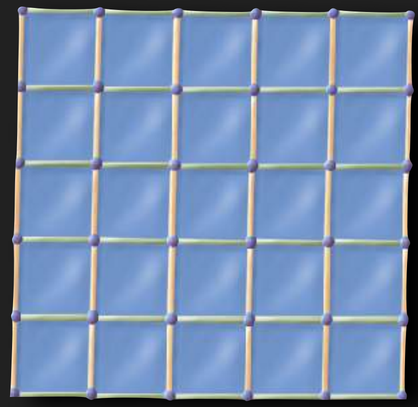
Part II

Loops and Tilings

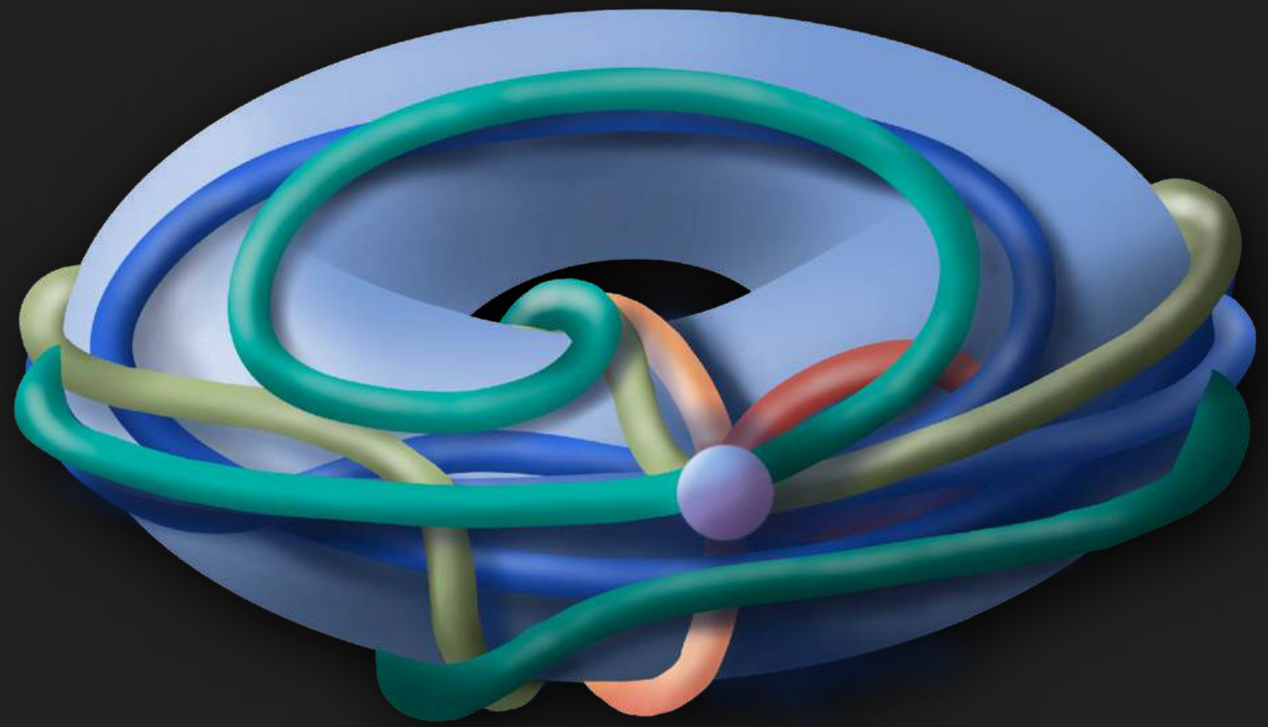






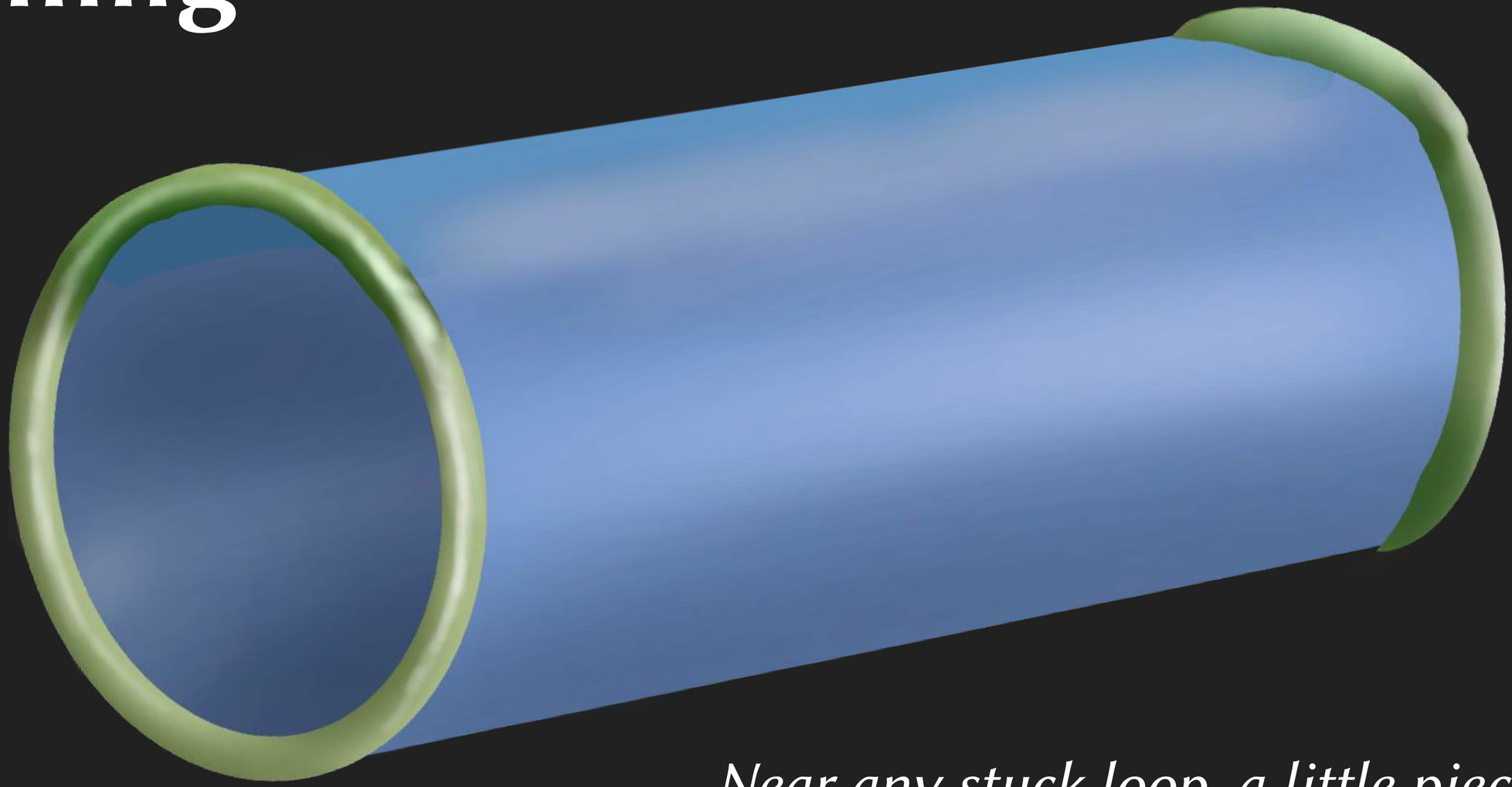


Computing The Loop Invariant



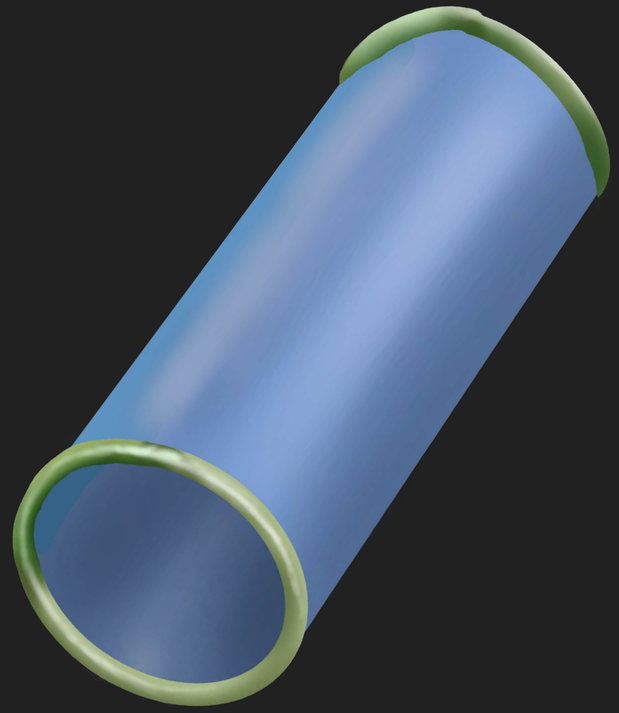
How to understand the collection of **all stuck loops** on a manifold?

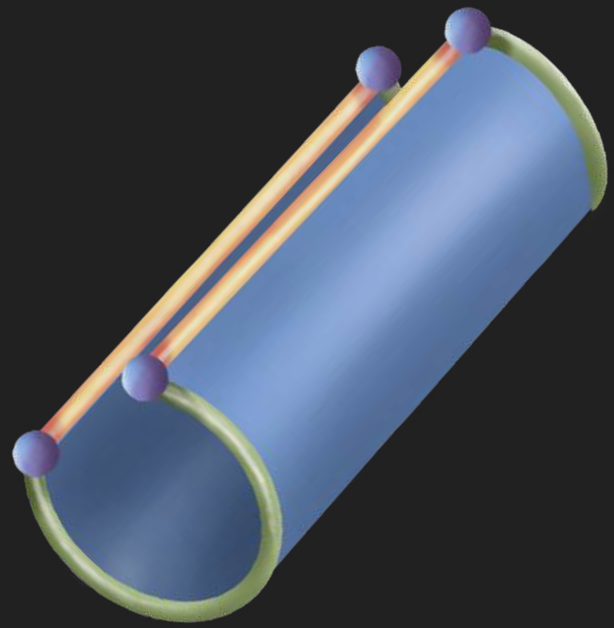
Unrolling *along a* Loop



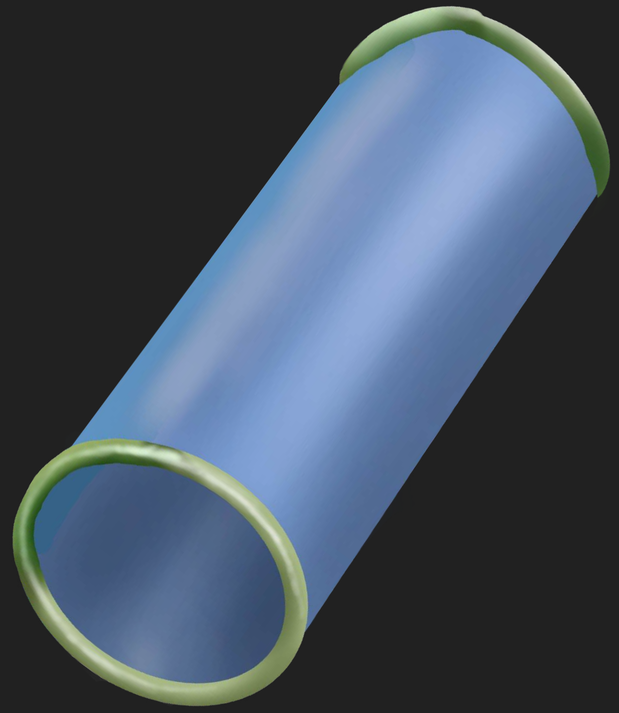
Near any stuck loop, a little piece of the surface looks like a cylinder.

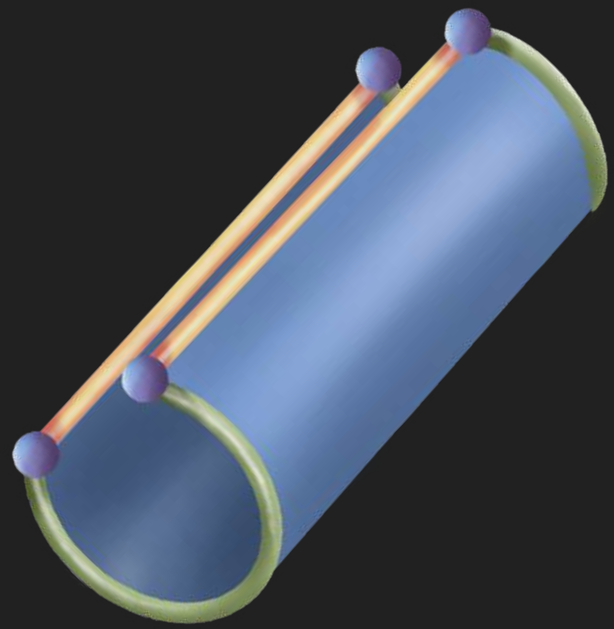
Unrolling *along a* Loop

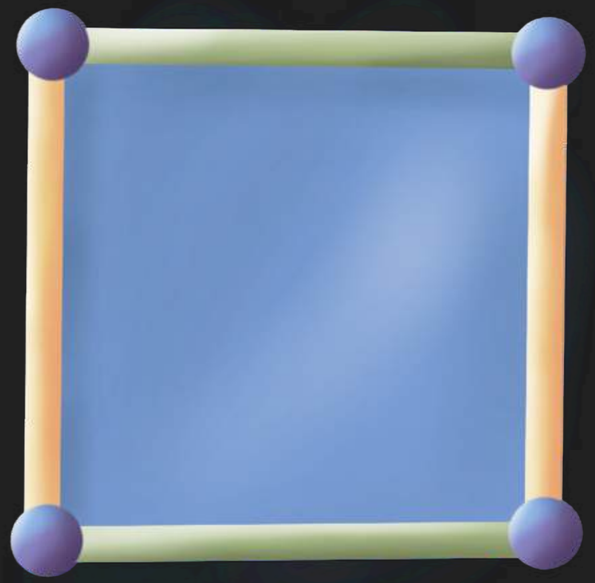




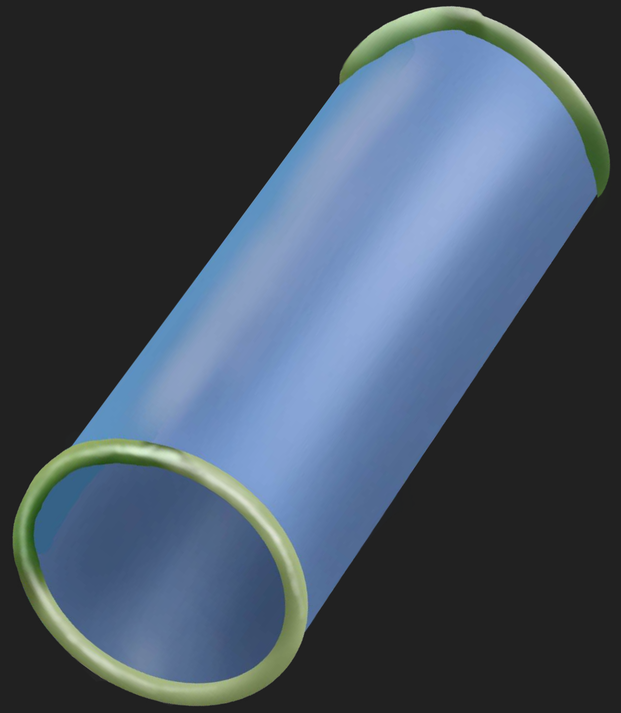
Unrolling *along a* Loop

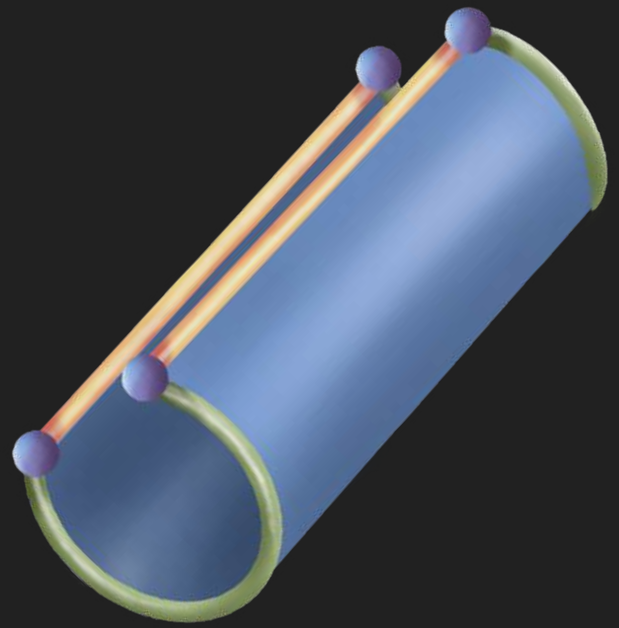


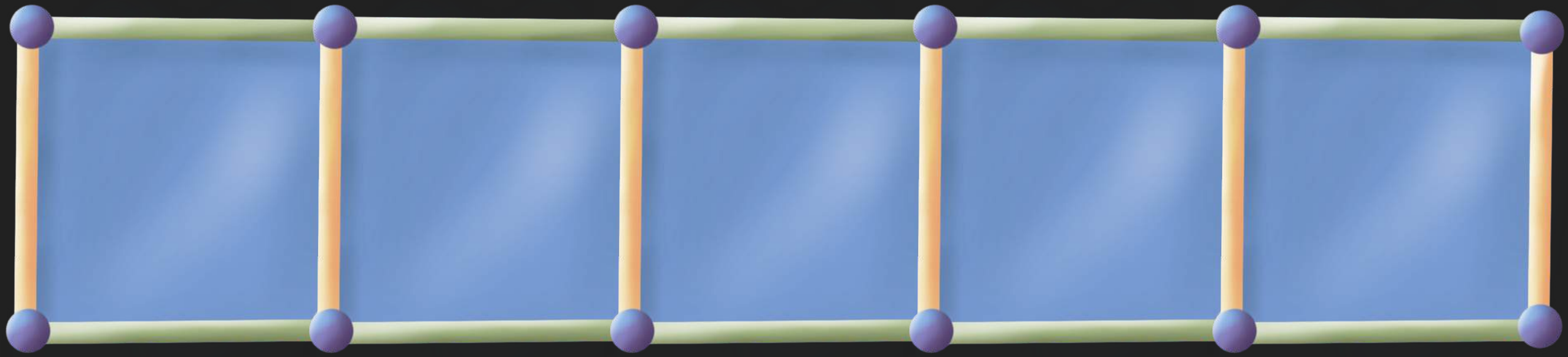


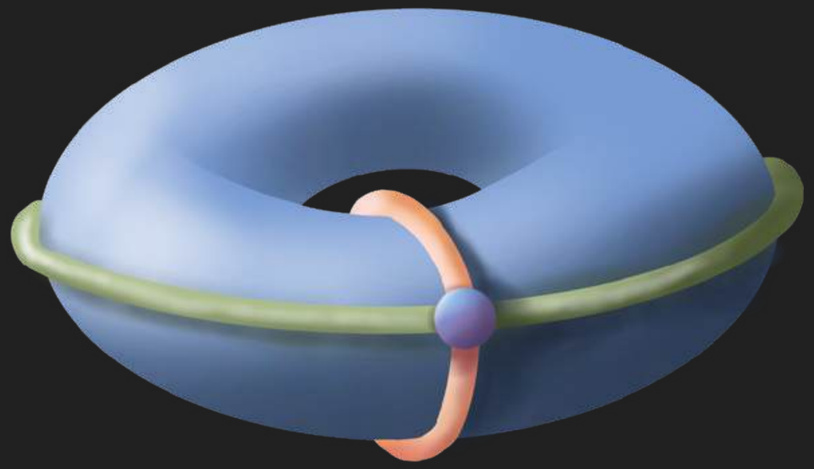


Unrolling *along a* Loop



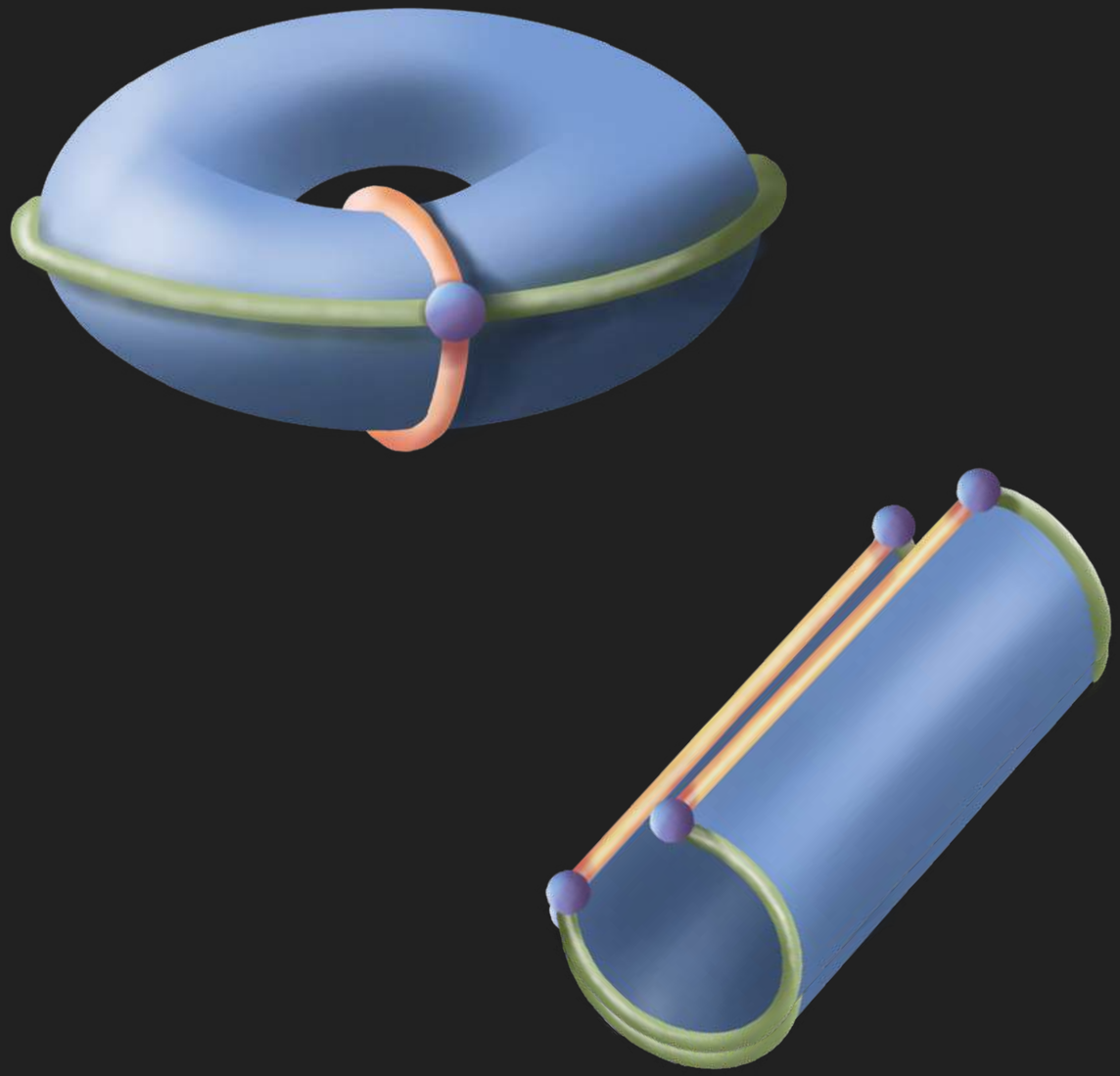






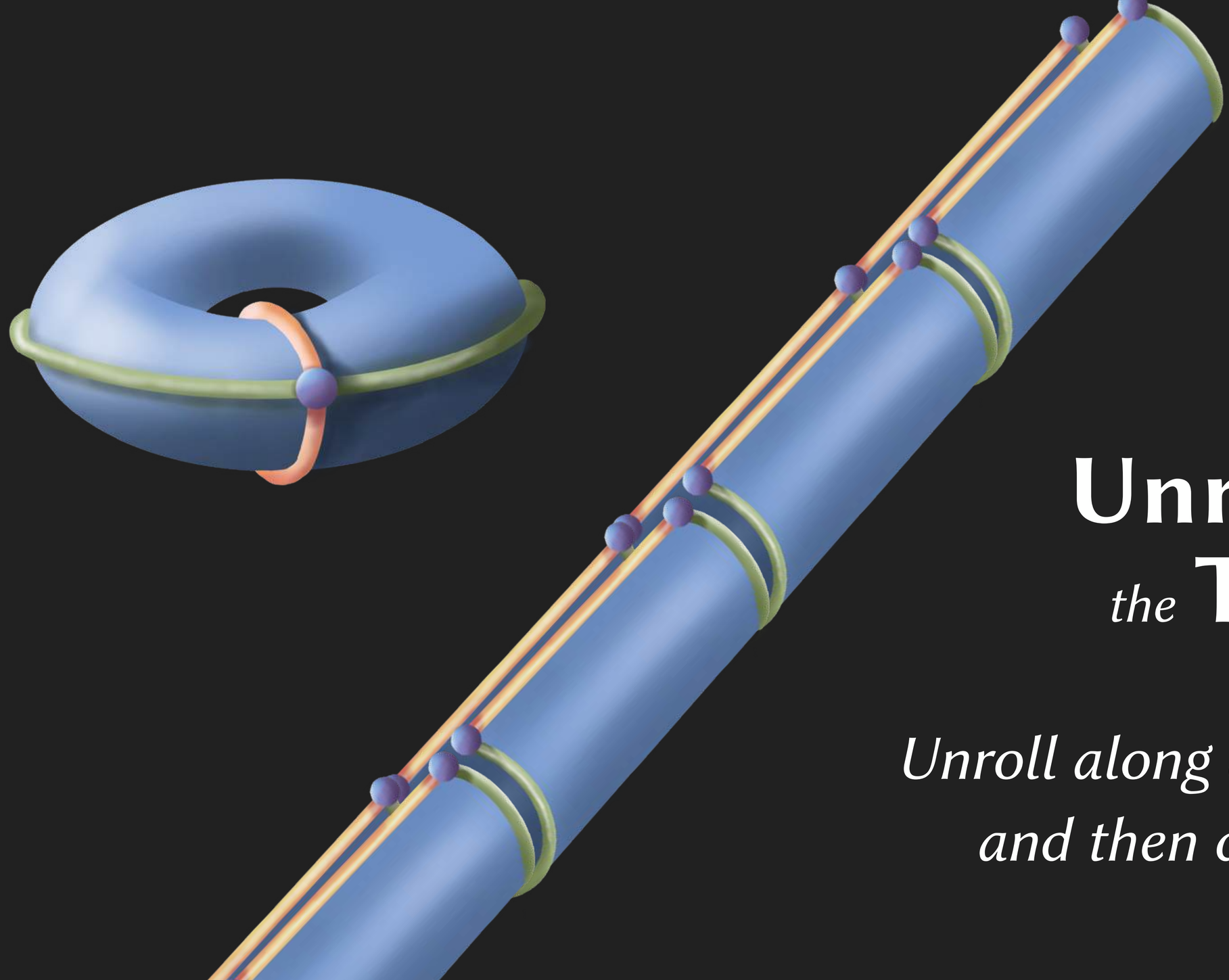
Unrolling *the* Torus

Unroll along a stuck loop...



Unrolling *the* Torus

Unroll along a stuck loop...



Unrolling *the* Torus

Unroll along a stuck loop...
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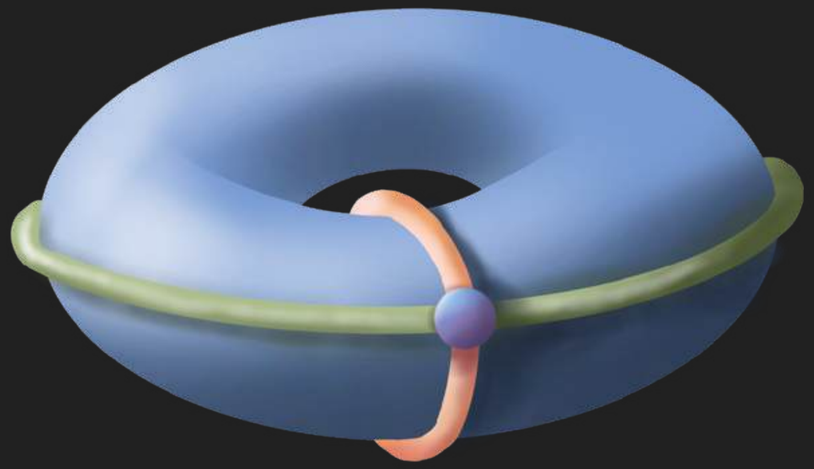

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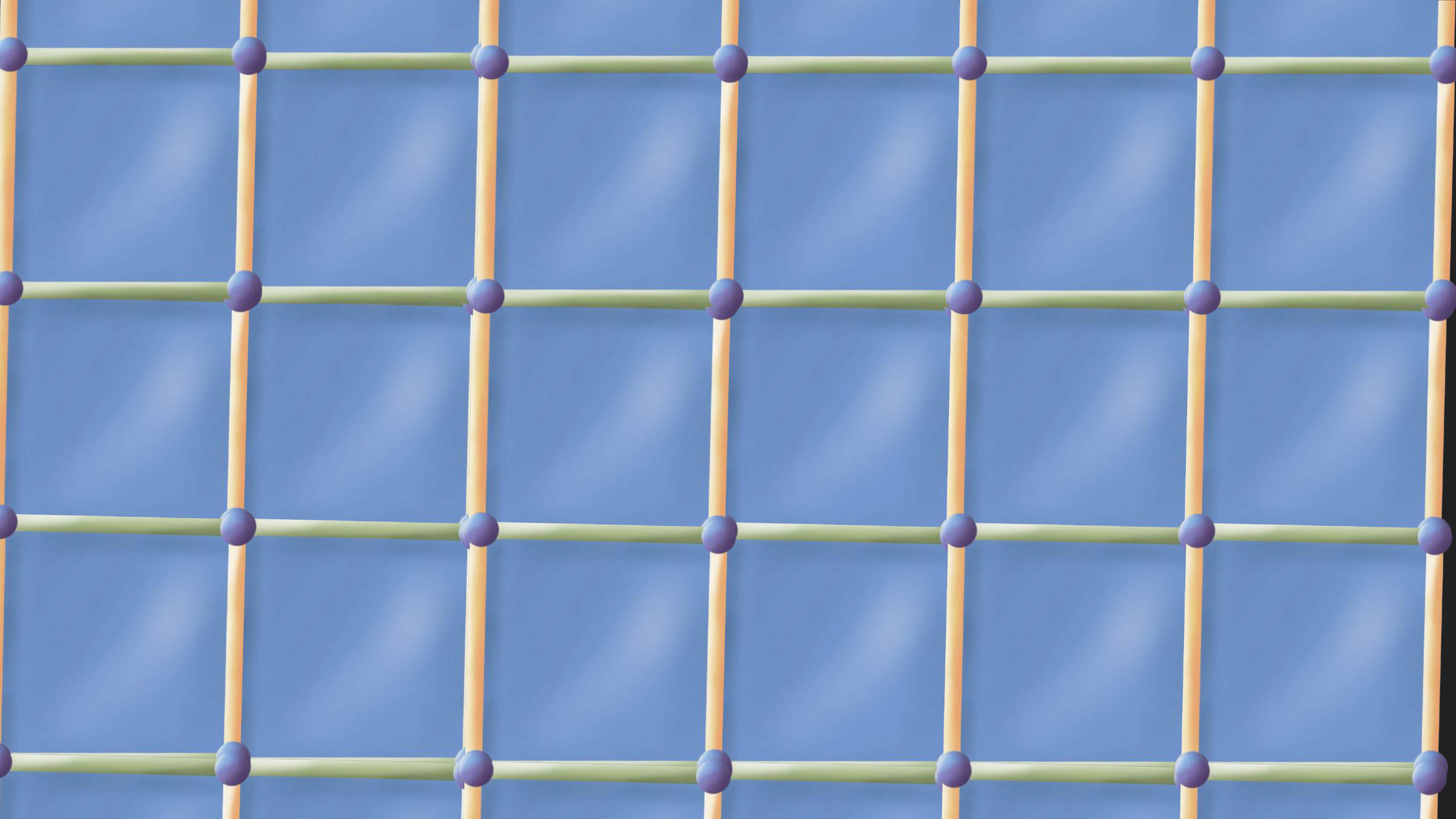
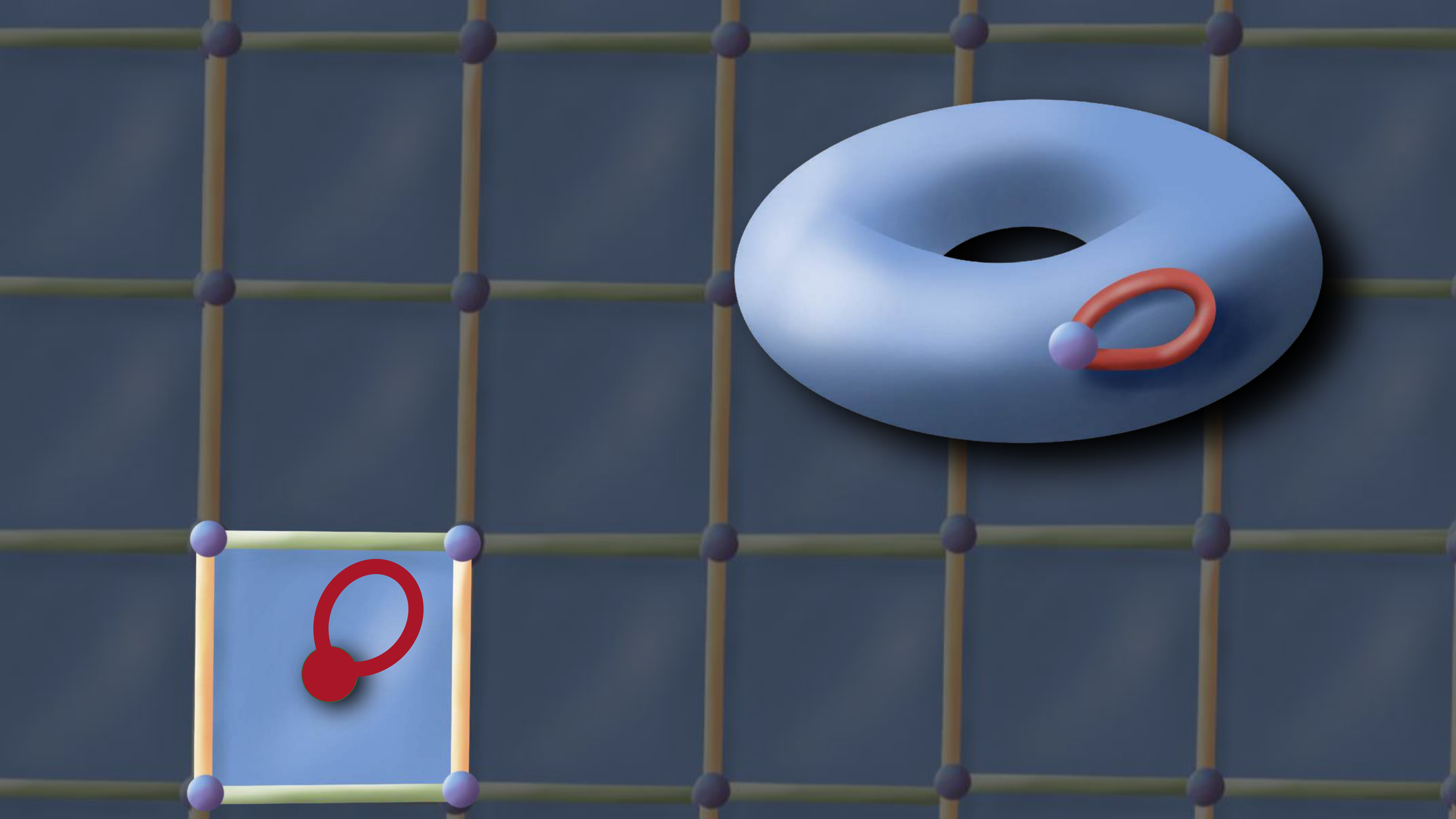
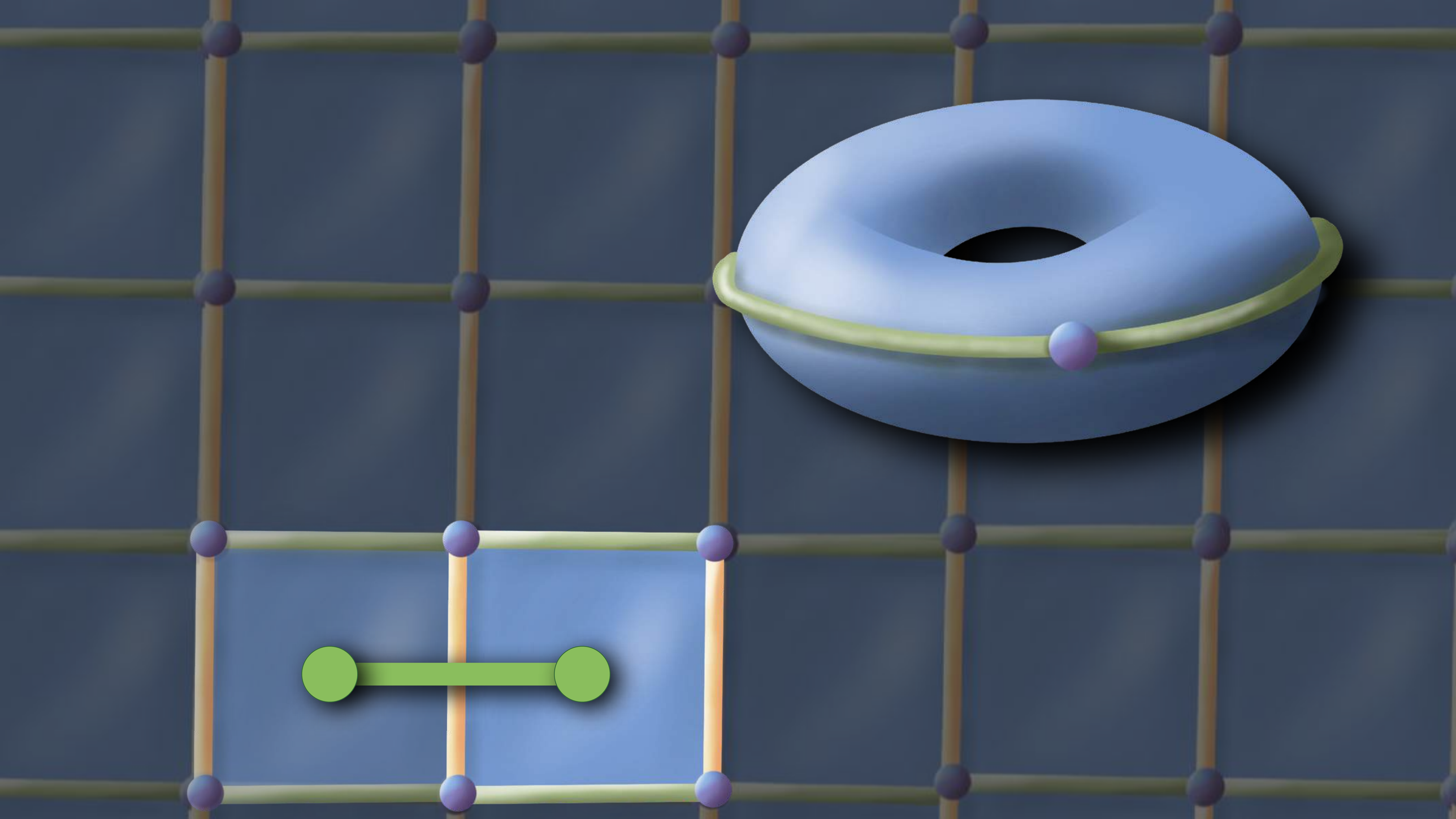
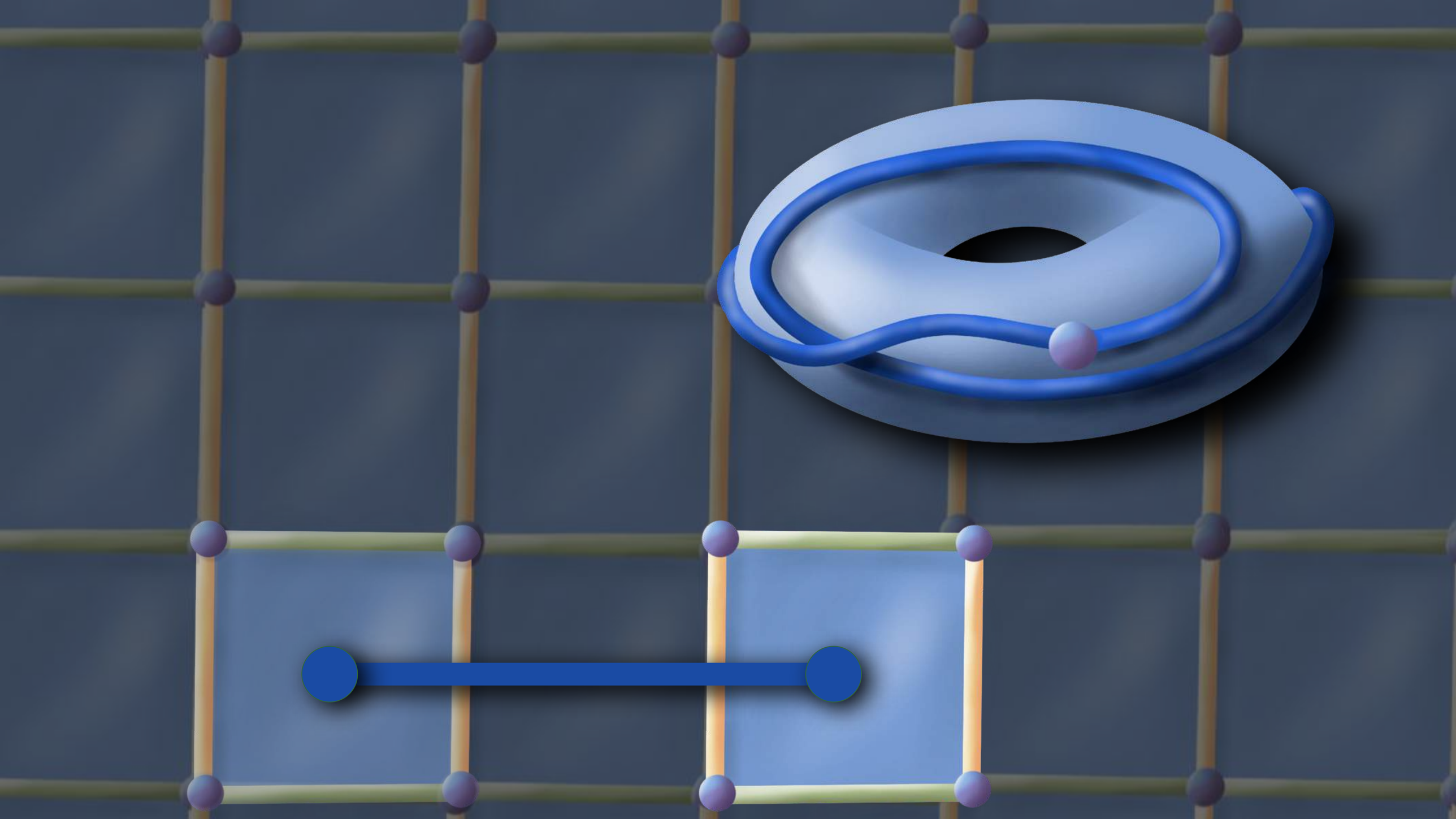
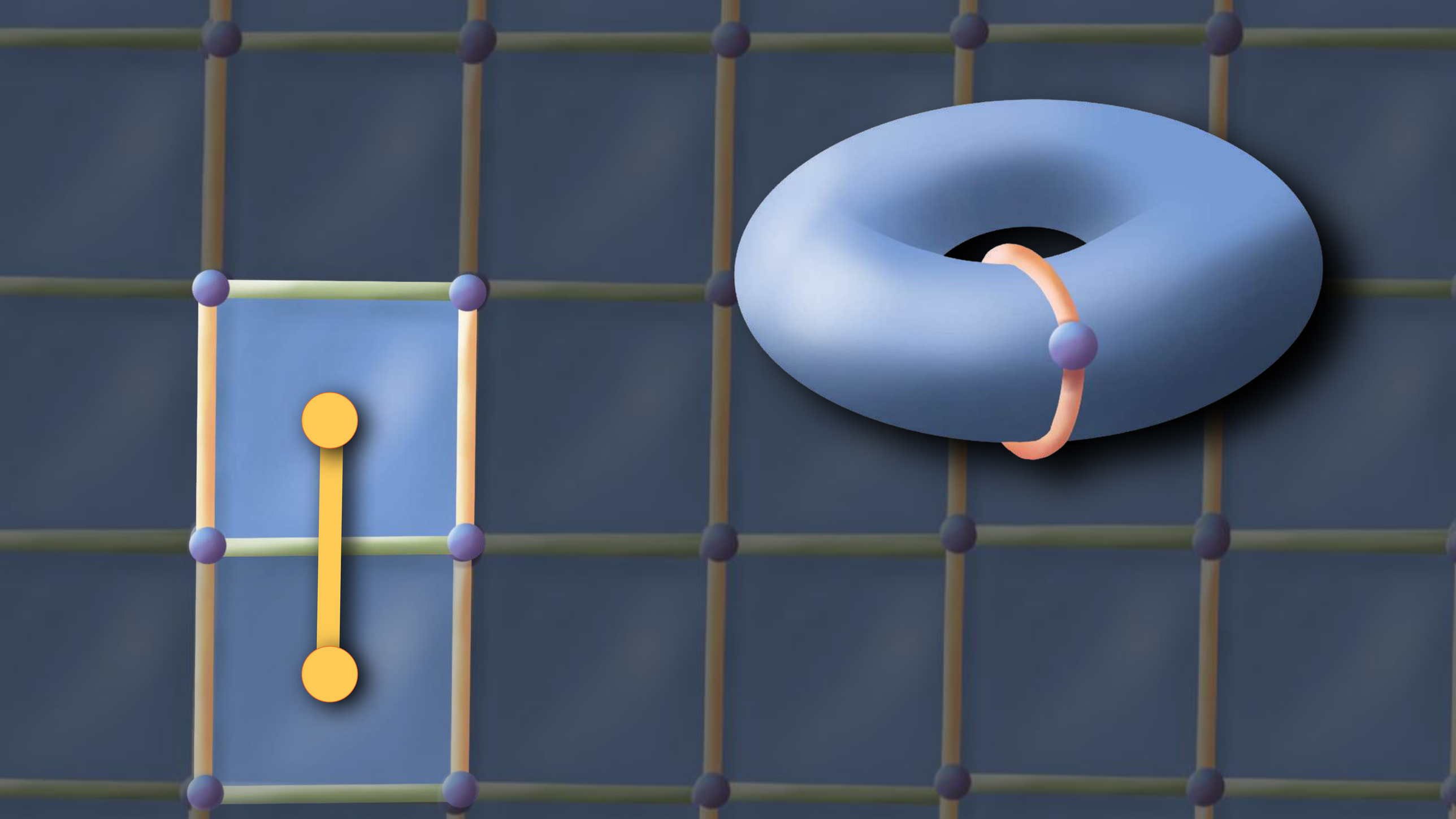
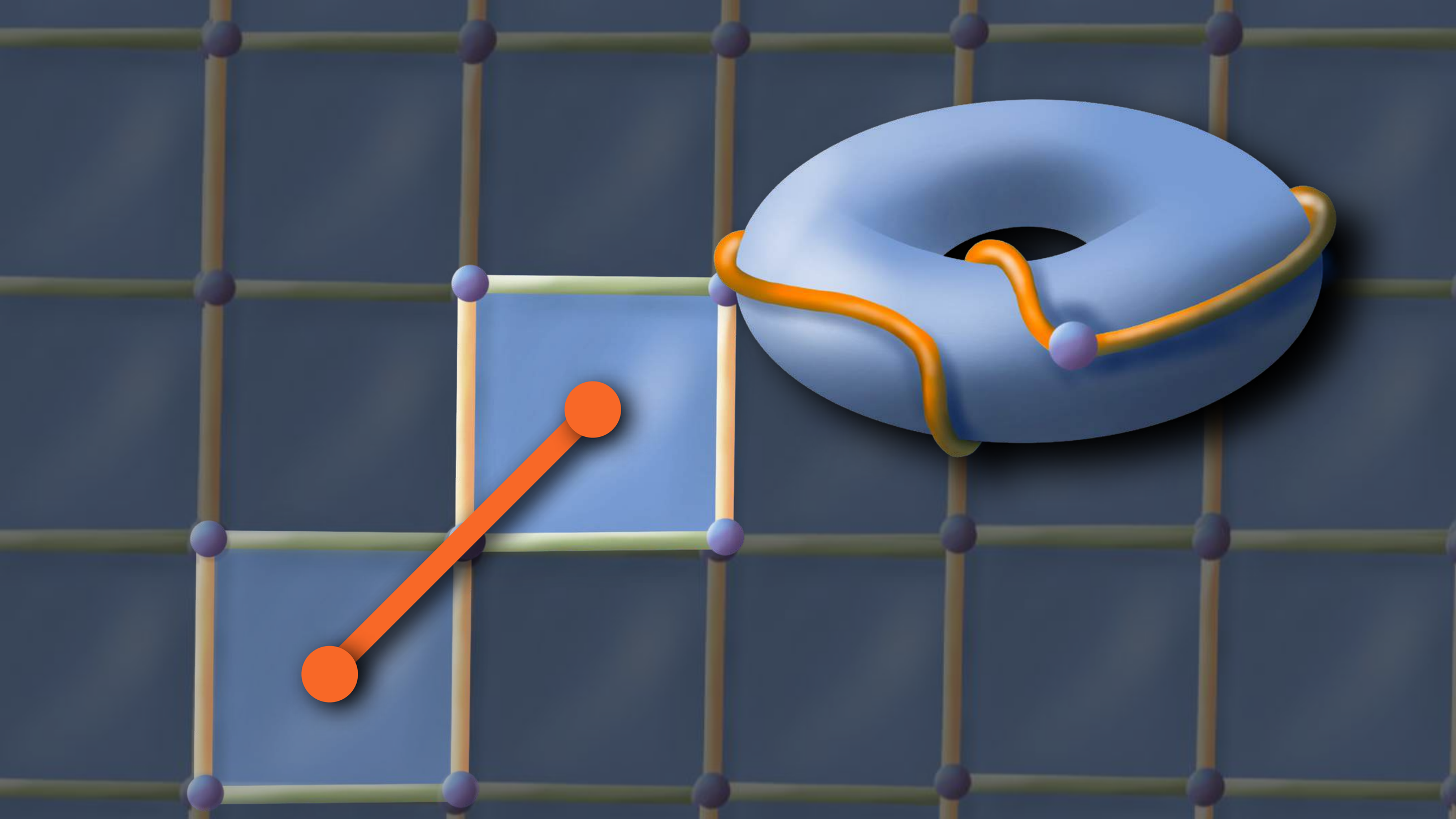


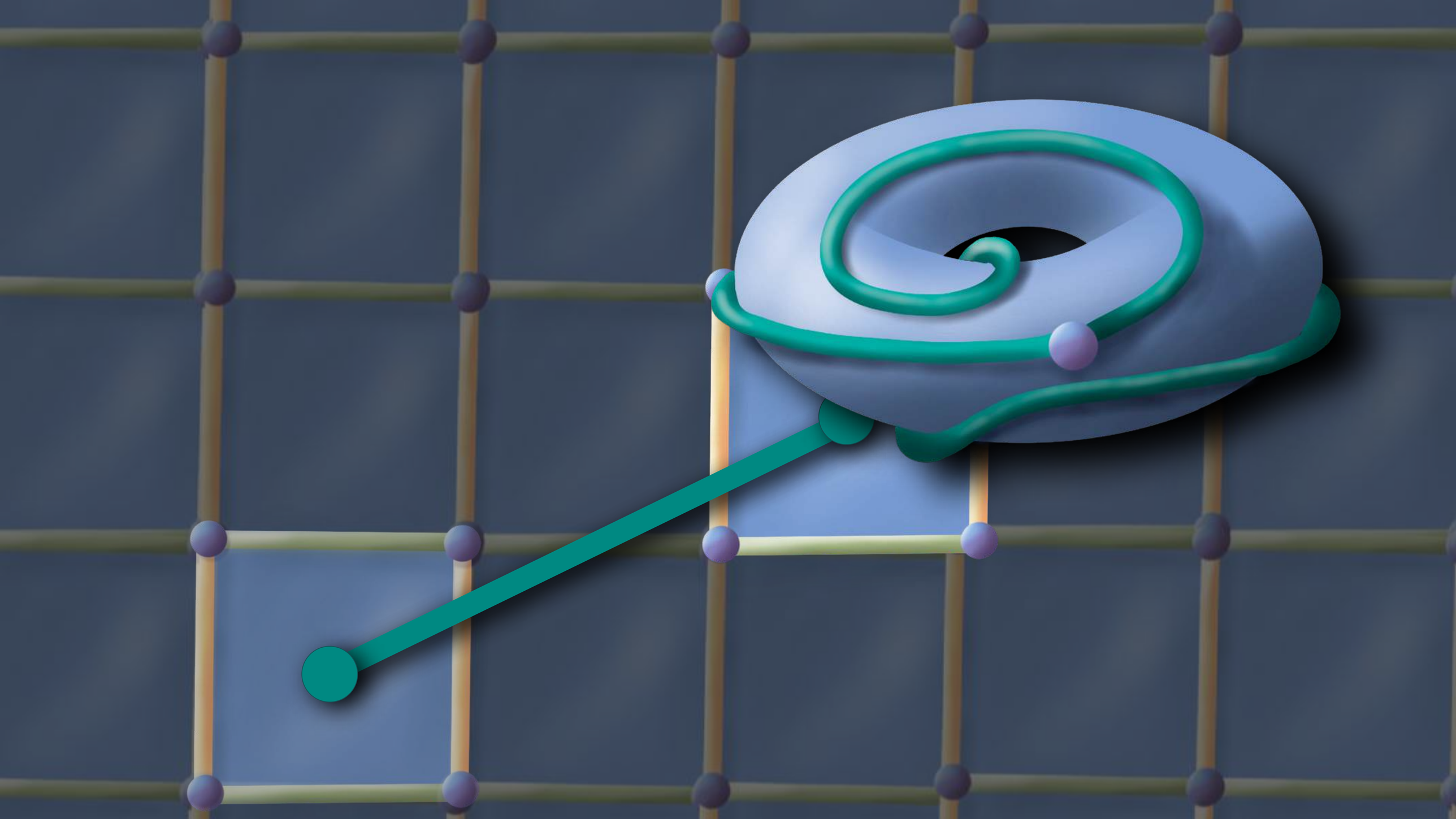






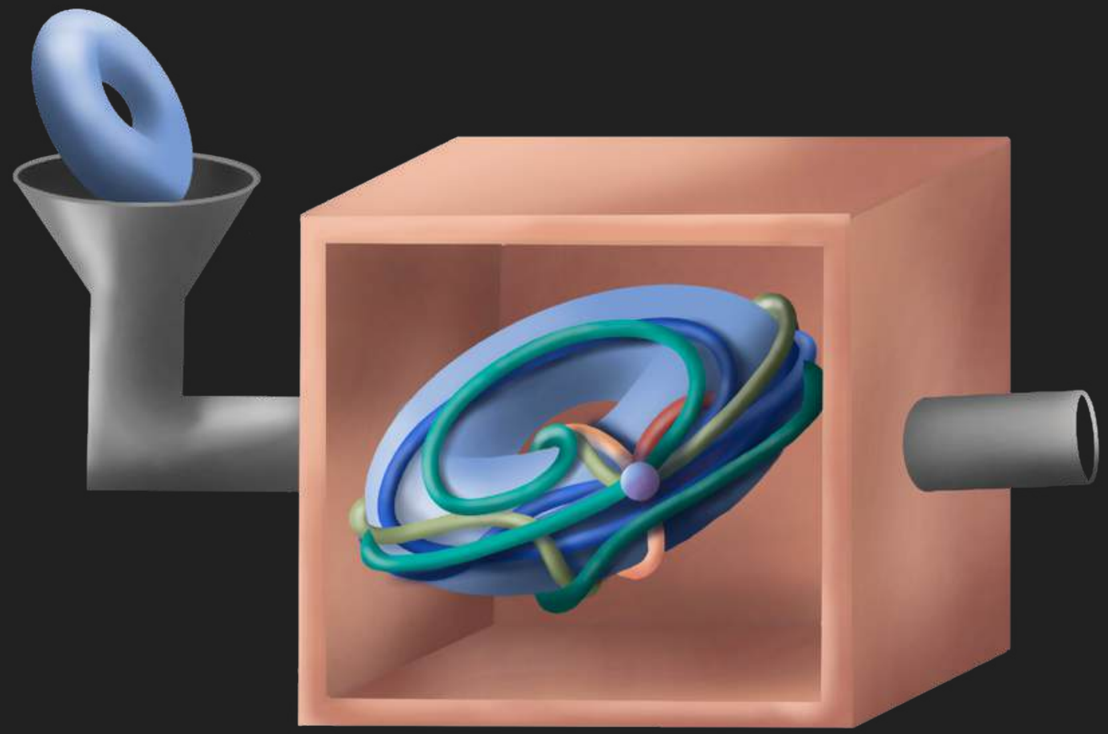


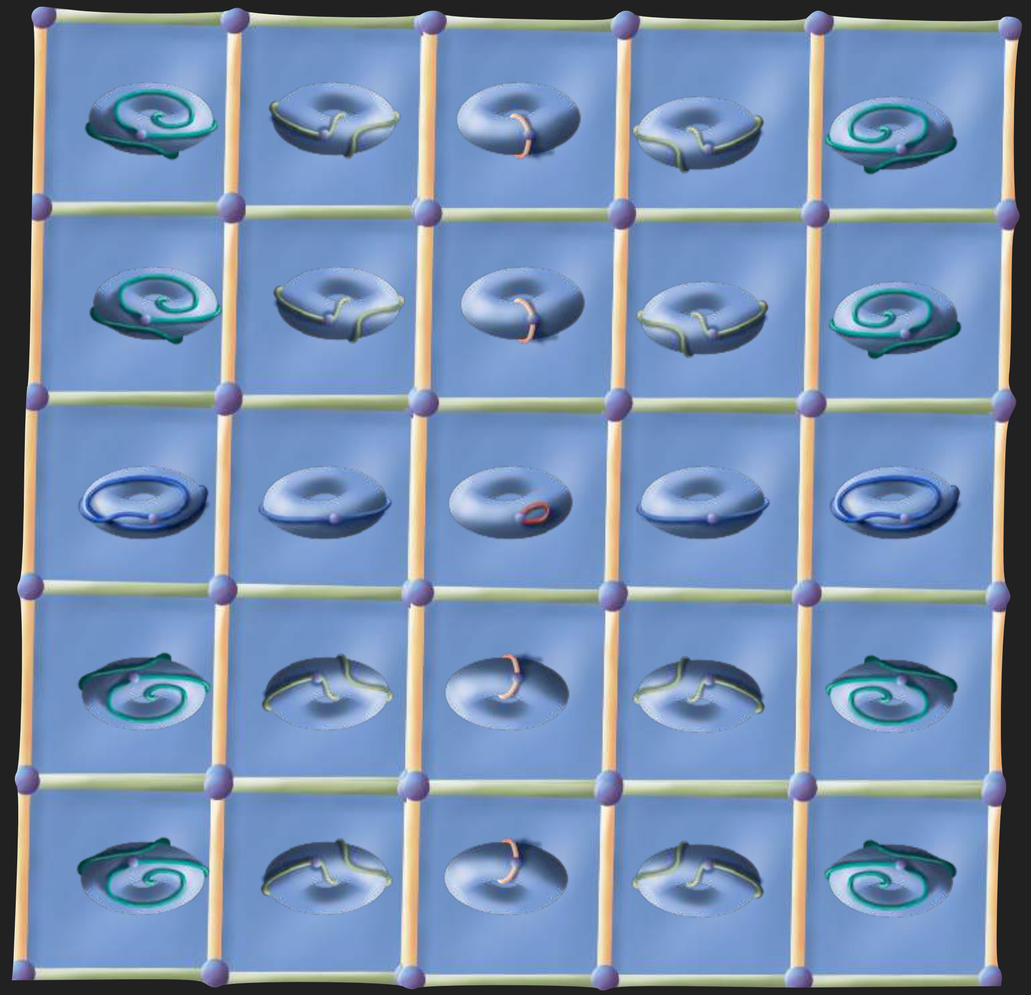




The Loop Invariant

The collection of loops & their structure is the fundamental group.





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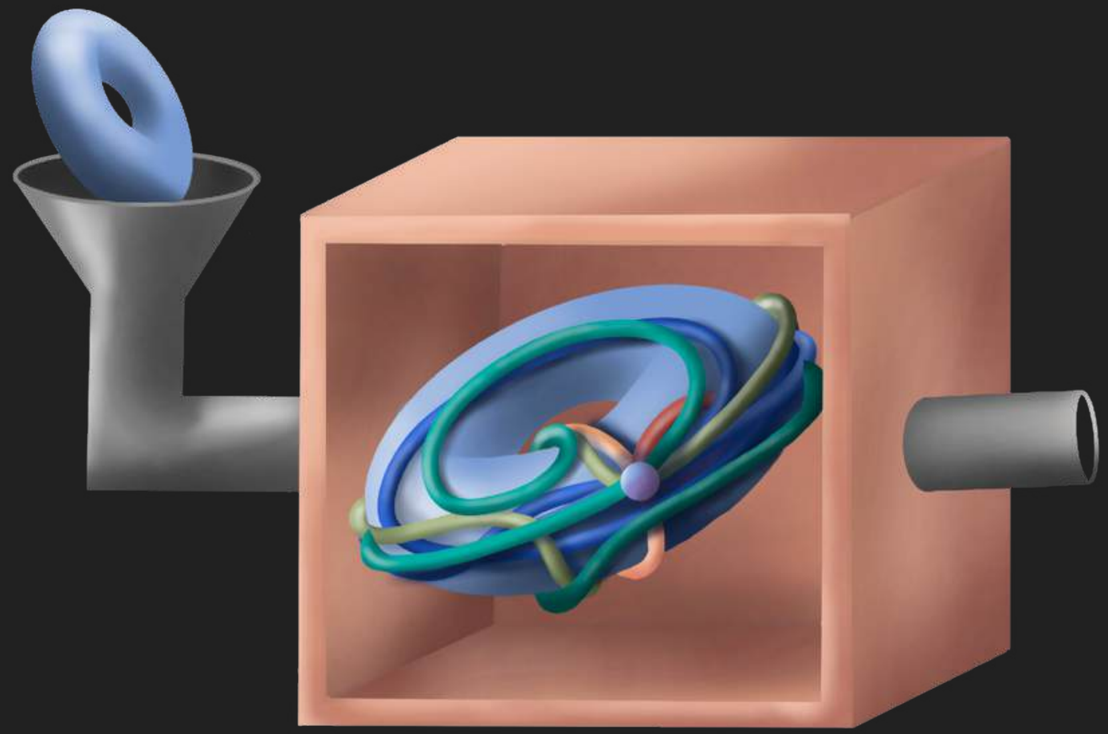

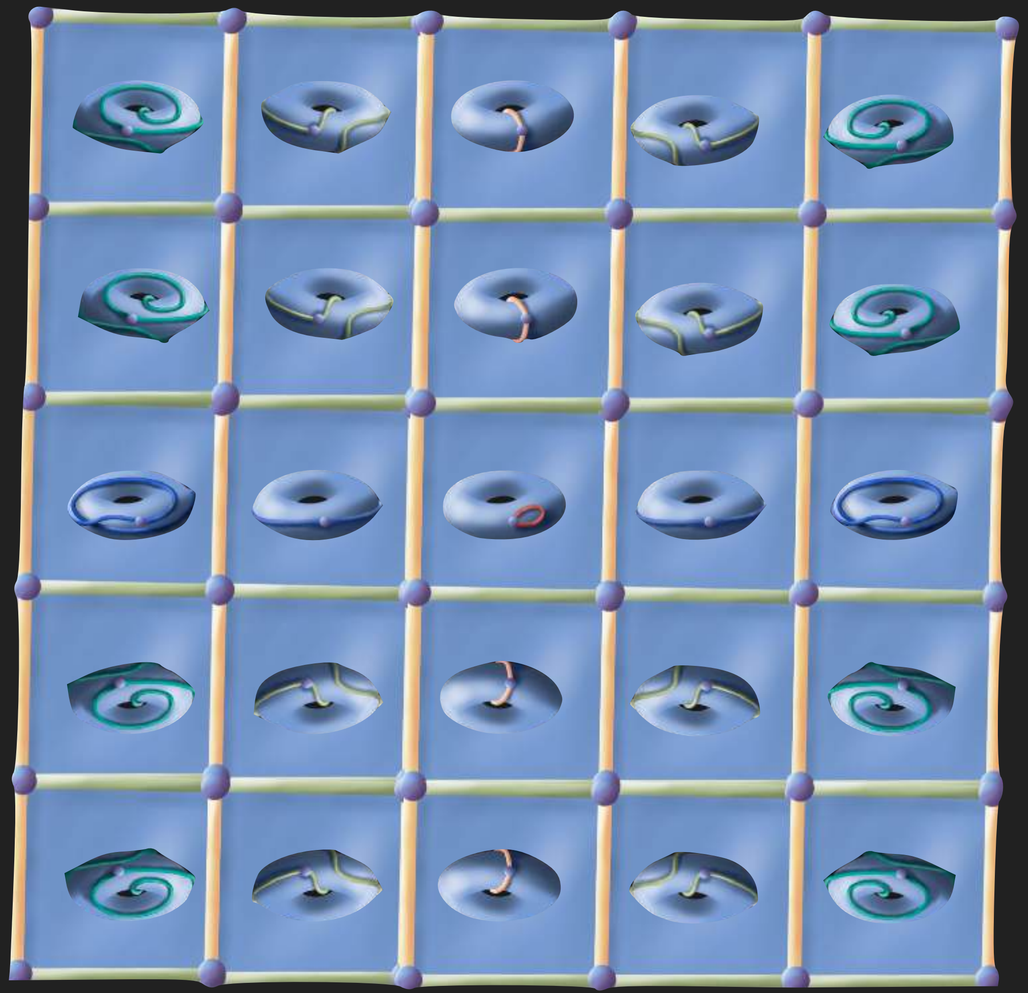

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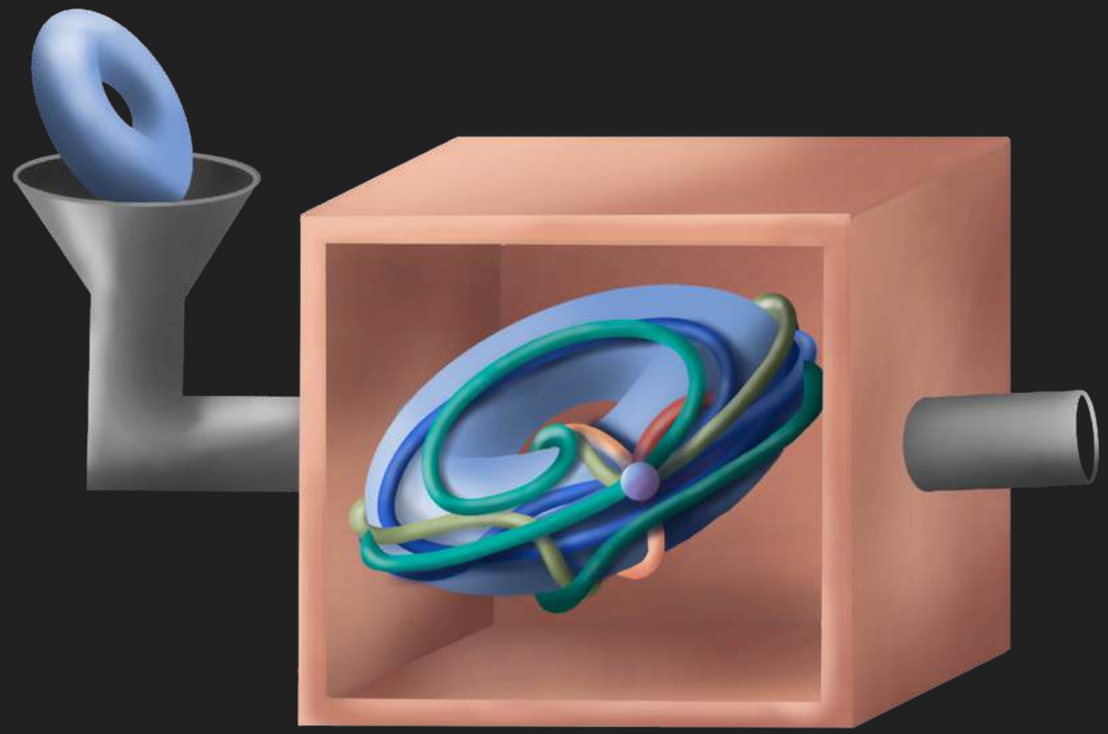

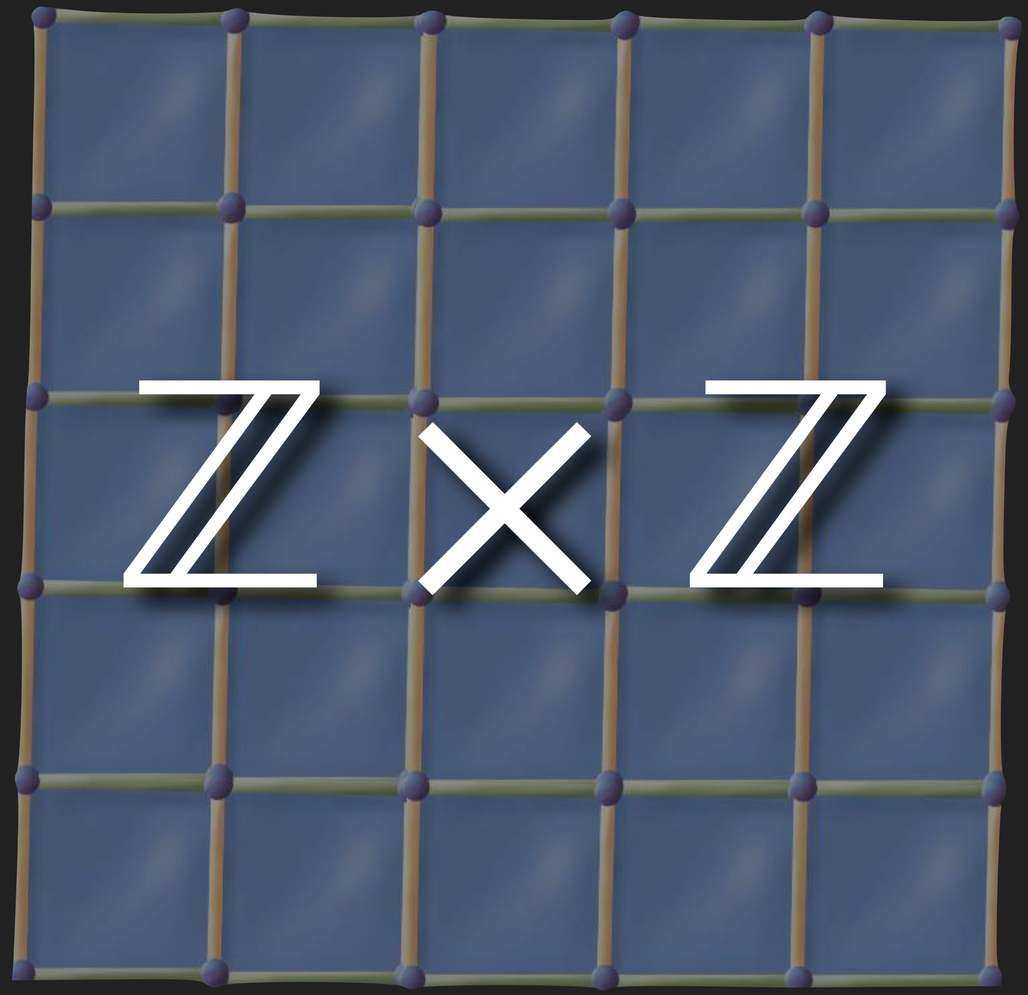

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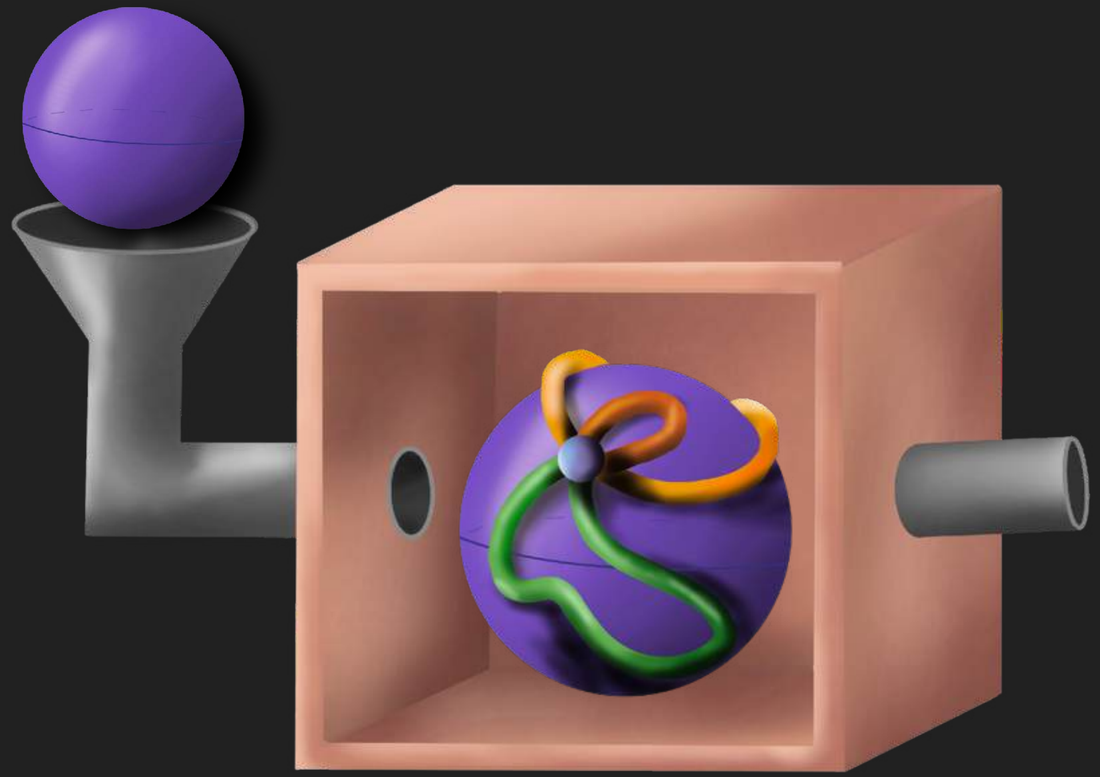
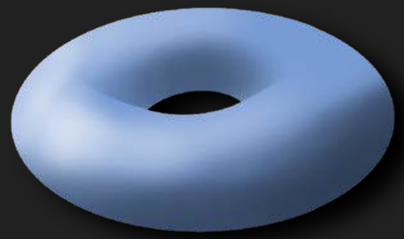


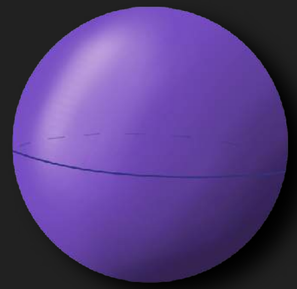
The sphere has no stuck loops, so it doesn't unwrap.

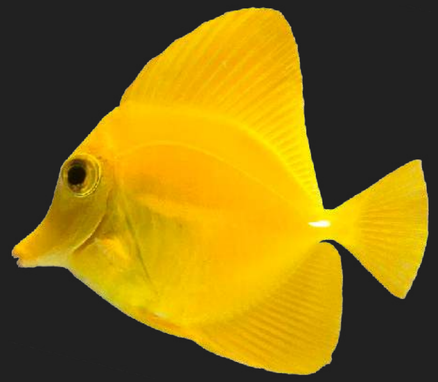
$\{0\}$

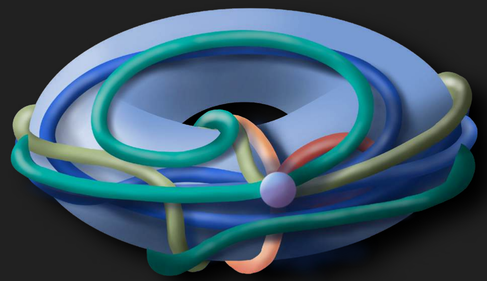
Manifolds have DNA:

A discrete code, built out of loops.









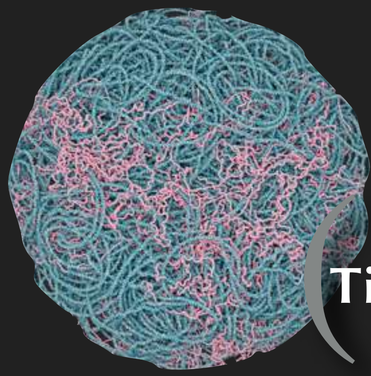
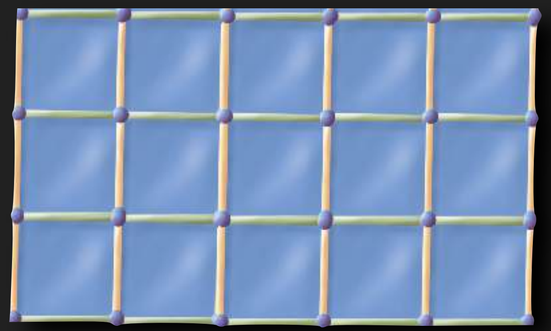


Image Credit:
Tim Davidson

$\mathbb{Z} \times \mathbb{Z}$



$\{0\}$

GCCTAGCCAAAGCTCTT
CCAAGGTGACTCTCAGT
TCAAGCTGCCTAGCCCG
CAGTTCAAGCTGCCTAG

The

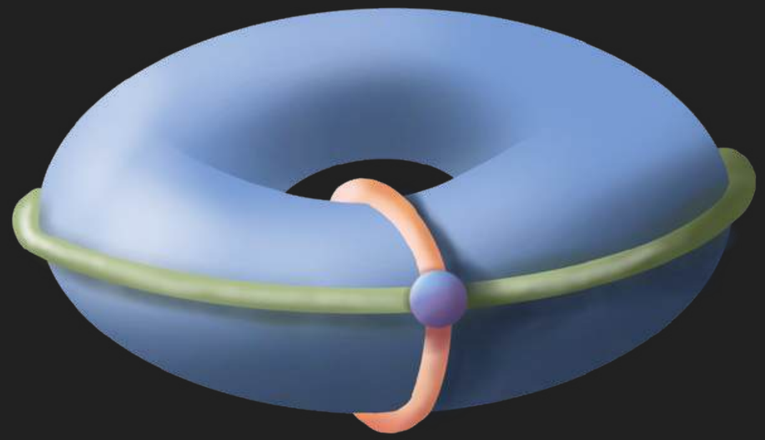
Torus

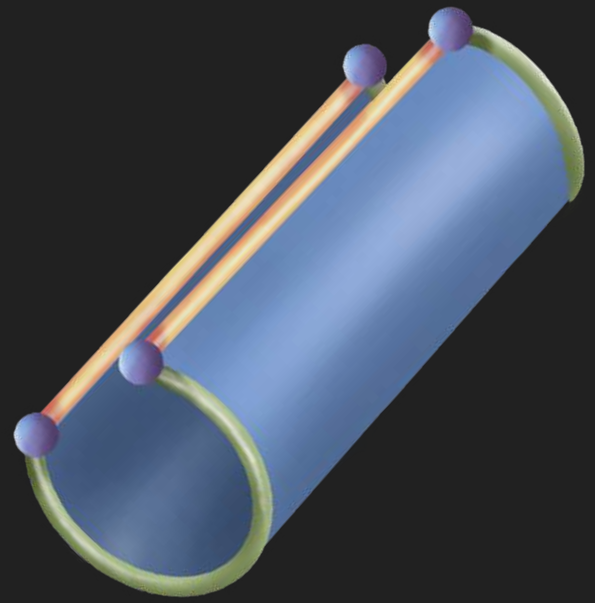
unrolls to

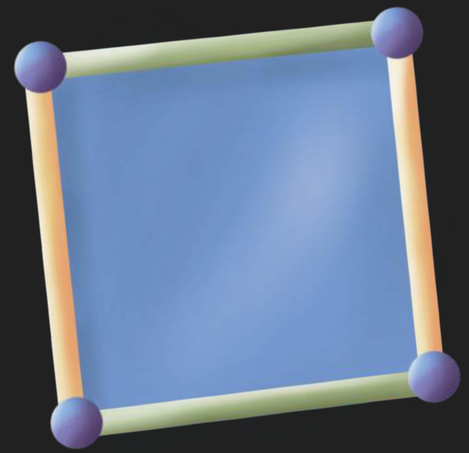
Squares

on the

Plane







But
How
do we
Unroll
this
Surface?

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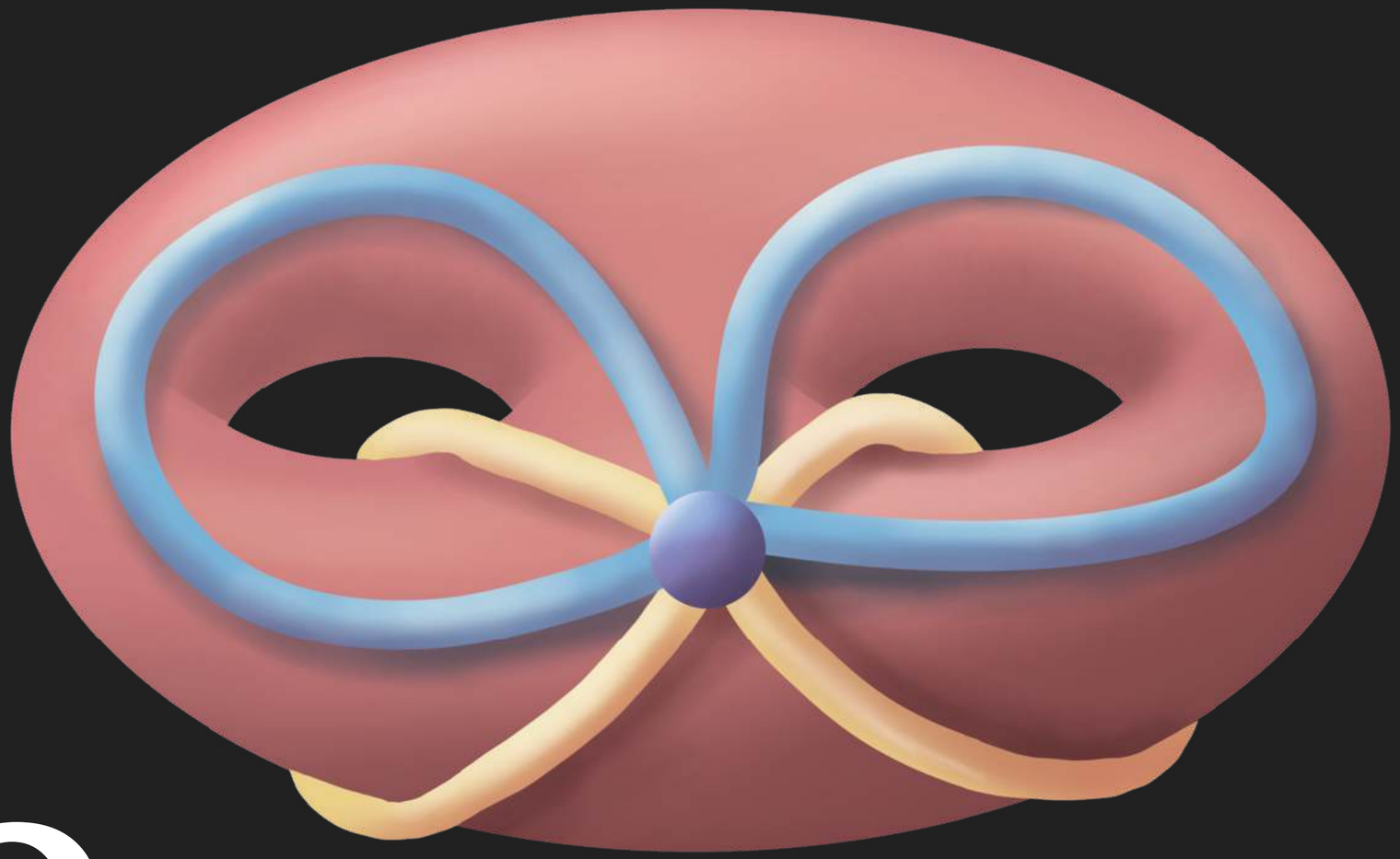

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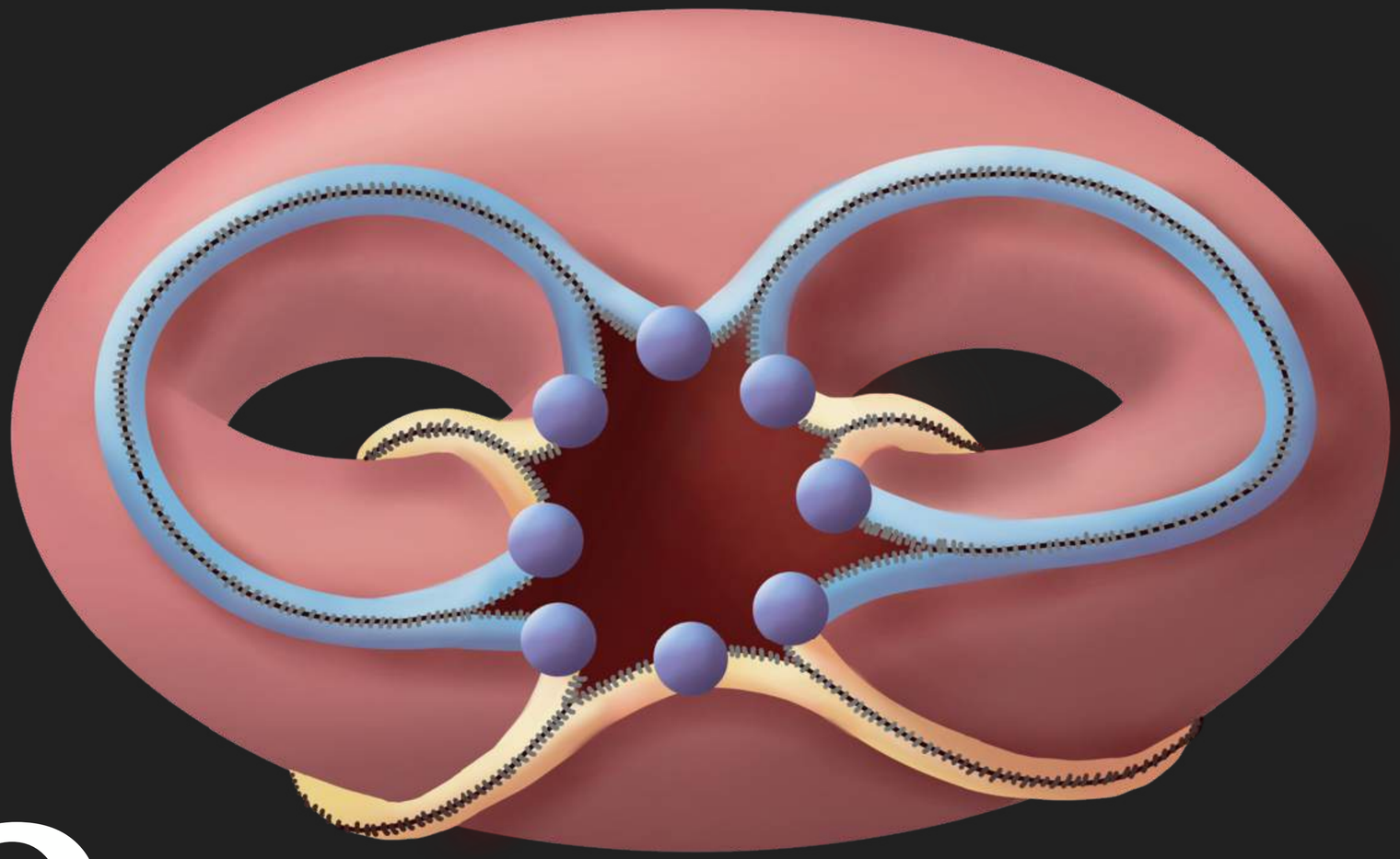

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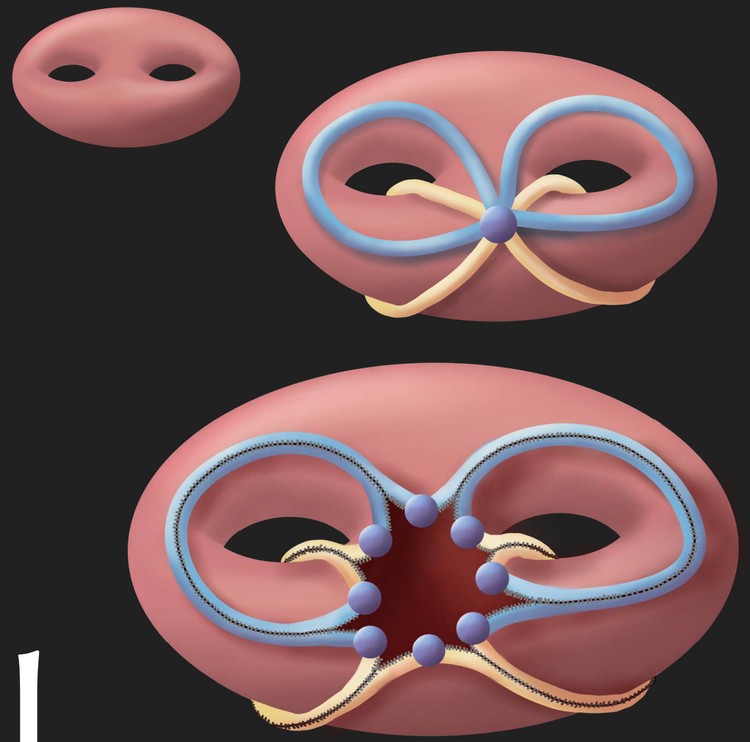

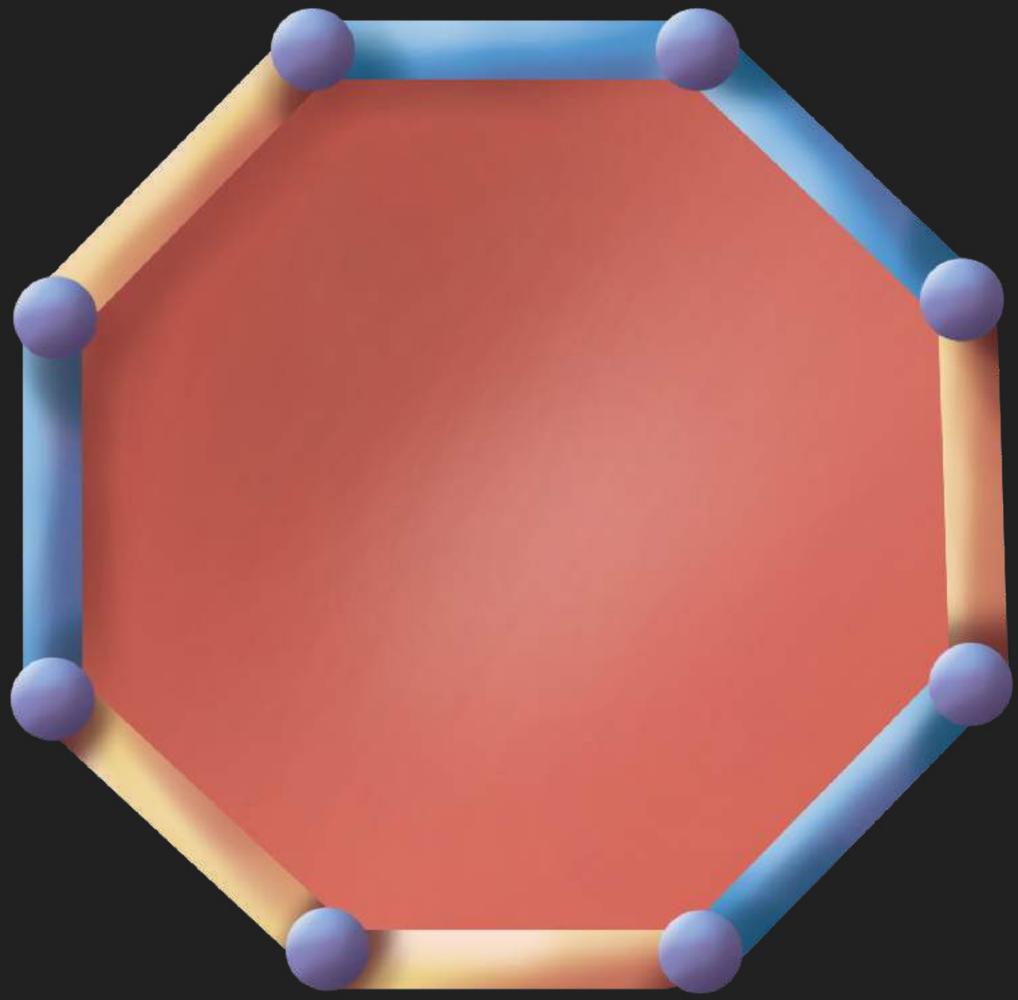

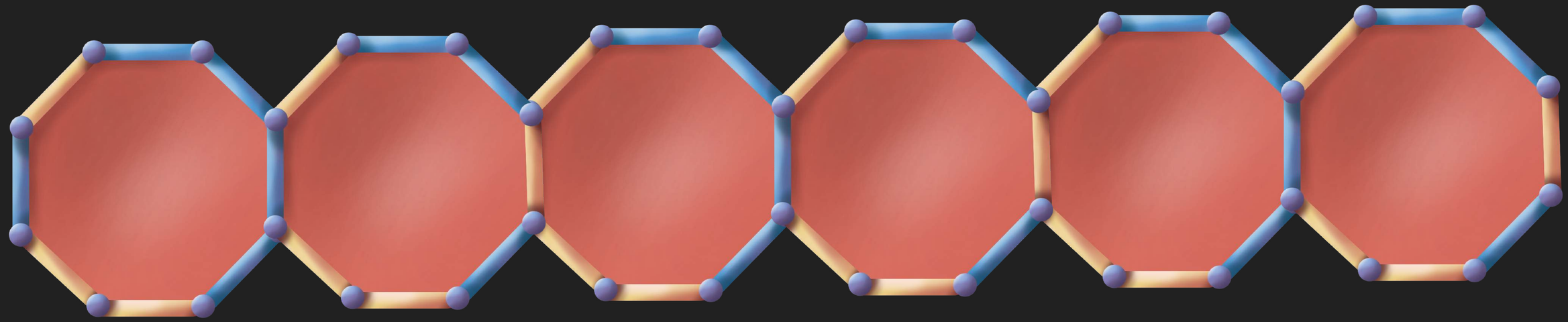

Unrolling Loops

should produce a

Tiling by Octagons

Unrolling Loops

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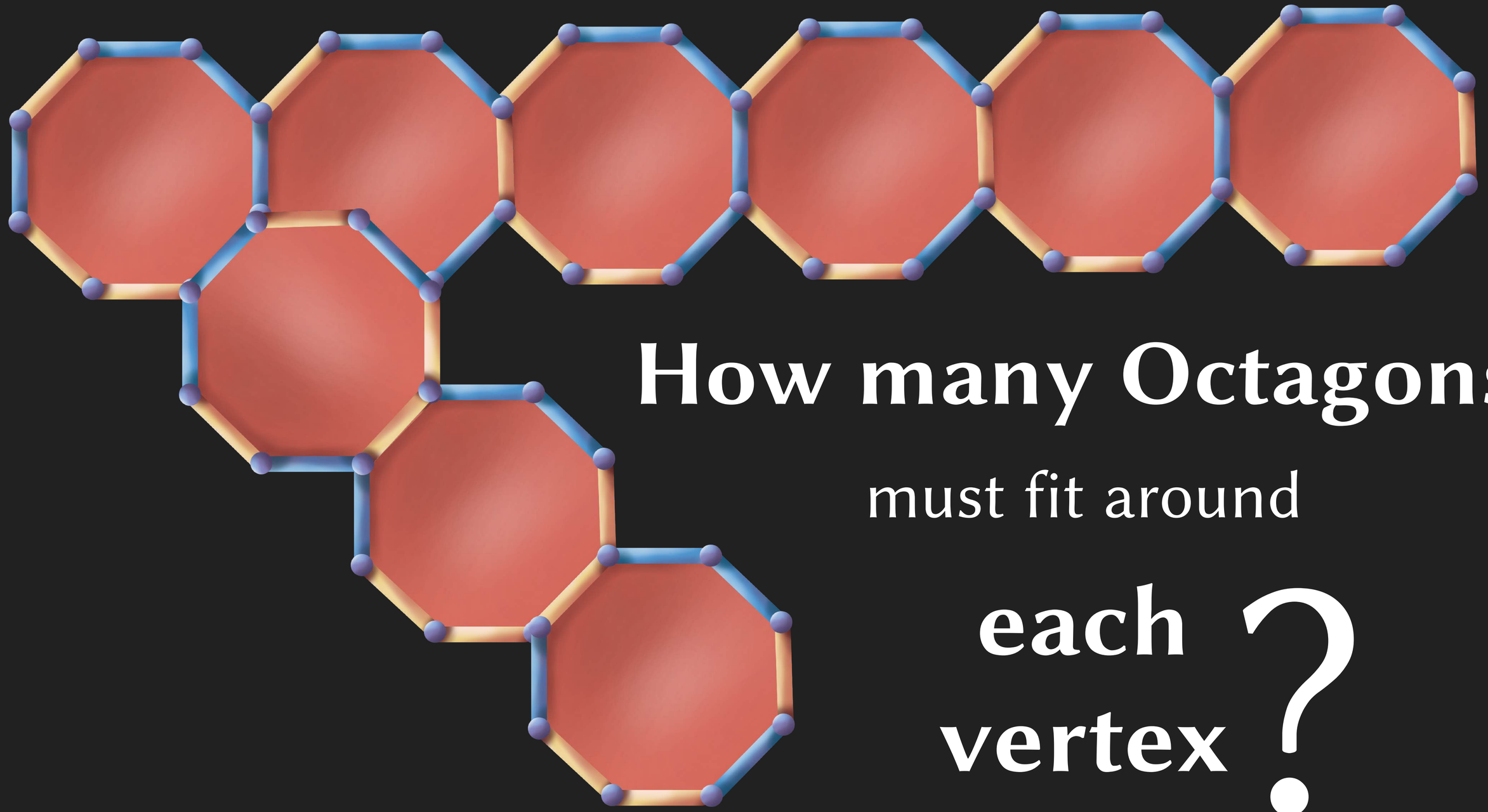
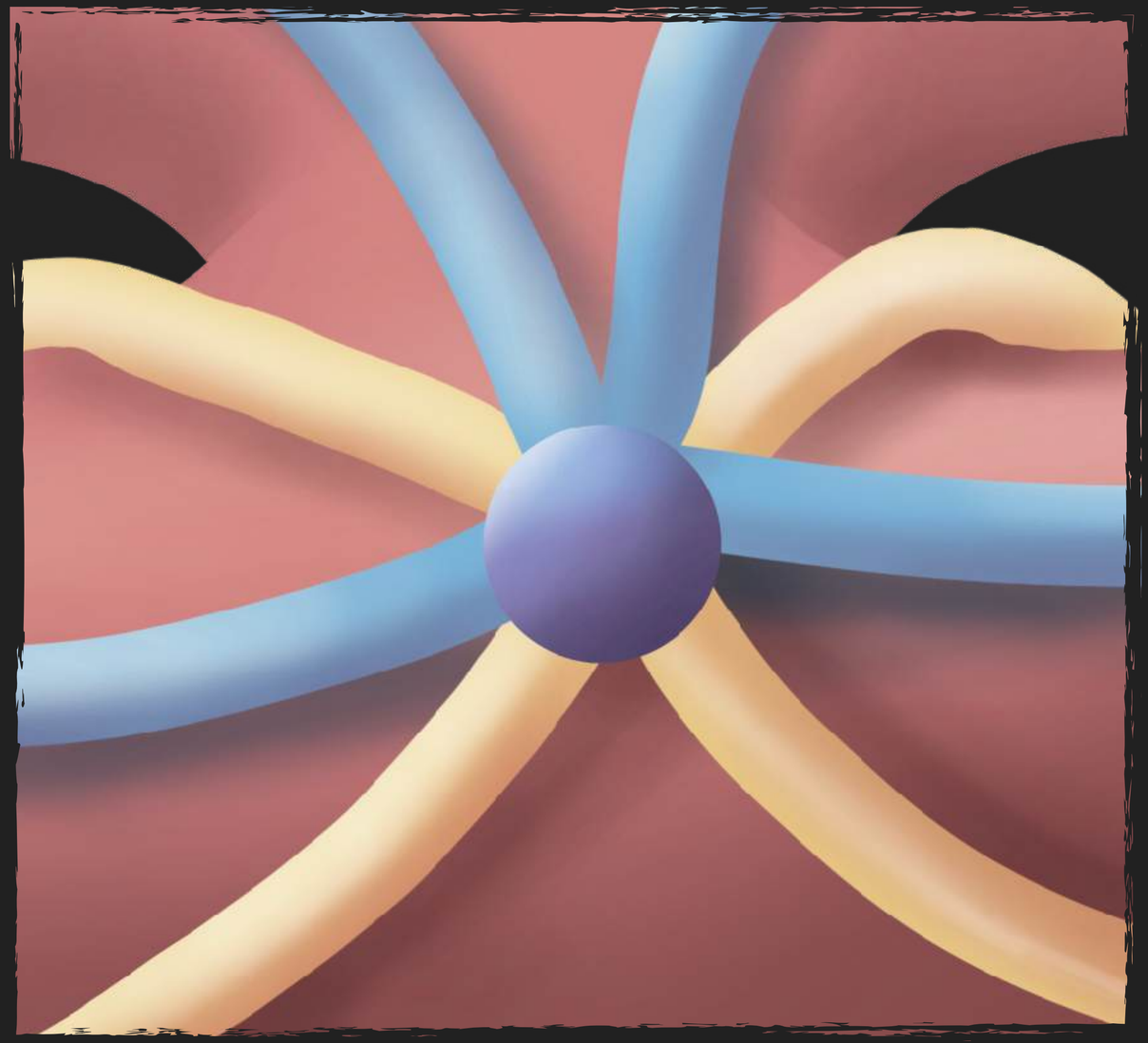
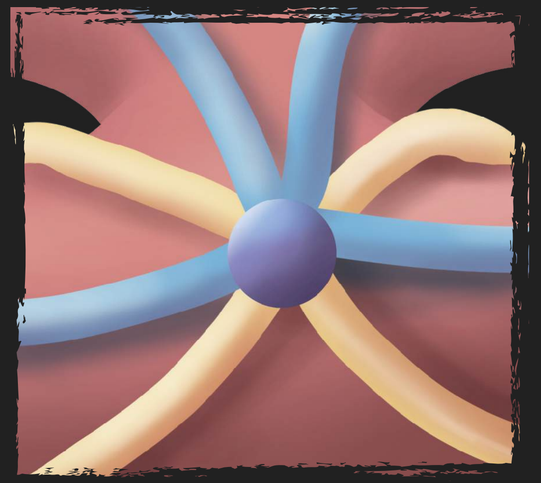
Tiling by Octagons



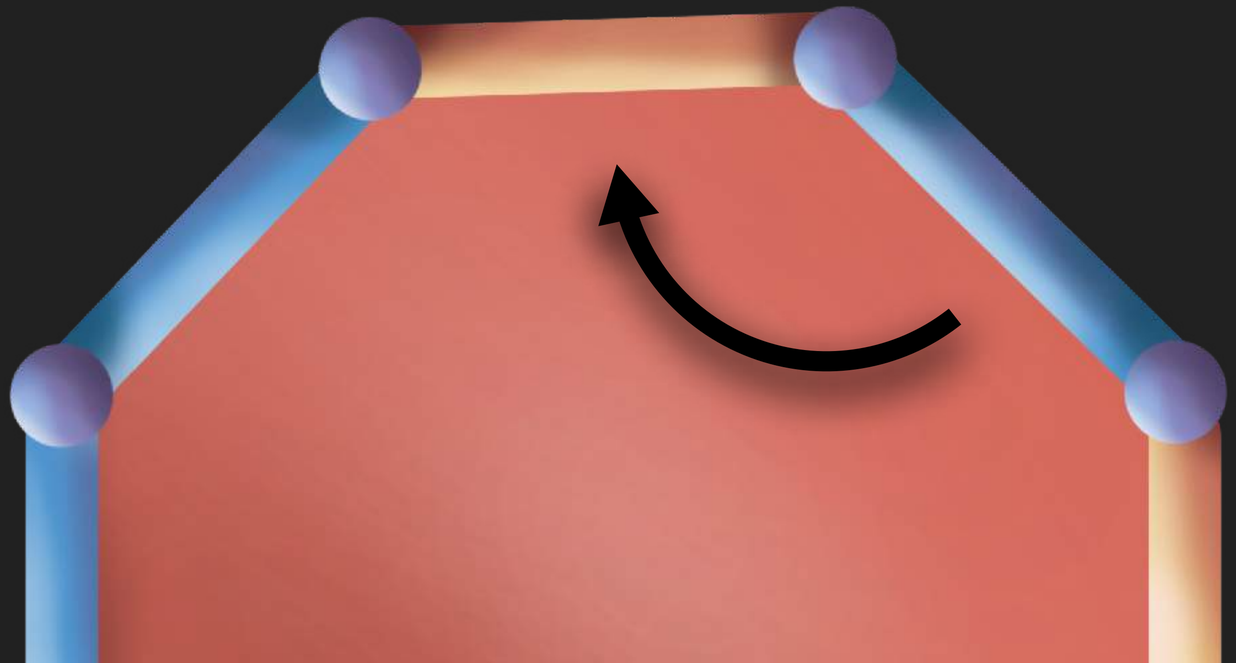
How many Octagons must fit around each vertex ?



Eight Octagons must fit around each vertex?!



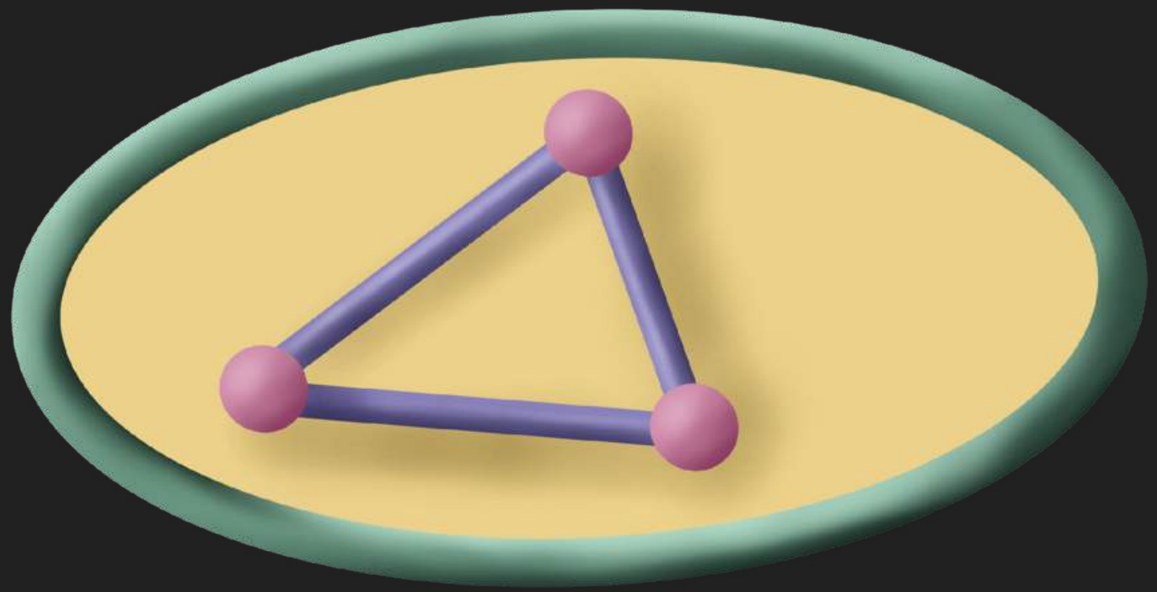
This angle isn't 45° 😭

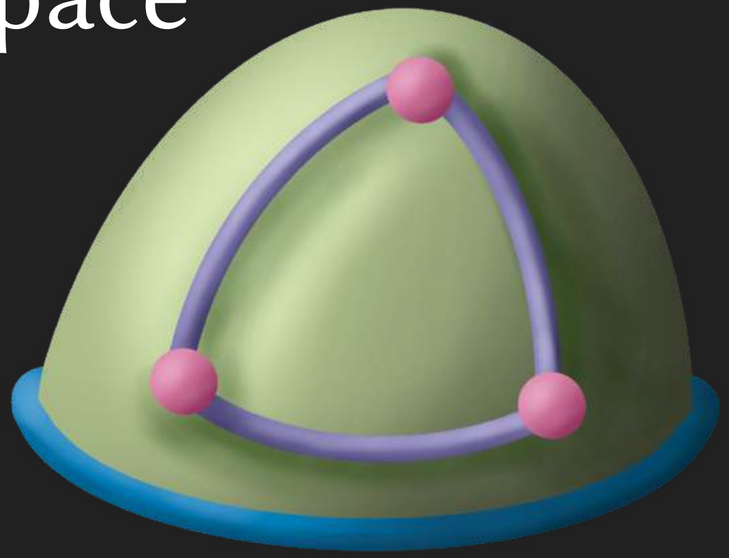


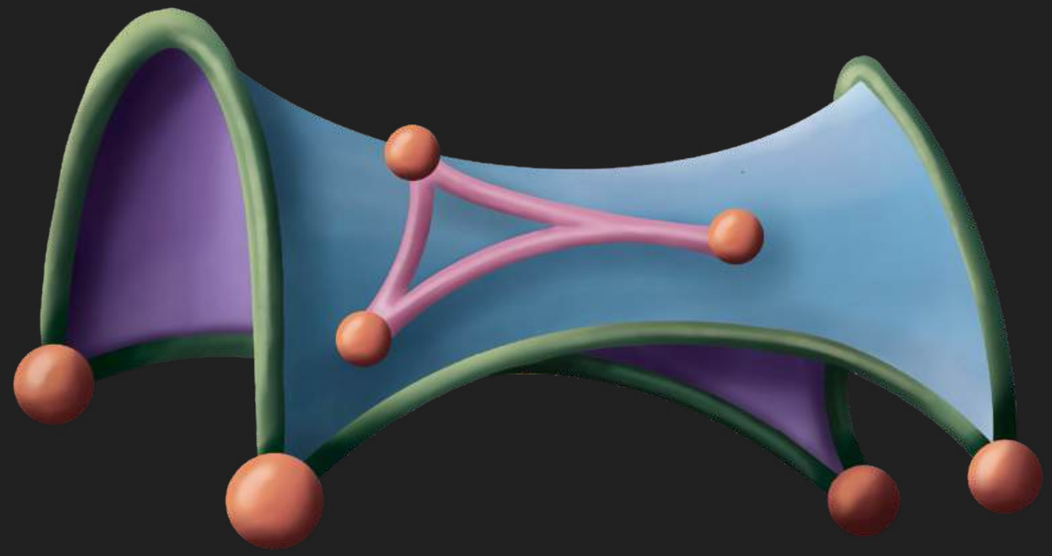
Eight Octagons must fit around each vertex ?!

But...

Angles change
if you let space
Curve







If you curve space

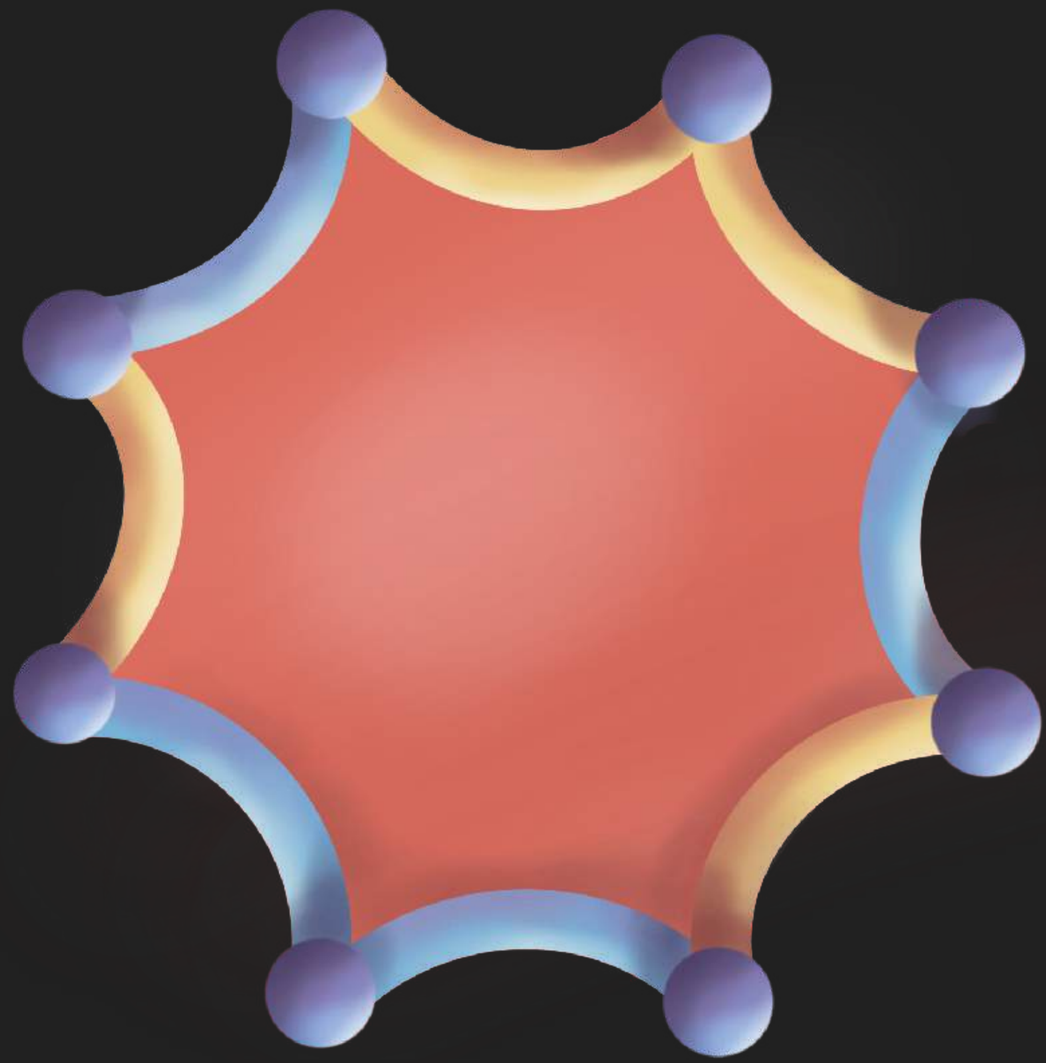
Just Right

you can make an

Octagon

with

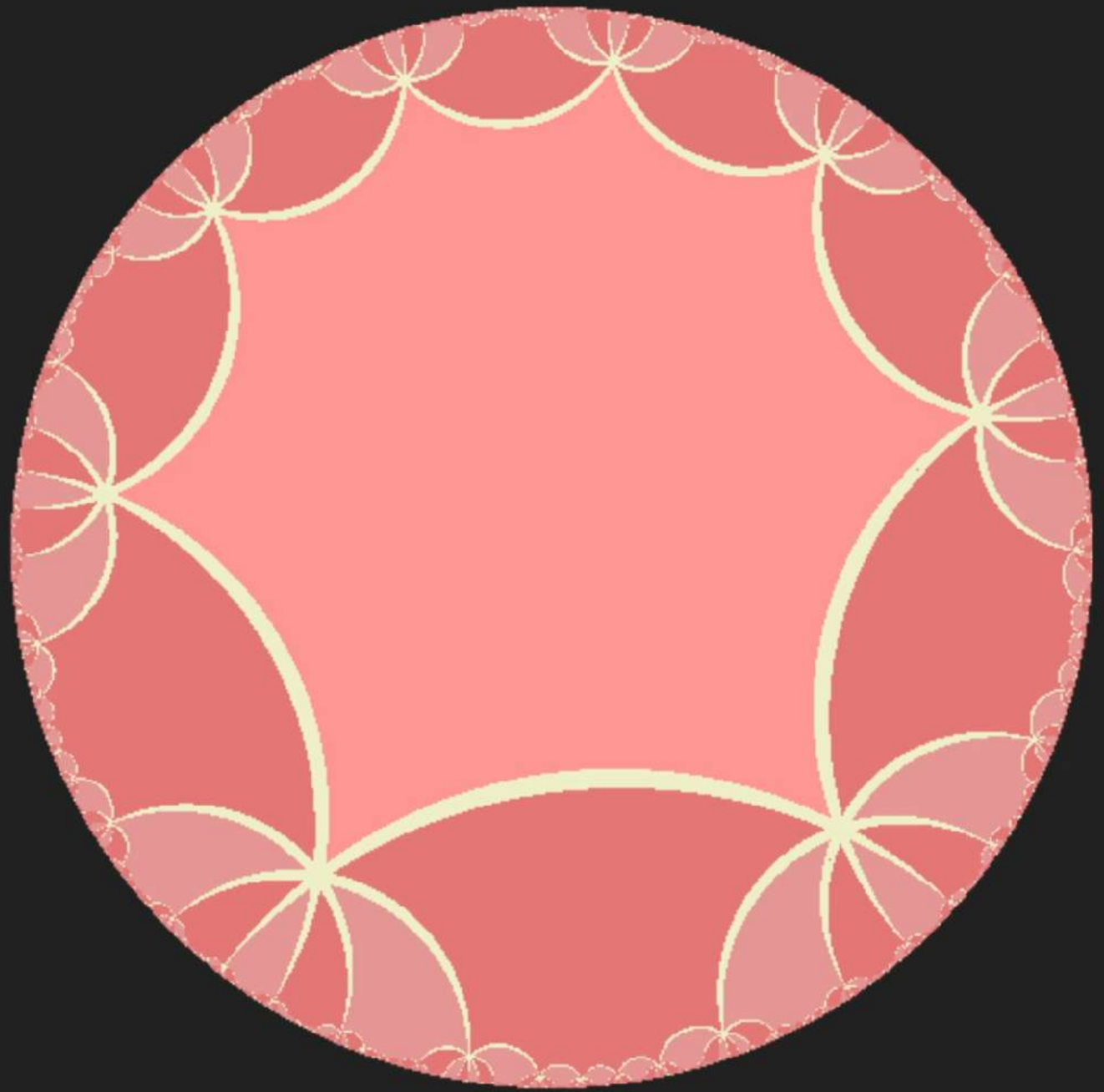
45° angles



Nikolai Lobachevsky
1829 CE

Janos Bolyai
1832 CE

Carl Friedrich Gauss
Early 1800s CE



How about other surfaces?

More complicated surfaces cut into polygons with more sides.

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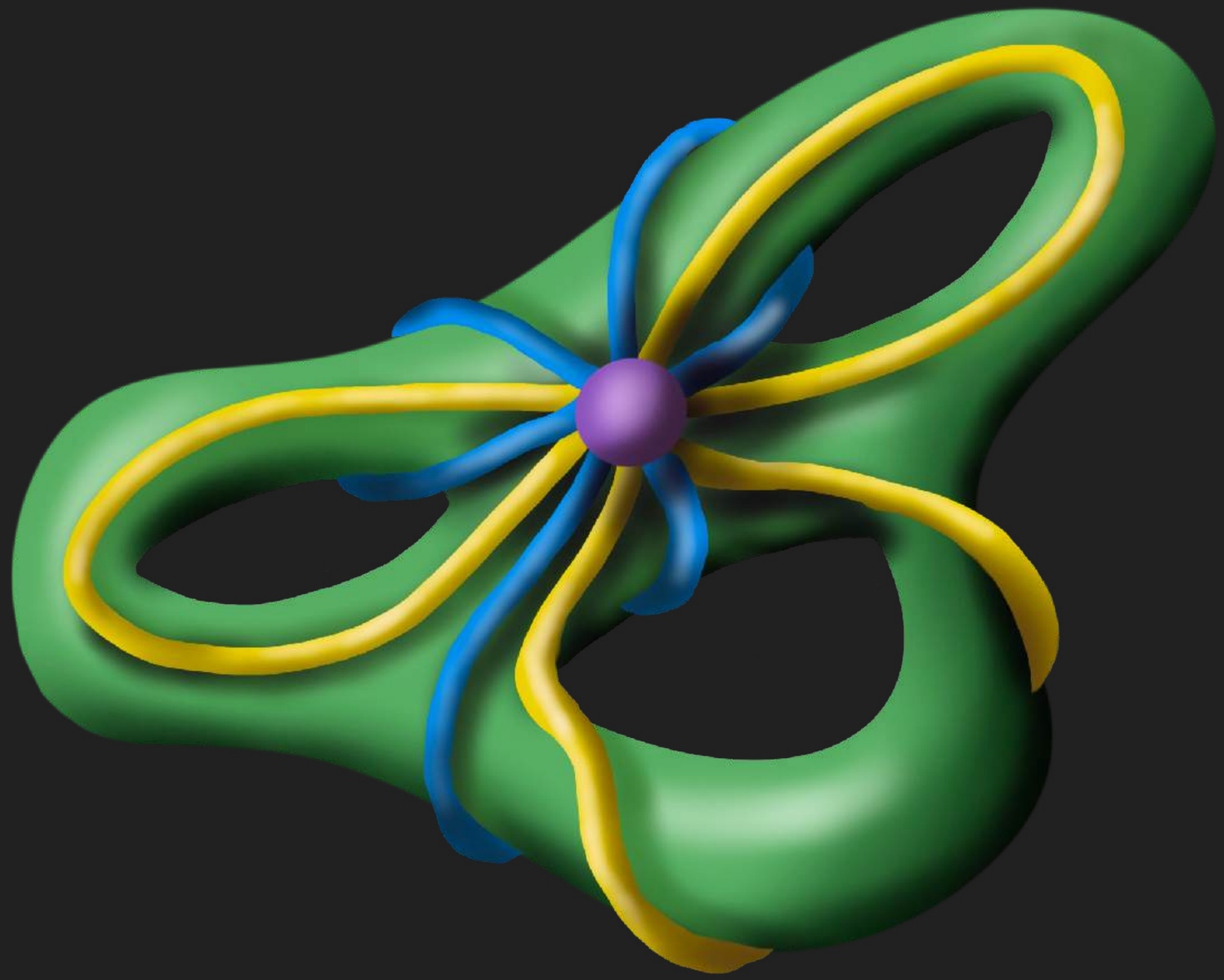

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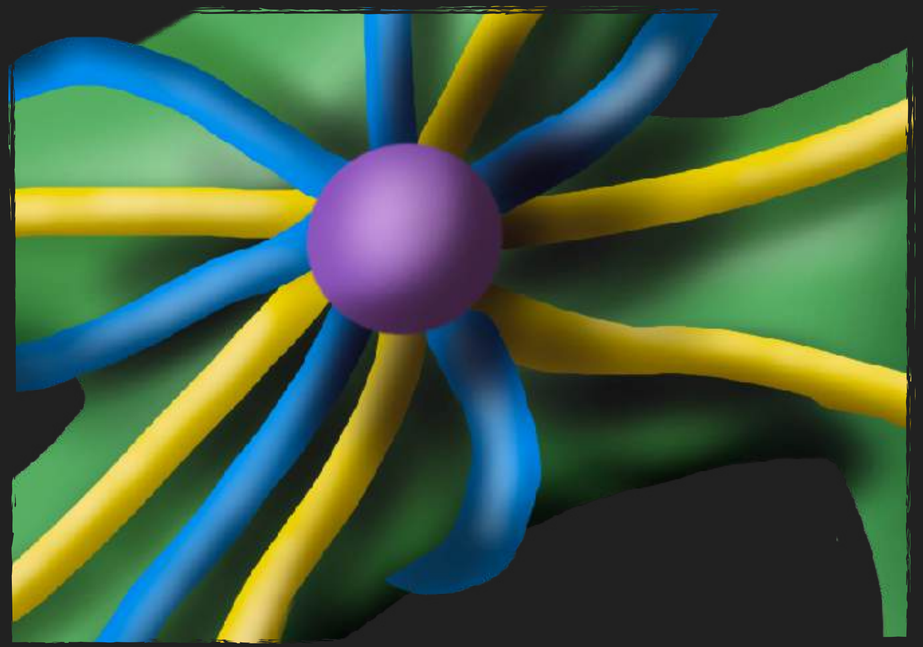
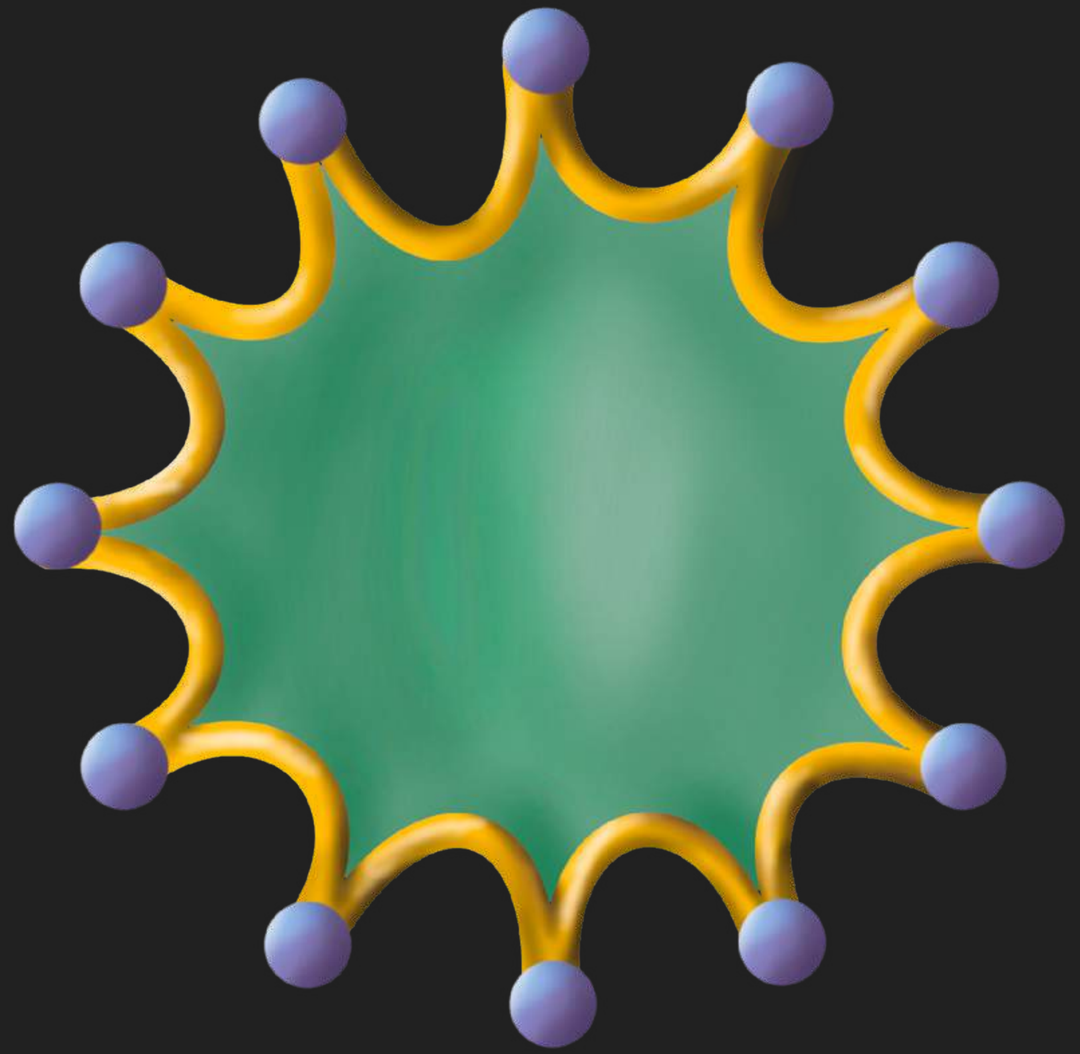
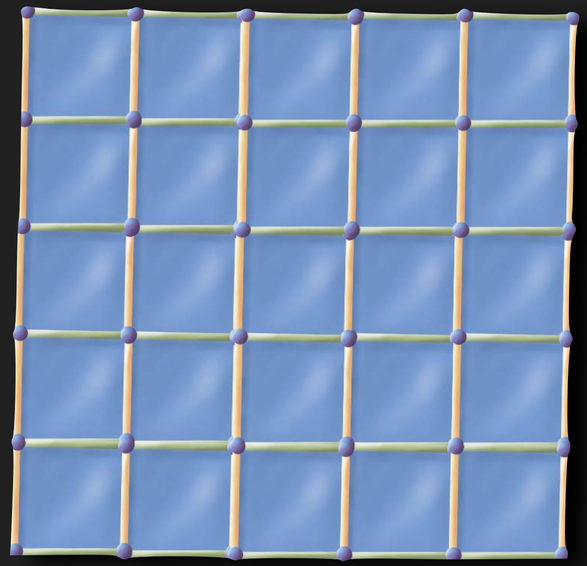




With negative curvature, its still possible to make these have small enough angles!

Every surface *can be unrolled onto* **the sphere, the Euclidean plane, or the hyperbolic disk.**

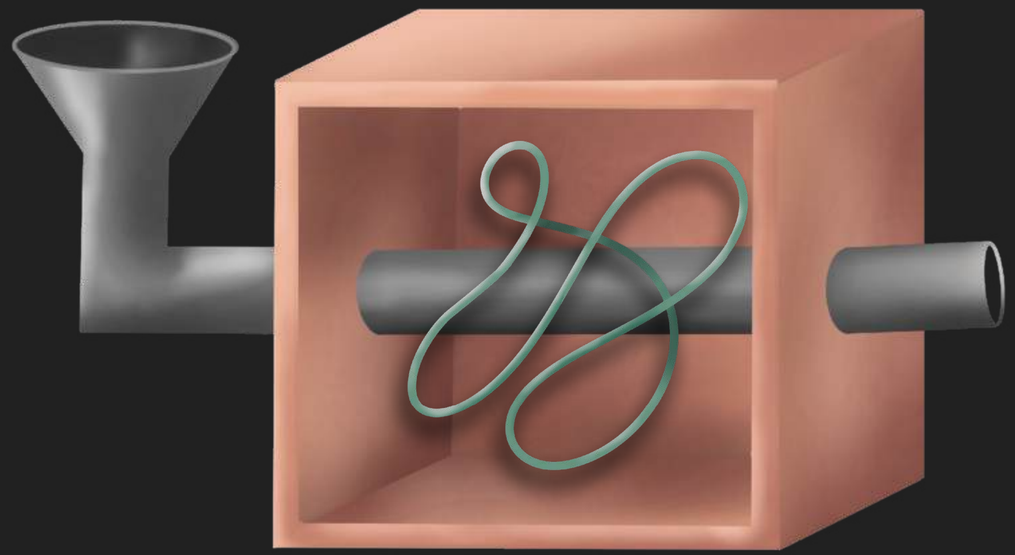


Uniformization Theorem (1907)

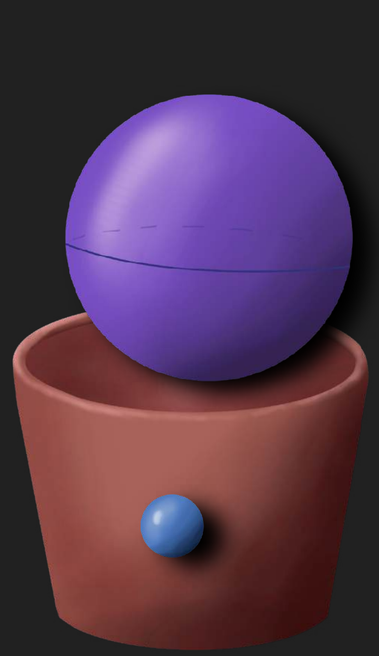
Felix Klein

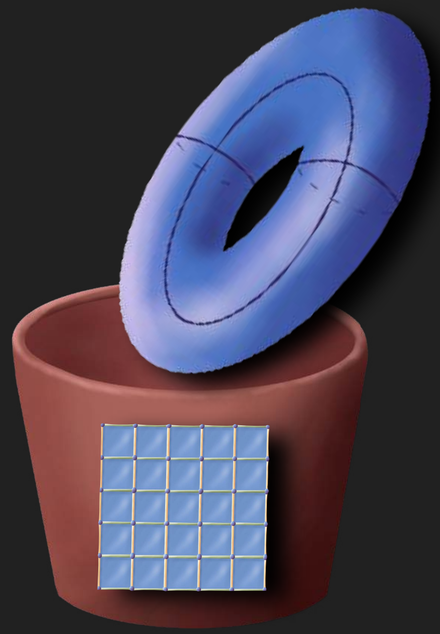
Paul Koebe

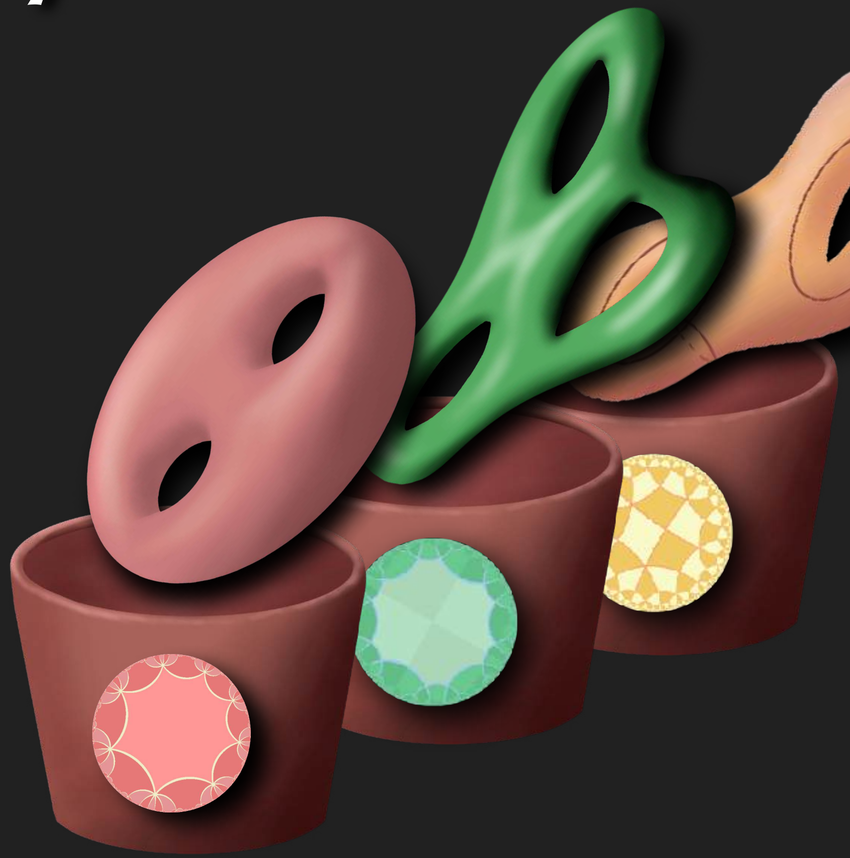
Henry Poincare

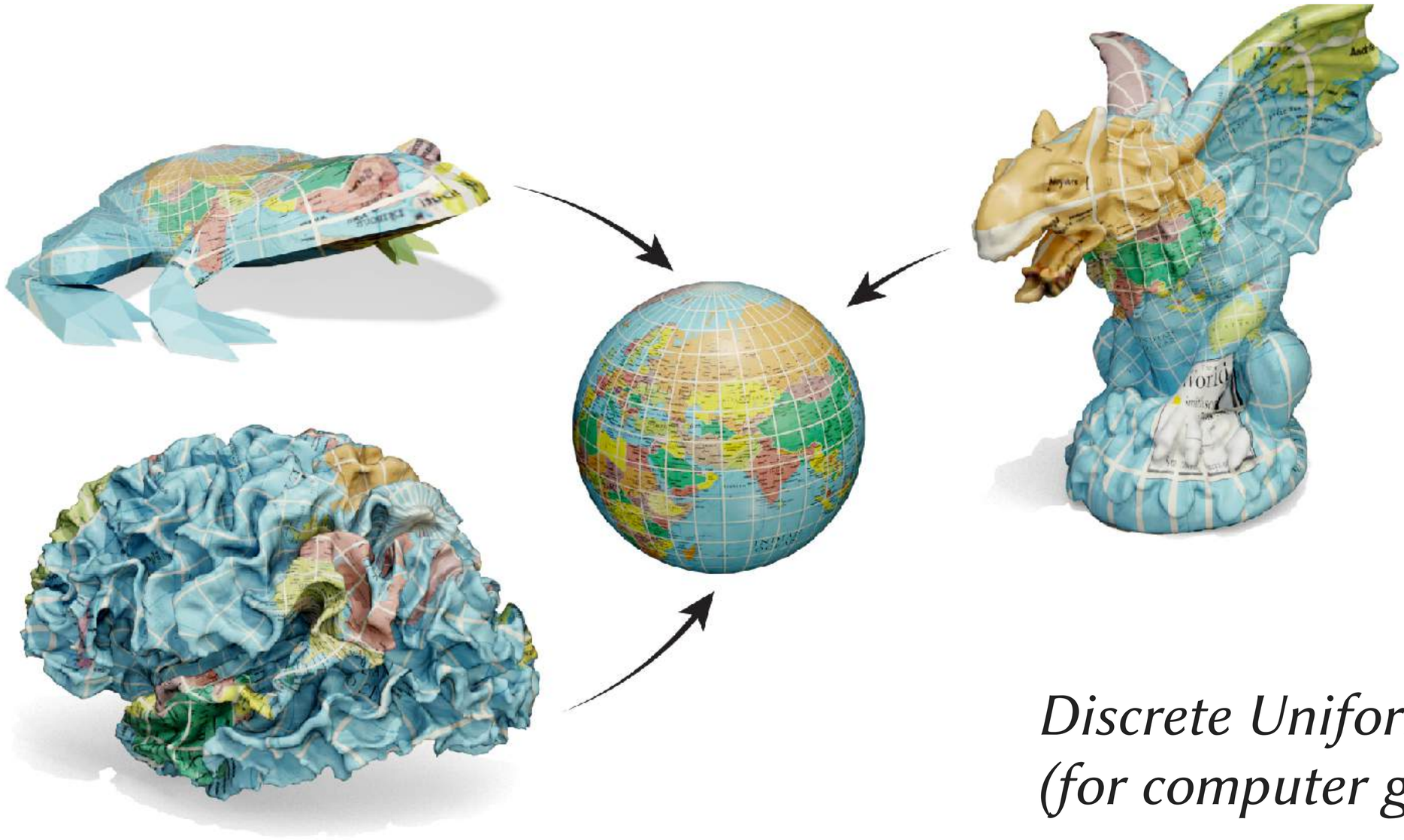


The loop invariant sorts surfaces by geometry







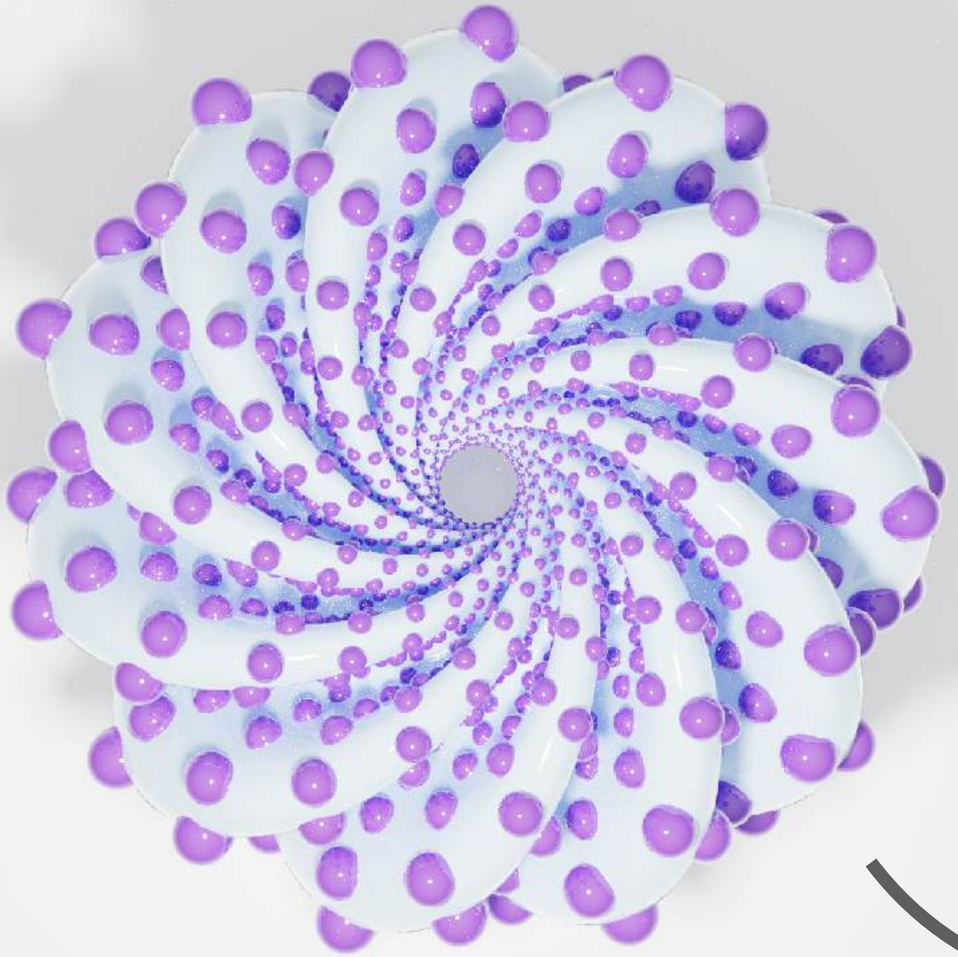


Discrete Uniformization
(for computer graphics)

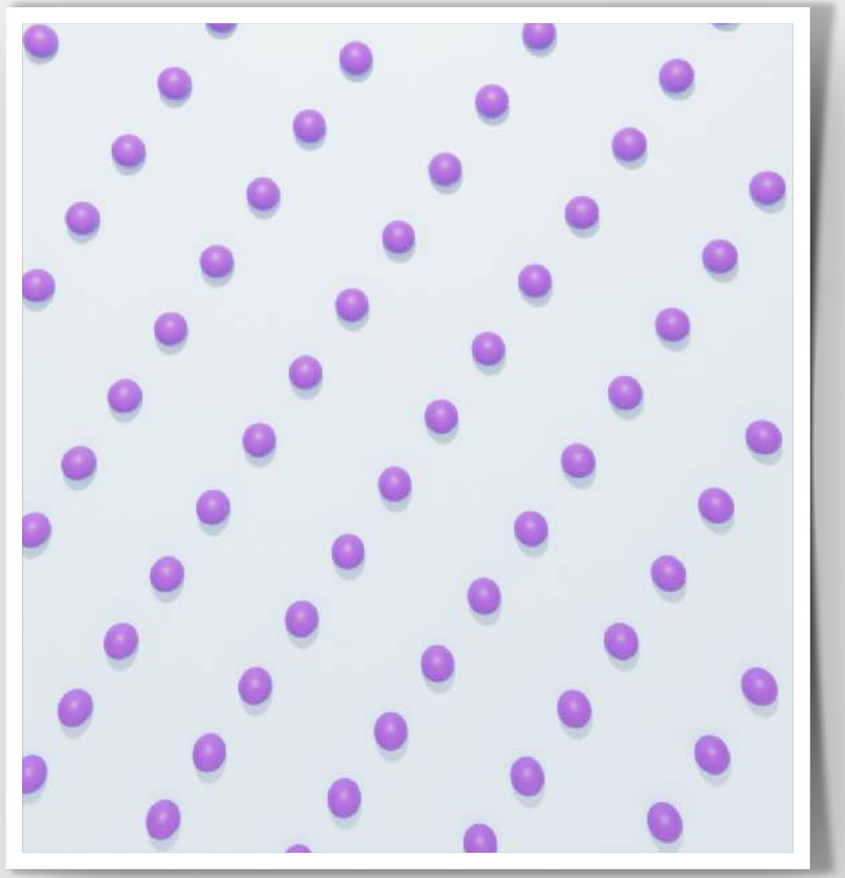
Keenan Crane, 2022

Visualizing Elliptic Curves over Finite Fields

joint with Nadir Hajouji, 2025



Uniformization



Part III

Proving the Poincare Conjecture

(In 2 dimensions)

The Poincare Conjecture

If it

walks & quacks

like a duck, then

it is a duck

The Poincare Conjecture

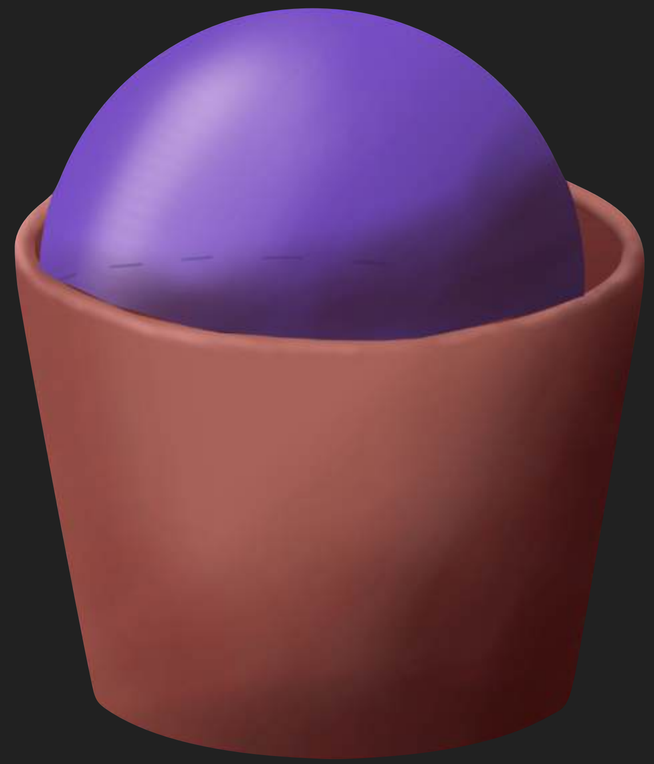
If a shape has the same

Loop Invariants

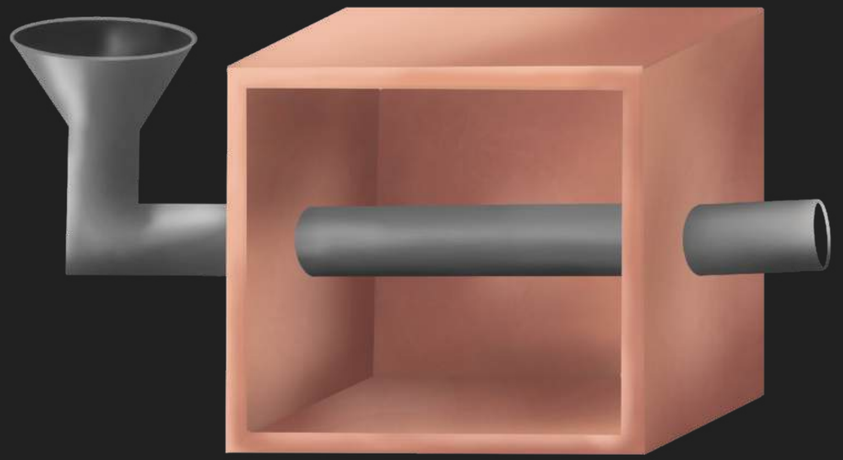
as the sphere, then

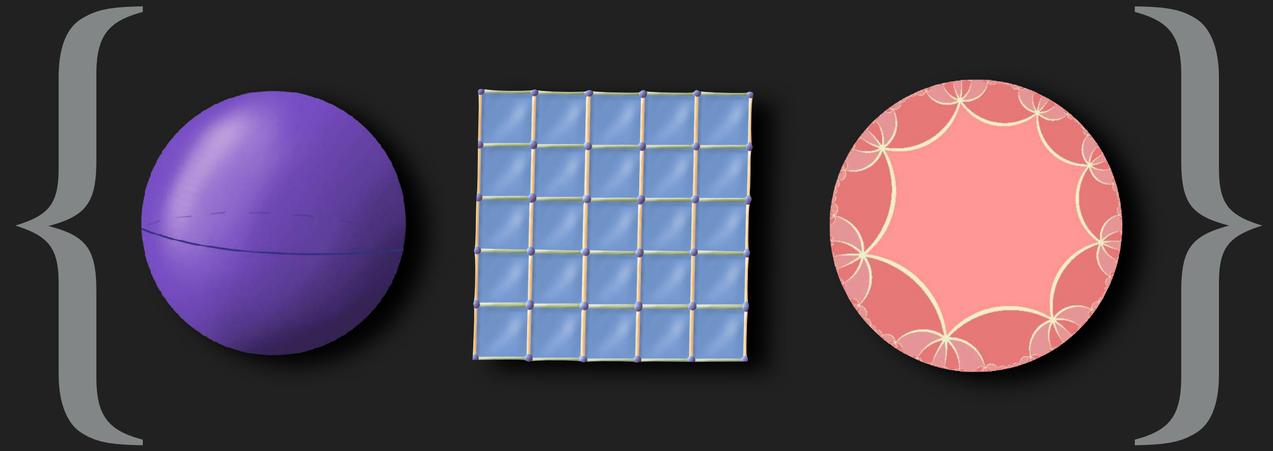
it is the sphere

-Poincaré, 1904



Proving the **Poincaré Conj.** for **Surfaces**

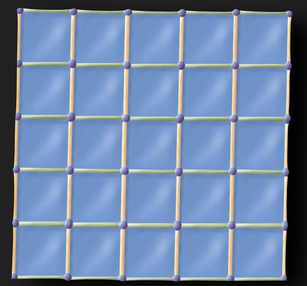




Start with a mystery surface.
With the **same loop invariants as the sphere.**

Feeding it into the uniformization machine unrolls it into one of the three geometries.

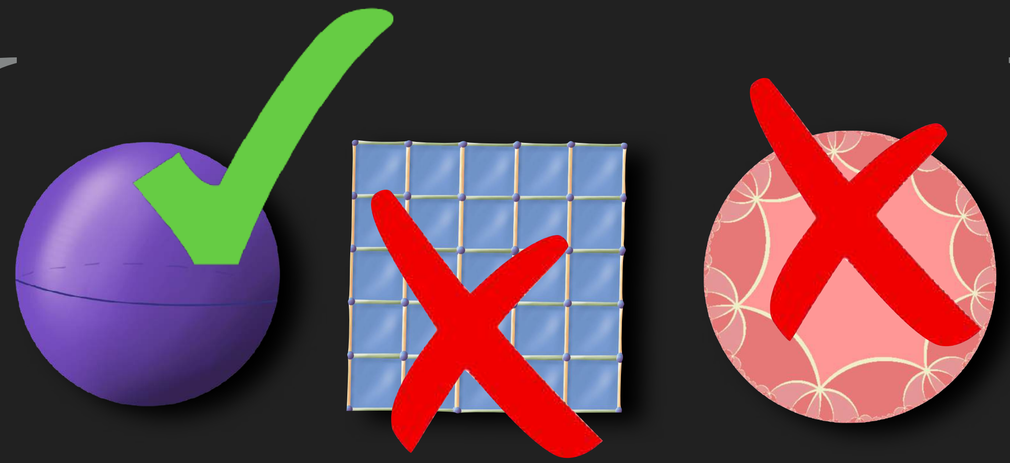
Proving the **Poincaré Conj.** for **Surfaces**

$\in$ {    }

Our surface is **simply connected**, so it didn't need any unrolling! Thus it IS one of the geometries.

Proving the **Poincaré Conj.** for **Surfaces**

$\in$



Our surface is **finite**, so it cannot equal one of the infinite geometries.

Proving the **Poincaré Conj.** for **Surfaces**

Knowing only that the mystery surface had the same **loop invariants** *as the sphere, told us it was the sphere!*

Thus, our surface must be the sphere.

Proving the **oincaré Conj.** for **Surfaces**

Knowing only that the mystery surface
had the same **loop in**
sphere, told us it w

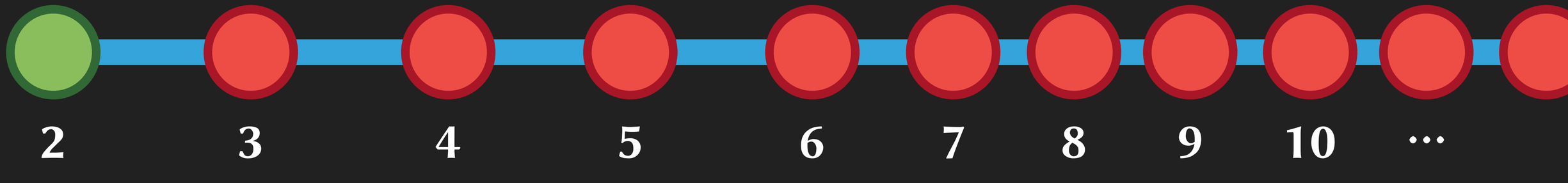
Thu *here.*

Where things go **From Here**

Poincare's original question was about *three dimensional spaces*, not two dimensional surfaces.

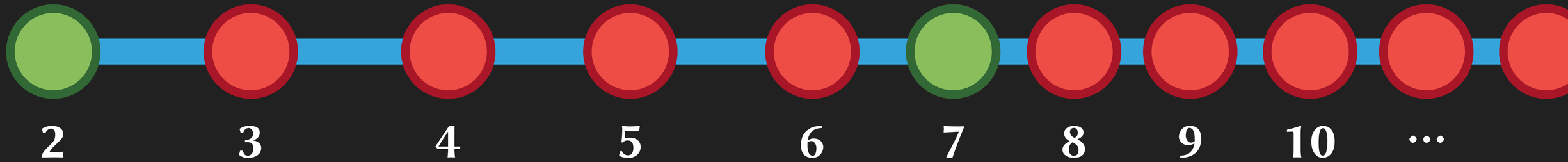
But the possibility of wormholes makes this difficult

In Any Dimension...



In Any Dimension...

In 1960, Stephen Smale proved the Poincare Conjecture in dimension 7



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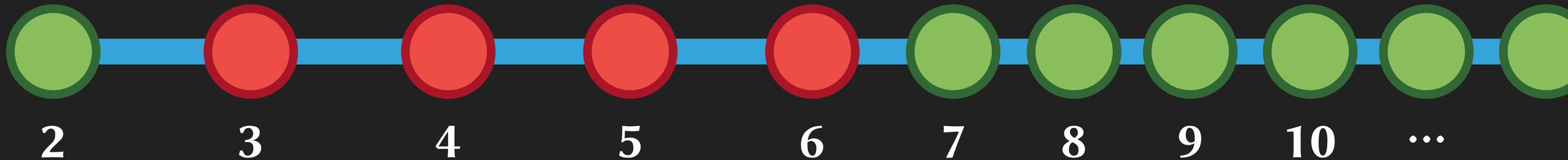
...and showed the same proof worked in all higher dimensions!



In Any Dimension...

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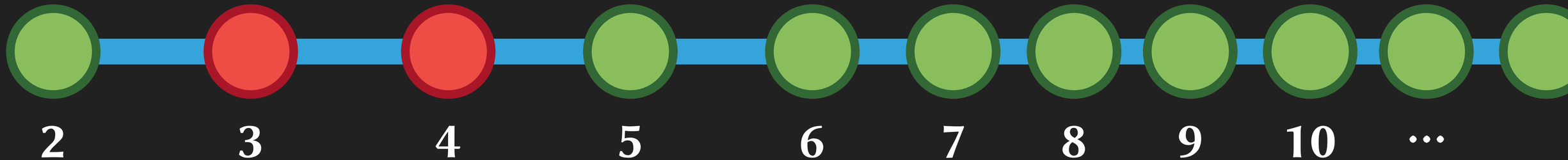
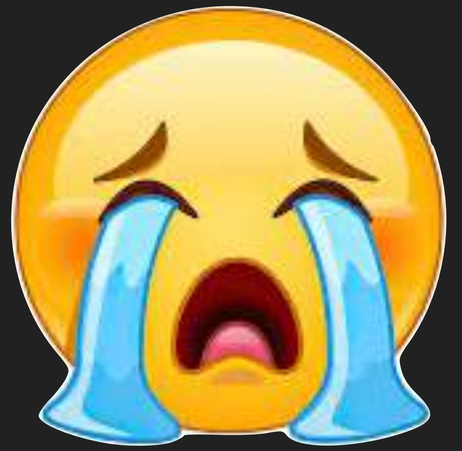
In 1961, Smale showed his techniques also work in dimension 5 and 6

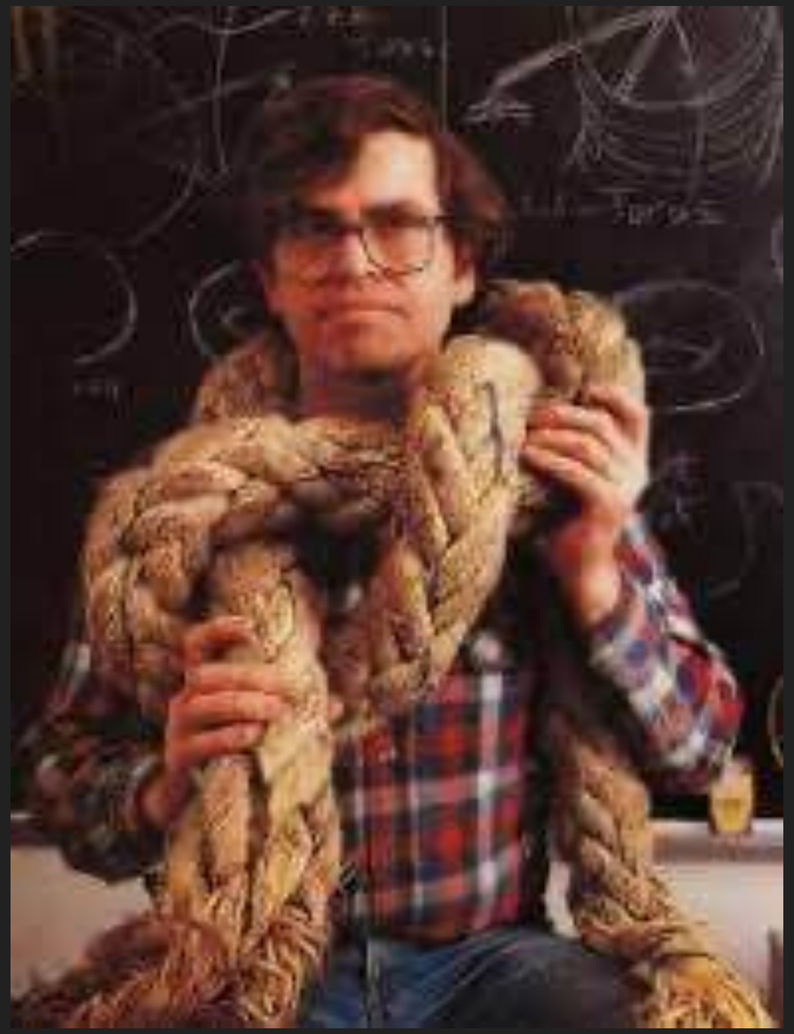


In Any Dimension...

In 1982, Michael Freedman used *totally different techniques* to prove the Poincare Conjecture in dimension 4.

But none of the ideas from dimension 4 and above help at all in dimension 3.



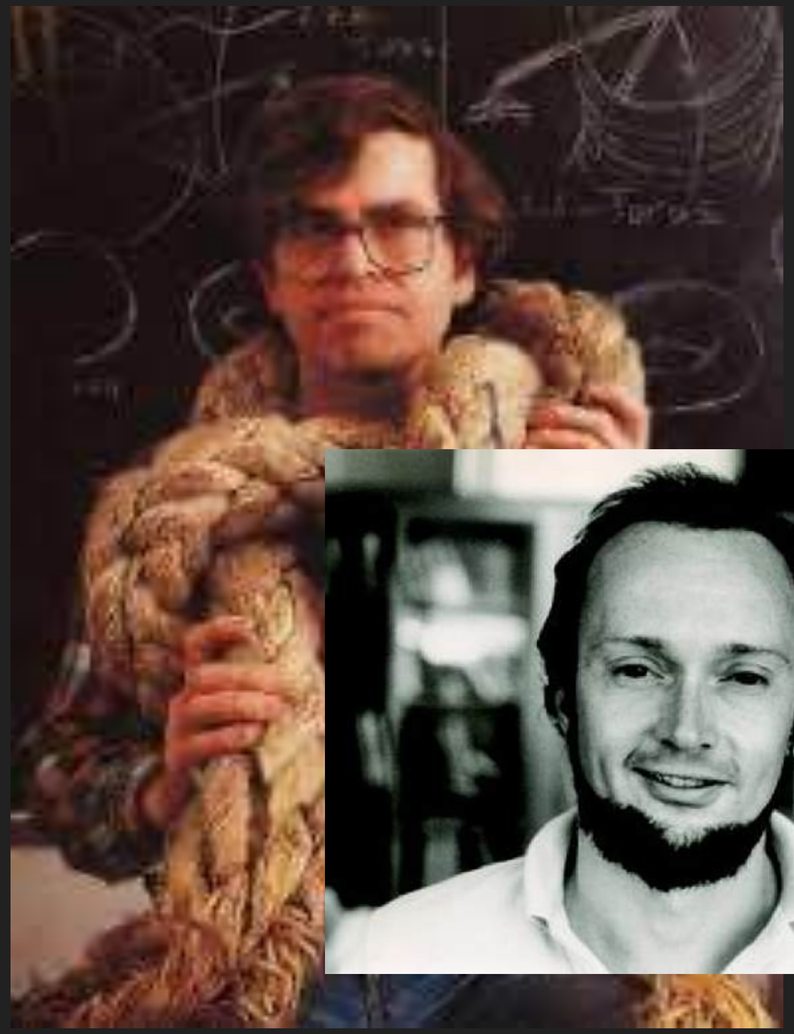


In Any Dimension...

In the 1980s, William Thurston laid out a vision for how to modify the geometric ideas from dimension 2 to accommodate wormholes.

And Richard Hamilton suggested a way to prove it using differential equations

In Any Dimension...



In 2002, Grigori Perleman succeeded in making this proof strategy work! **After nearly 100 years, the 3 Dimensional Poincare Conjecture is solved!**