

The
Linear
Algebra of

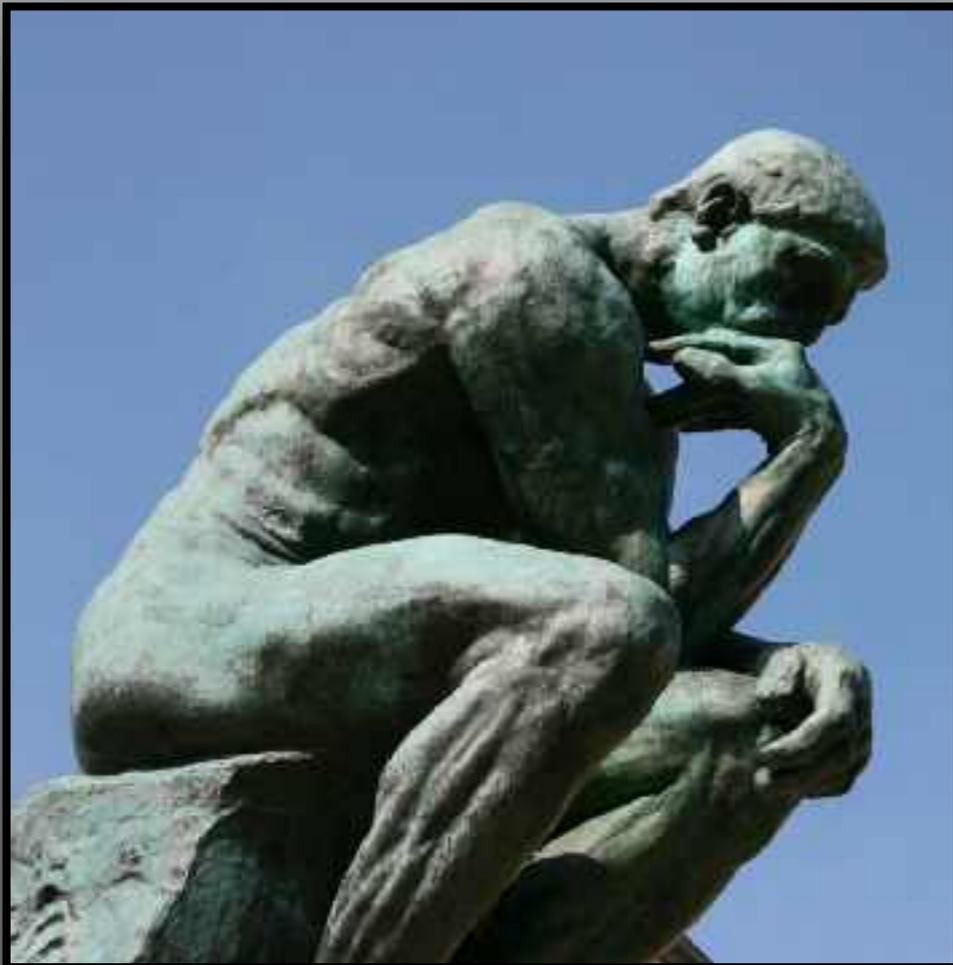
Color Vision

Steve Trettel, USF



The Phenomenology of Color

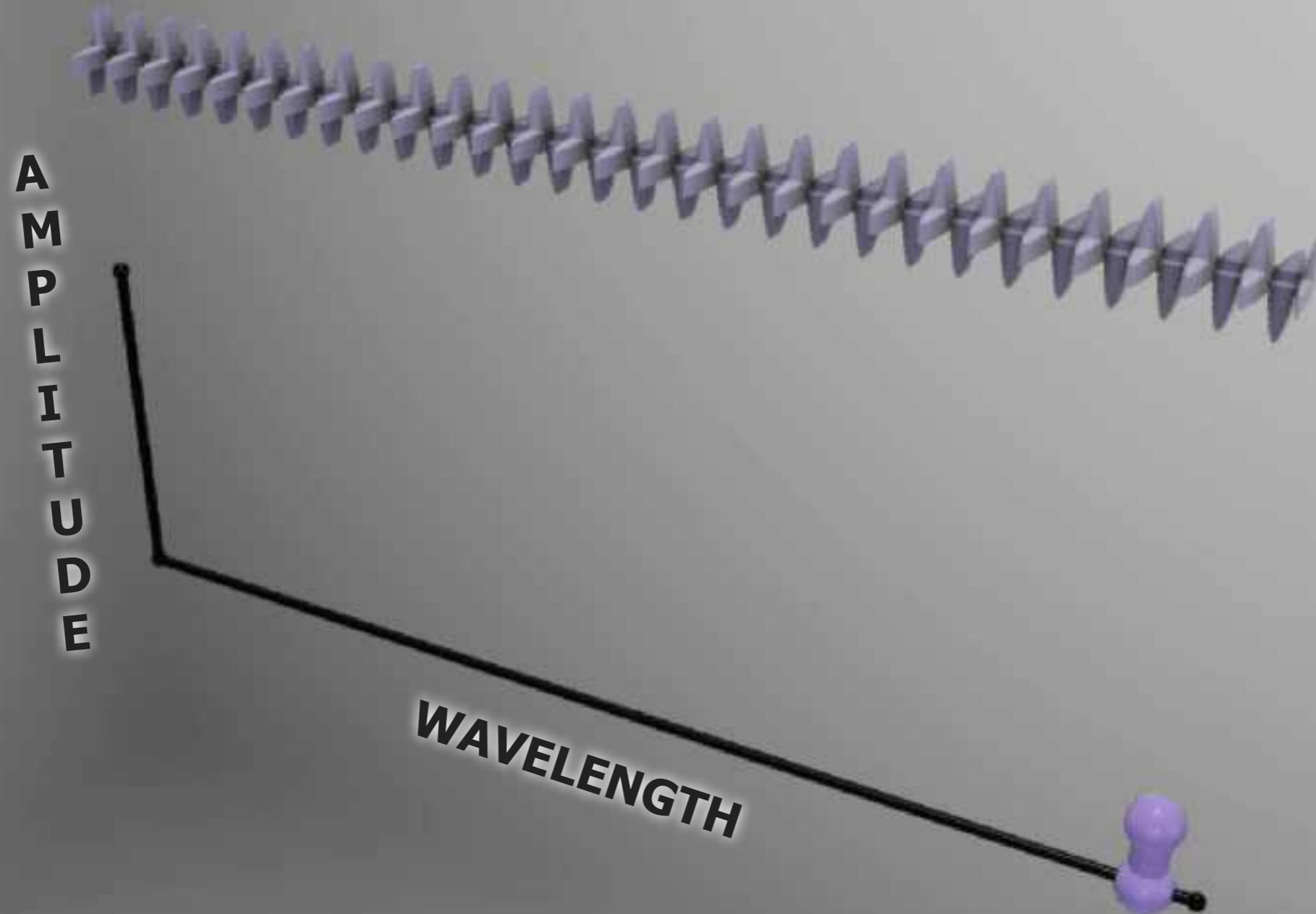
Why are there
three primary
colors?



Does TV look
realistic to dogs?

The Physics of Color:

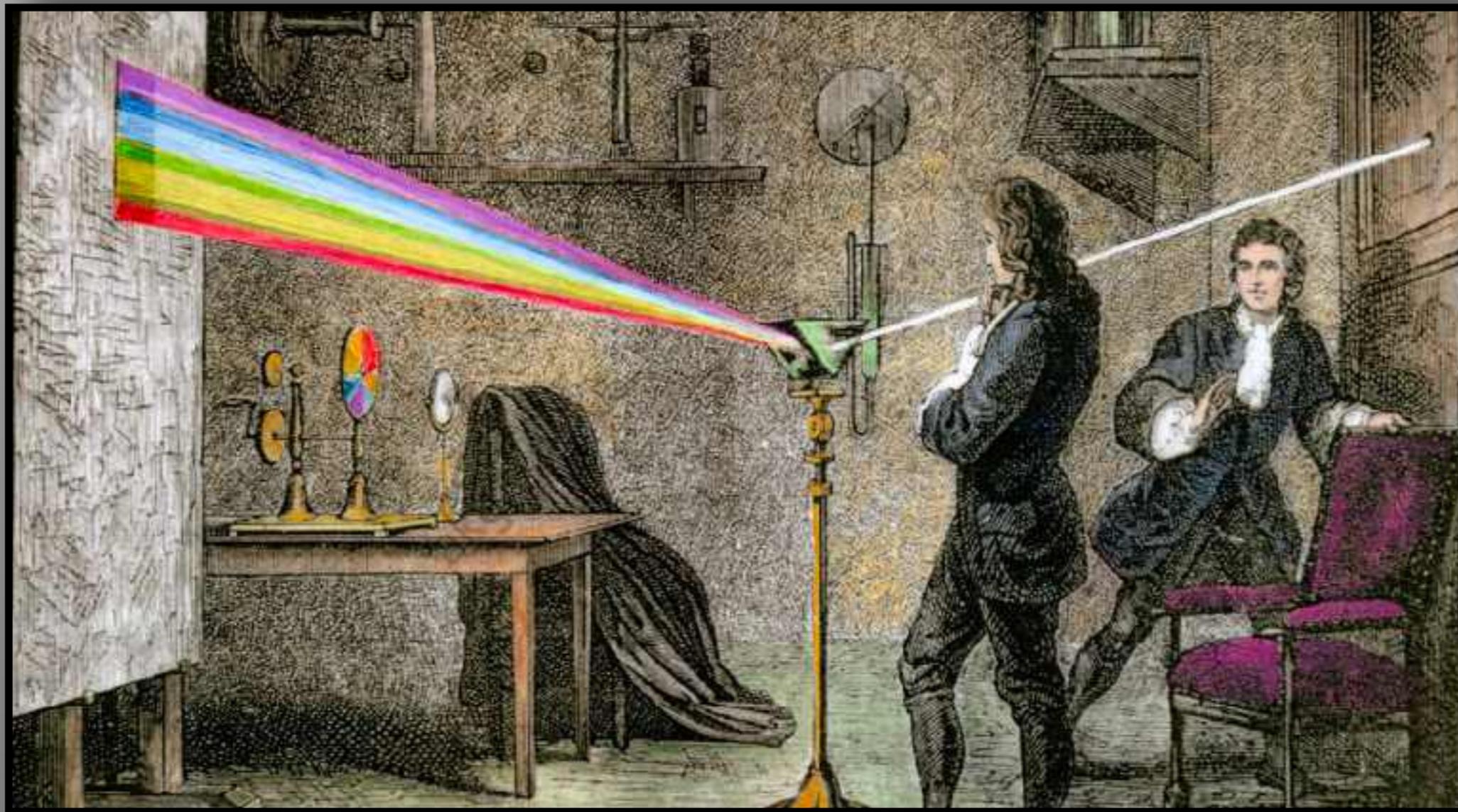
Light is an Electromagnetic Wave



*Phase won't concern us

The Physics of Color:

A single color is made of many different wavelengths.

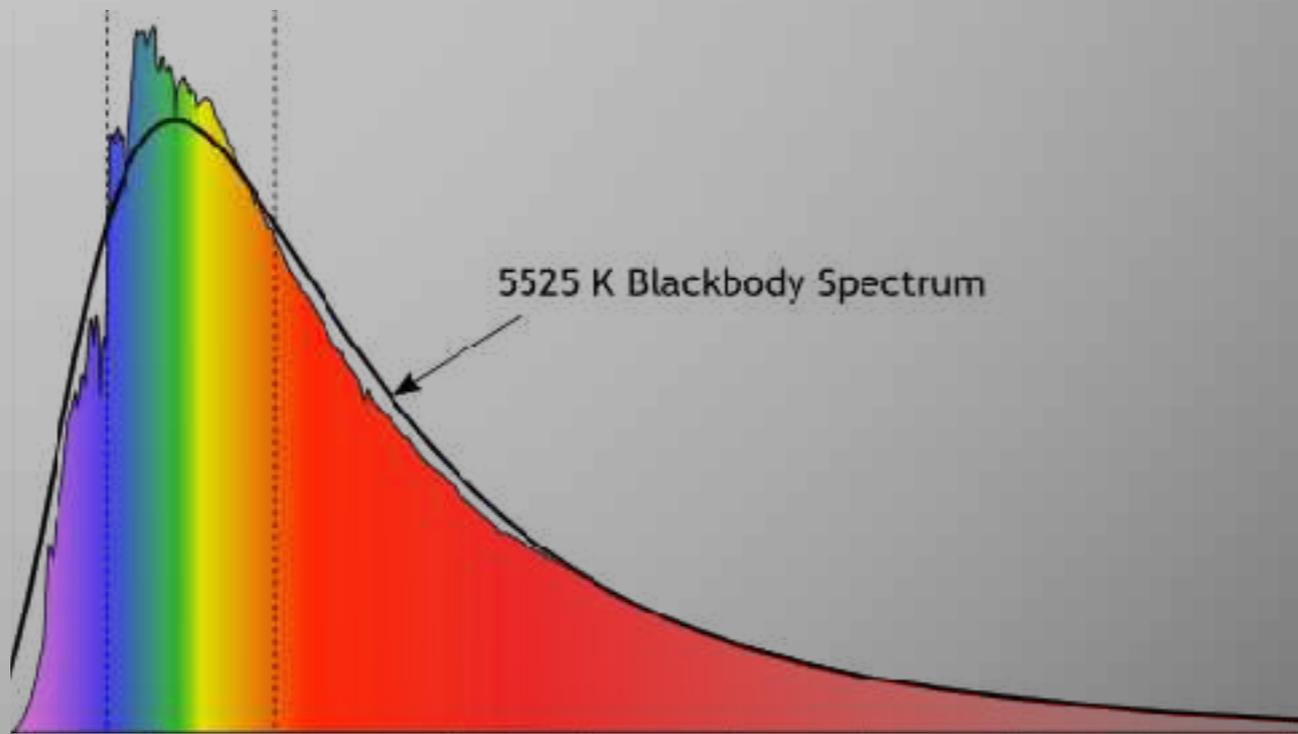


The Physics of Color:

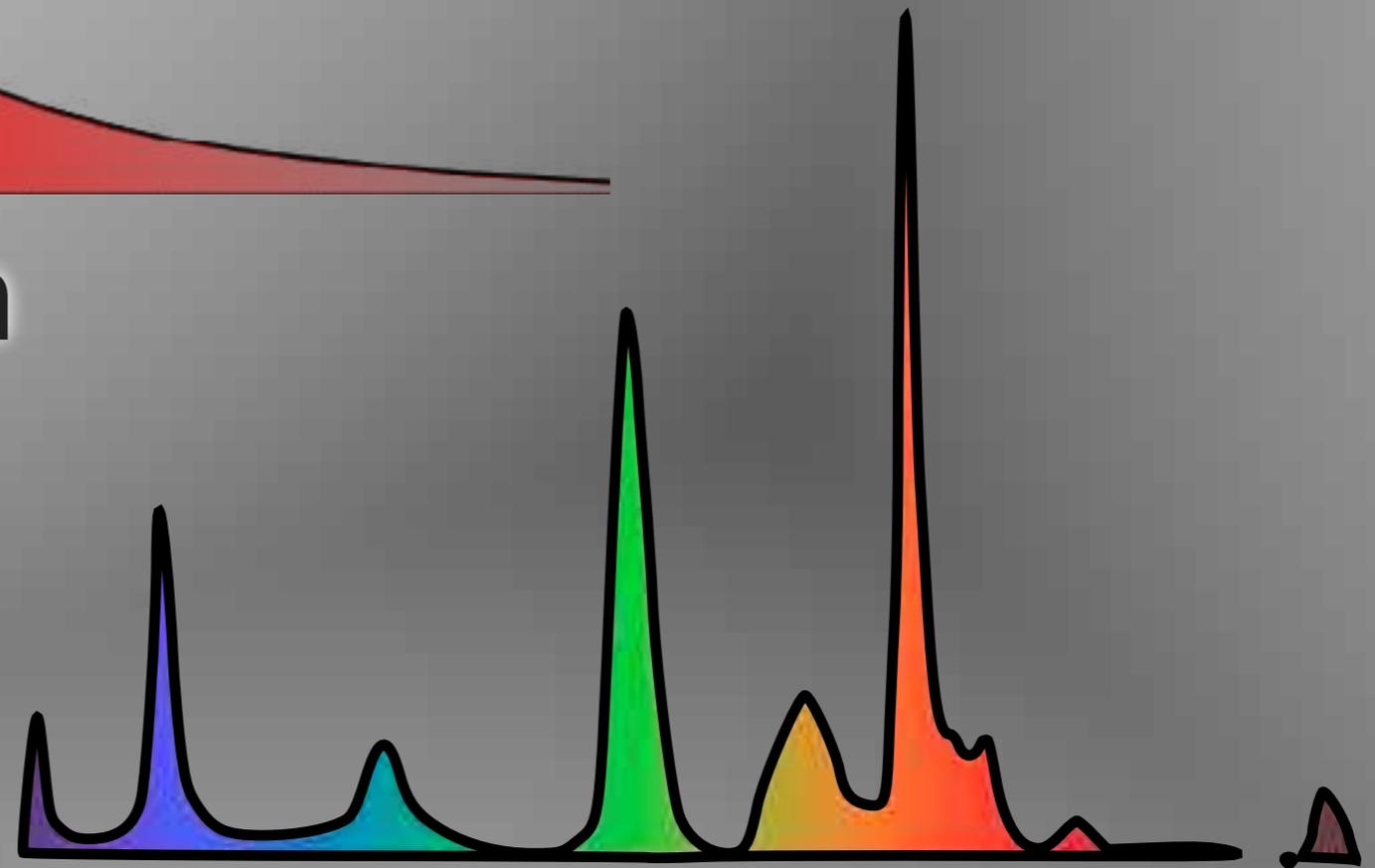


Definition: Spectrum (Spectral Radiance/Reflectivity)

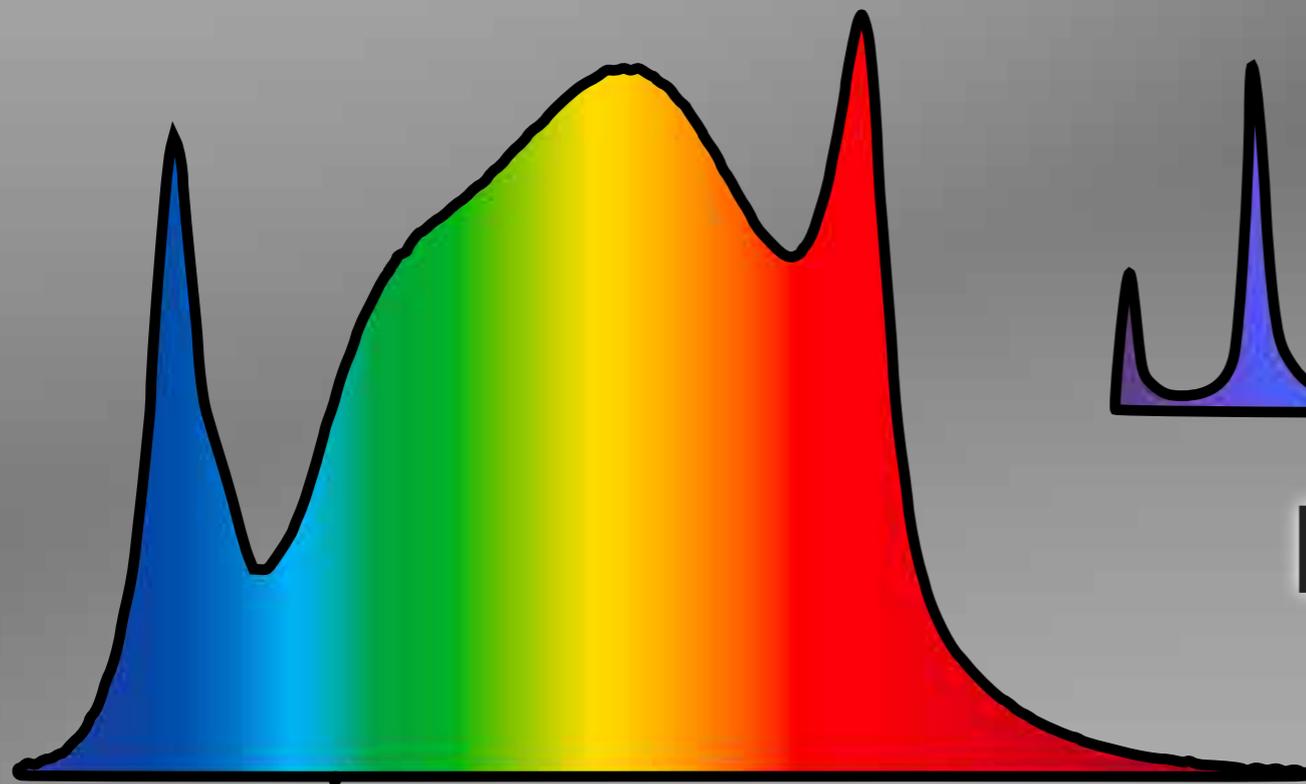
A function $\sigma: \mathbb{R}_{>0} \rightarrow \mathbb{R}_{\geq 0}$ giving the amount $\sigma(\lambda)$ of each wavelength λ in an electromagnetic wave (light beam).



The Sun

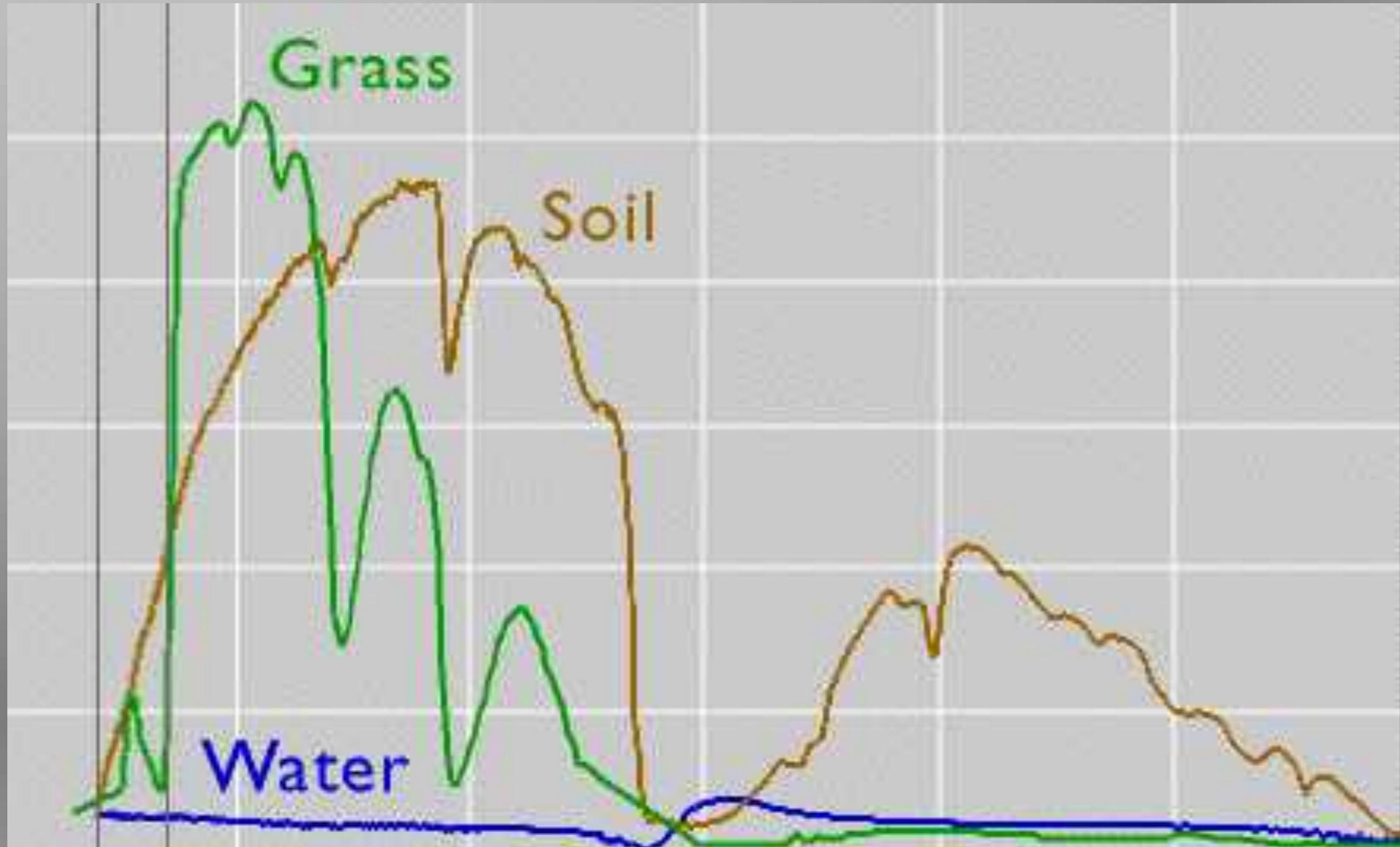


Fluorescent Lamp



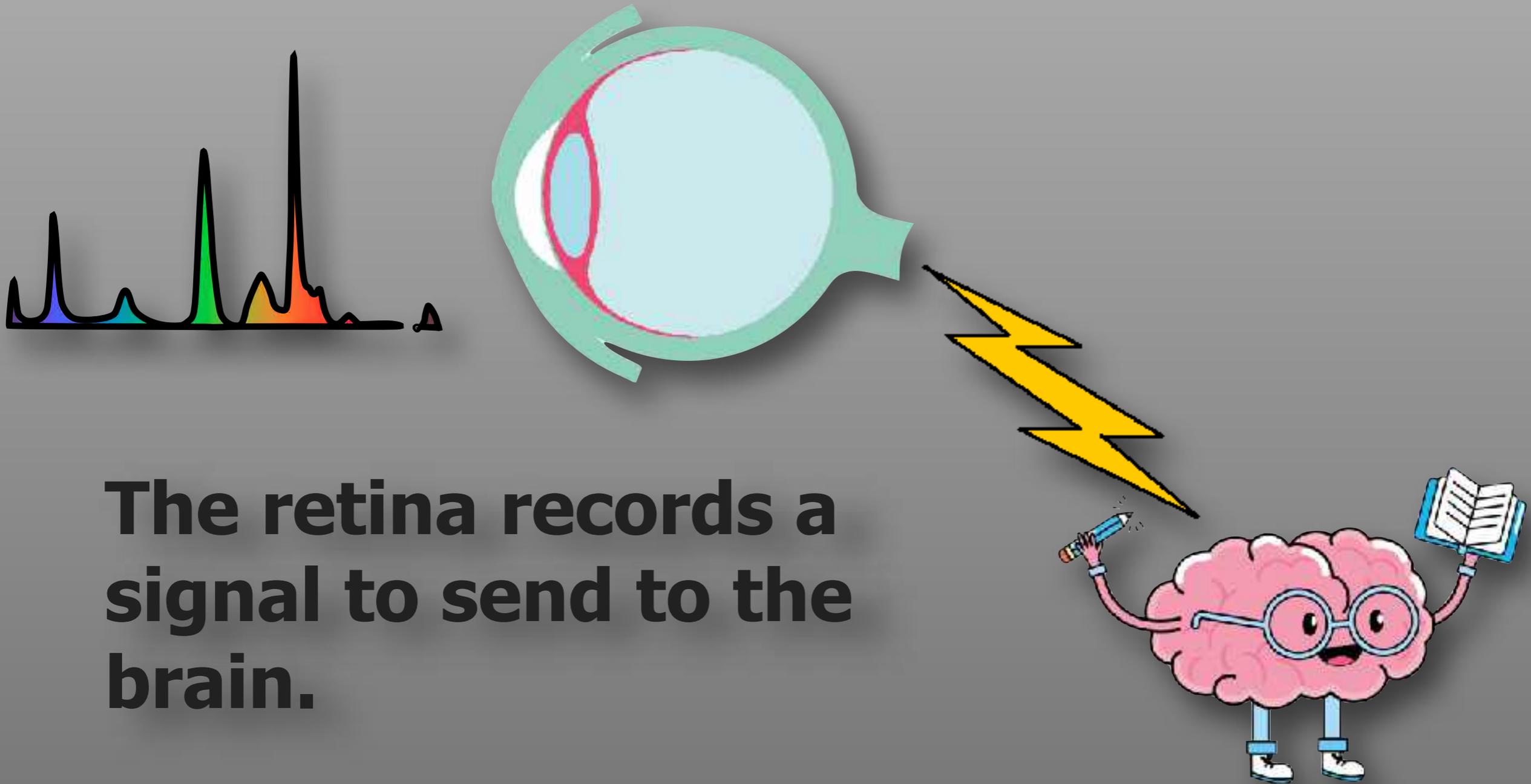
White LED

Spectra of grass, soil and water:



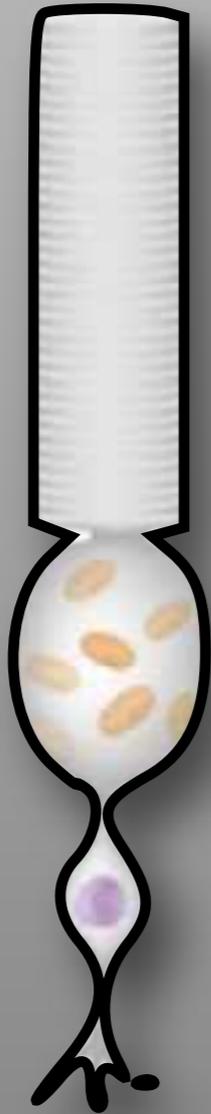
wavelength (nm)

The Biology of Color



The retina records a signal to send to the brain.

The Biology of Color

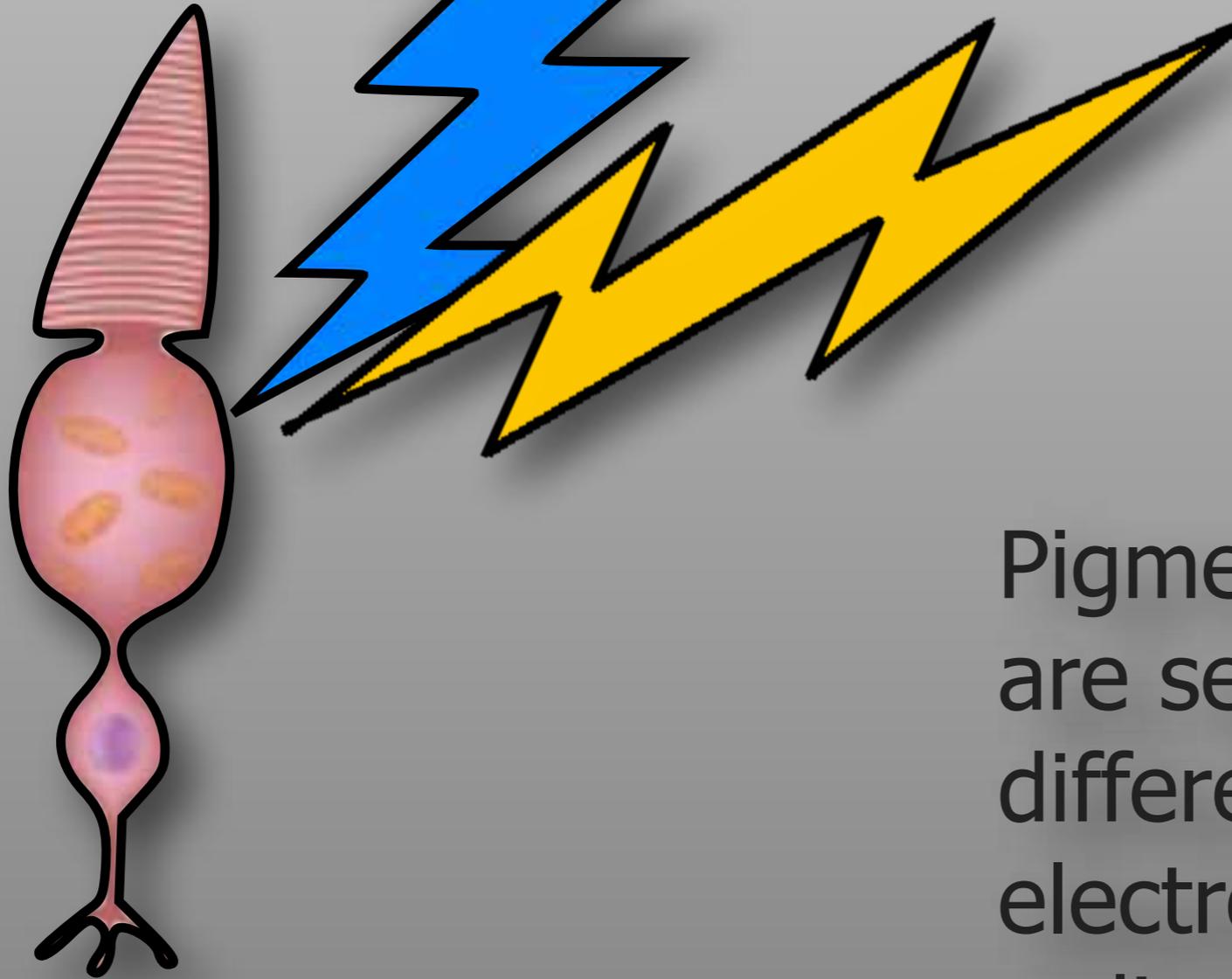


Rods sense
luminosity and
movement



Cones sense color

The Biology of Color



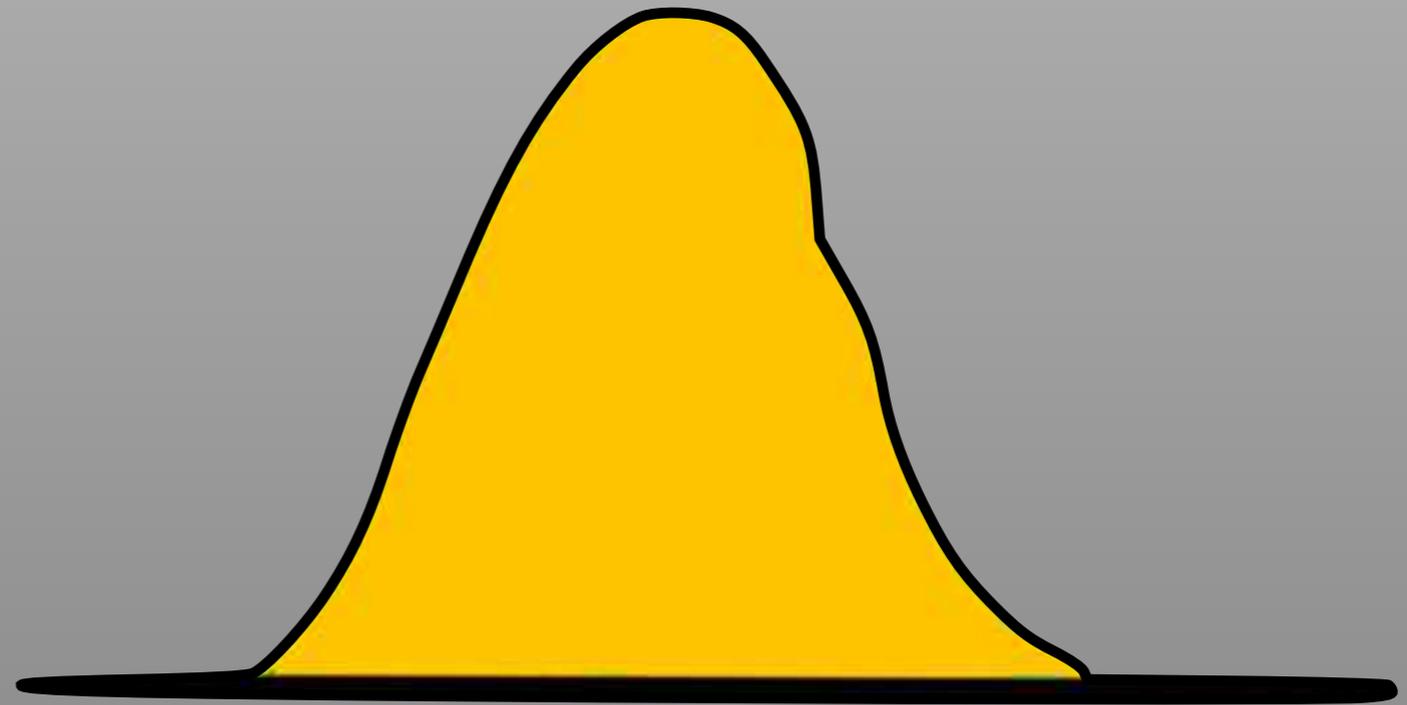
Pigments in the cell are sensitive to different types of electromagnetic radiation

Cones sense color

The Biology of Color



Cones sense color



The **Response Curve** shows how sensitive a given cone is to each wavelength.

The Mathematics of Color Vision

The Eye is a Function!



Physical
Color

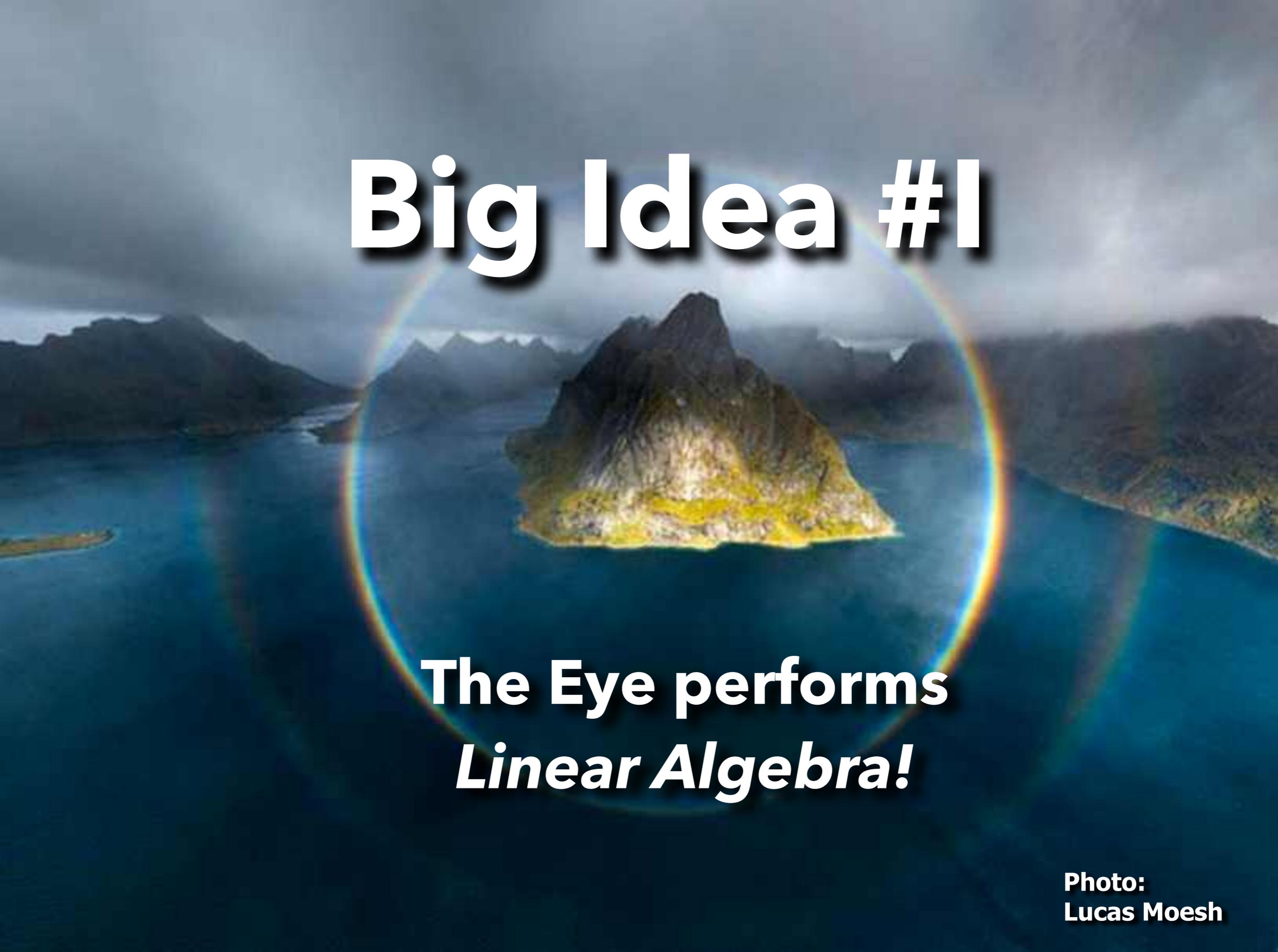
Signals to
Brain

Fixing Notation:

The eye takes as input a spectrum of light, and outputs a response to the brain.



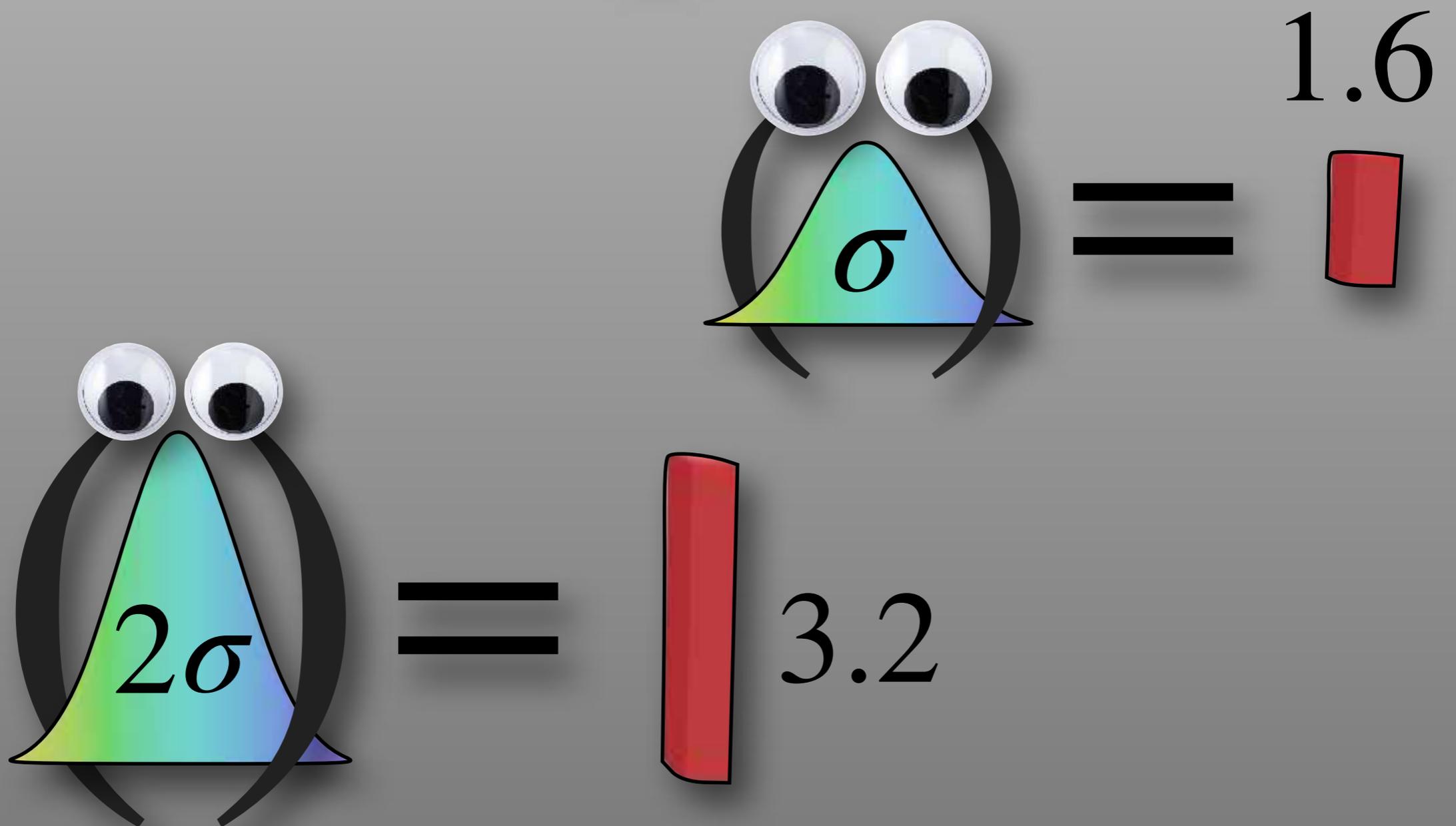
Big Idea #1



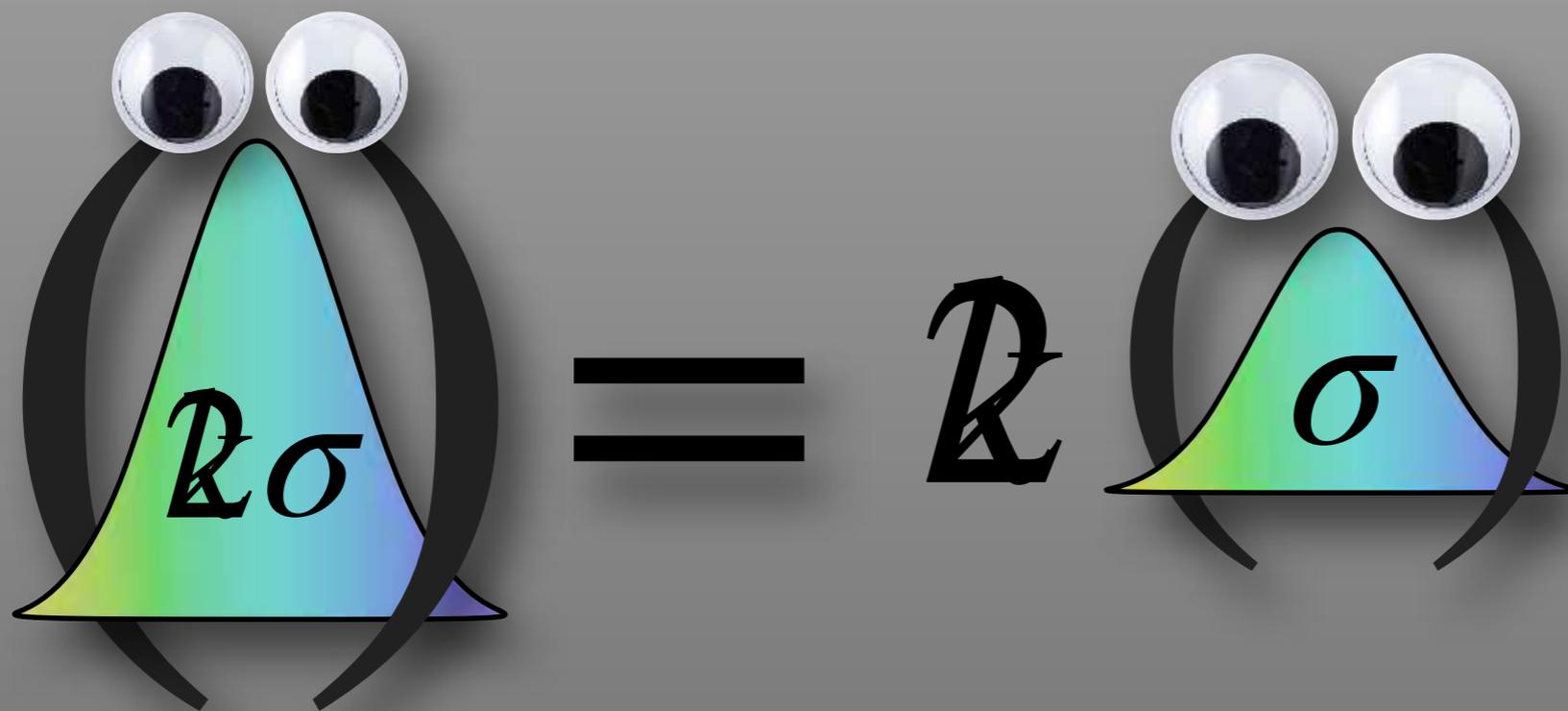
**The Eye performs
*Linear Algebra!***

**Photo:
Lucas Moesh**

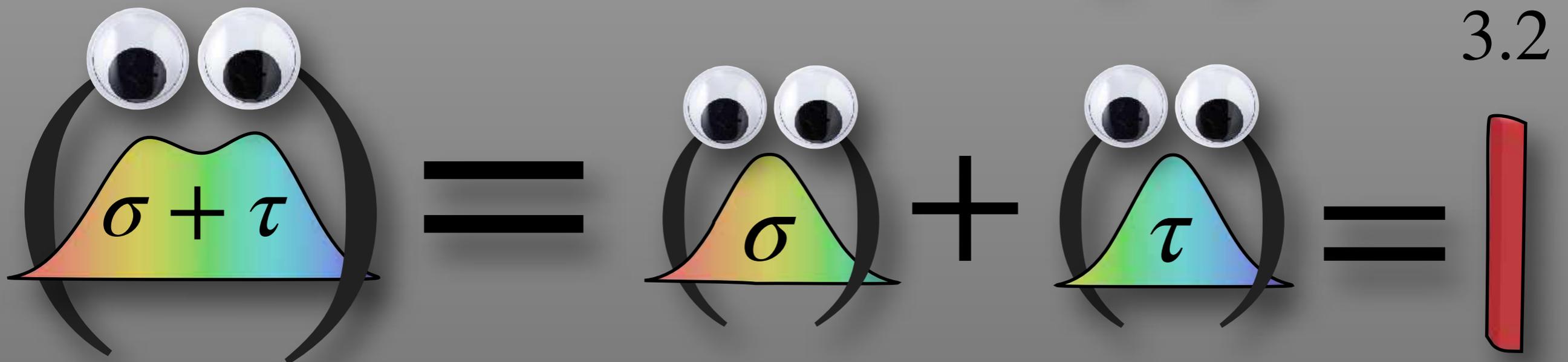
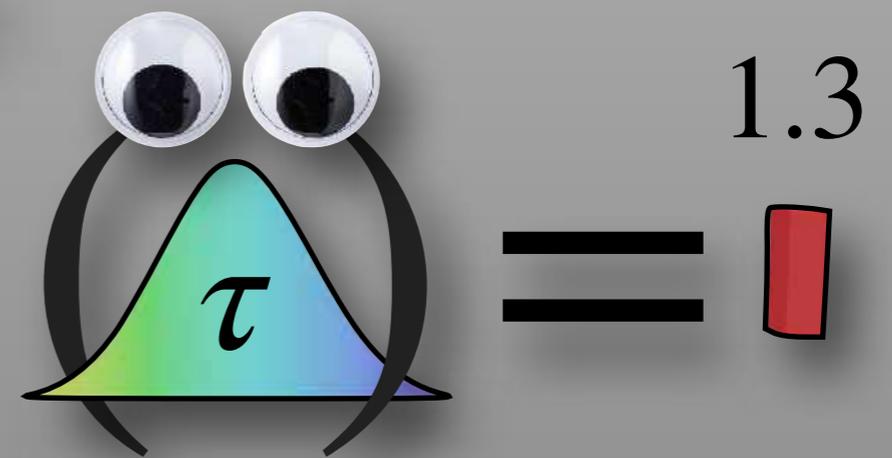
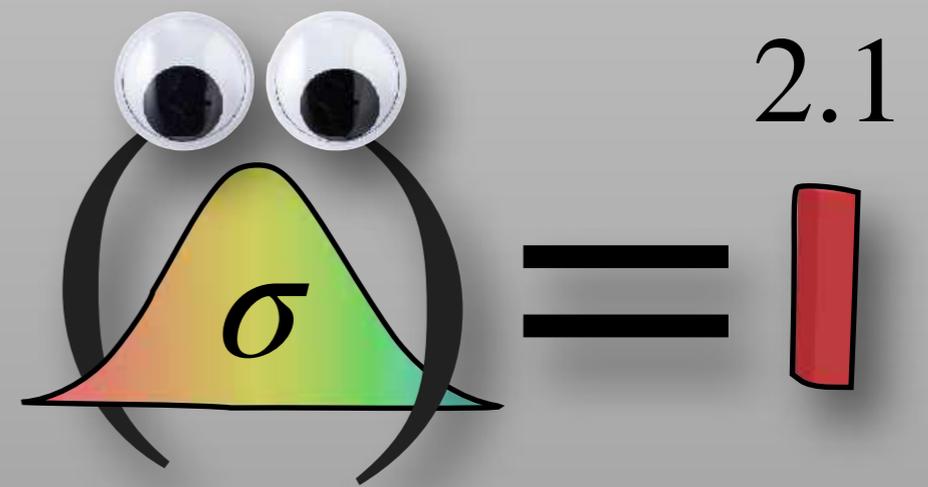
“Twice as much light is
twice as bright”



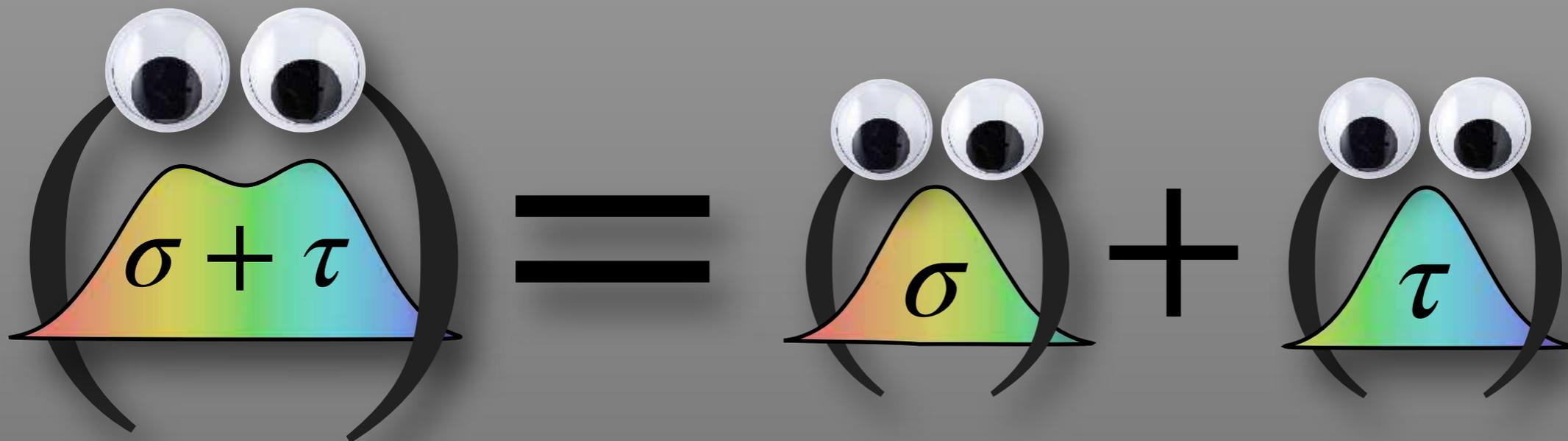
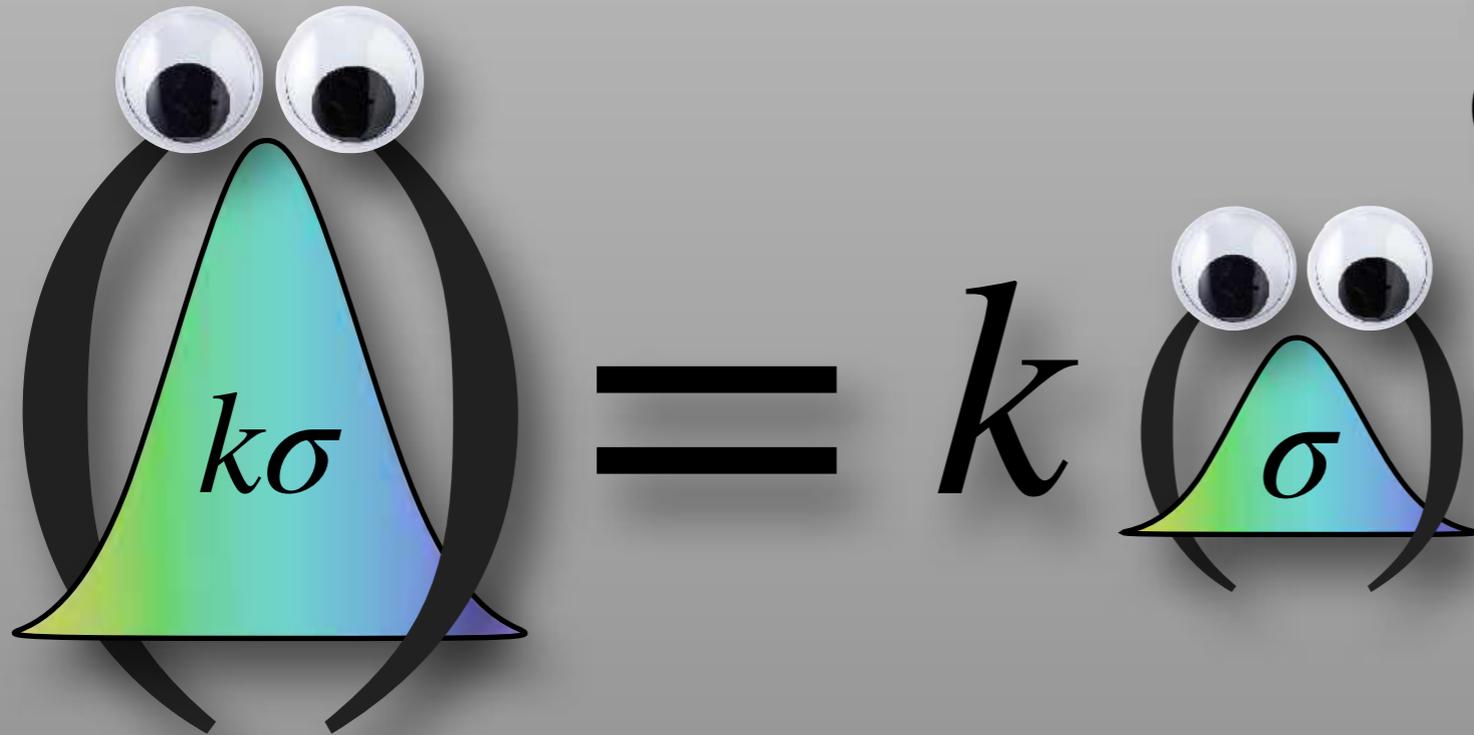
“Twice as much light is
twice as bright”



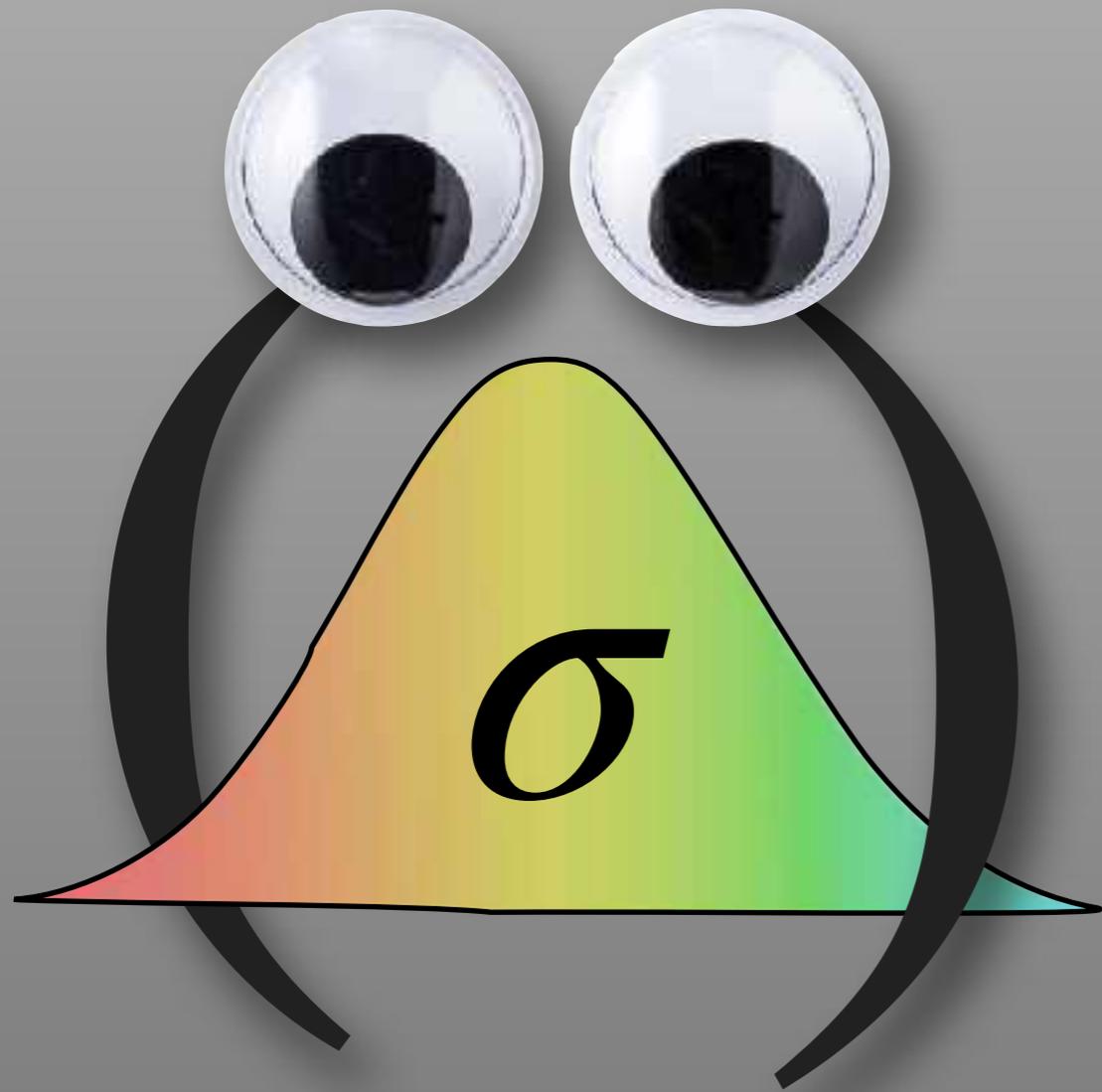
“If you see two lights at once, your eye responds to both”



**These are the
properties
of a linear
map!**



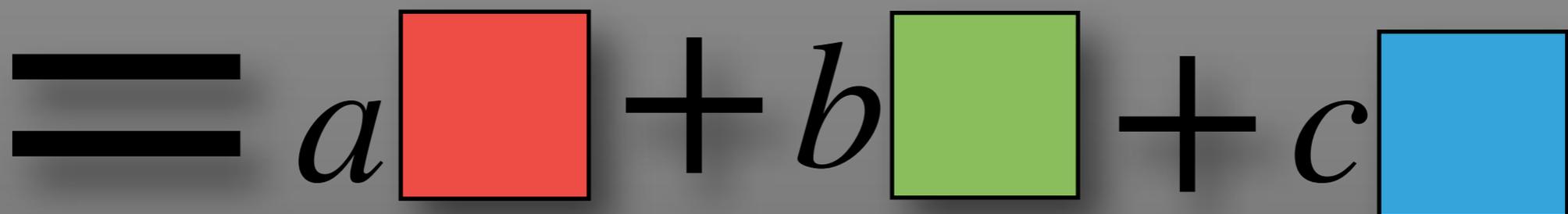
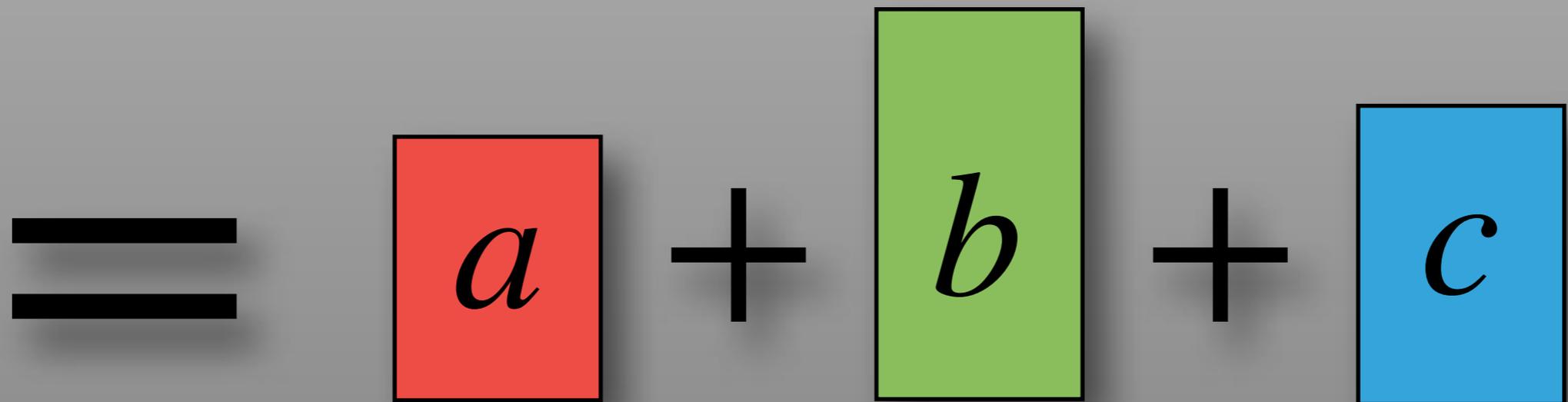
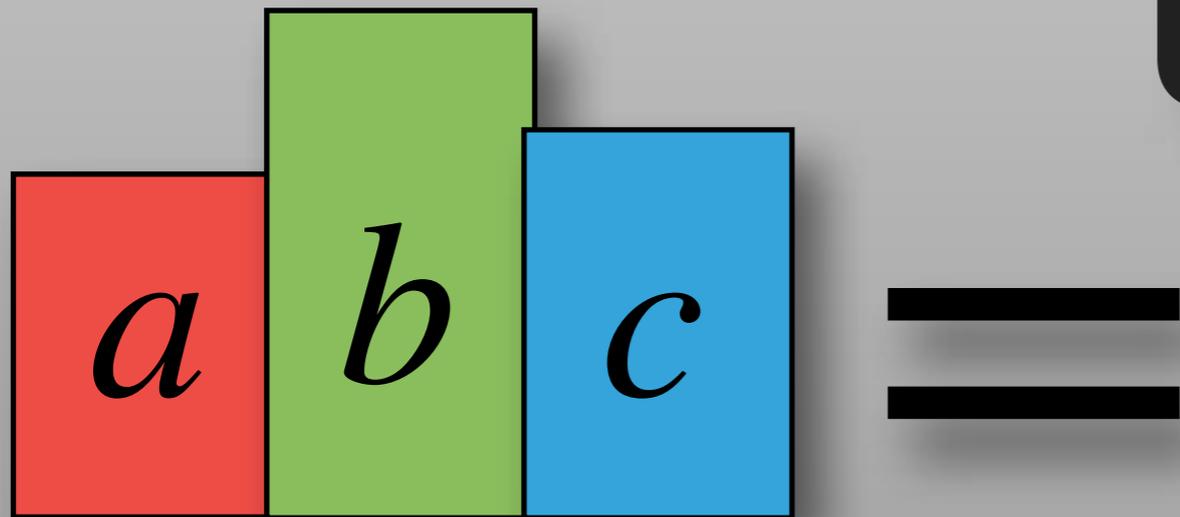
**Can we use this...
to get a formula?**

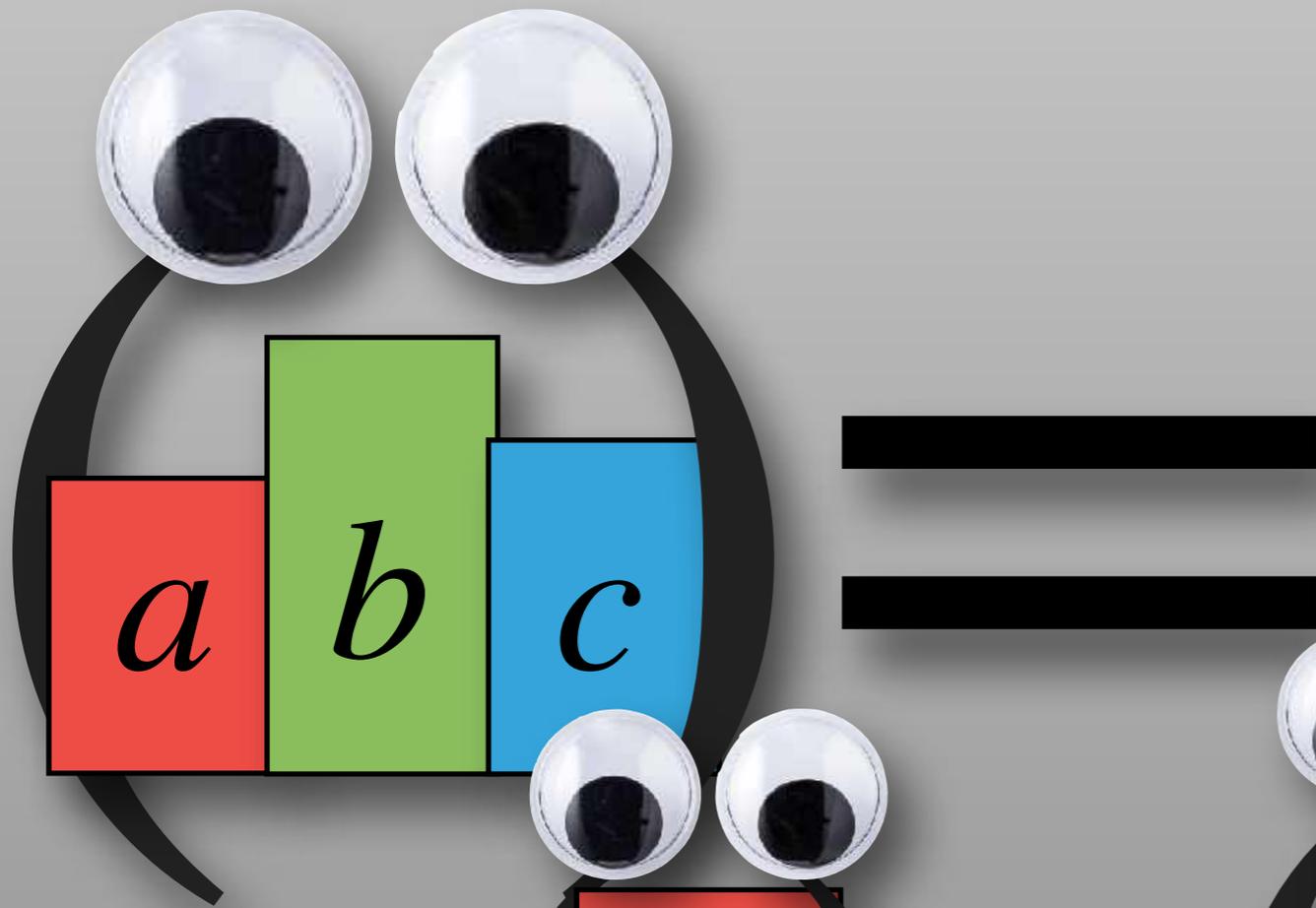


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Use Linearity:

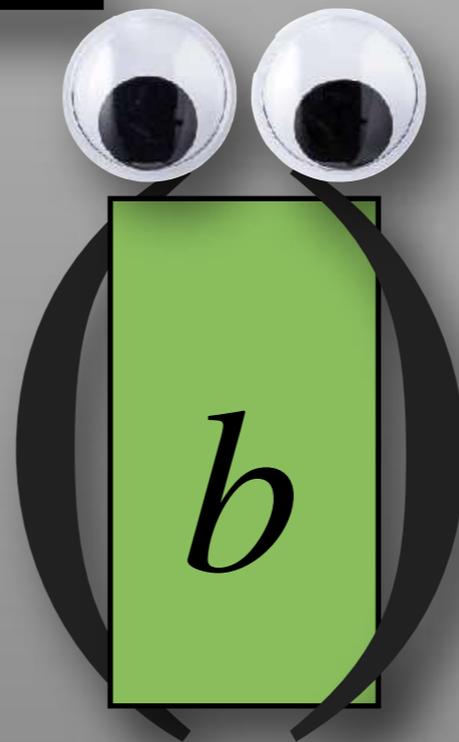




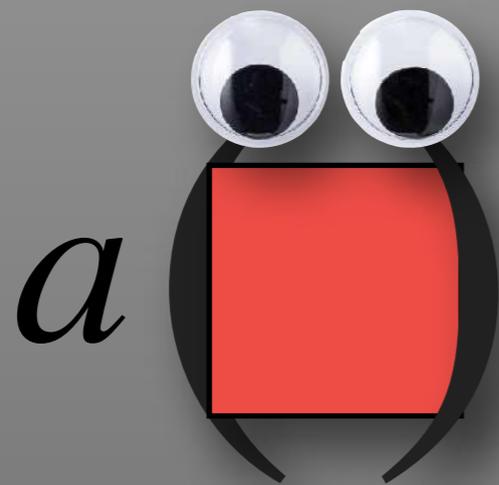
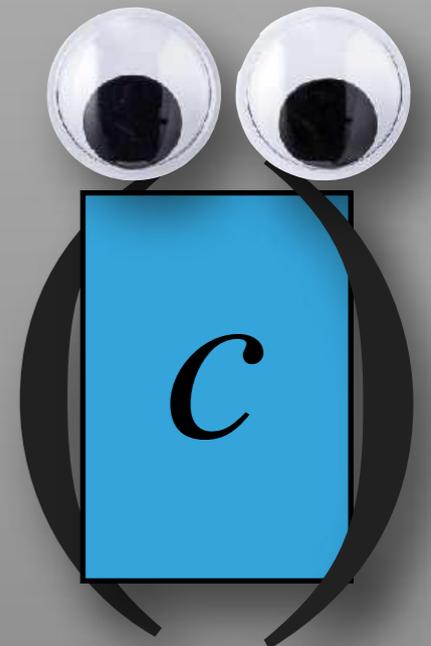
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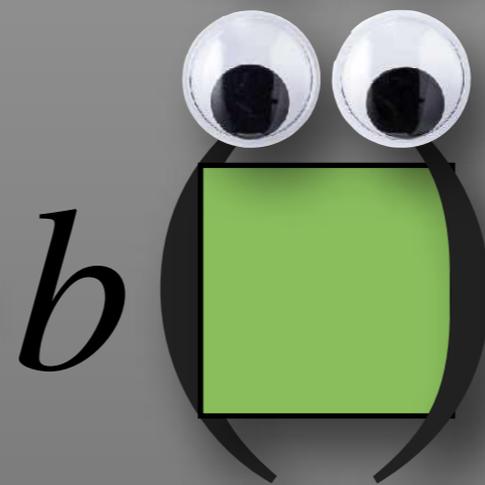
+



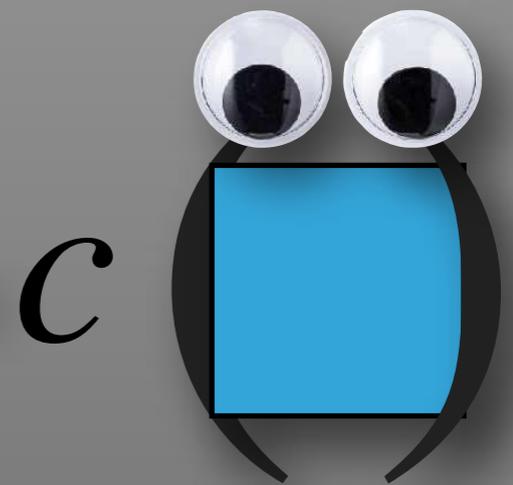
+



+

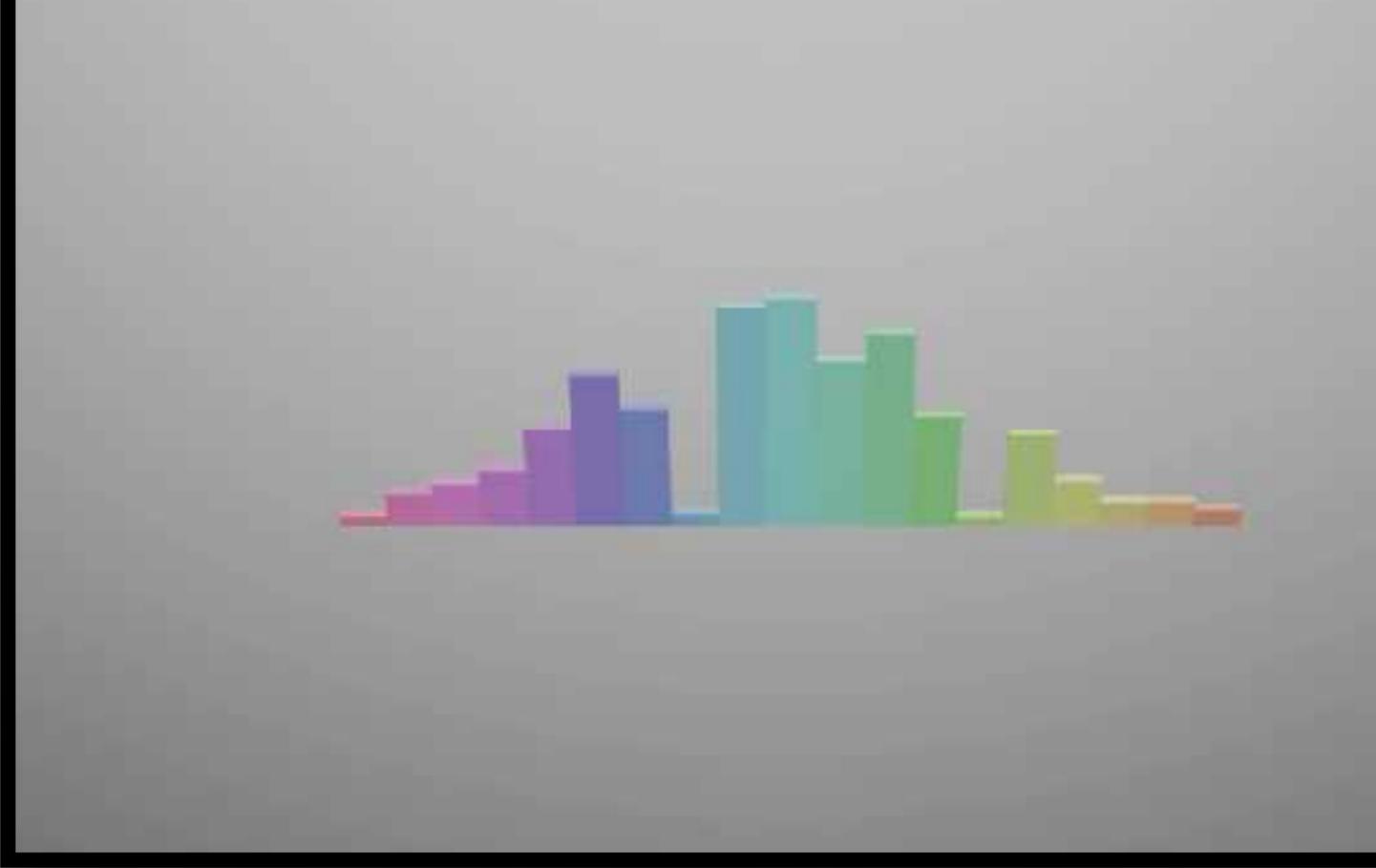


+



$\sum_i \left(\sigma_i \lambda_i \right) =$

$\sigma_i \left(\lambda_i \right)$



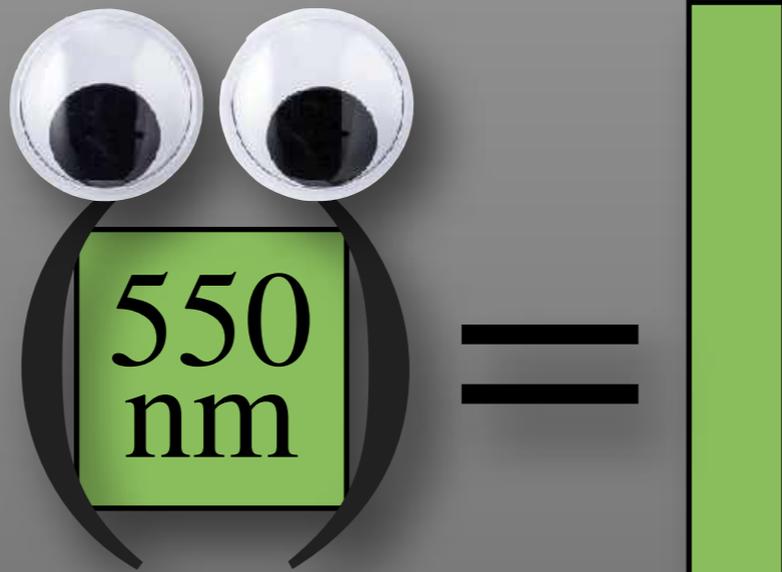
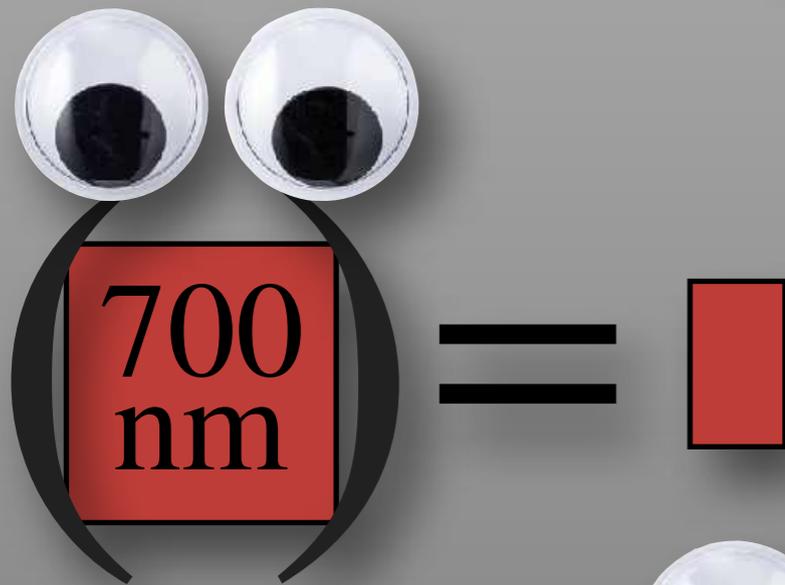
$$\int \sum_i \sigma_i \left(\lambda_i \right) d\lambda$$

What
is



“The Response
Curve”

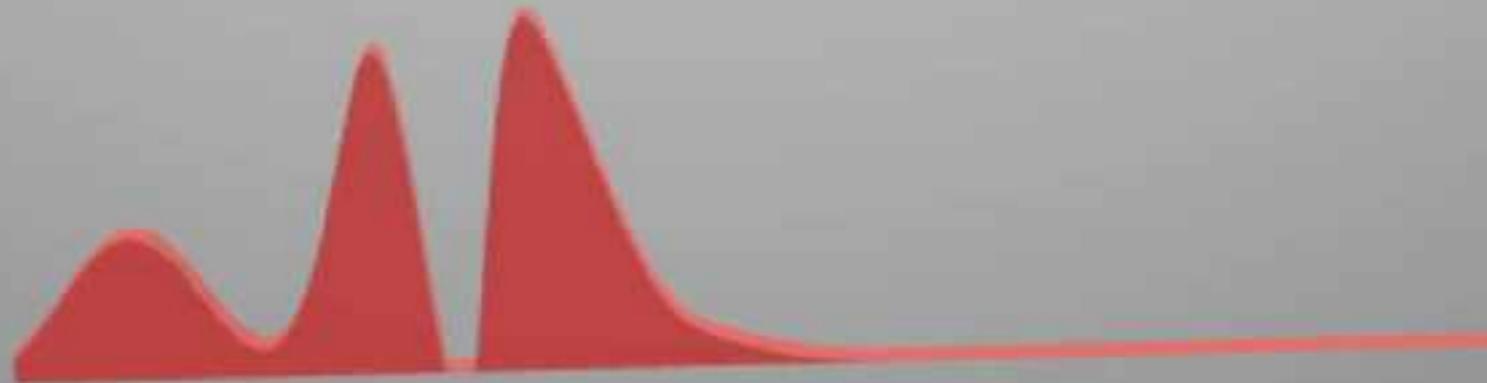
Sensitivity of
the eye to the
wavelength λ



The Response to a general spectrum.



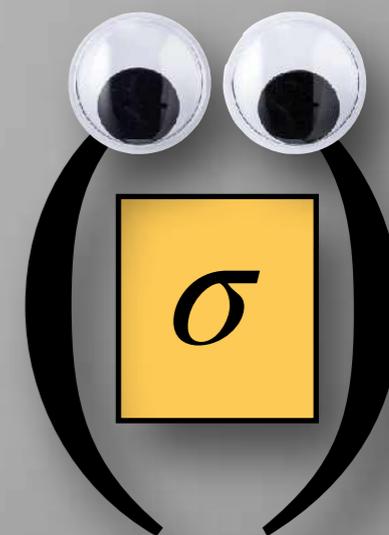
σr



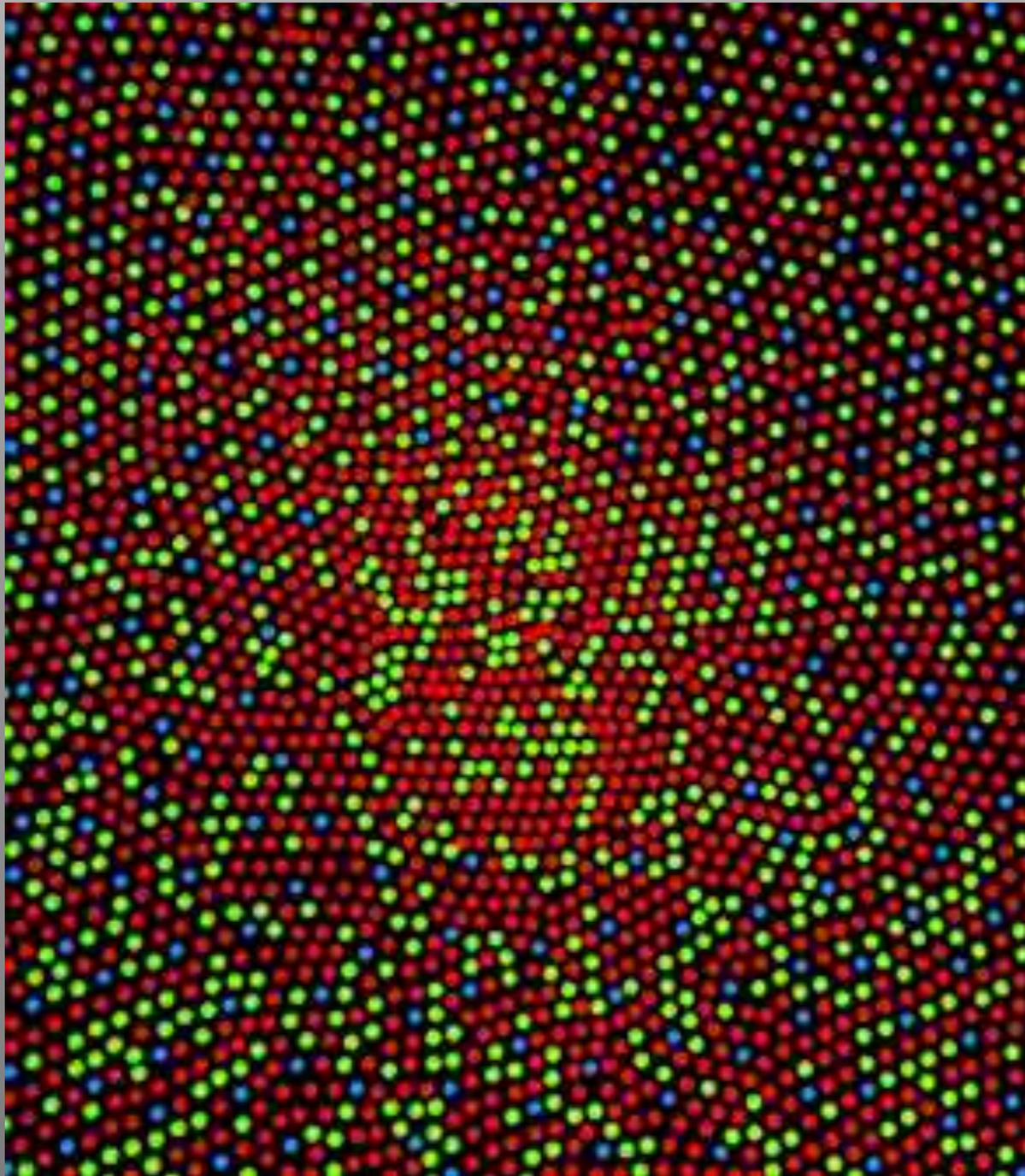
$\int \sigma r d\lambda$



A red rectangular block is positioned to the left of the integral symbol in the equation.

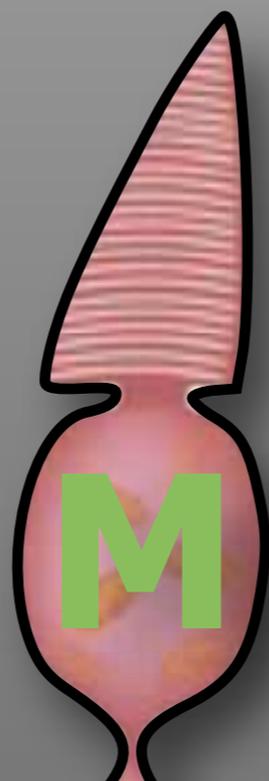
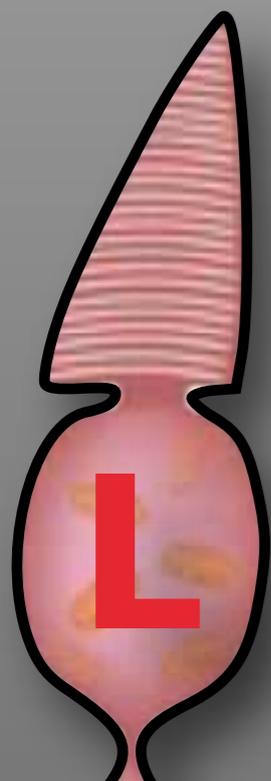


A Fact from Biology:



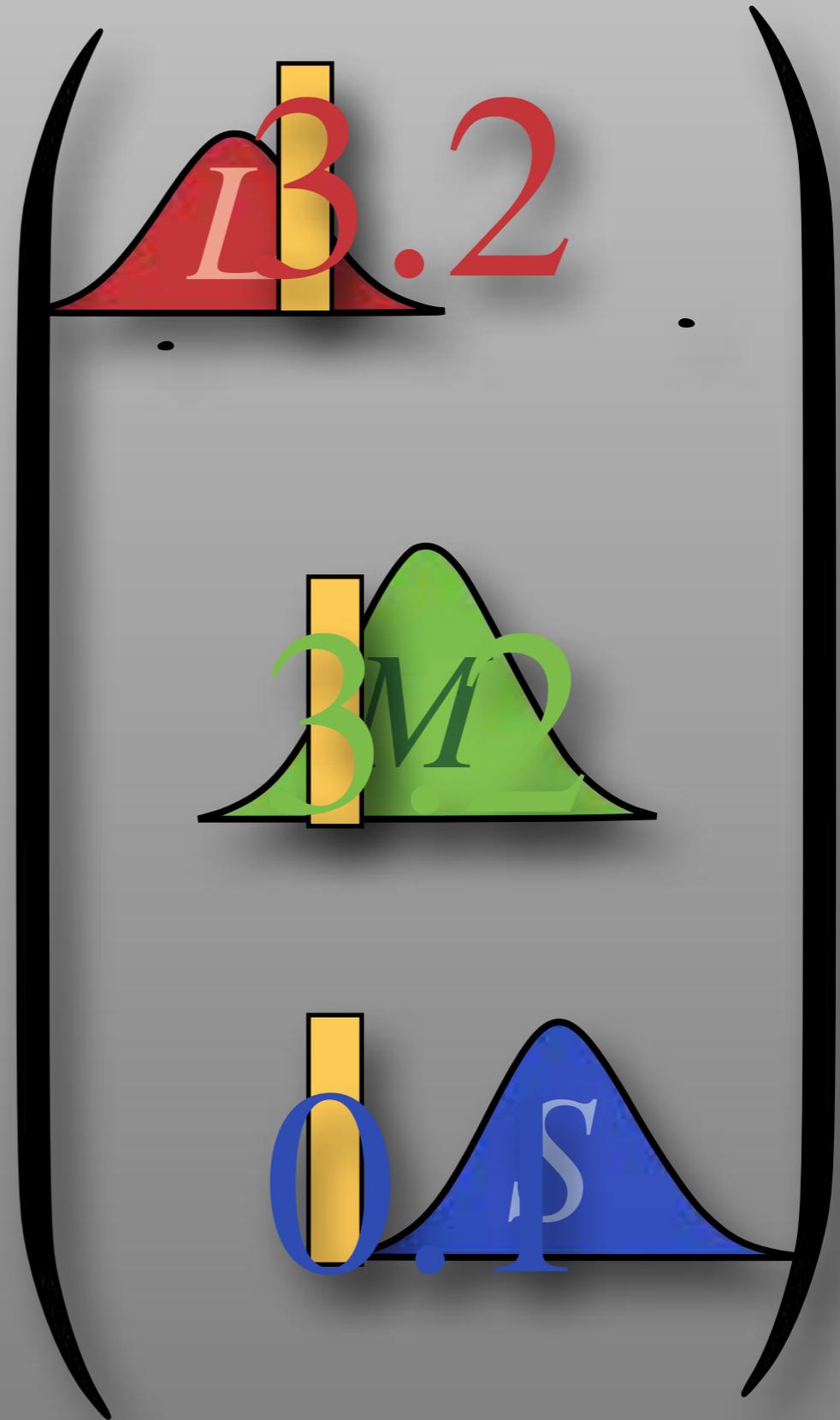
The human eye has three different photosensitive pigments.

These have three different response curves.





=



is
vector-
valued!

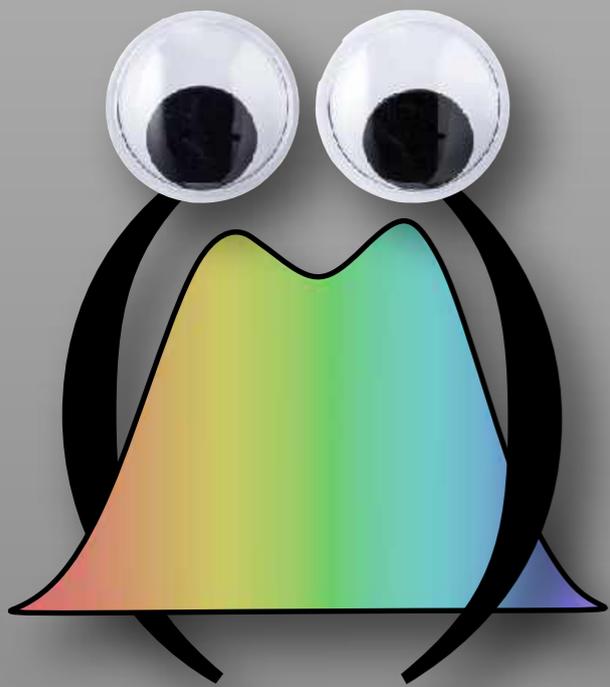
Big Idea #11

A vibrant rainbow arches across a clear blue sky, spanning from the left side of the frame to the right. Below the rainbow, a lush green lawn stretches across the foreground. In the middle ground, several trees are visible, including a large, leafless tree in the center and several evergreen trees on the right. On the far left, a portion of a red barn with a white door is visible. The overall scene is bright and colorful, with the rainbow being the central focus.

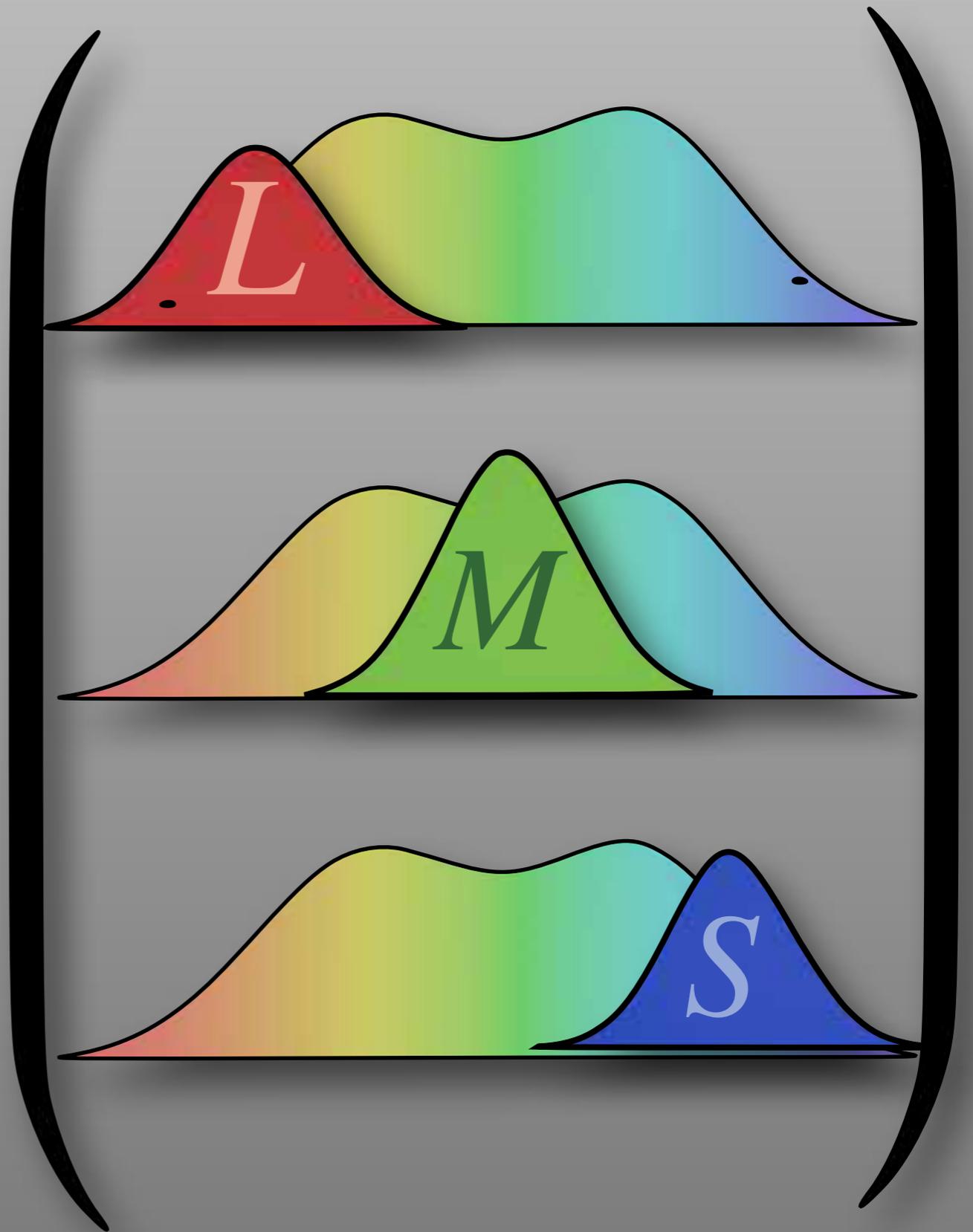
**Perceptual Color is a
*3-Dimensional Vector Space***

Photo: My Brother

Alex Trettel!

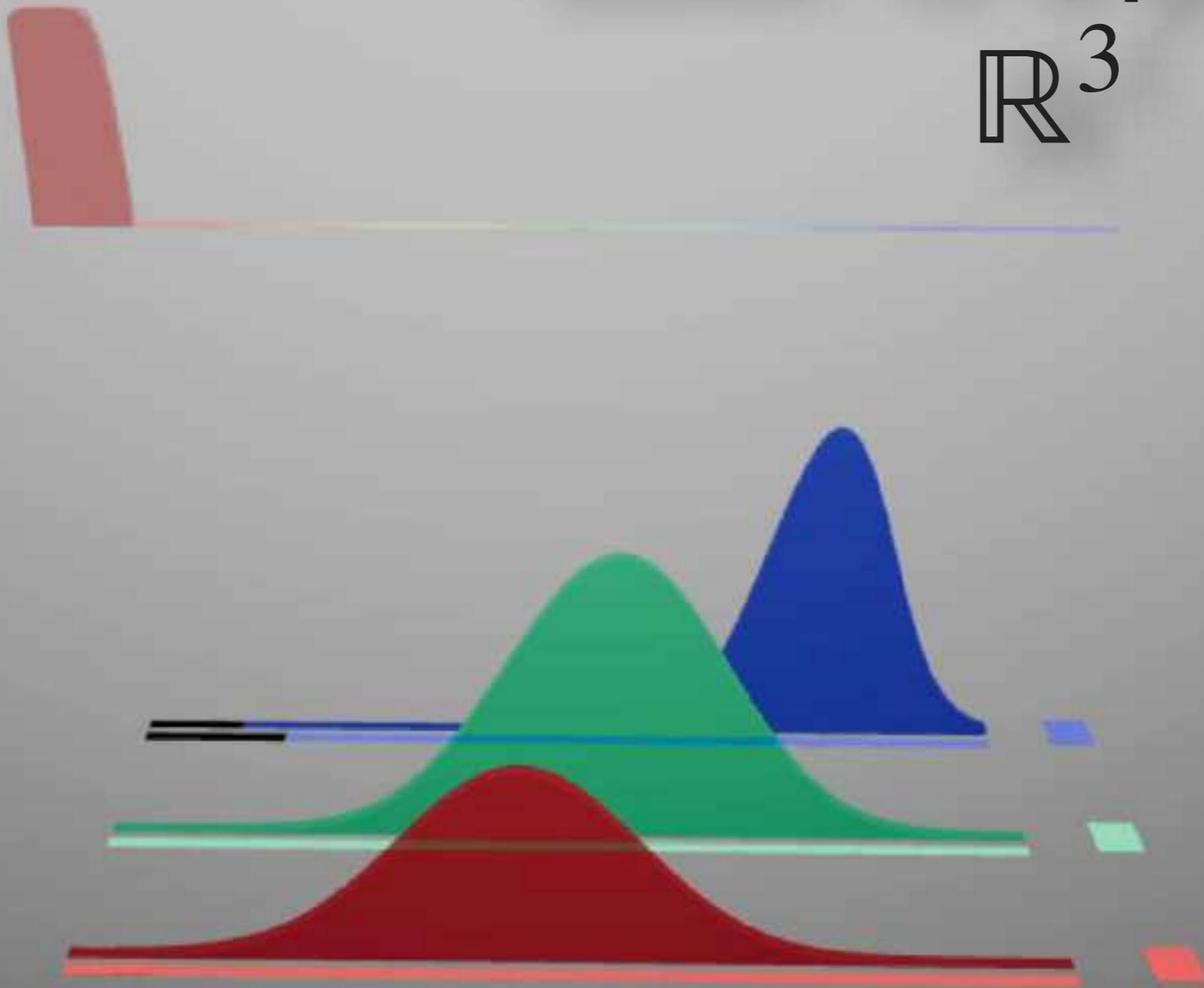


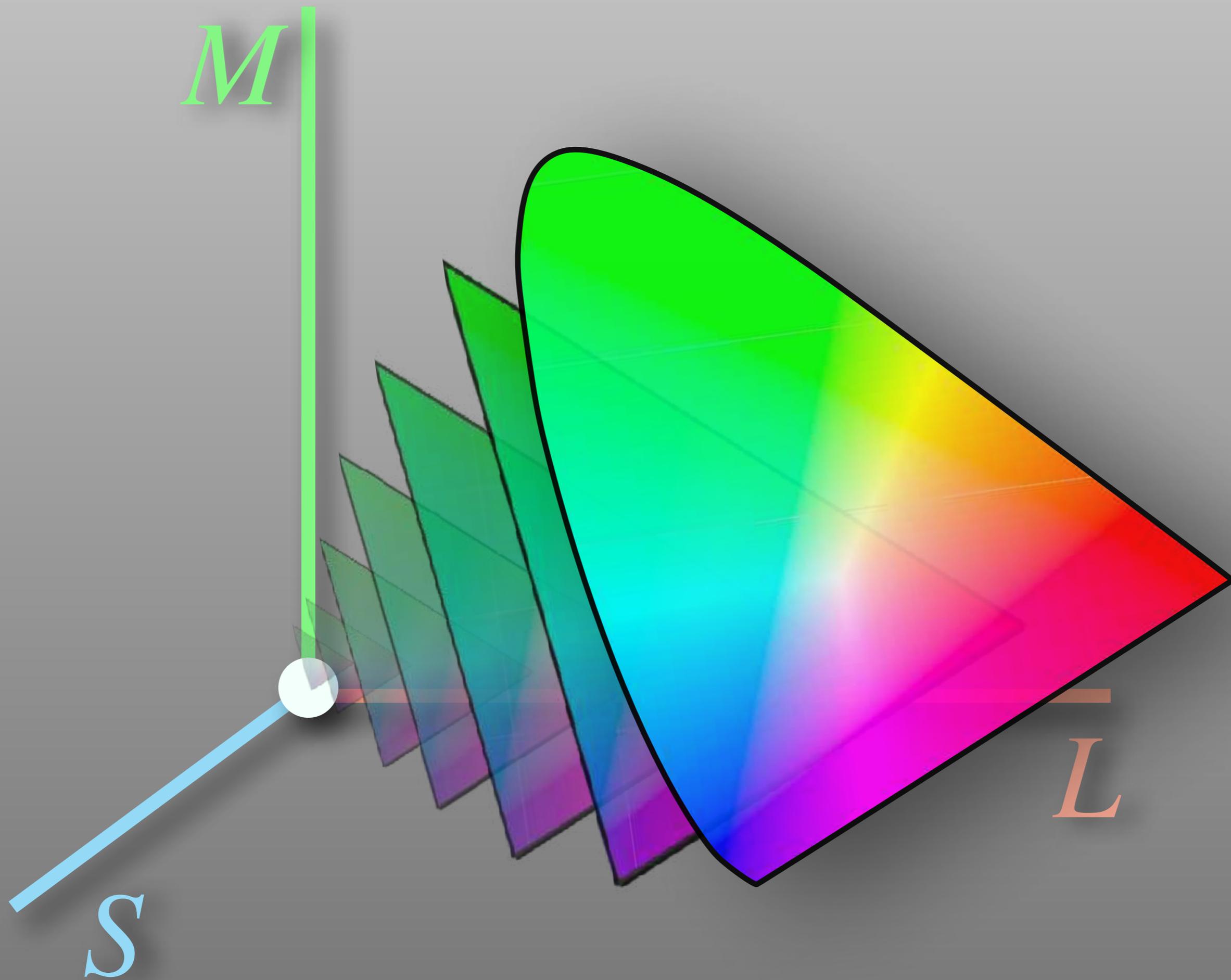
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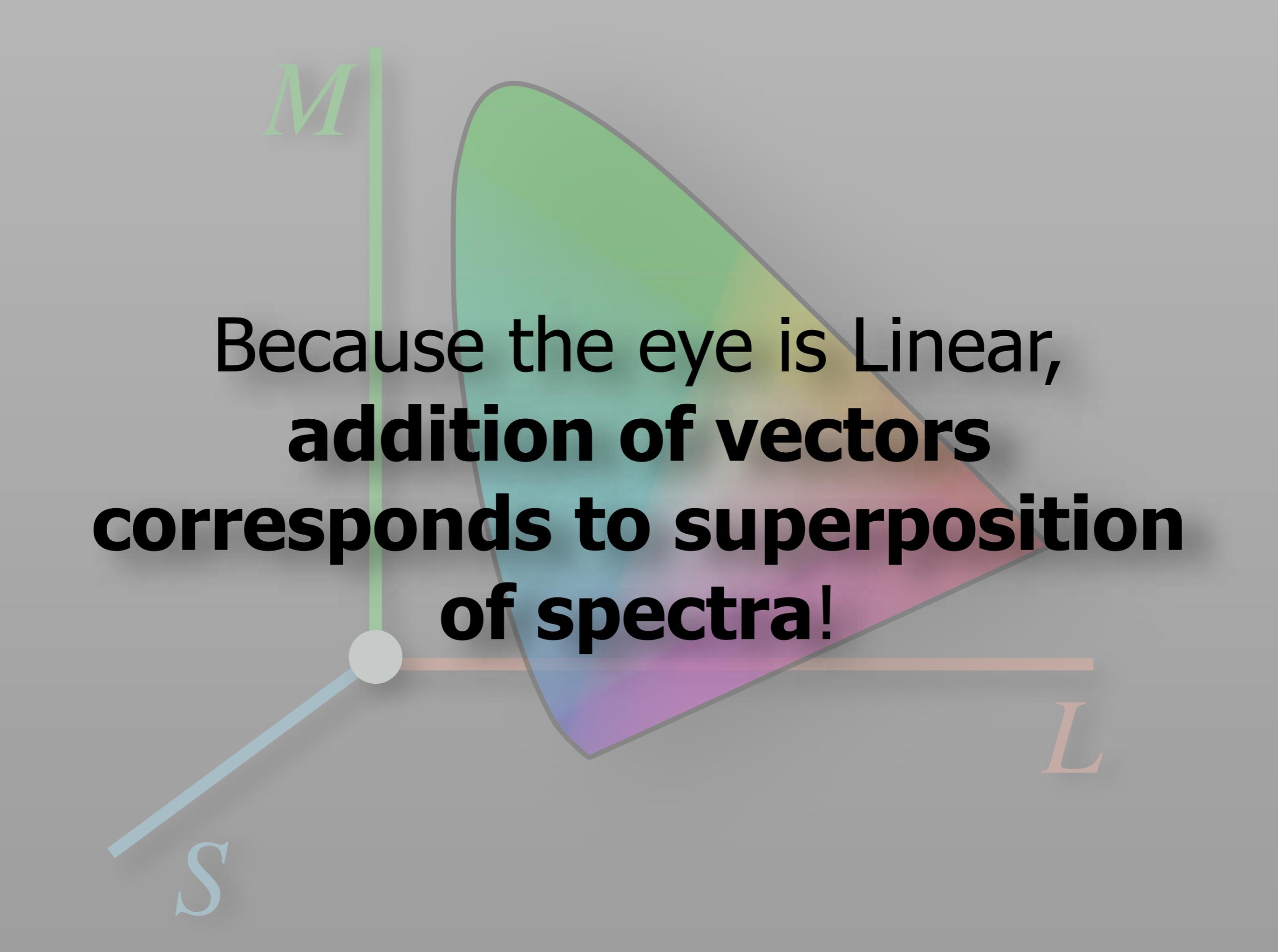


**This is all the
brain gets!!!**

Every perceptual
color is a point in
 \mathbb{R}^3





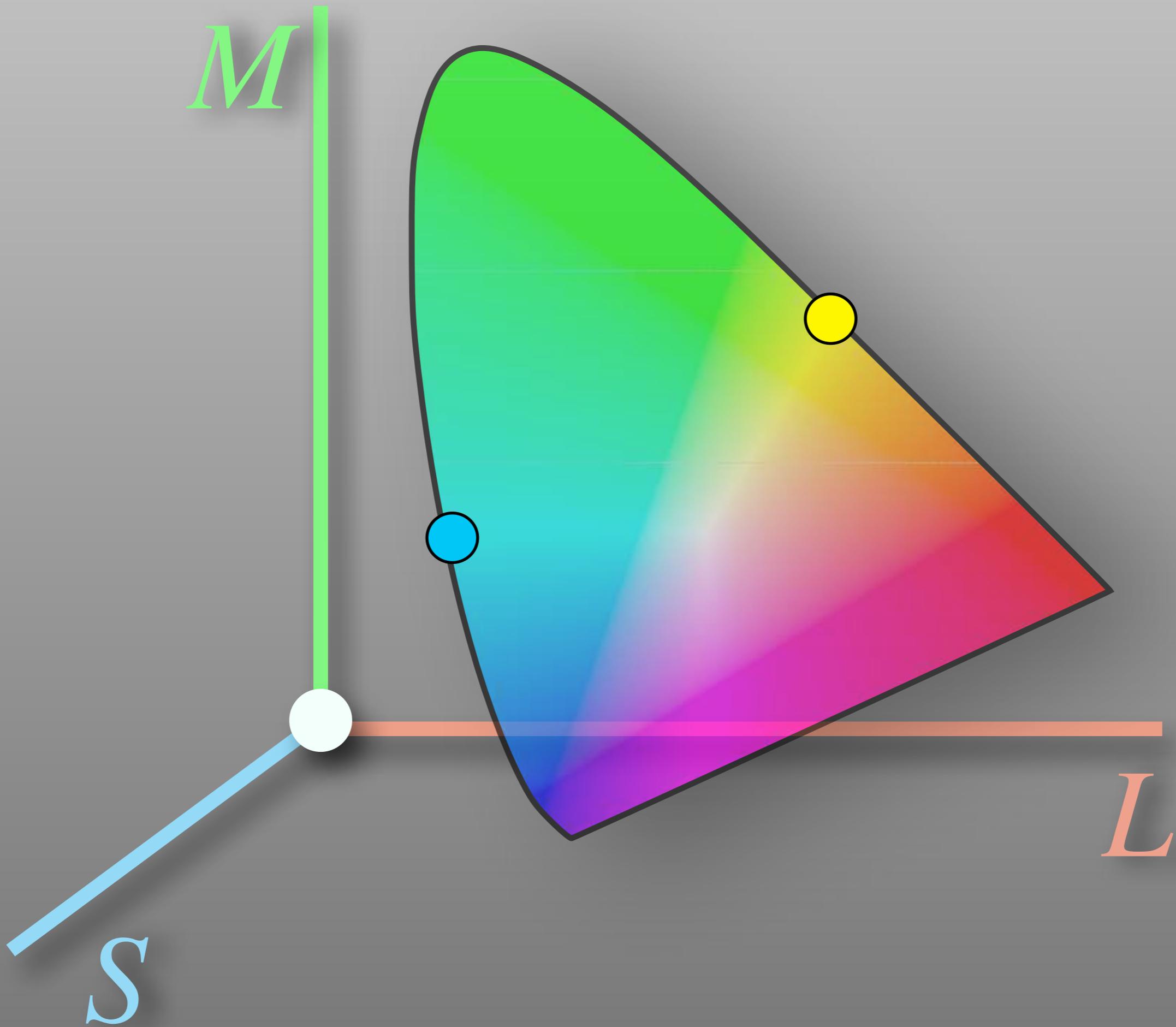


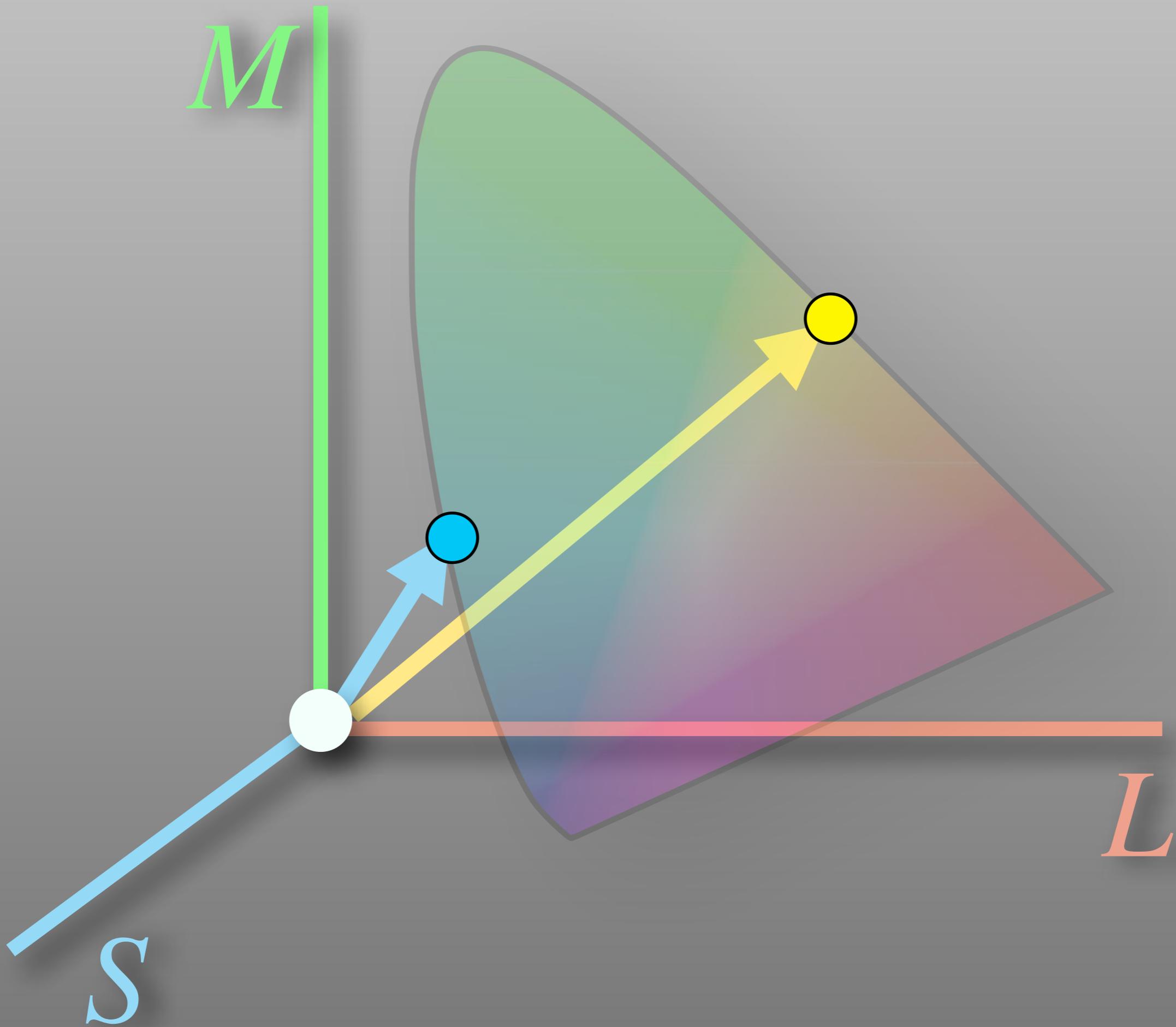
Because the eye is Linear,
addition of vectors
corresponds to superposition
of spectra!

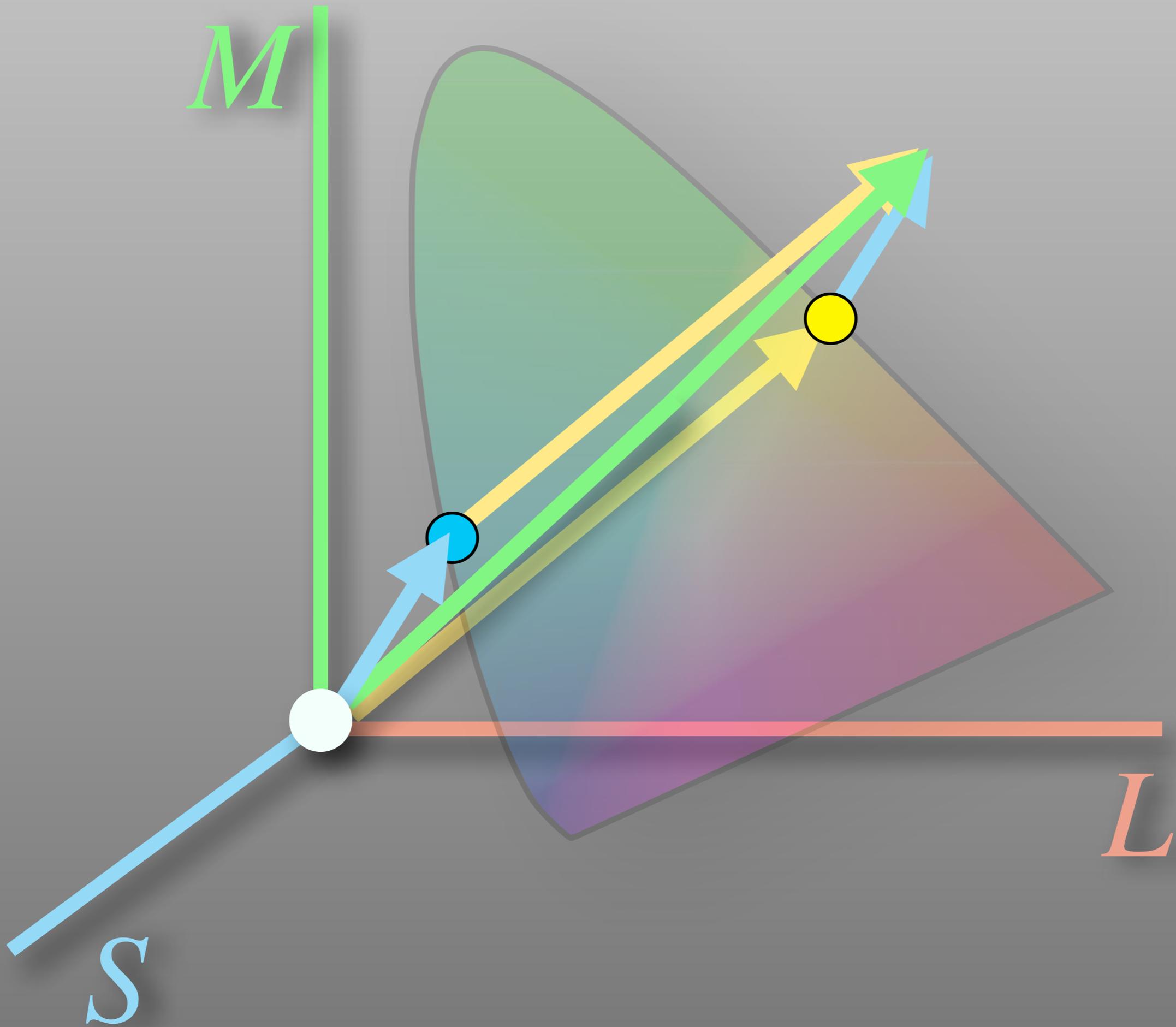
S

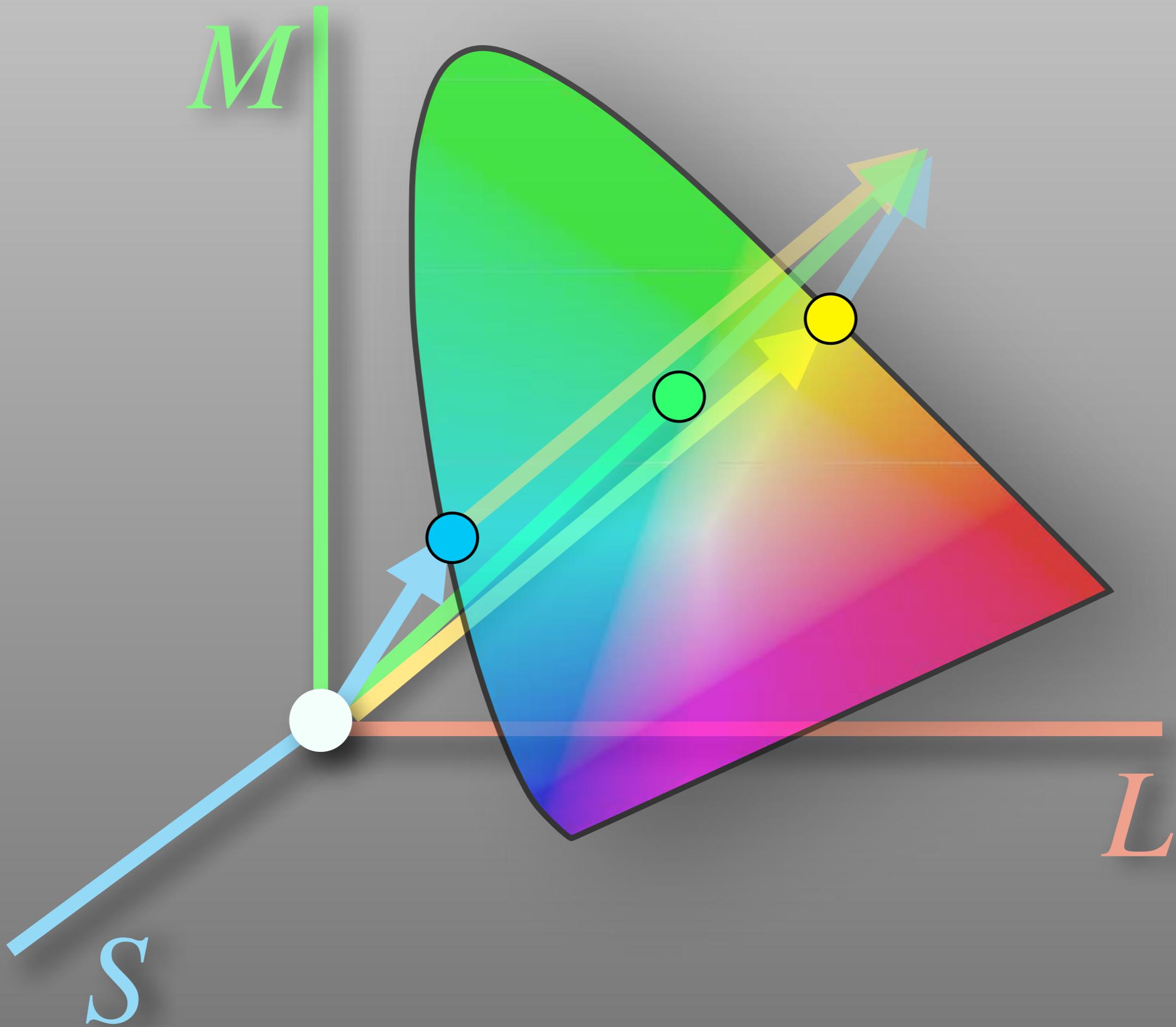
L

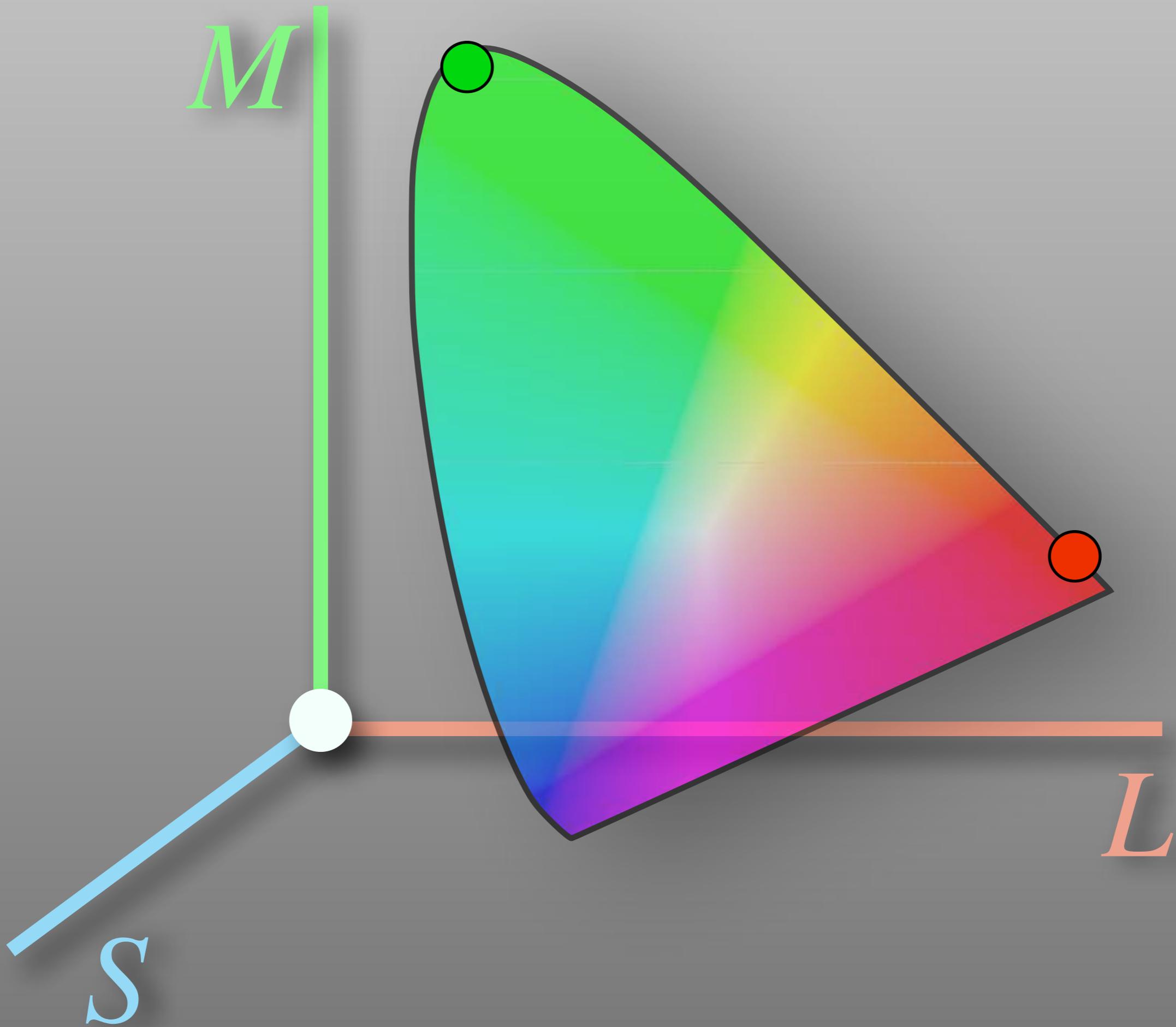
M

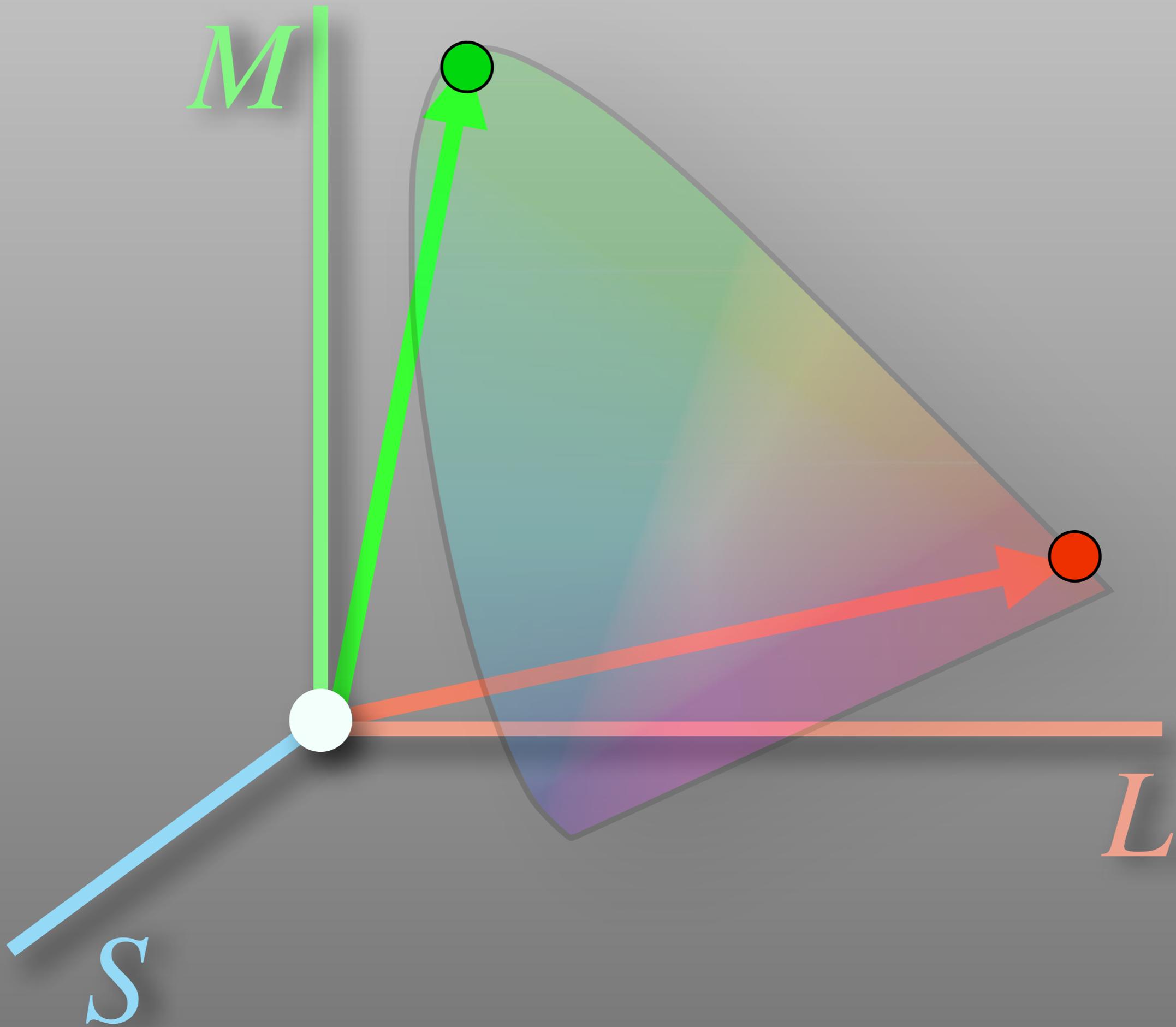


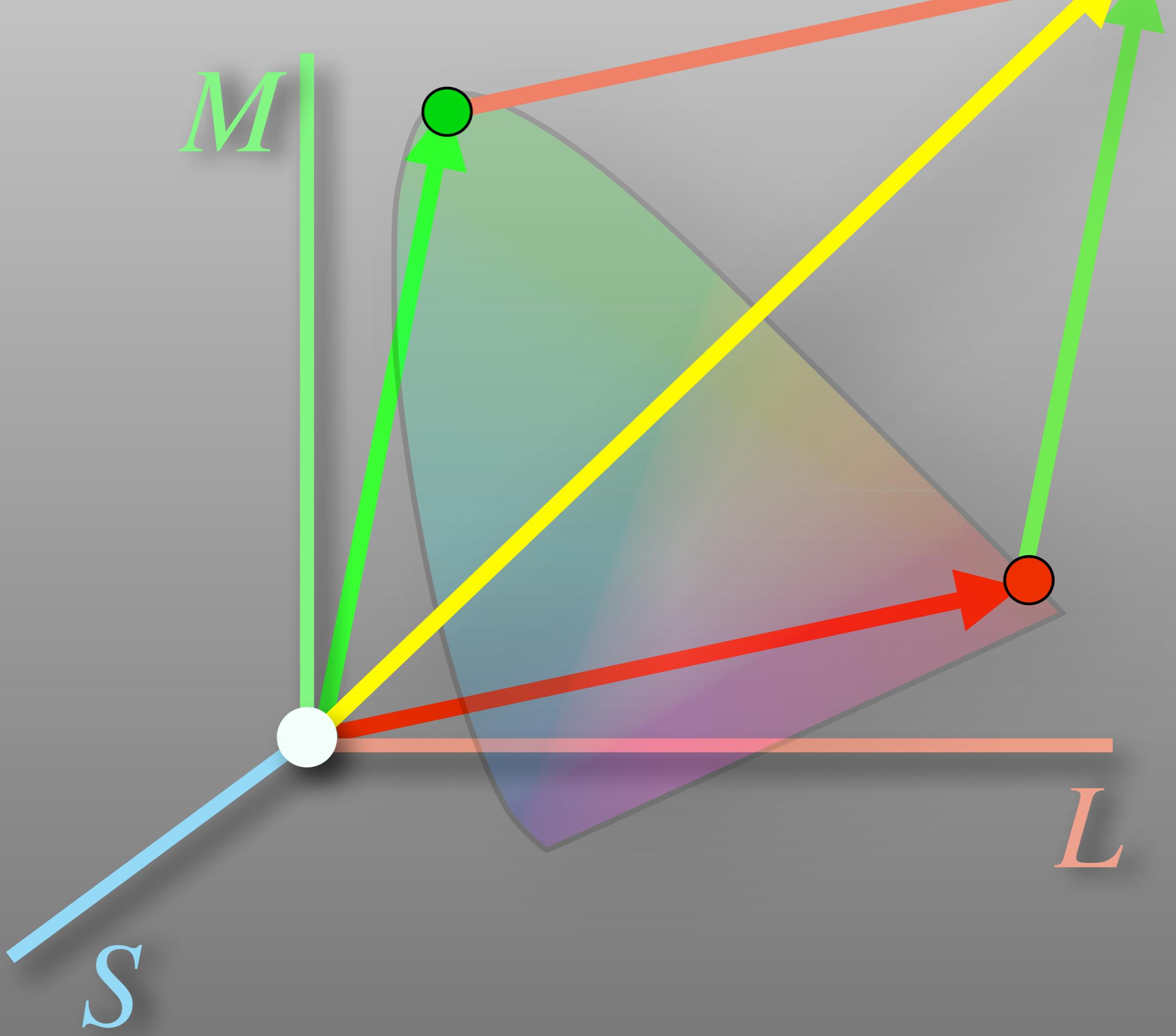


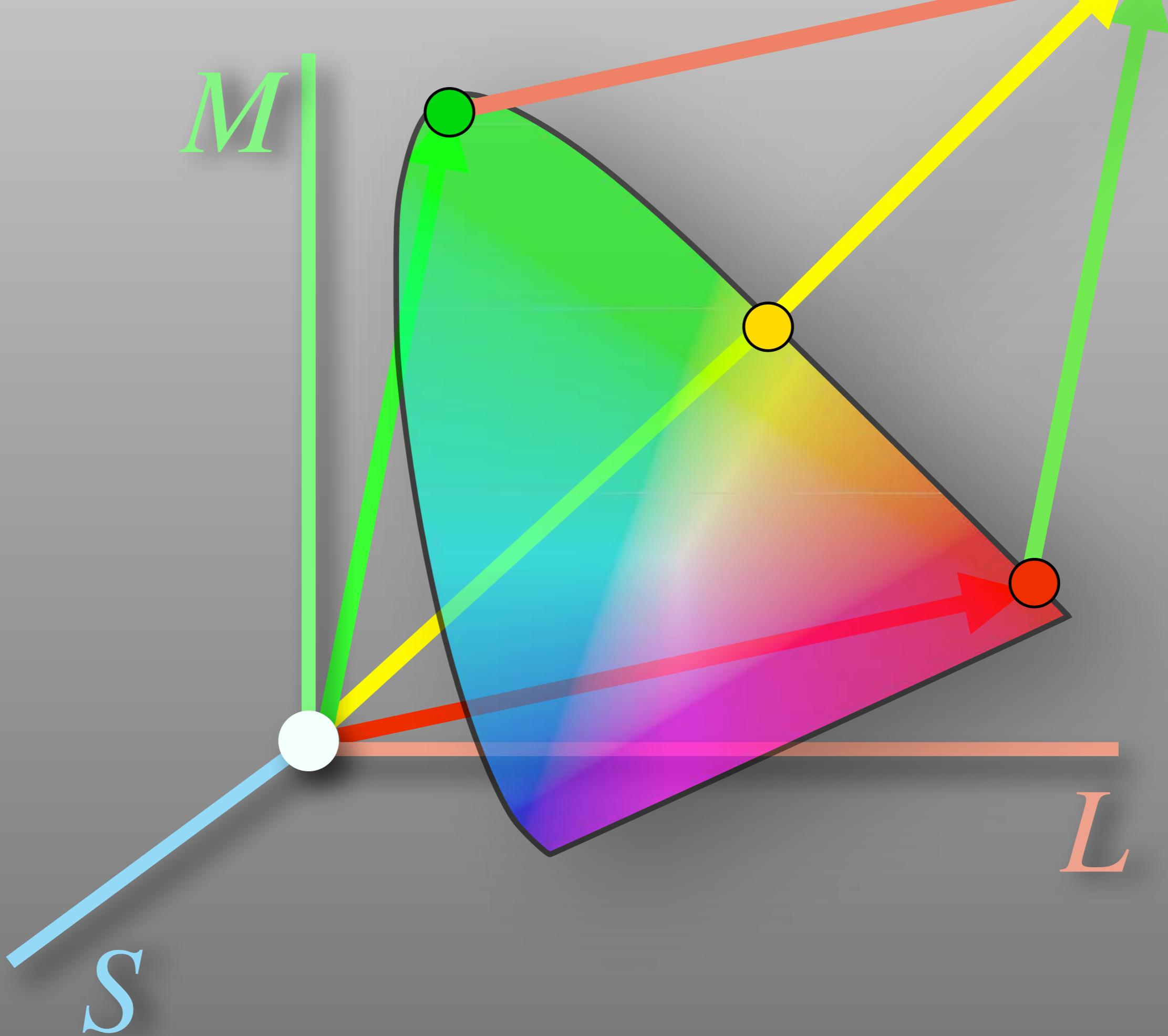






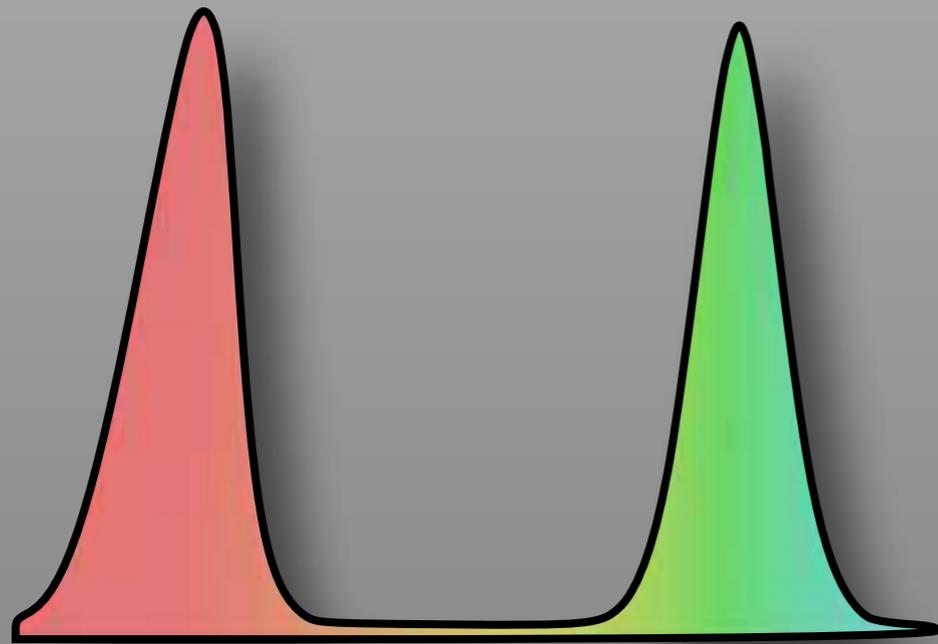




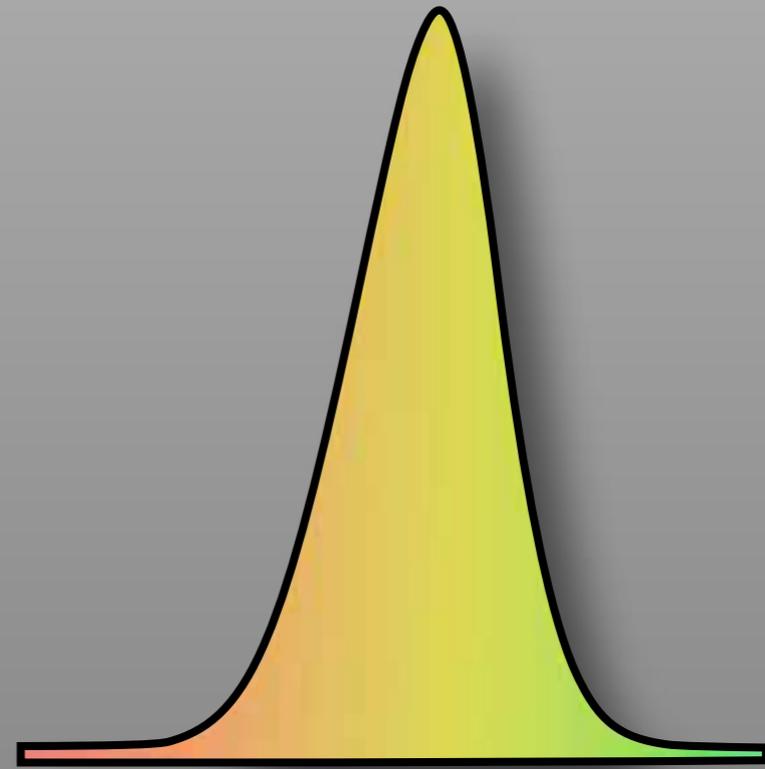


What is going on?

As spectra, these could not be more different.



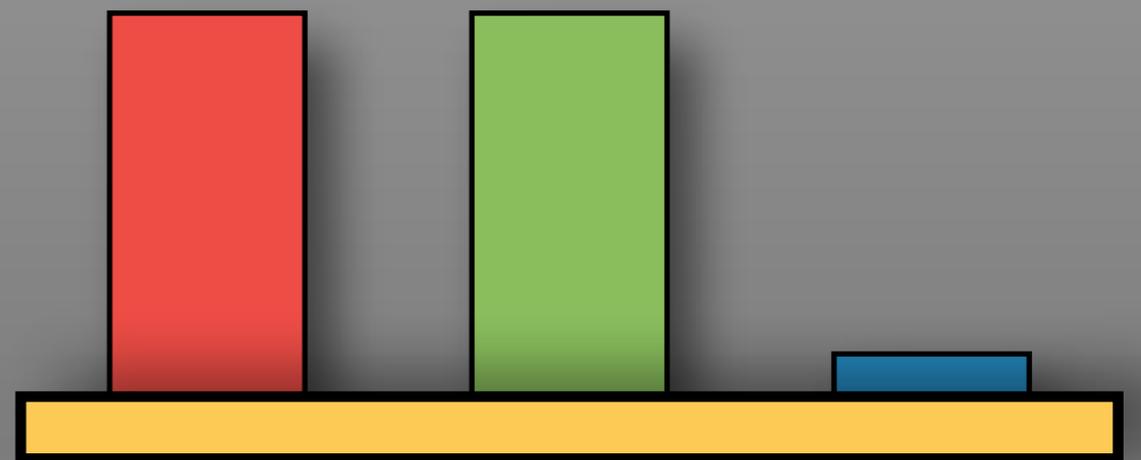
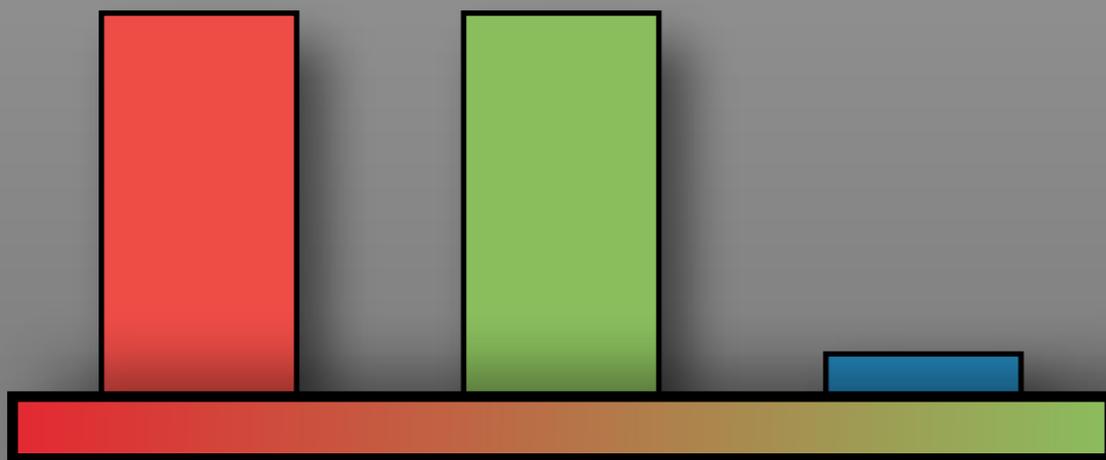
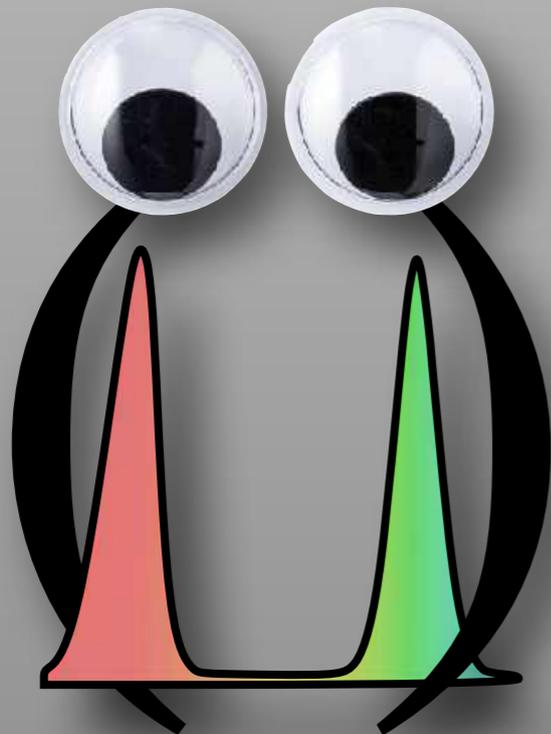
Red-Green

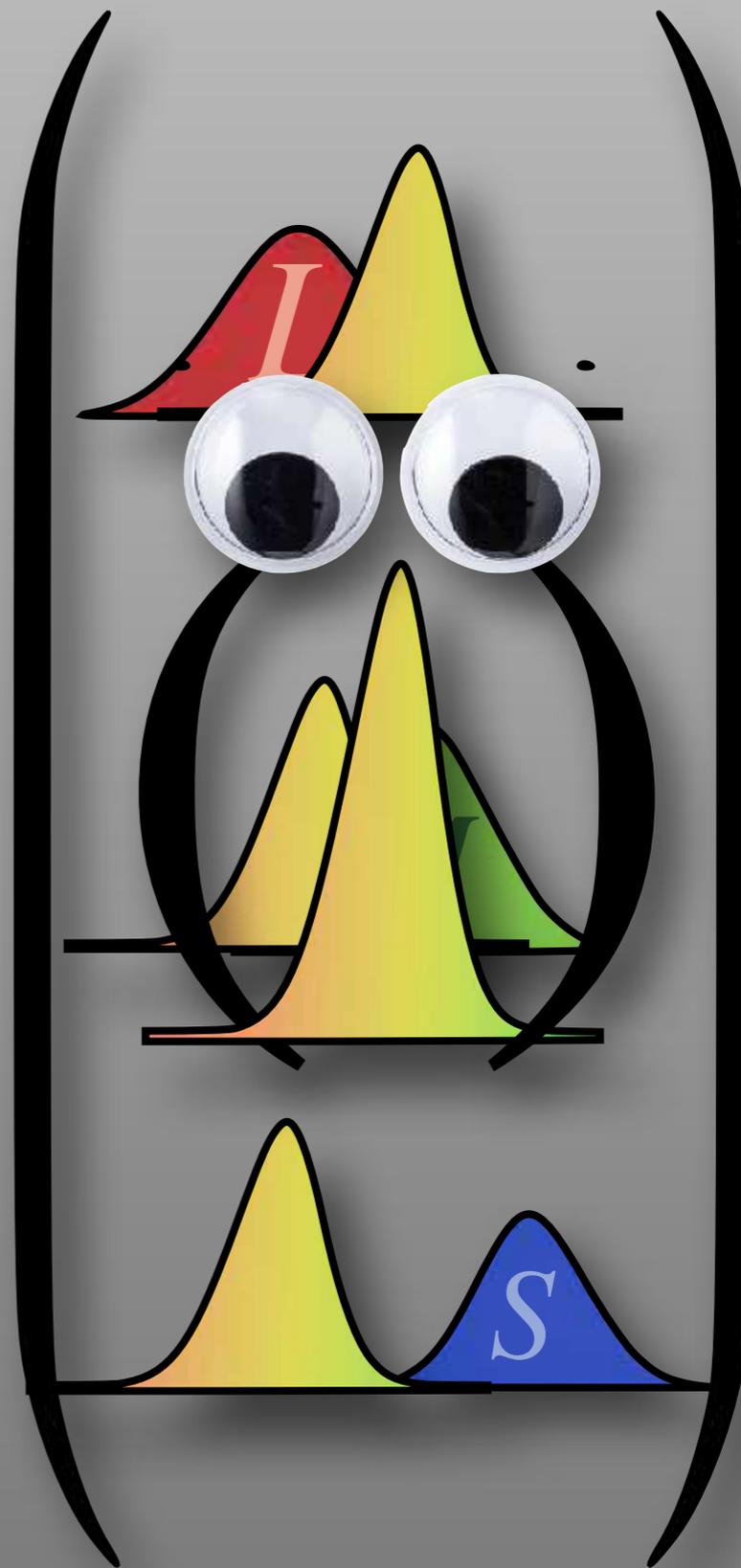
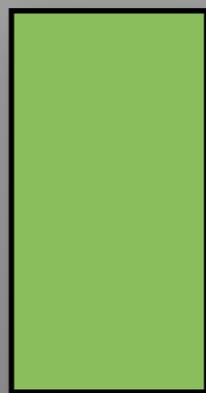
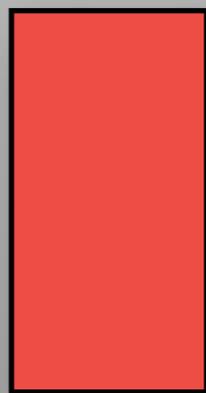
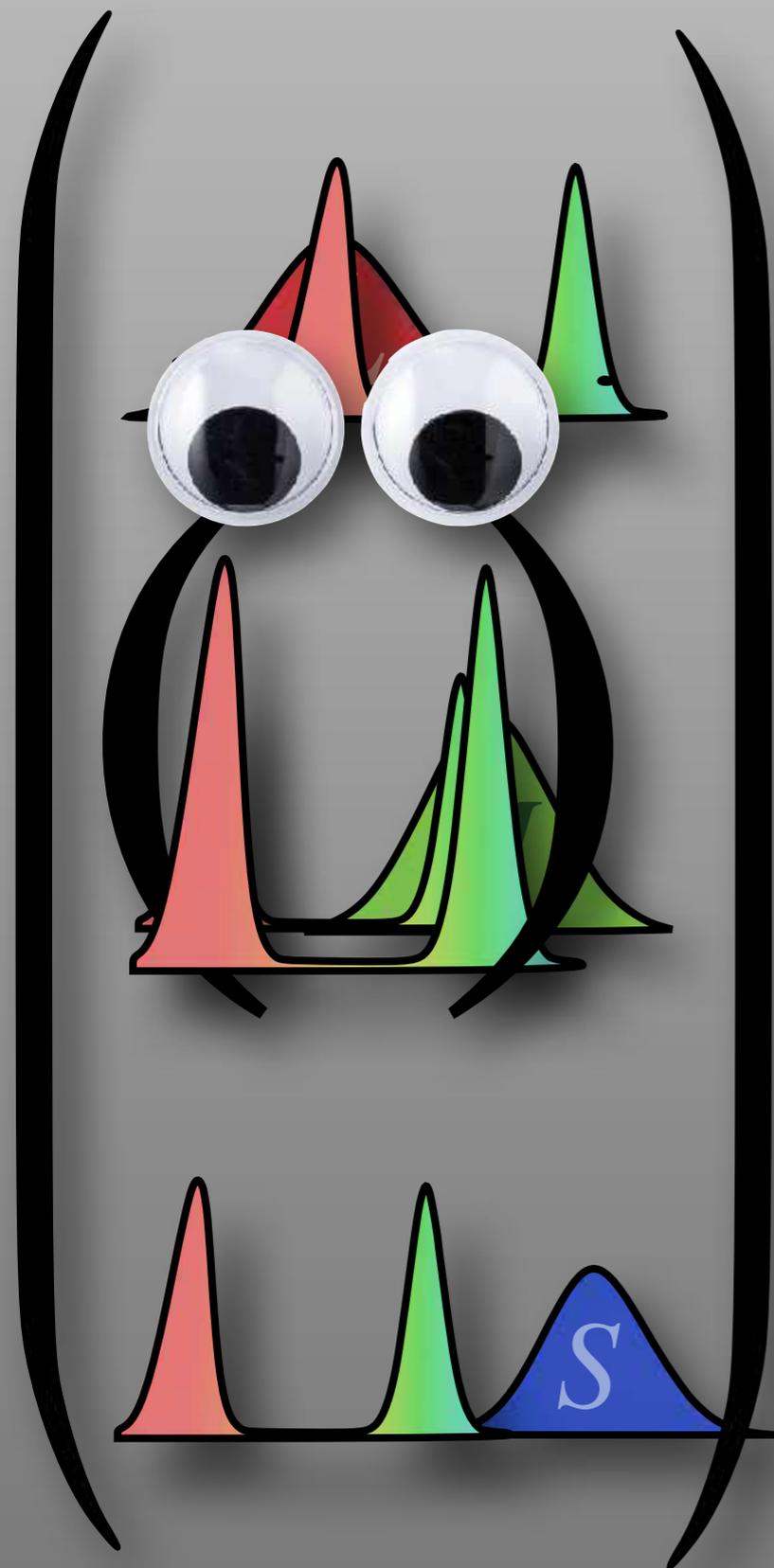


Yellow

What is going on?

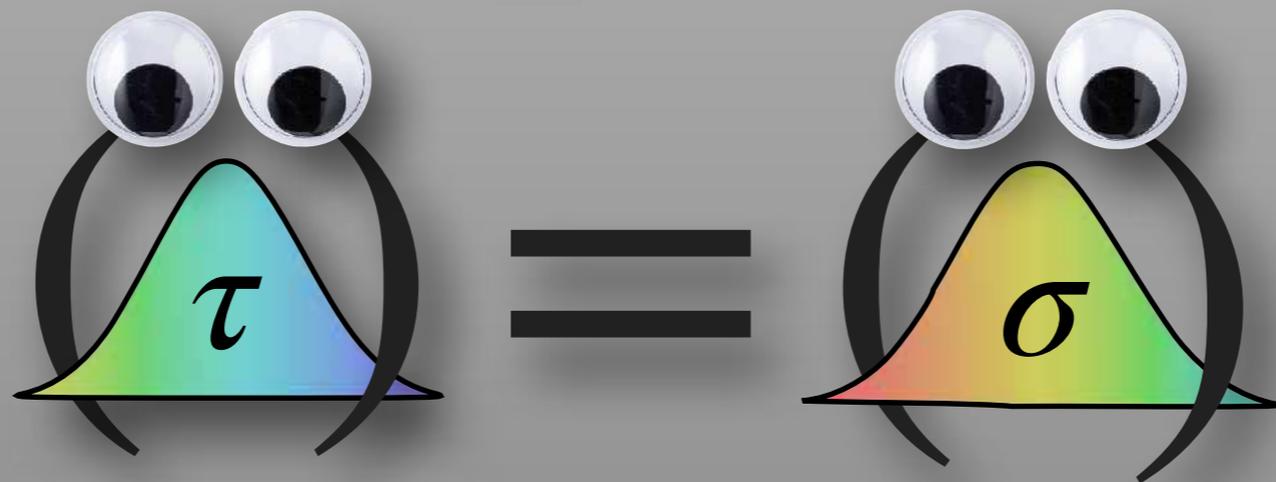
BUT: the LMS response to both are *equal*!





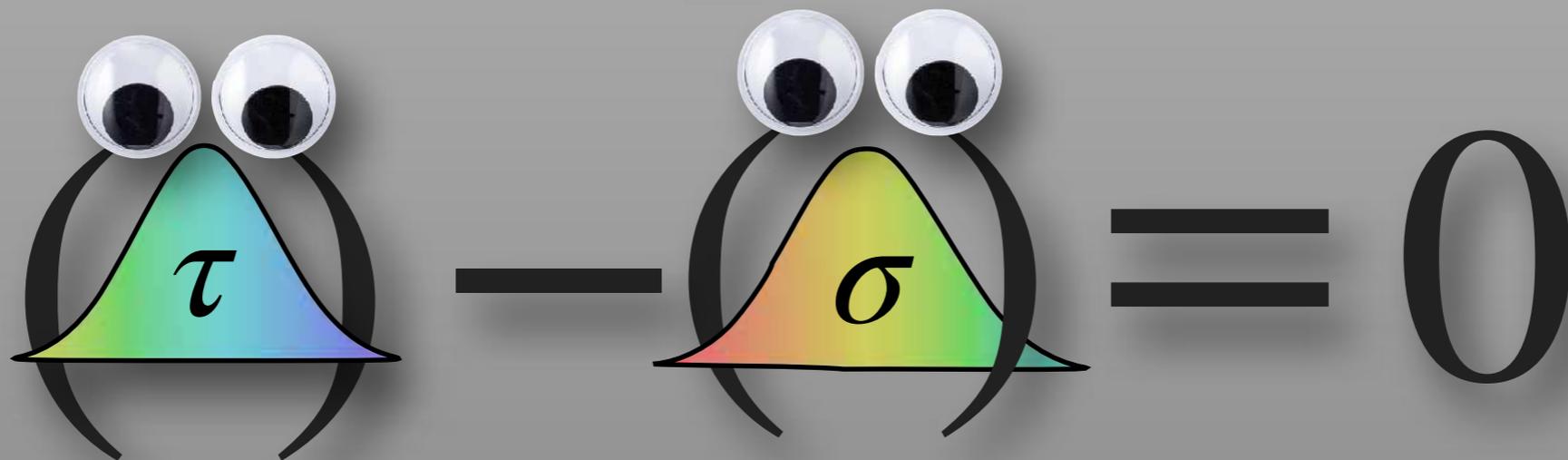
Metamer:

Two spectra are **metamers** if they look the same to the eye.



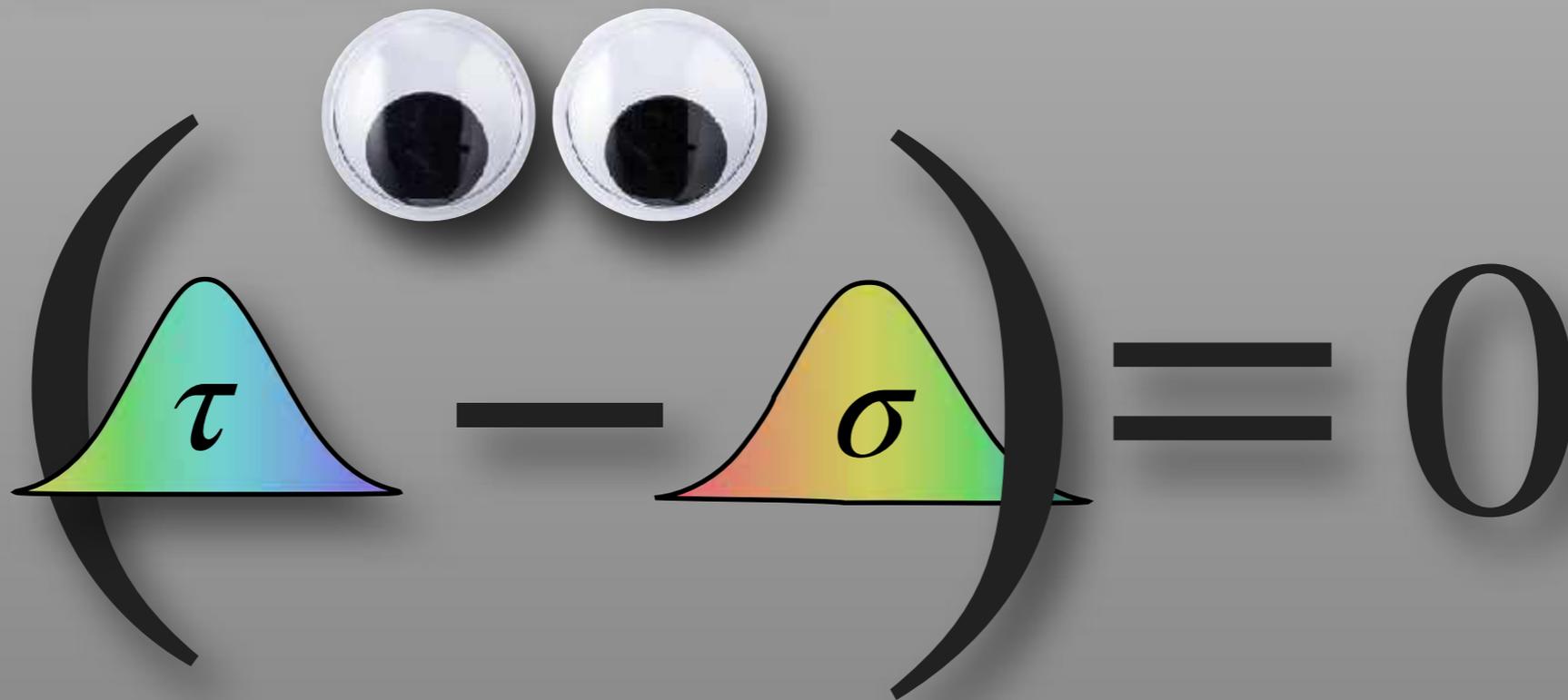
Metamer:

Two spectra are **metamers** if they look the same to the eye.



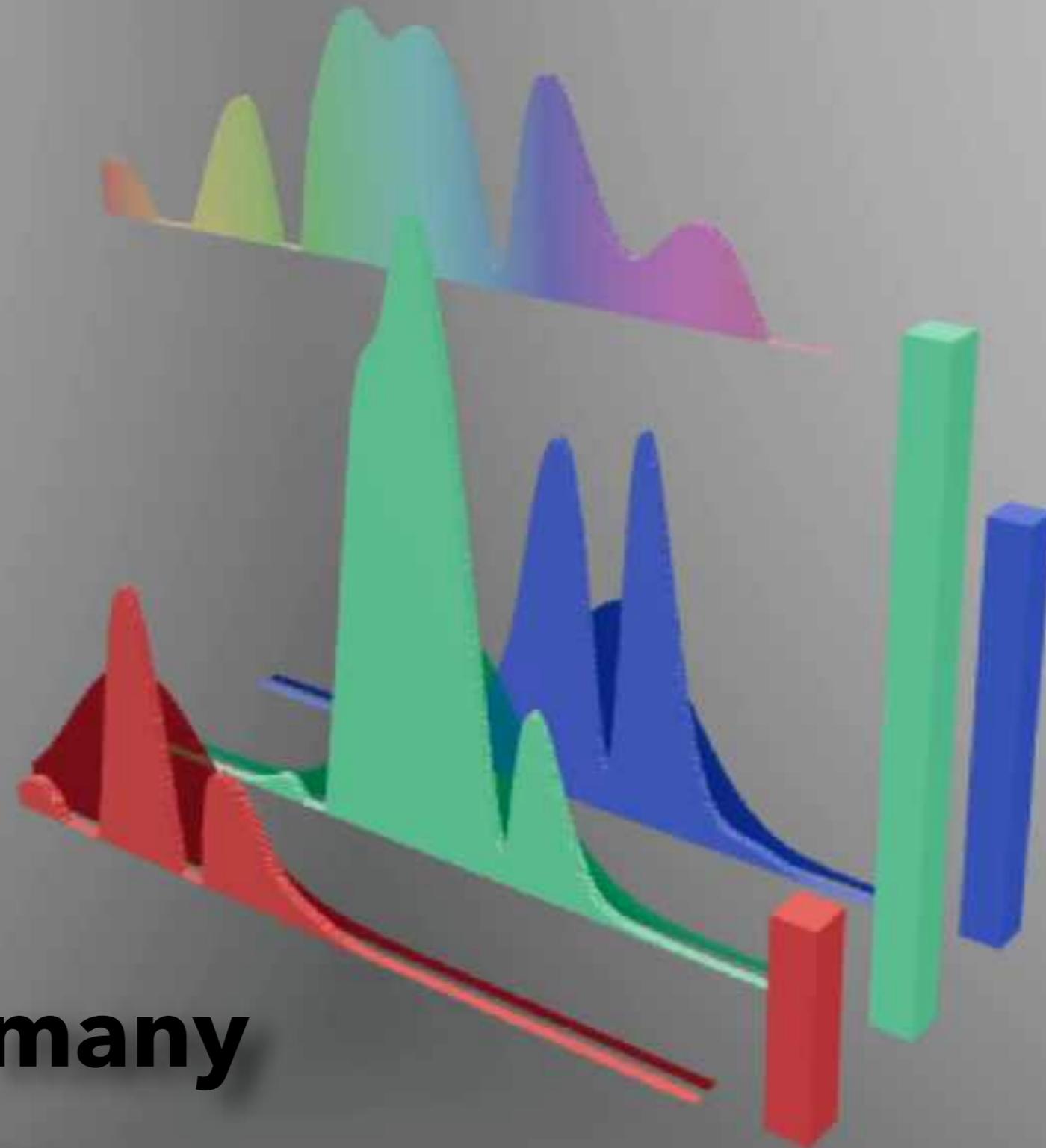
Metamer:

Two spectra are **metamers** if they look the same to the eye.



Two spectra are metamers if **their difference is in the null space of**





**Infinitely many
metamers:**

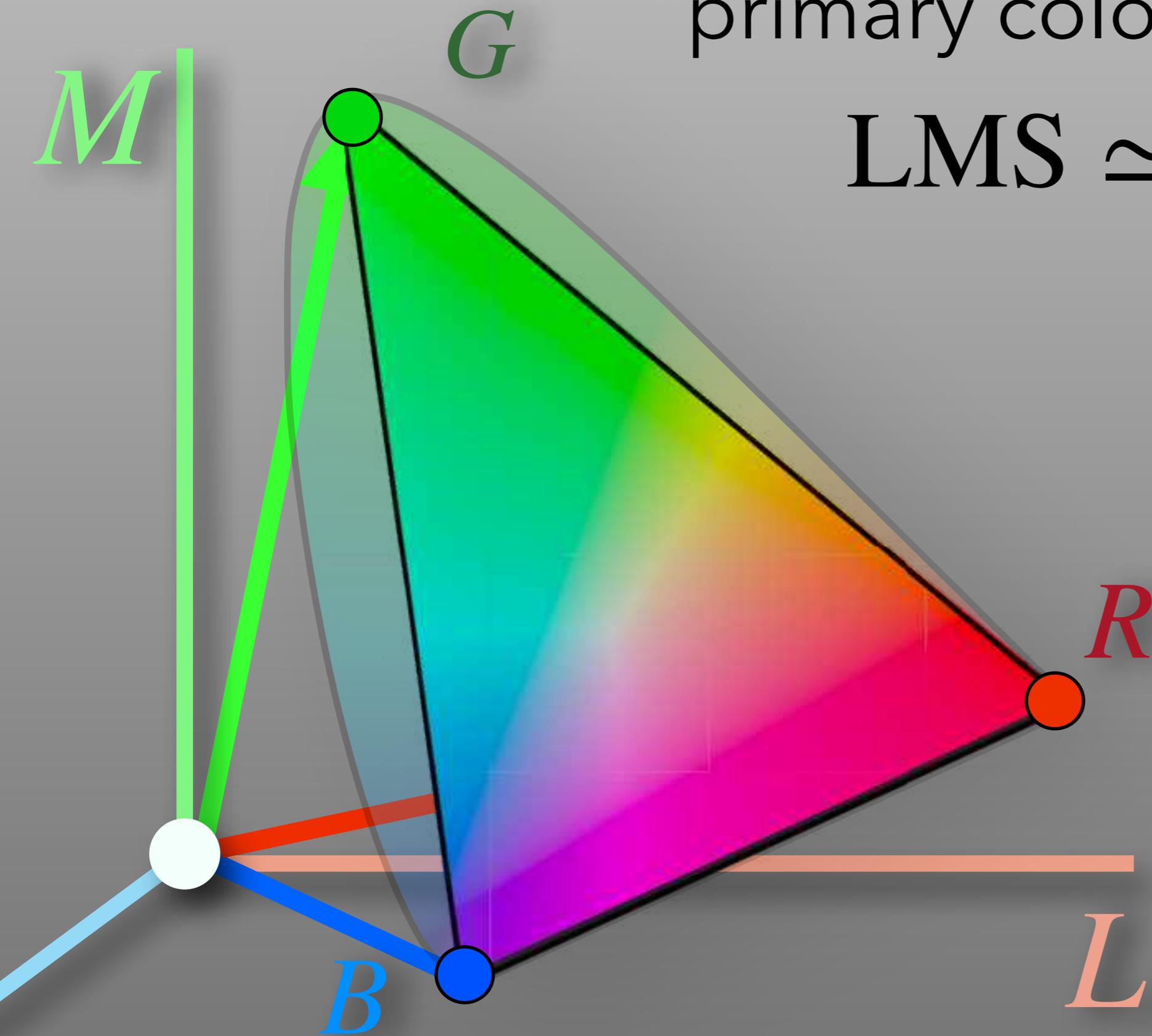


IDEA!

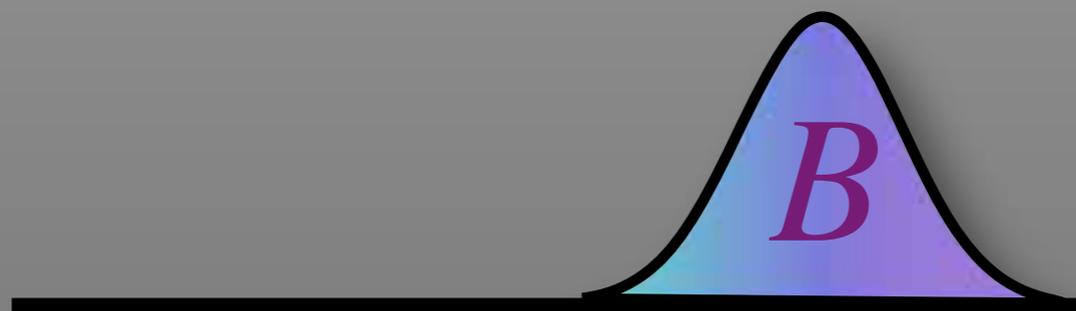
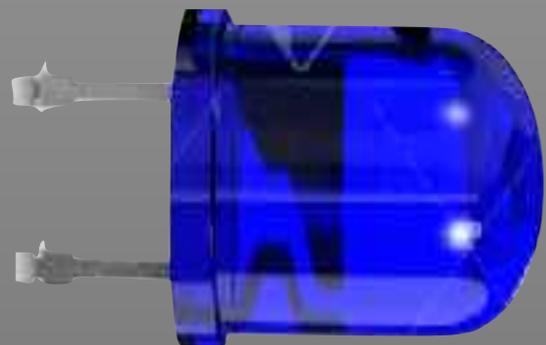
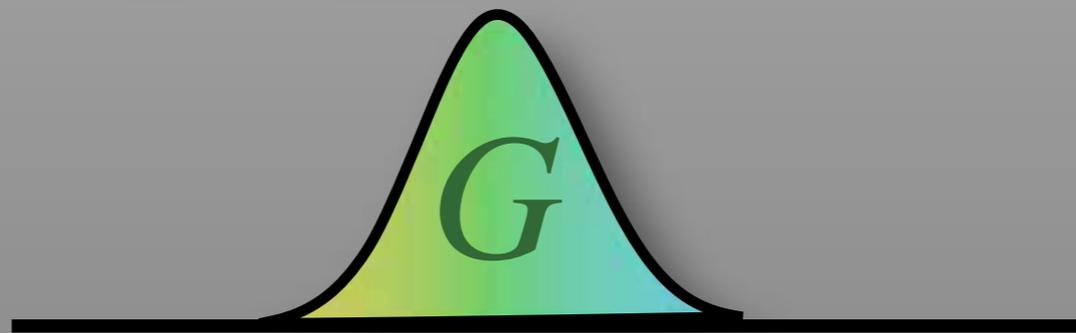
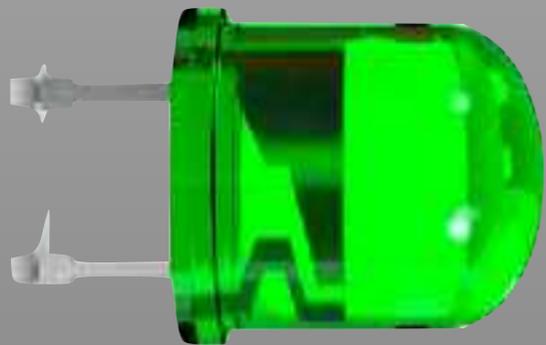
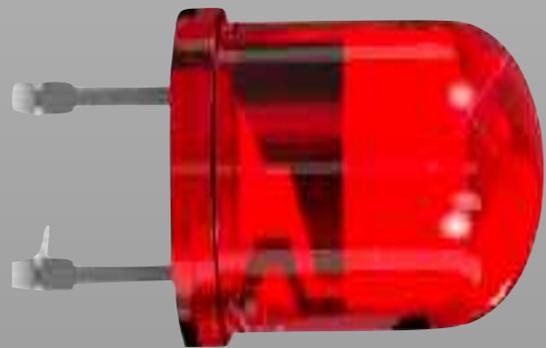
Use this imprecision to our
(technological) advantage

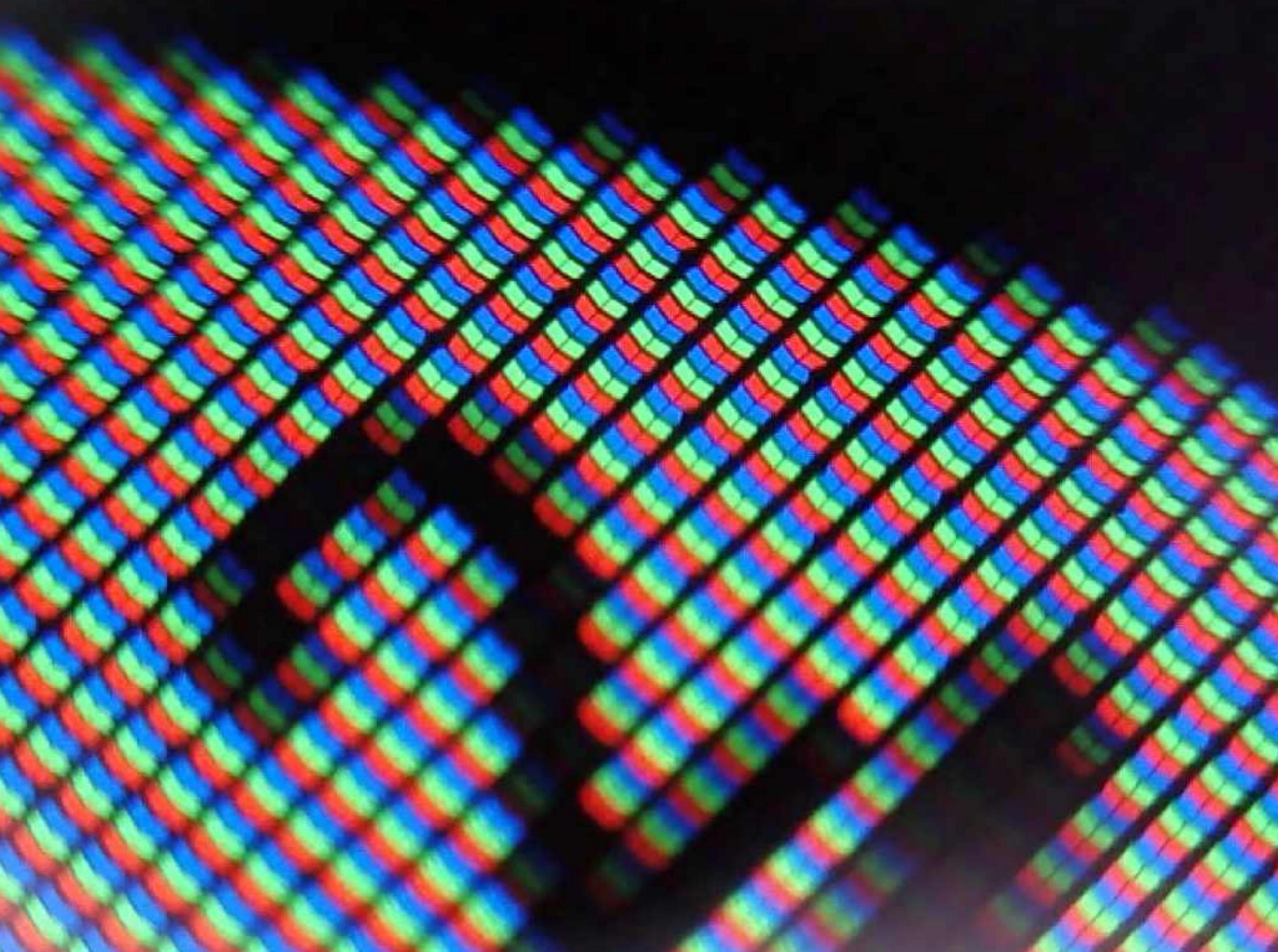
We can choose three
"primary colors" since

$$\text{LMS} \simeq \mathbb{R}^3$$



Choose lights producing these LMS values:
combinations of these produce (almost) all colors!





Big Idea #III

Other Color Spaces

Exist

Photo:
Midjourney AI

Monochromats: Color Space $\cong \mathbb{R}$



Owl Monkey



Cetaceans

Dichromats: Color Space $\cong \mathbb{R}^2$

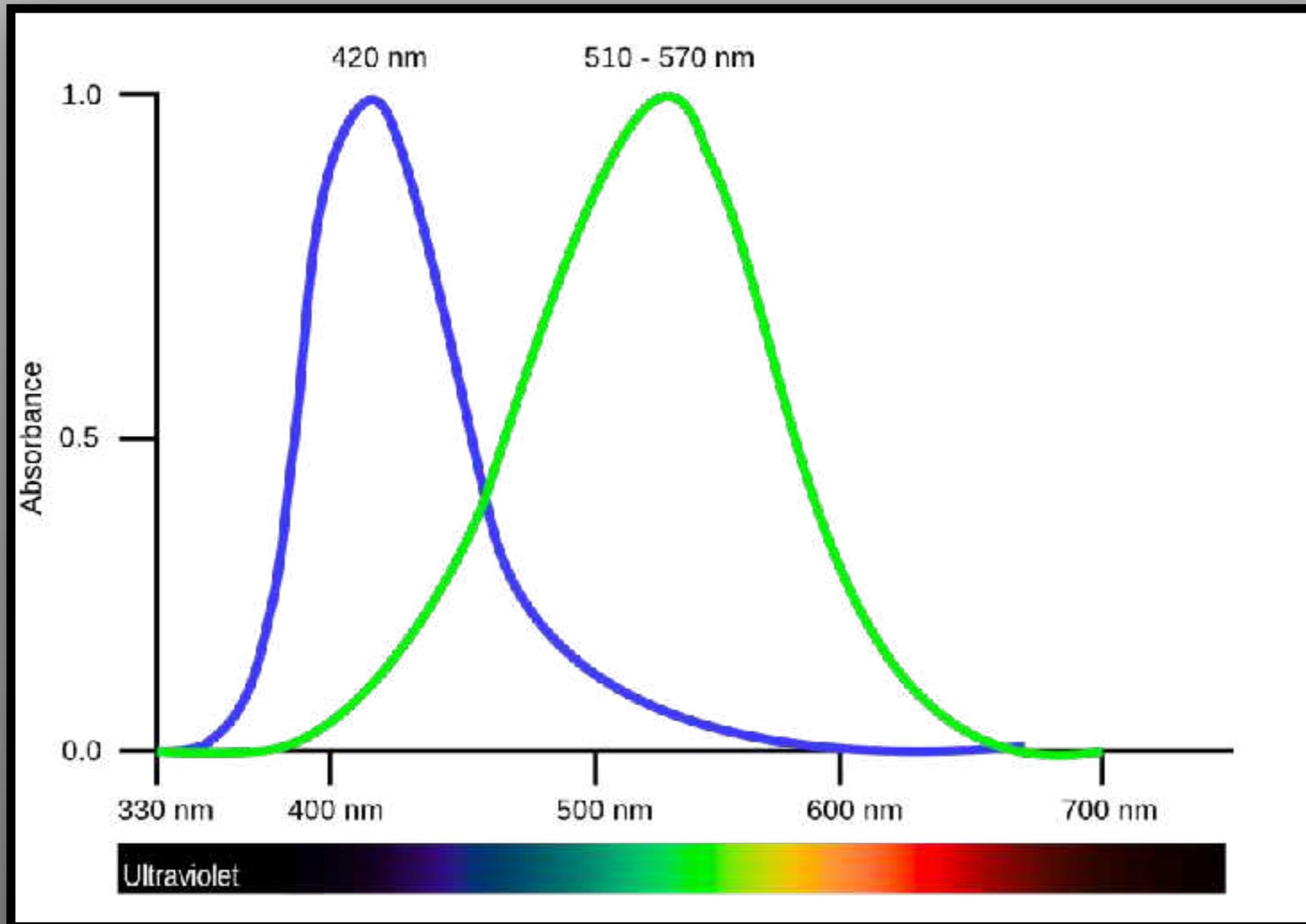


Dogs



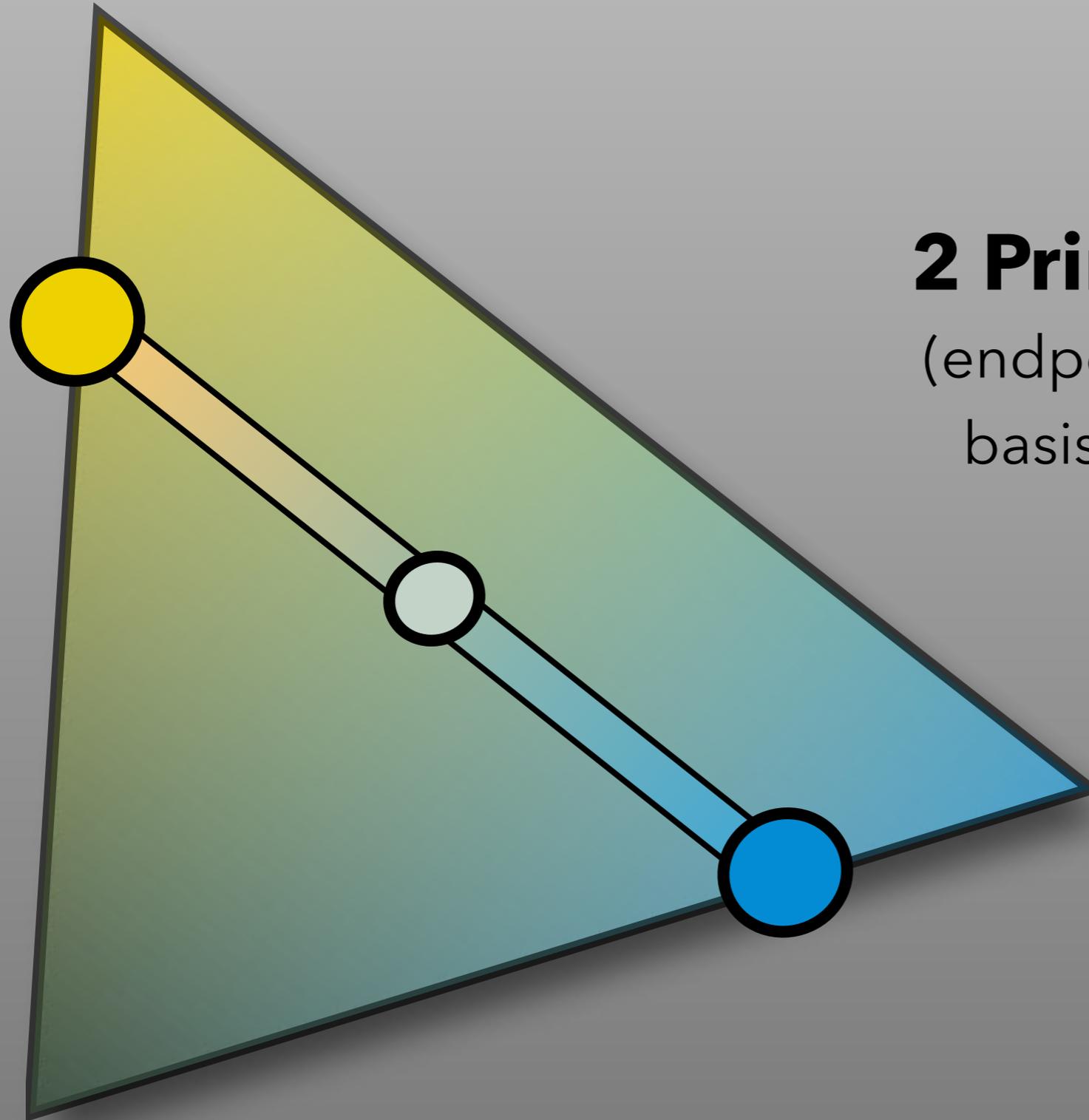
Most Mammals

Dichromats: Color Space $\cong \mathbb{R}^2$



Color space is a cone in 2D.

Cross section is an interval of hues.



2 Primary colors

(endpoints of interval:
basis vectors in \mathbb{R}^2)

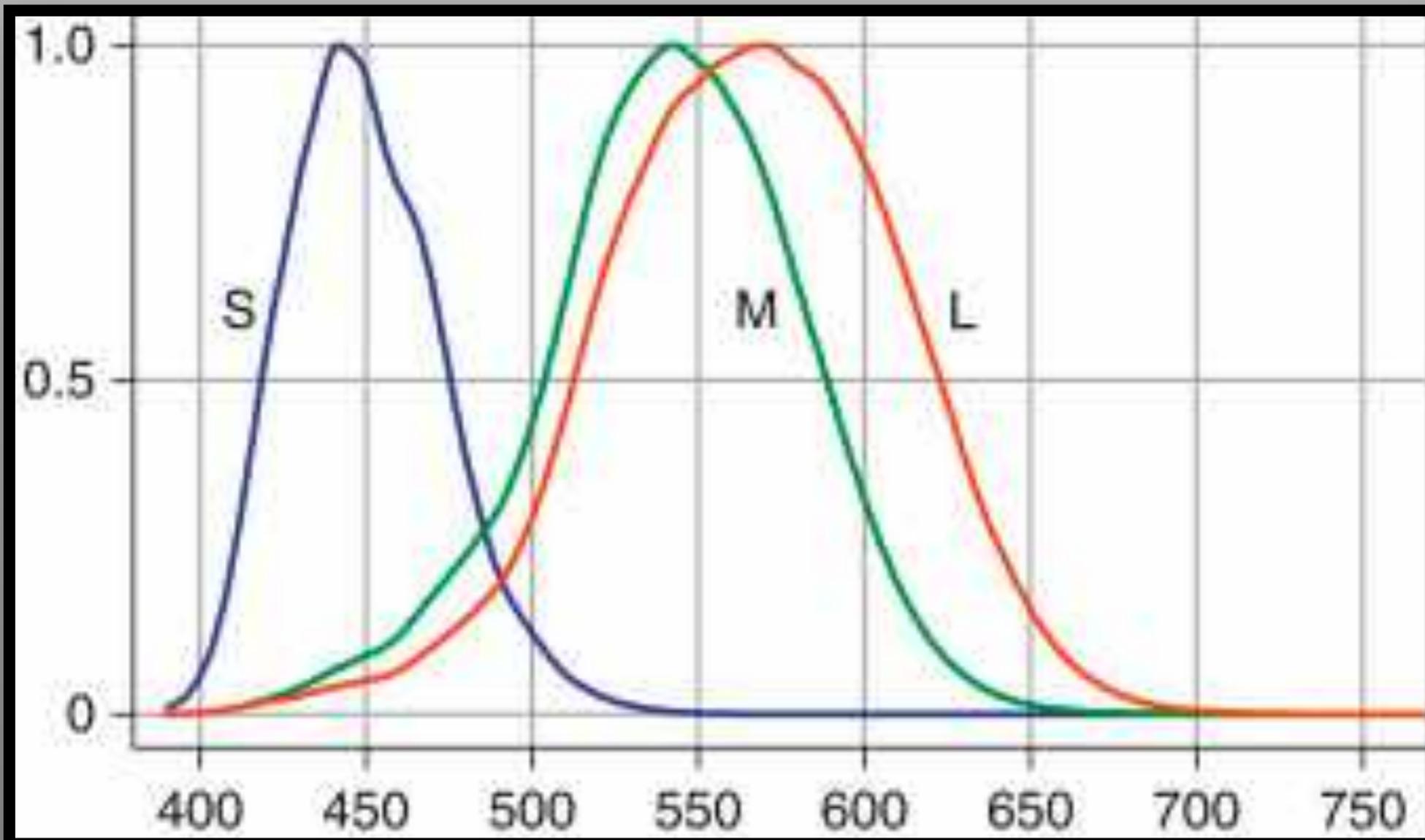
Mixing primaries
tends towards
white.

Trichromats: Color Space $\cong \mathbb{R}^3$



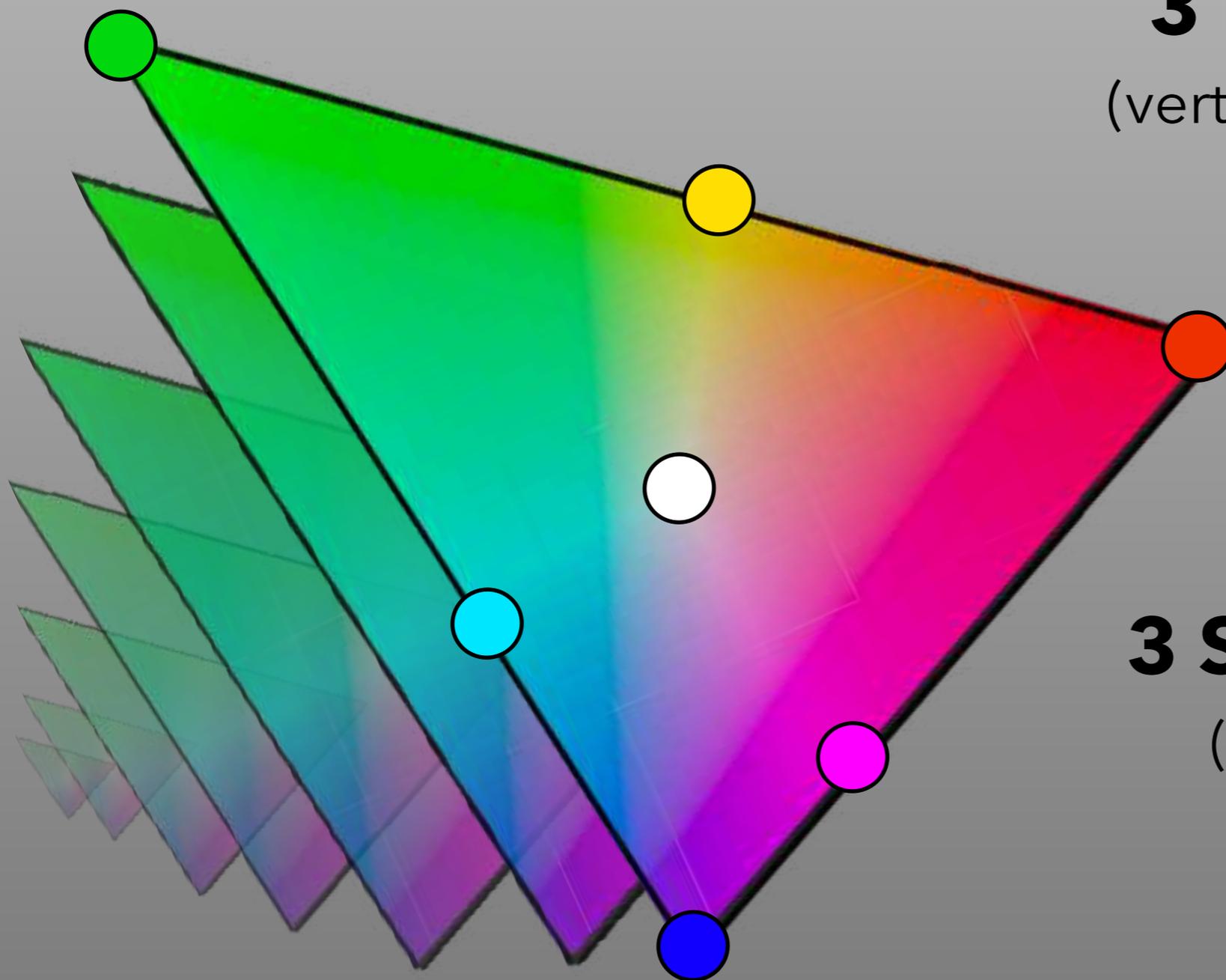
Most Primates

Trichromats: Color Space $\cong \mathbb{R}^3$



Color space is a cone in 3D.

Approximately triangular cross-section.



3 Primary colors

(vertices of triangle: basis vectors in \mathbb{R}^3)

3 Secondary colors

(midpoints of edges: combination of two primaries)

Tetrachromats: Color Space $\cong \mathbb{R}^4$

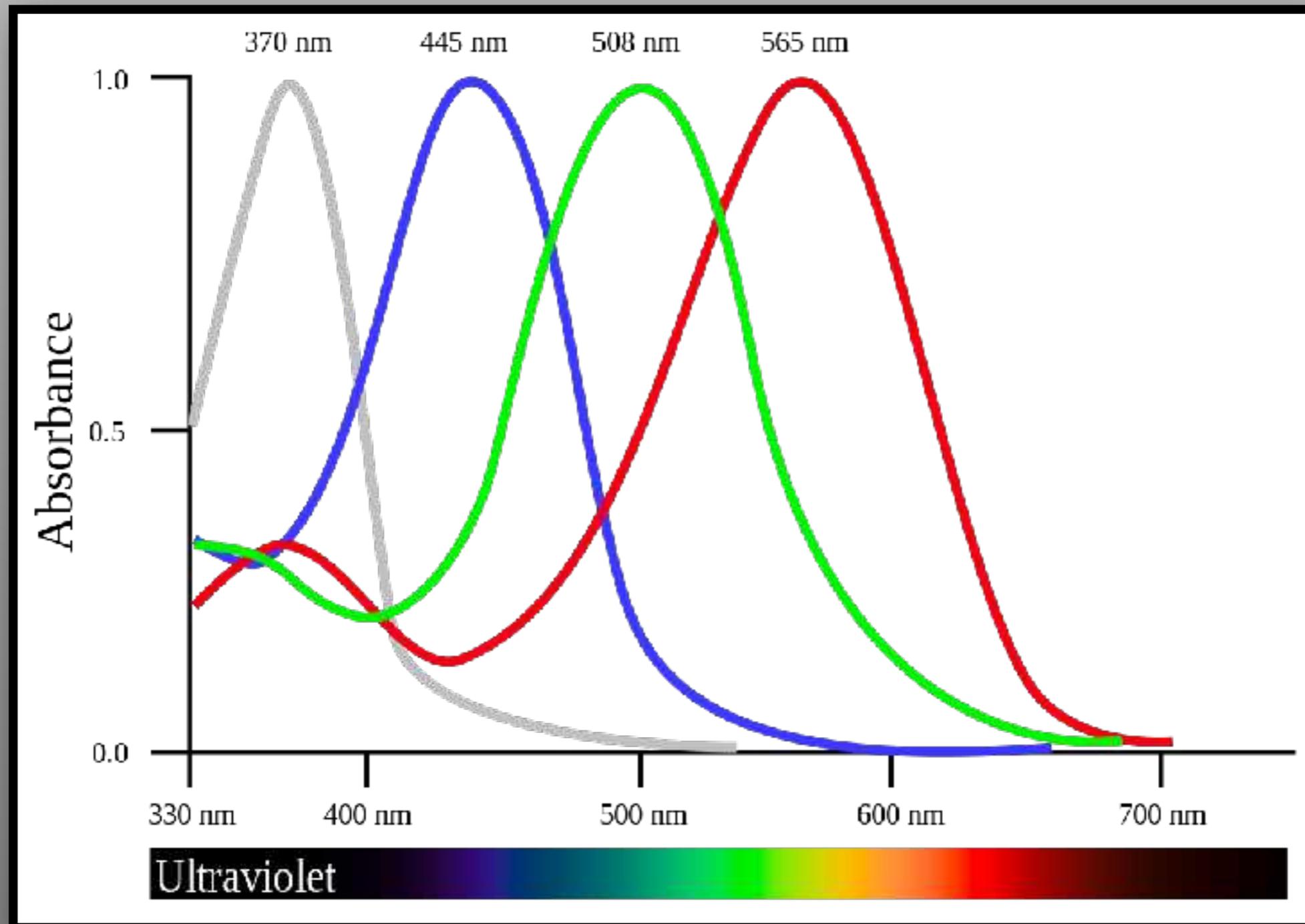


Fish



Reptiles

Tetrachromats: Color Space $\cong \mathbb{R}^4$



Color space is a cone in 4D.

Approximately *tetrahedral* cross-section.



Vertices of section =
4 Primary Colors

Six secondary colors from mixing two primaries

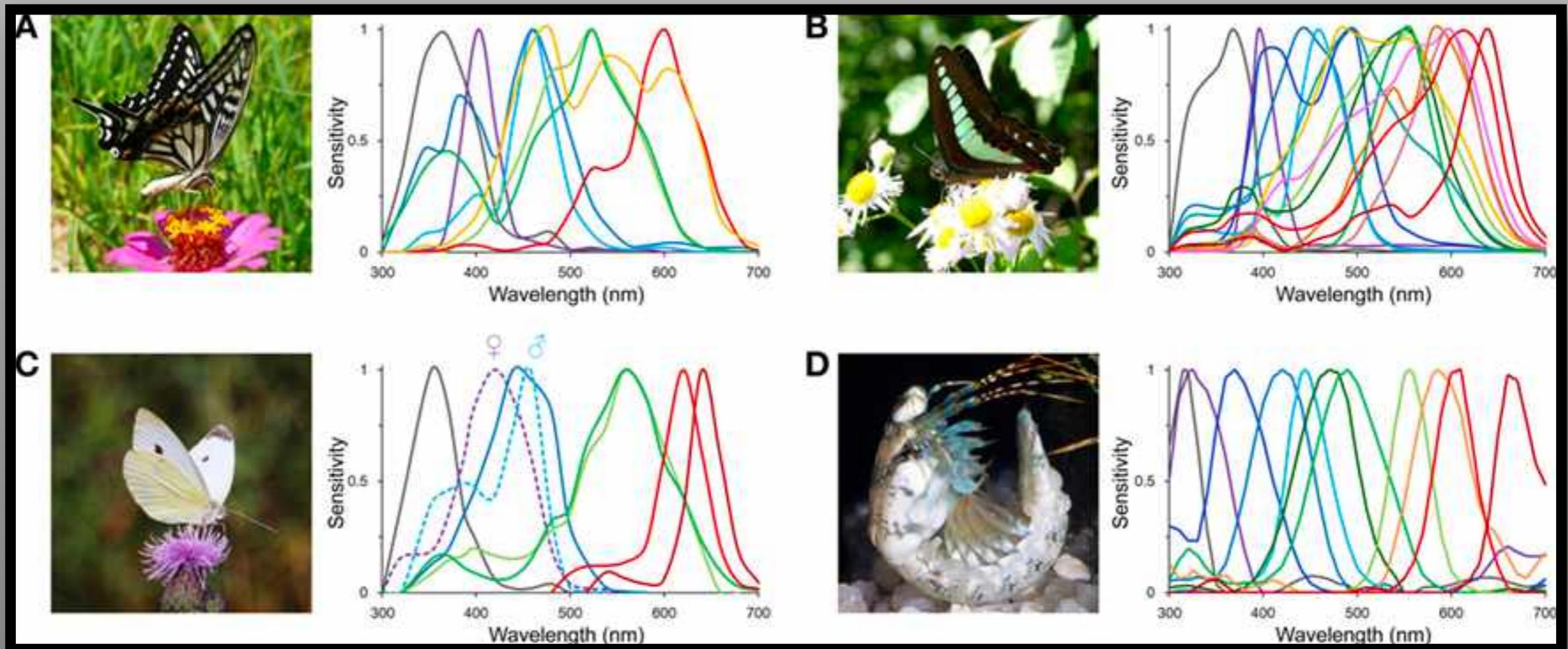
Four tertiary colors mixing three primaries at once!

N-Chromats: Color Space $\cong \mathbb{R}^n$



Butterflies

N-Chromats: Color Space $\cong \mathbb{R}^n$



**Color perception varies widely across insects:
butterflies have between 5 and 16 cones!**

Color Vision for an n-Chromat:

● n **Primary**

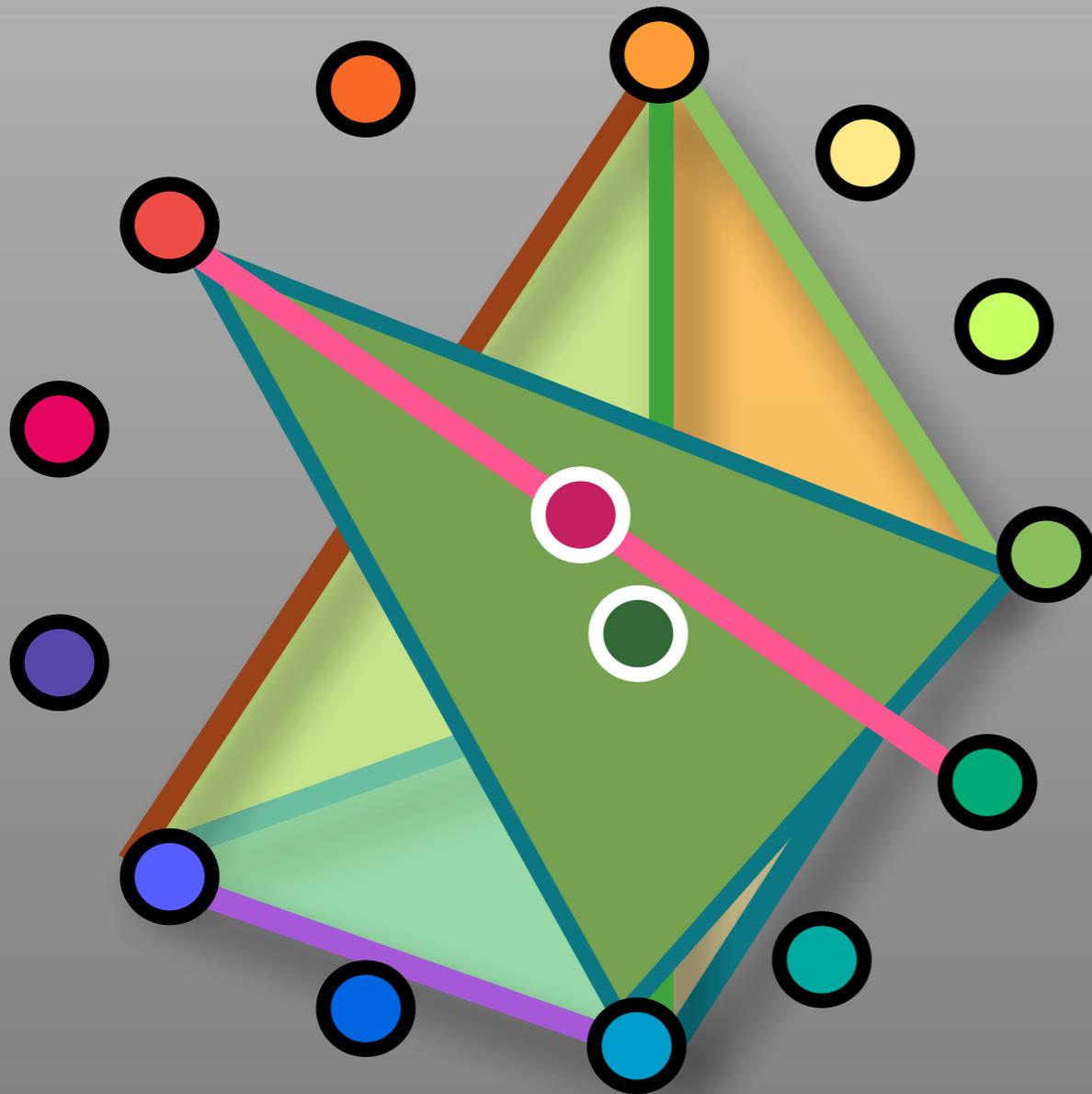
(from basis vectors in \mathbb{R}^n)

● $\binom{n}{2}$ **Secondary**

● $\binom{n}{3}$ **Tertiary** *How many secondary colors?*

● $\binom{n}{4}$ **Tetrinary**

●
●
●



Color Vision for an n-Chromat:



Simulation:
Jason Heise

This space is called the $(n-1)$ Simplex.

A higher-dimensional analog of the tetrahedron.

This seems familiar...

Dichromats

2

Trichromats

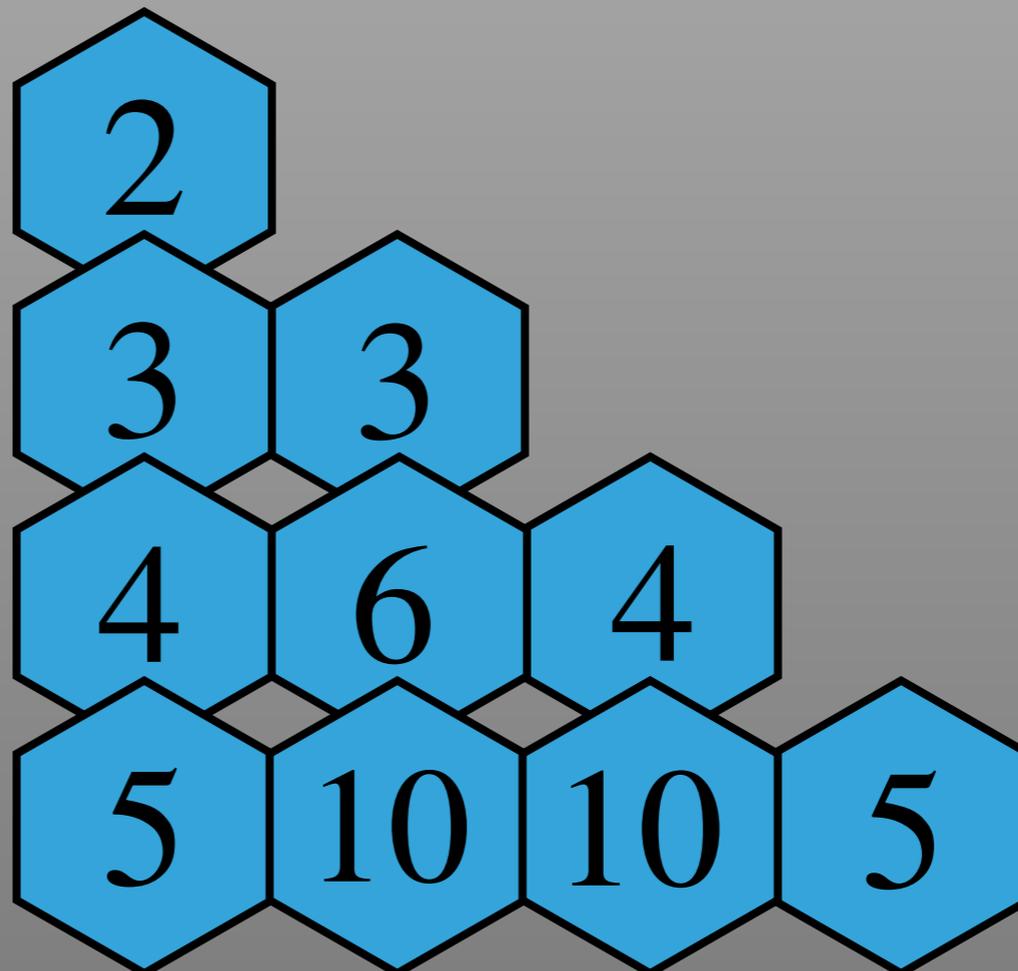
3 3

Tetrachromats

4 6 4

Pentachromats

5 10 10 5



This seems familiar...

ابو بكر محمد بن
الحسن الكرجى
(Al-Karaji)
953

عمر خیام
(Omar Khayyam)
1048

杨辉
(Yang Hui)
1275

Pascal
1623

