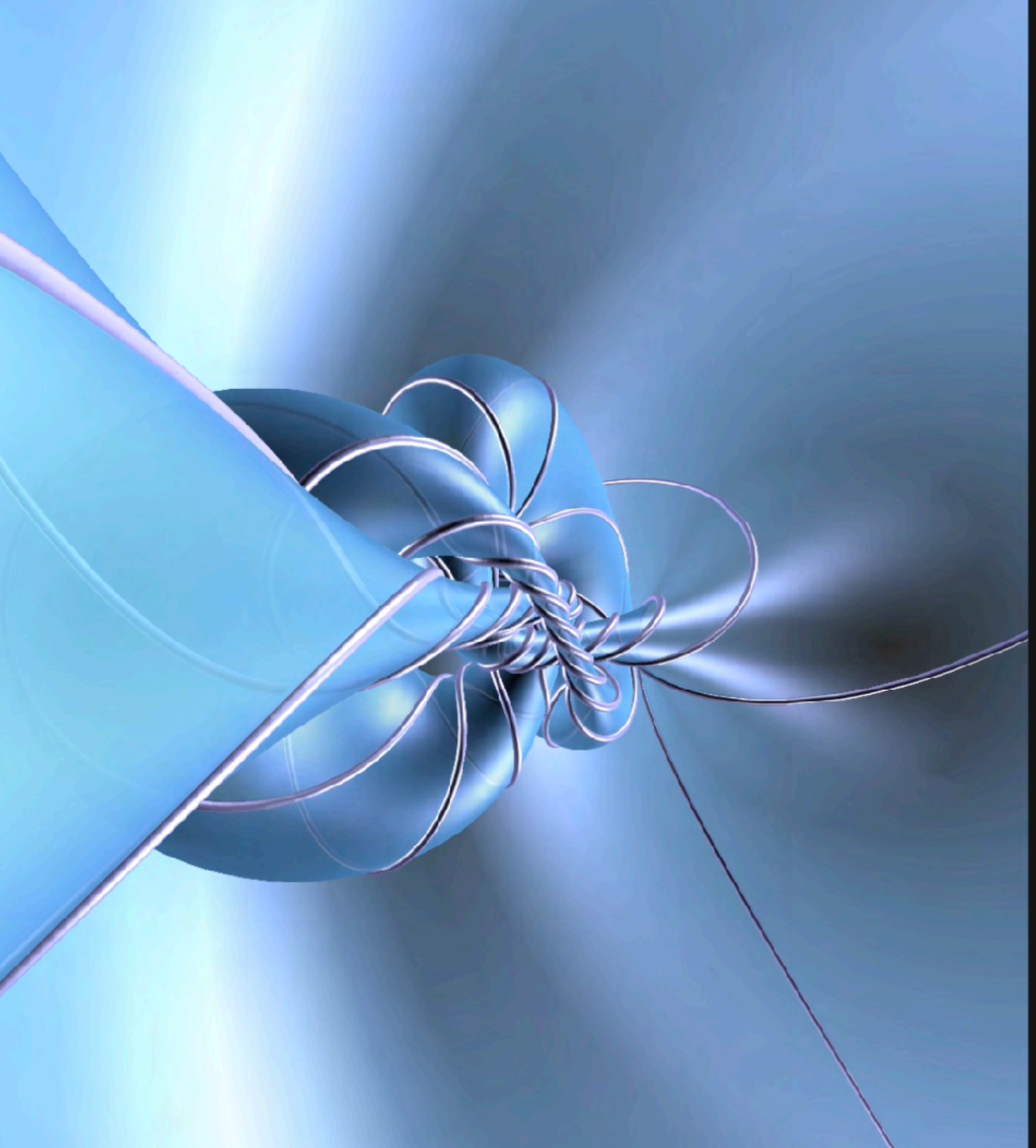
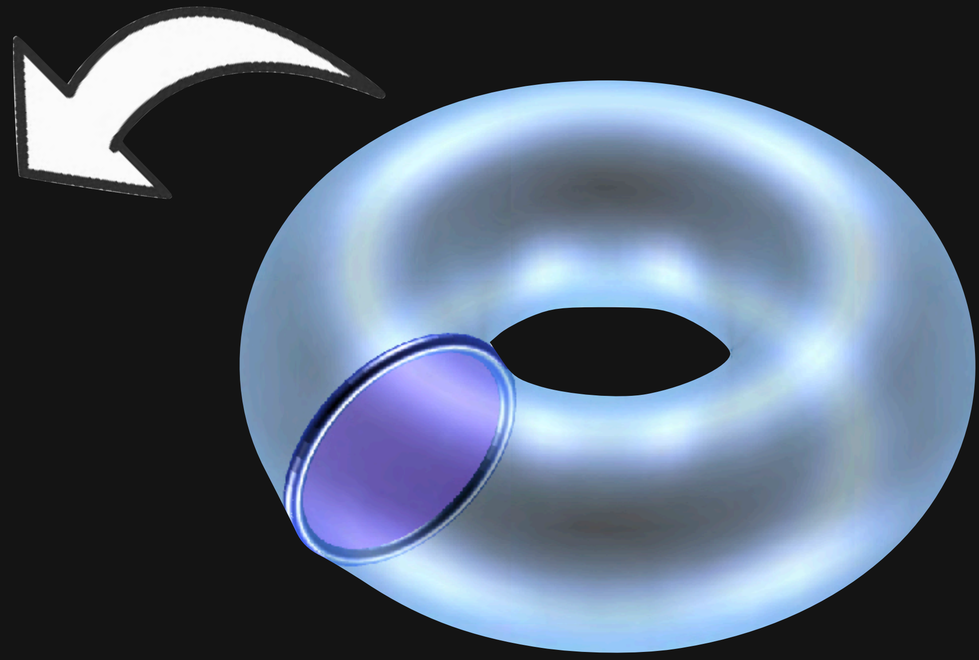


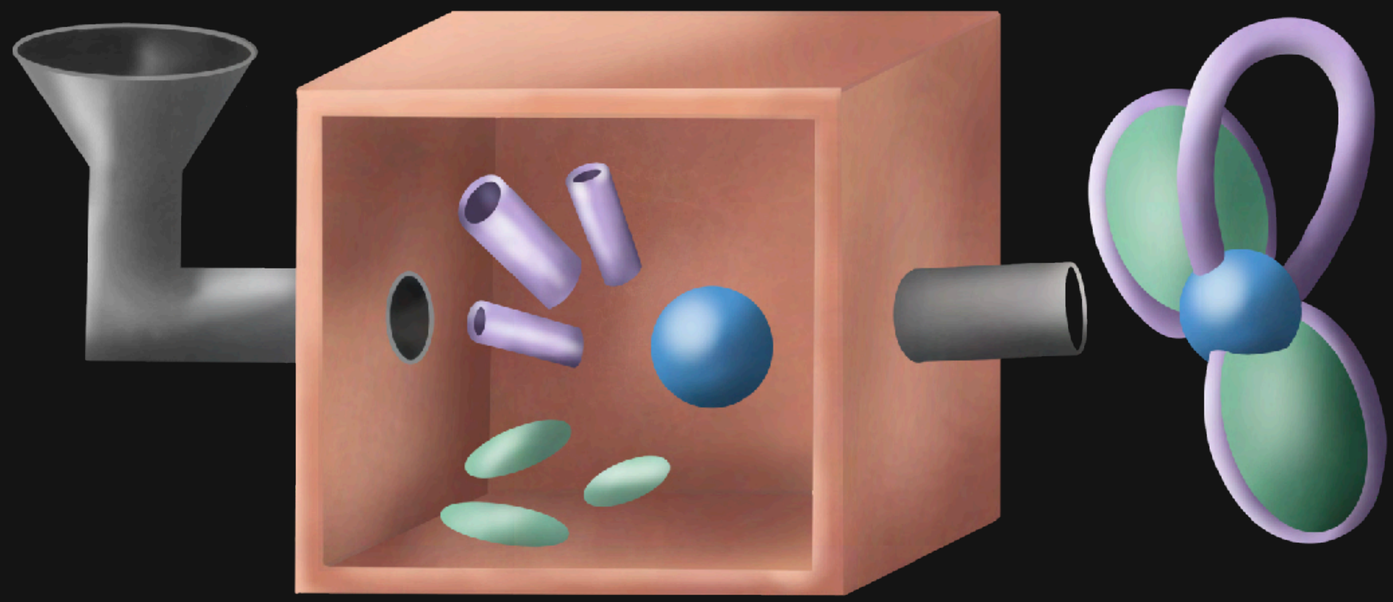


Some Groups Whose Boundaries Are Spheres

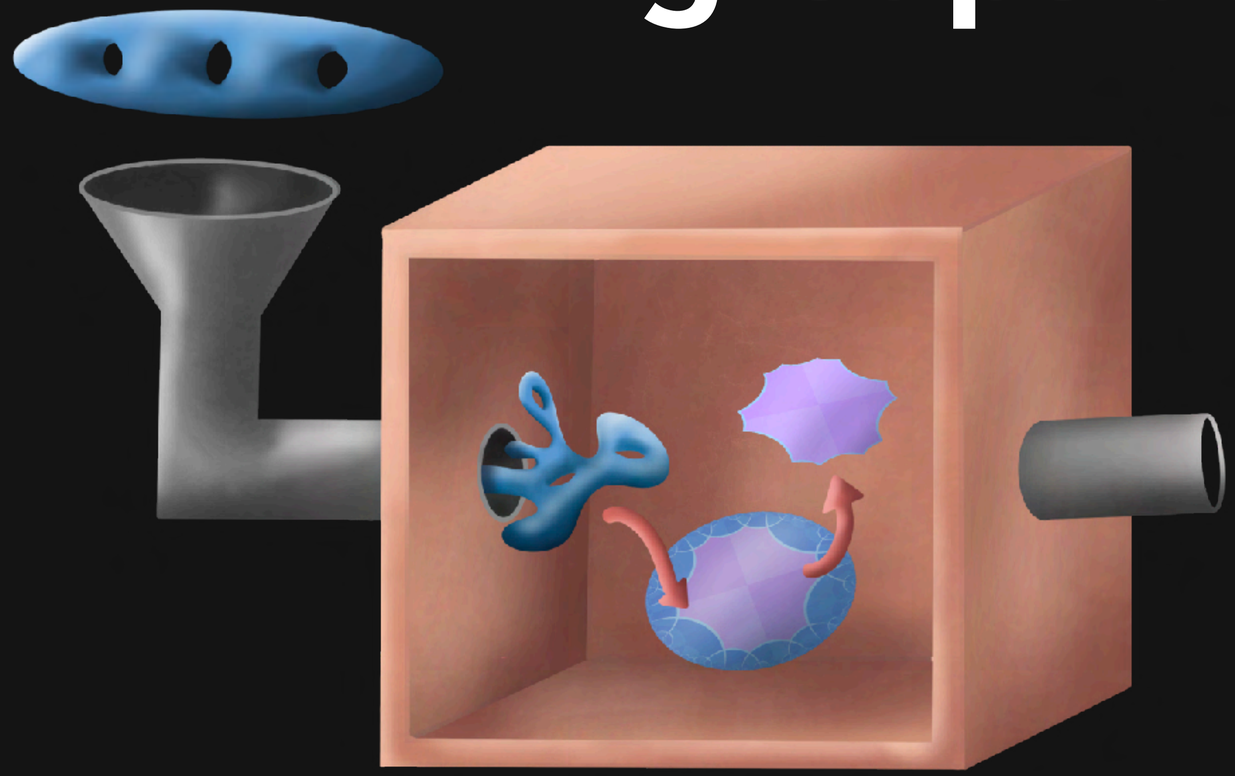
(Or....an excuse to talk about hyperbolic Dehn surgery)

Geometrizing a group to study it.

$\langle S \mid R \rangle$



Producing examples of groups using geometry.

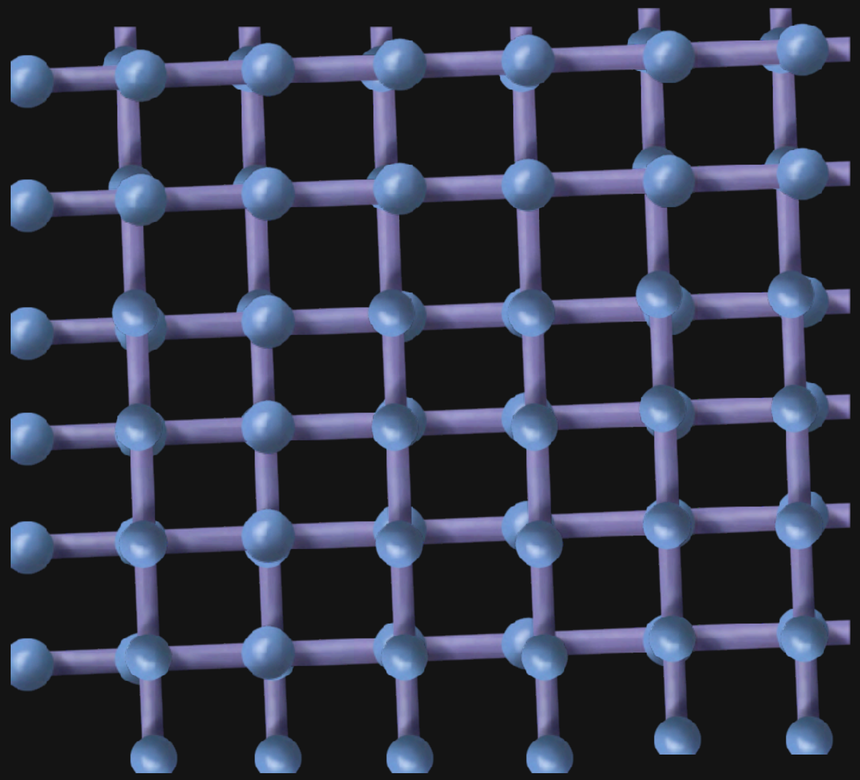


$\langle S \mid R \rangle$

Milnor-Švac

If a group G acts properly discontinuously and cocompactly on a space X, then G is quasi-isometric to X.

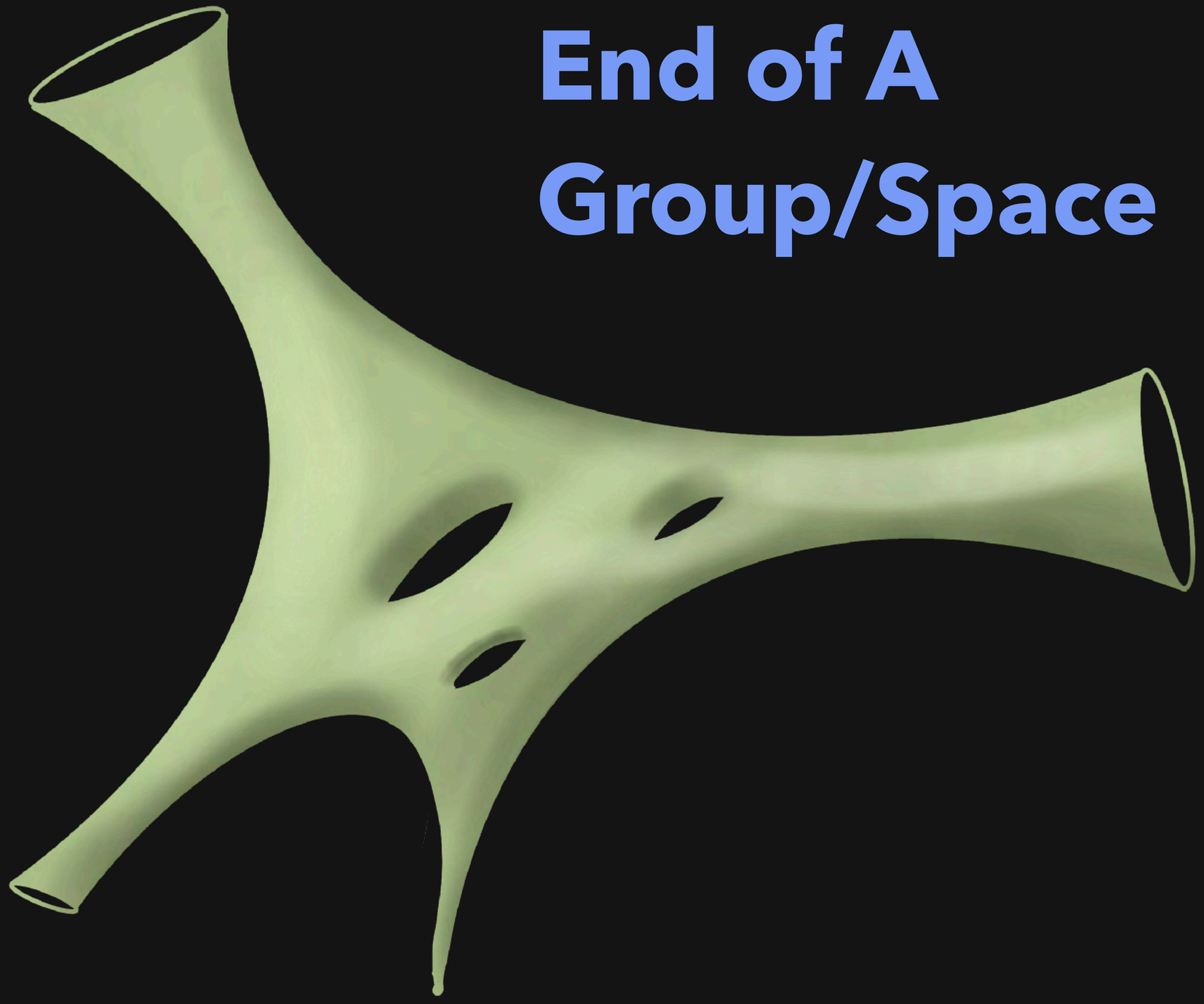
Quasi-Isometry



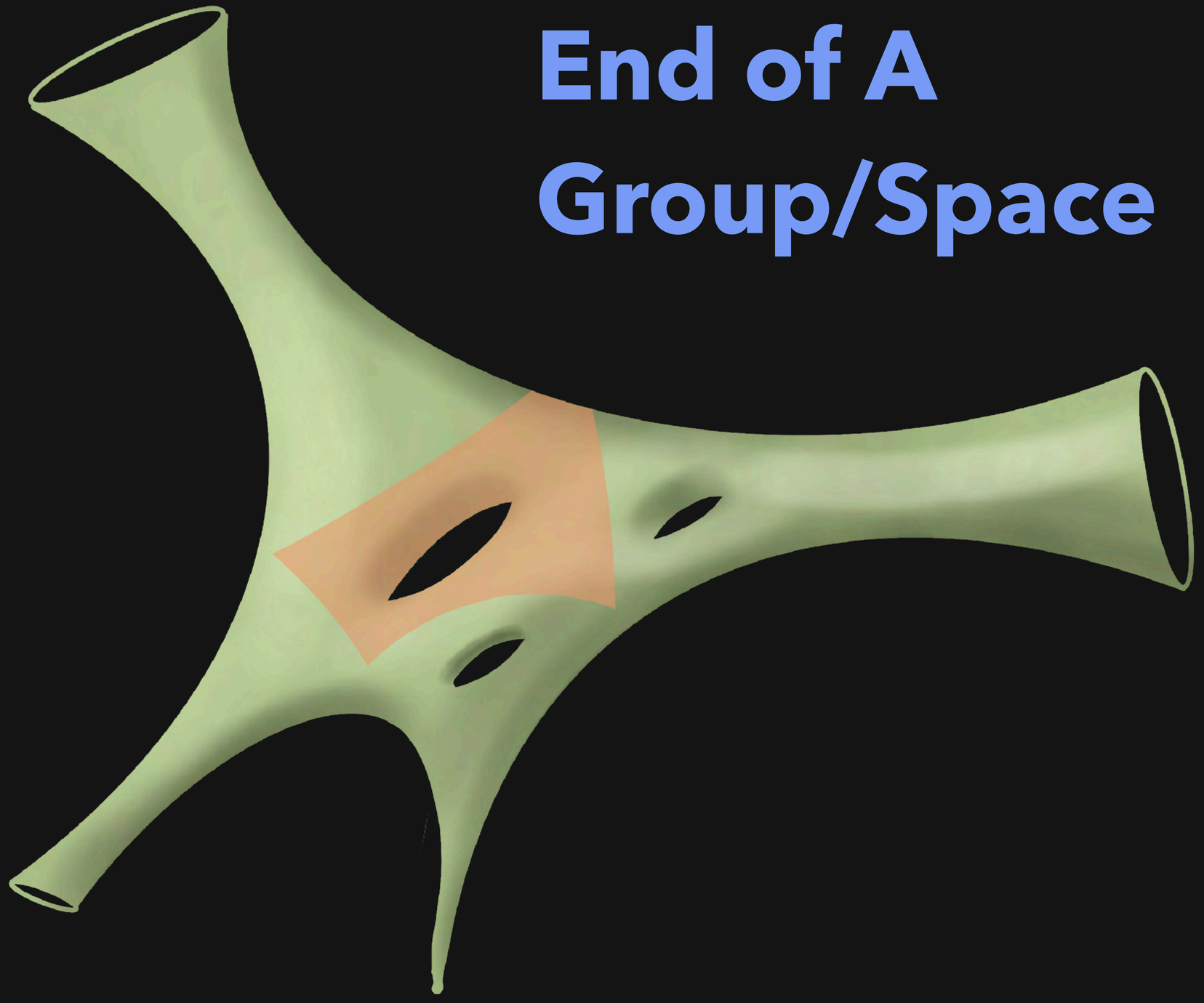
Our Goal Today:

Use Geometry to Produce groups with specified large-scale behavior

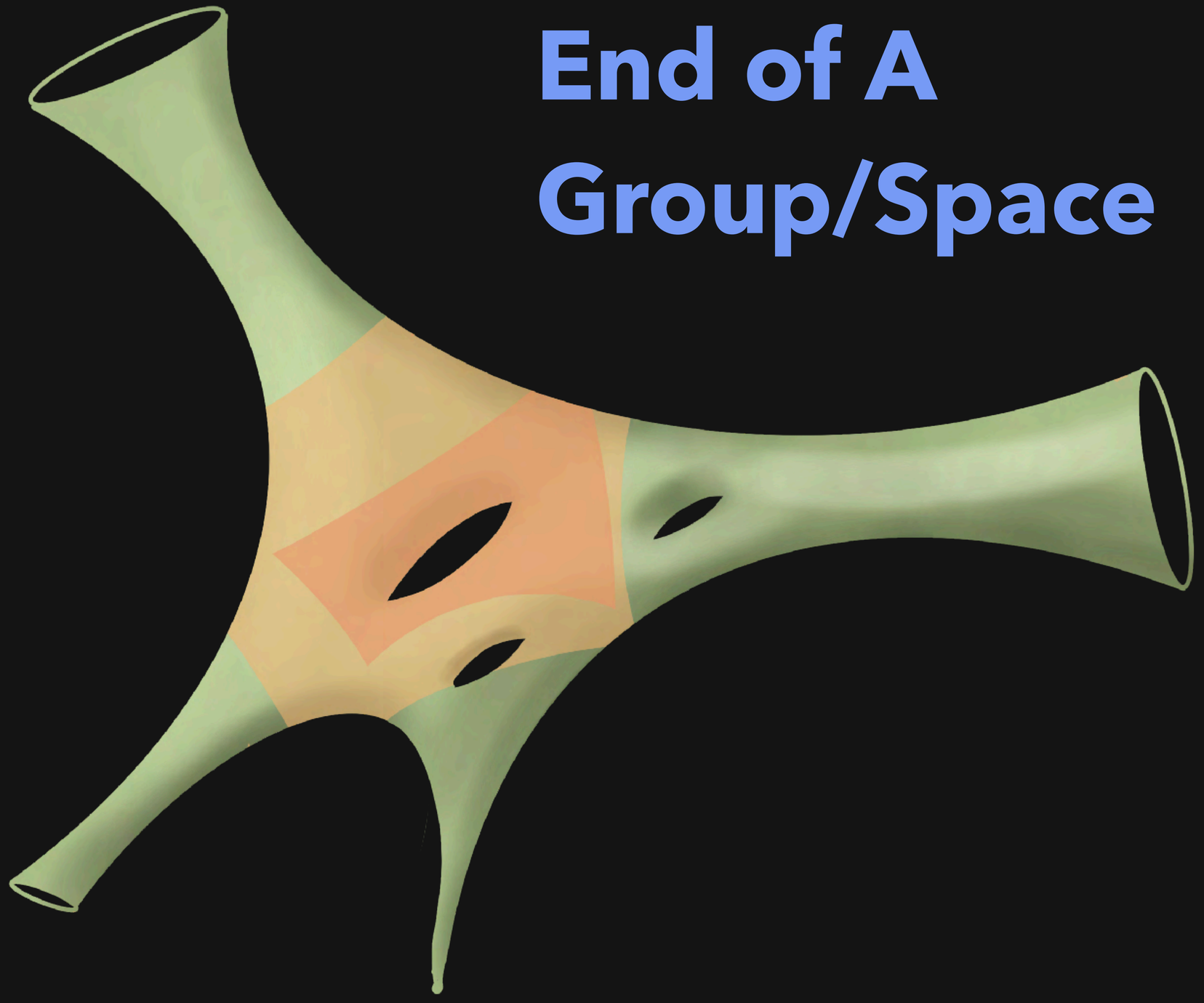
End of A Group/Space



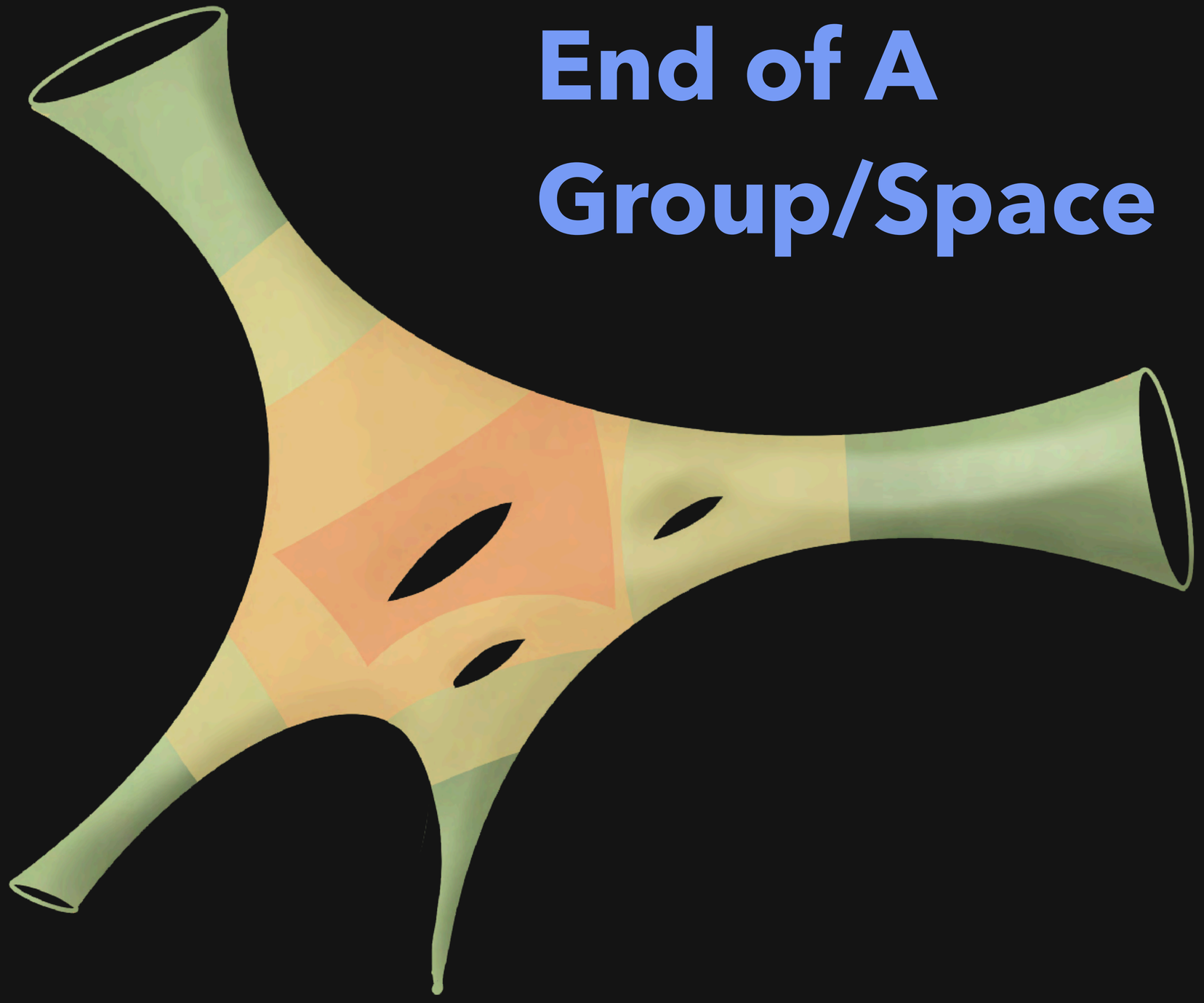
End of A Group/Space



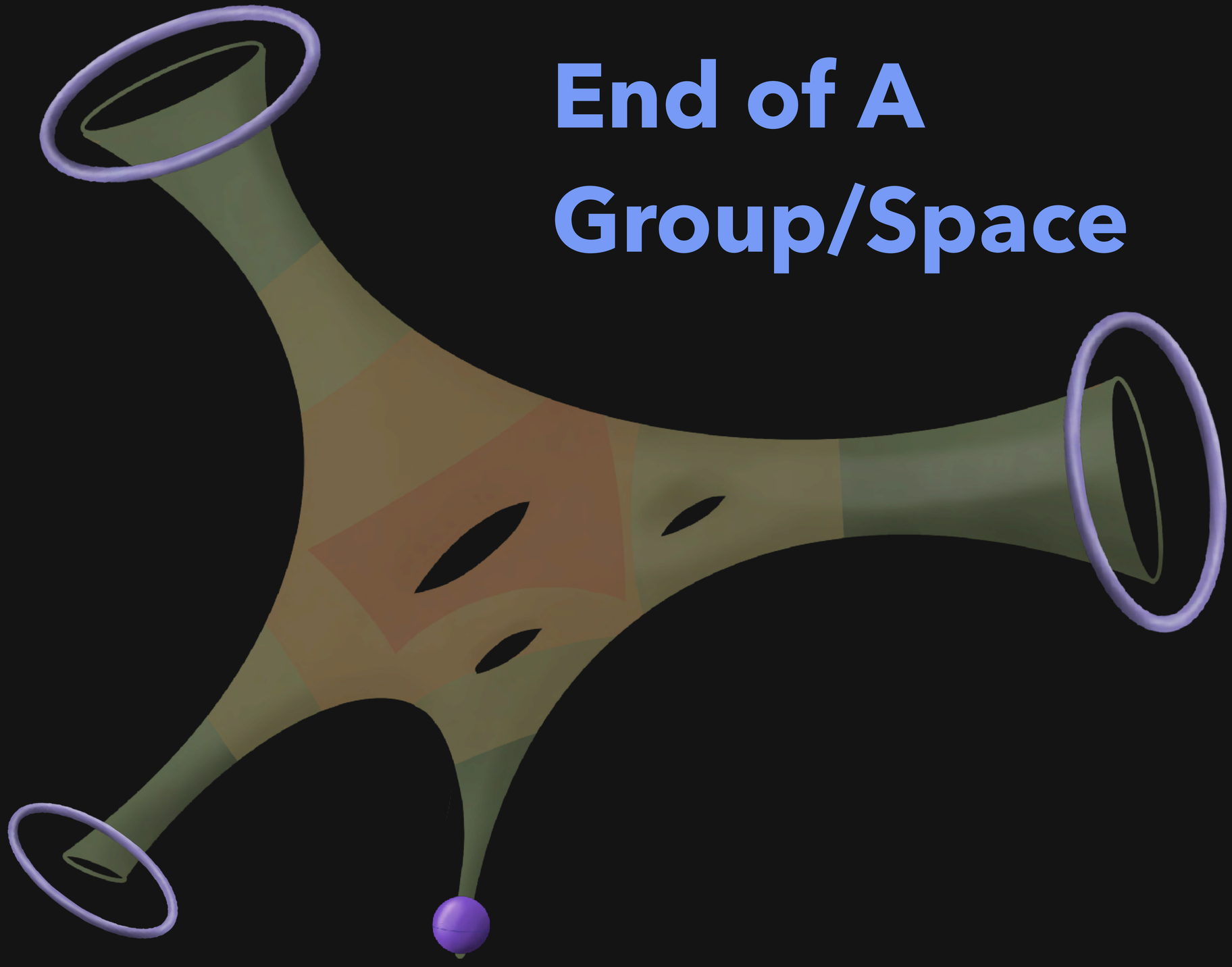
End of A Group/Space



End of A Group/Space



End of A Group/Space

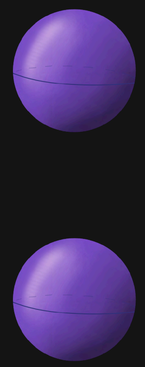


End of A Group/Space

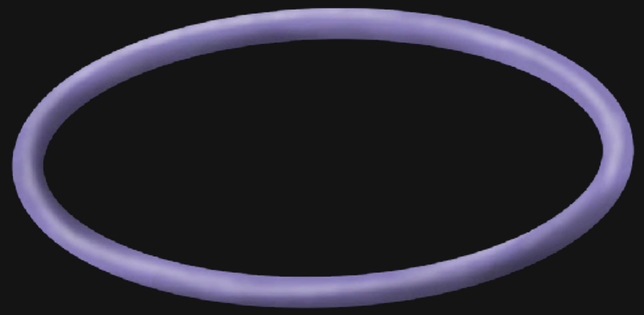
Ends are a QI - Invariant

If two spaces are quasi-isometric, their ends are homeomorphic.

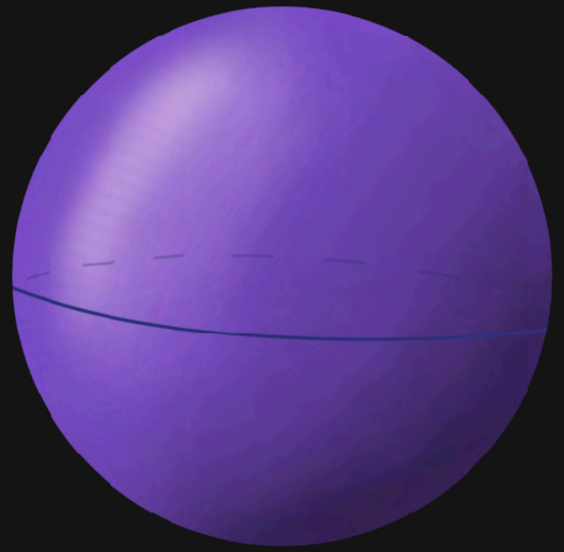
Building Groups with prescribed ends:



$\mathbb{S}^0$



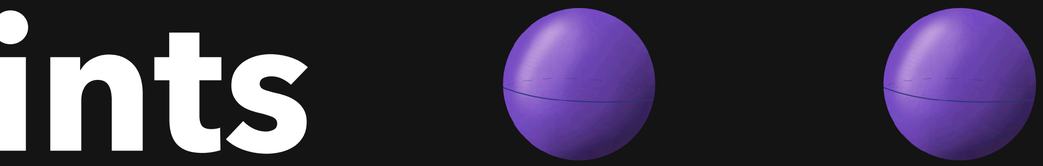
$\mathbb{S}^1$



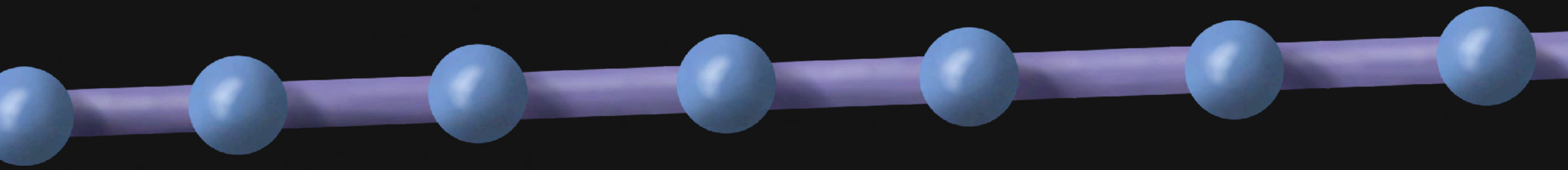
$\mathbb{S}^2$

Groups with boundary

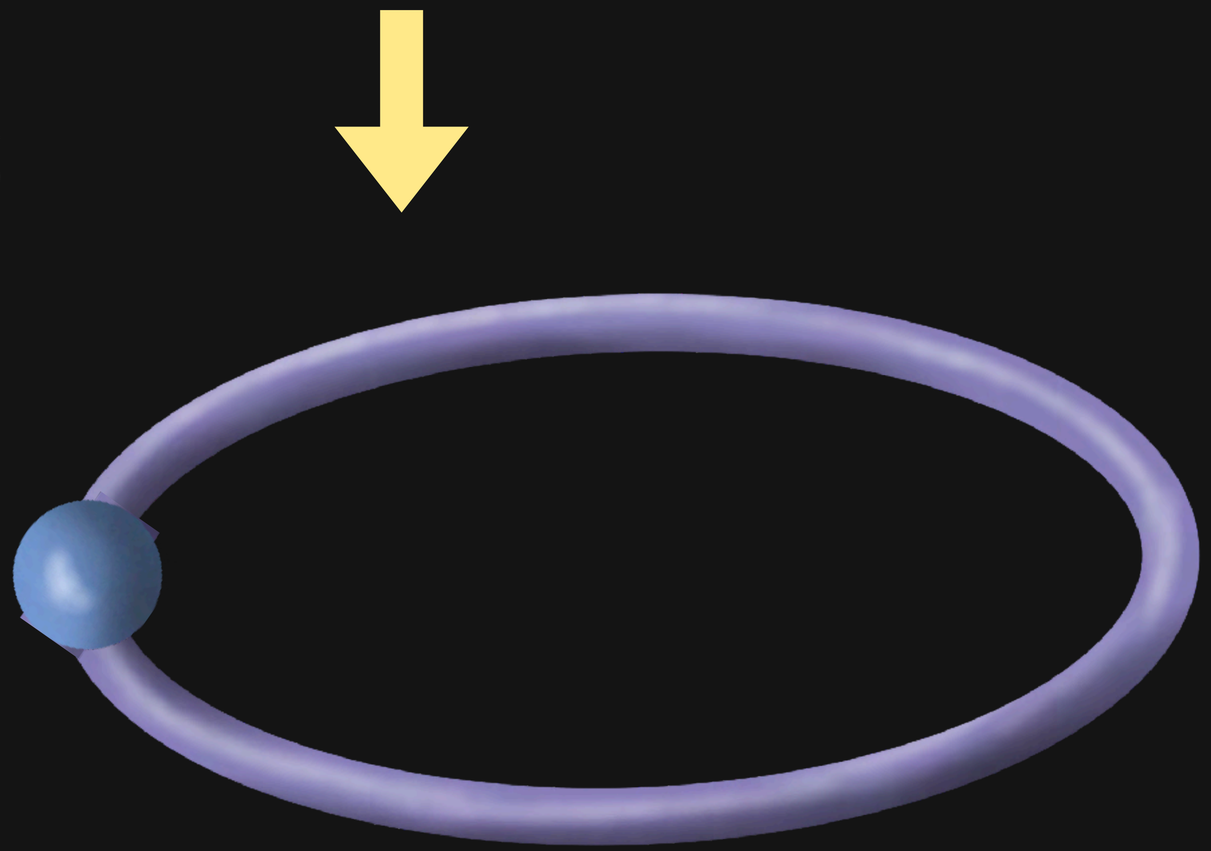
The 0-Dimensional Sphere

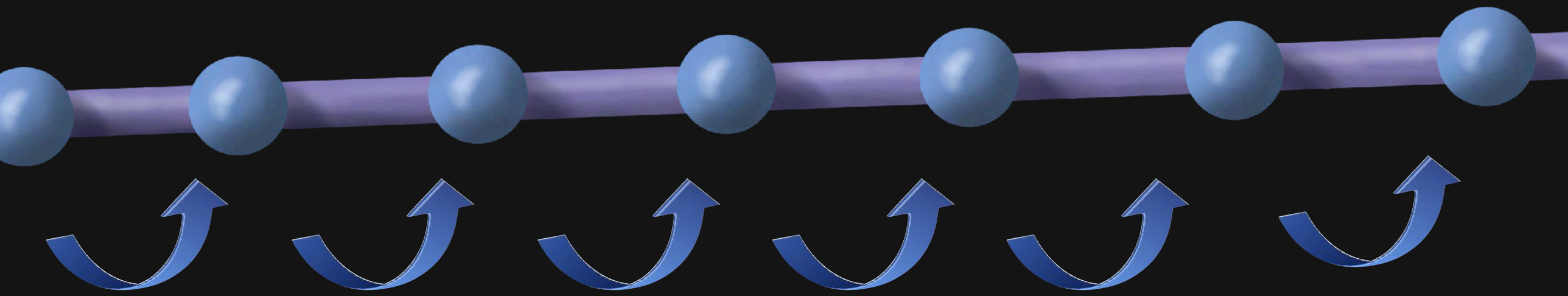
$\mathbb{S}^0$ is two points 

How can we construct a group with two ends?



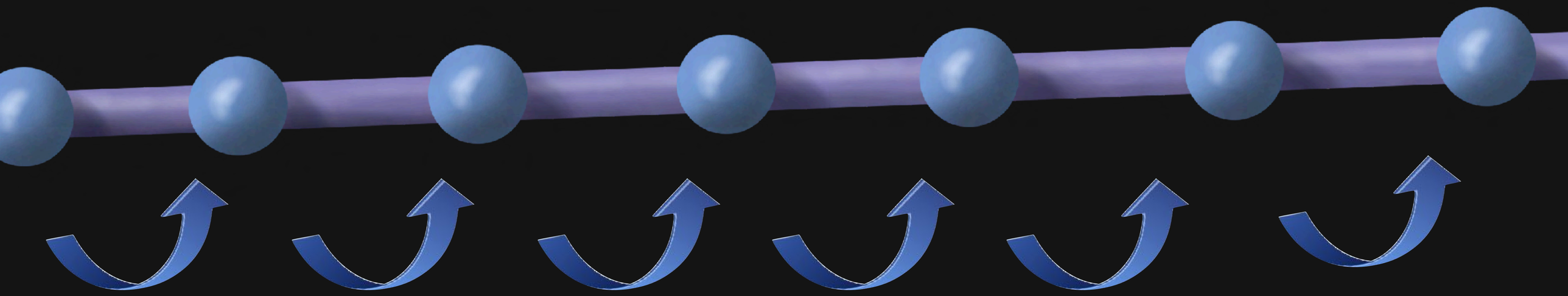
The circle's universal cover is a line





$$\pi_1(\mathbb{S}^1) \cong \mathbb{Z}$$

The deck group acts geometrically



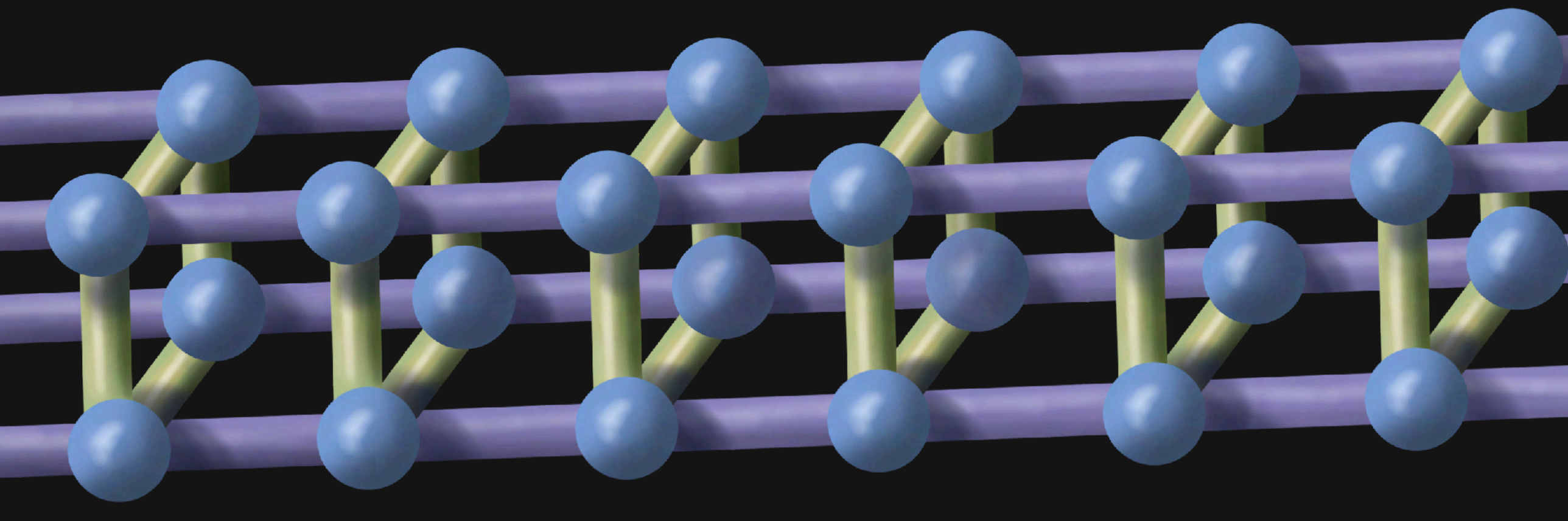
$$\pi_1(\mathbb{S}^1) \cong \mathbb{Z}$$

Thus, $\mathbb{Z}$ is QI to the line, which has two ends, so $\mathbb{Z}$ has two ends.

This is virtually it:

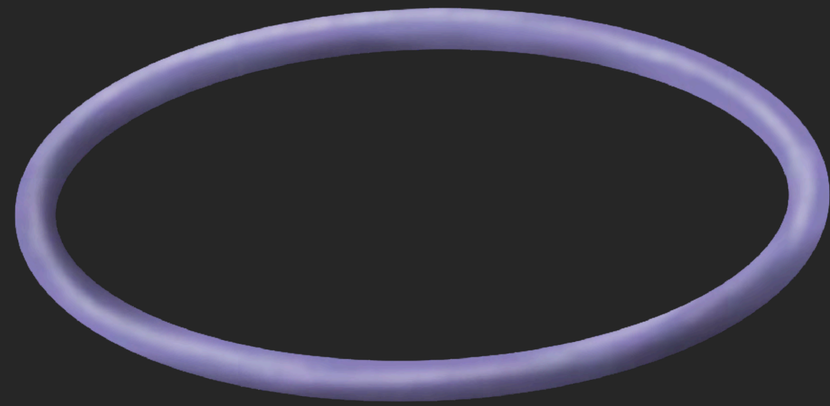
Freudenthal-Hopf

All other examples are virtually Z



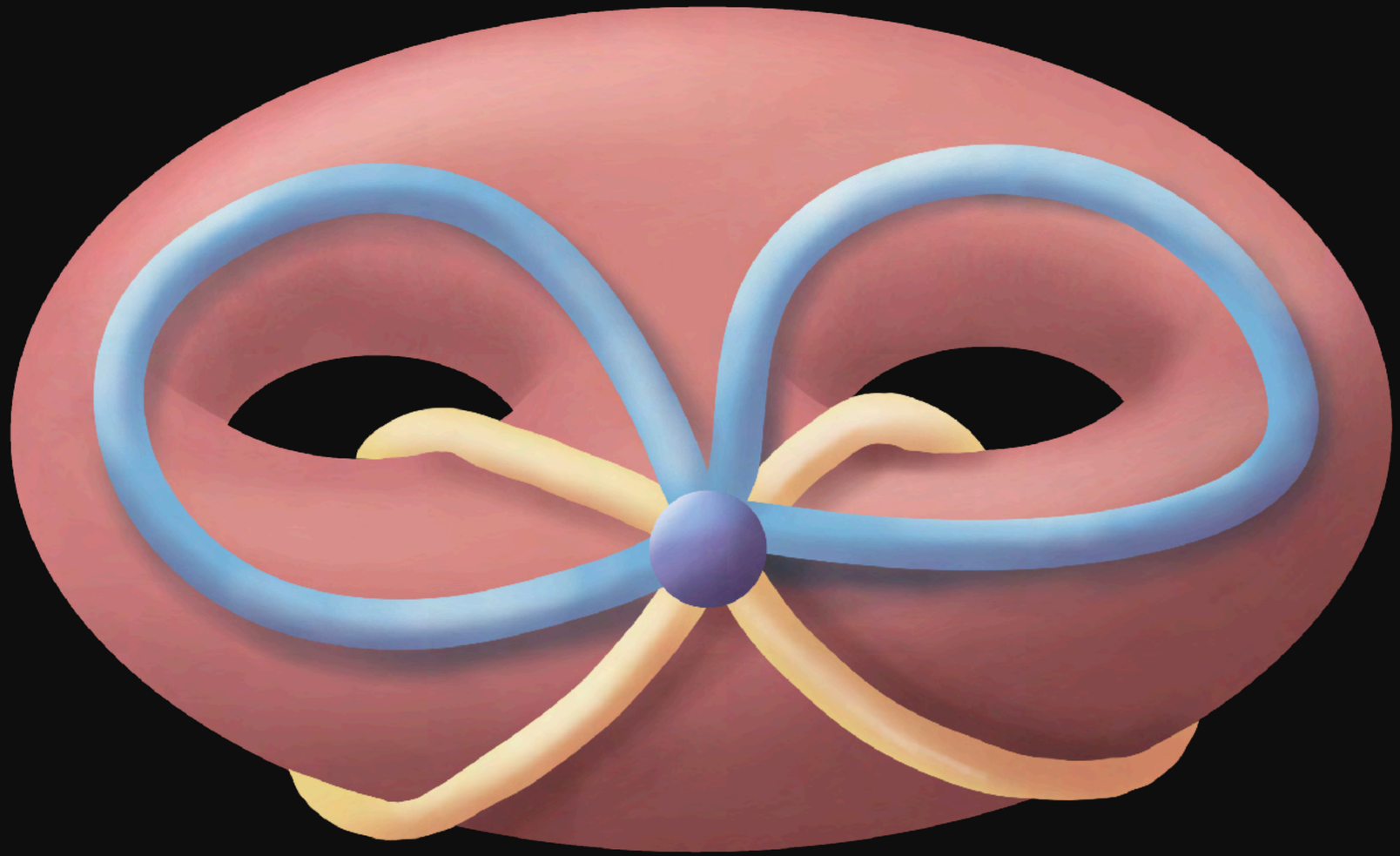
Groups with boundary

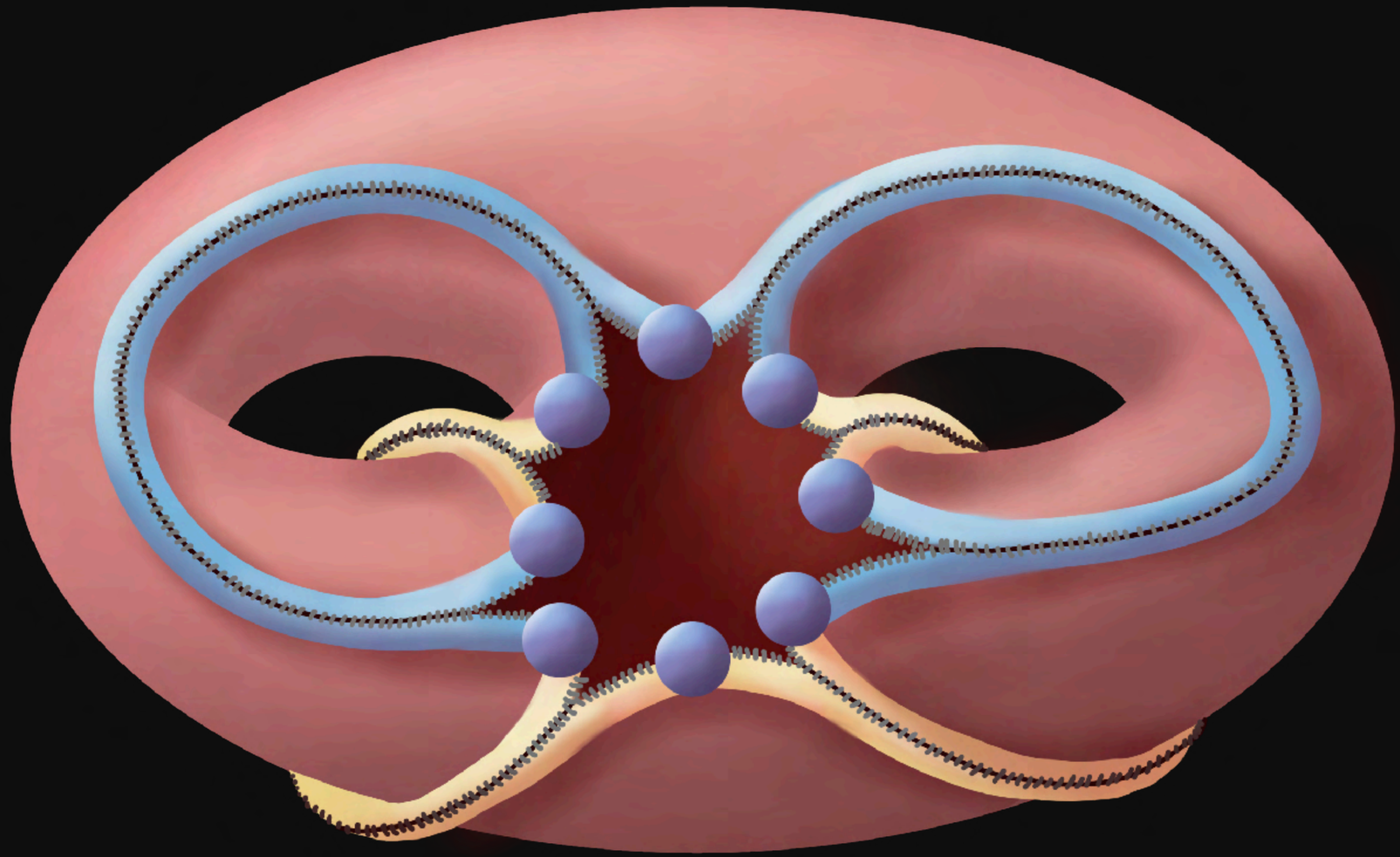
The 1-Dimensional Sphere

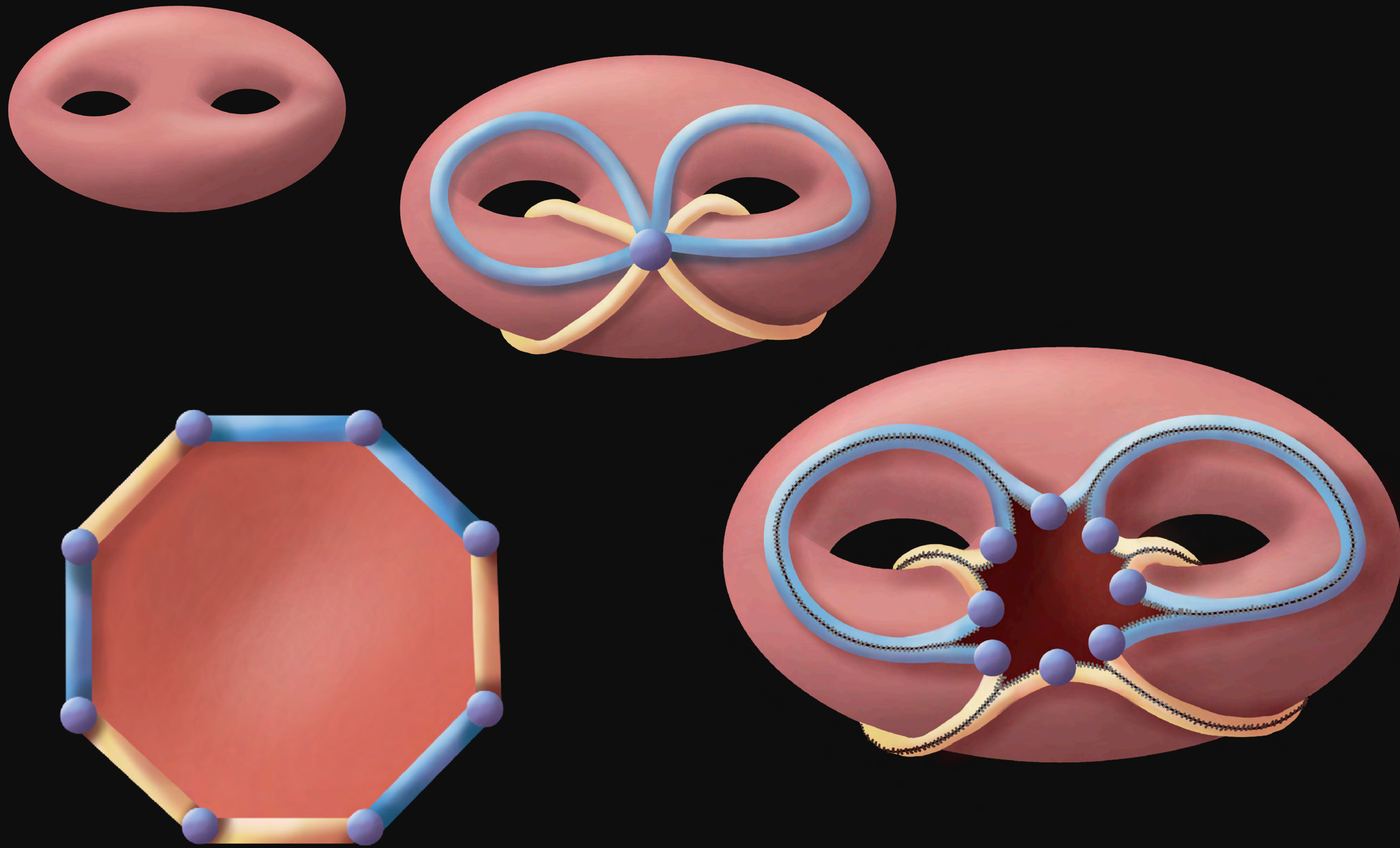


Use the geometry of surfaces to produce examples.

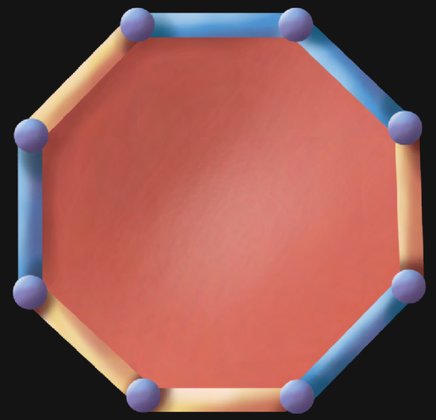
Σ_2

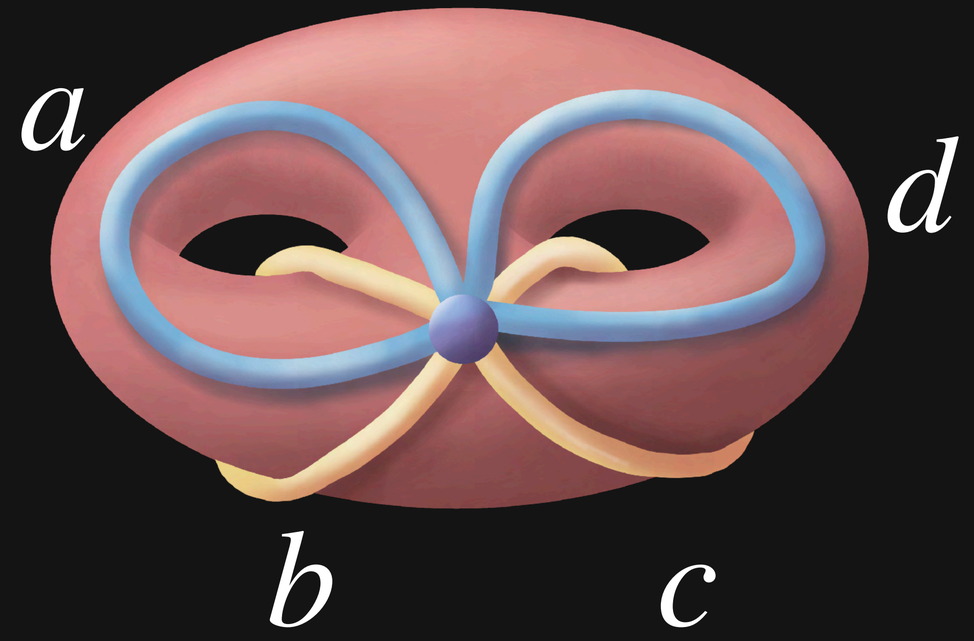




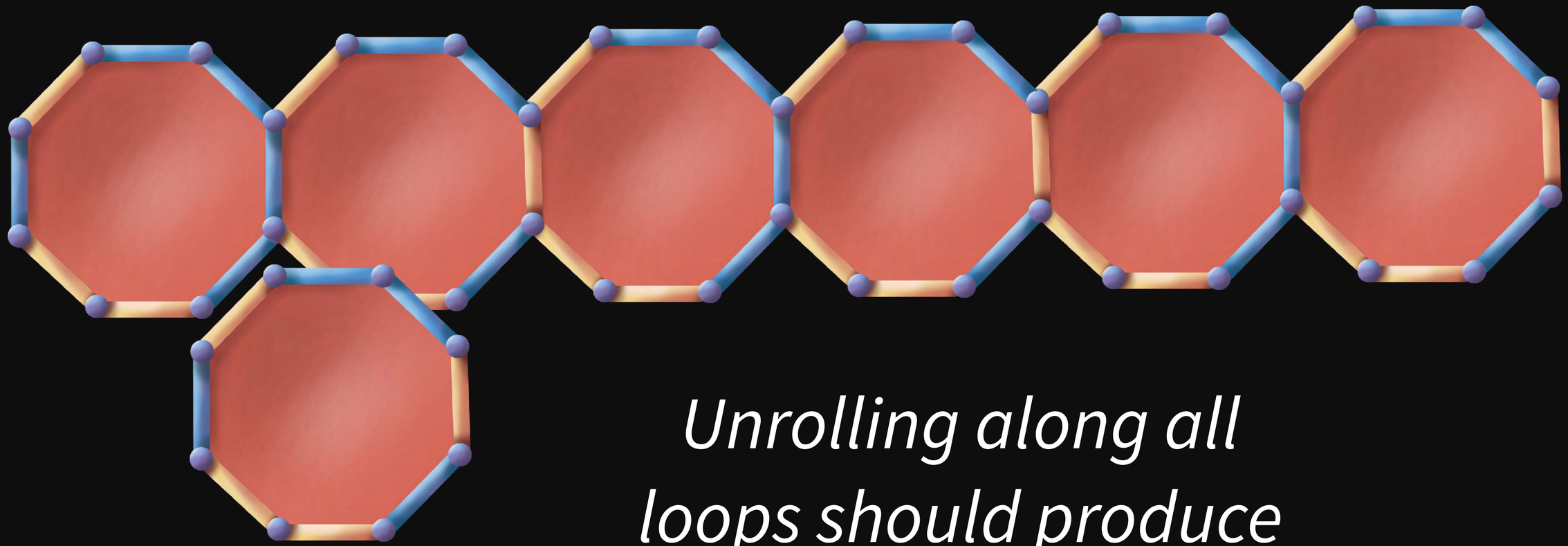


This gives us a presentation of $\pi_1(\Sigma_2)$





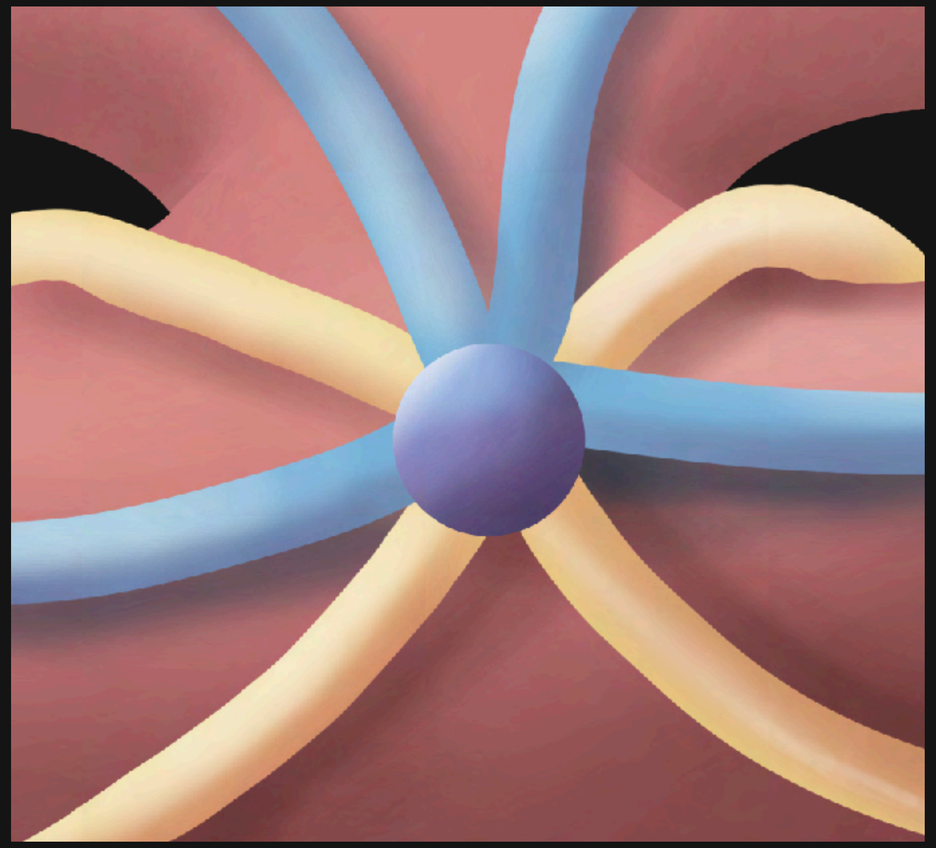
$$\langle a, b, c, d \mid aba^{-1}b^{-1}cdc^{-1}d^{-1} \rangle$$

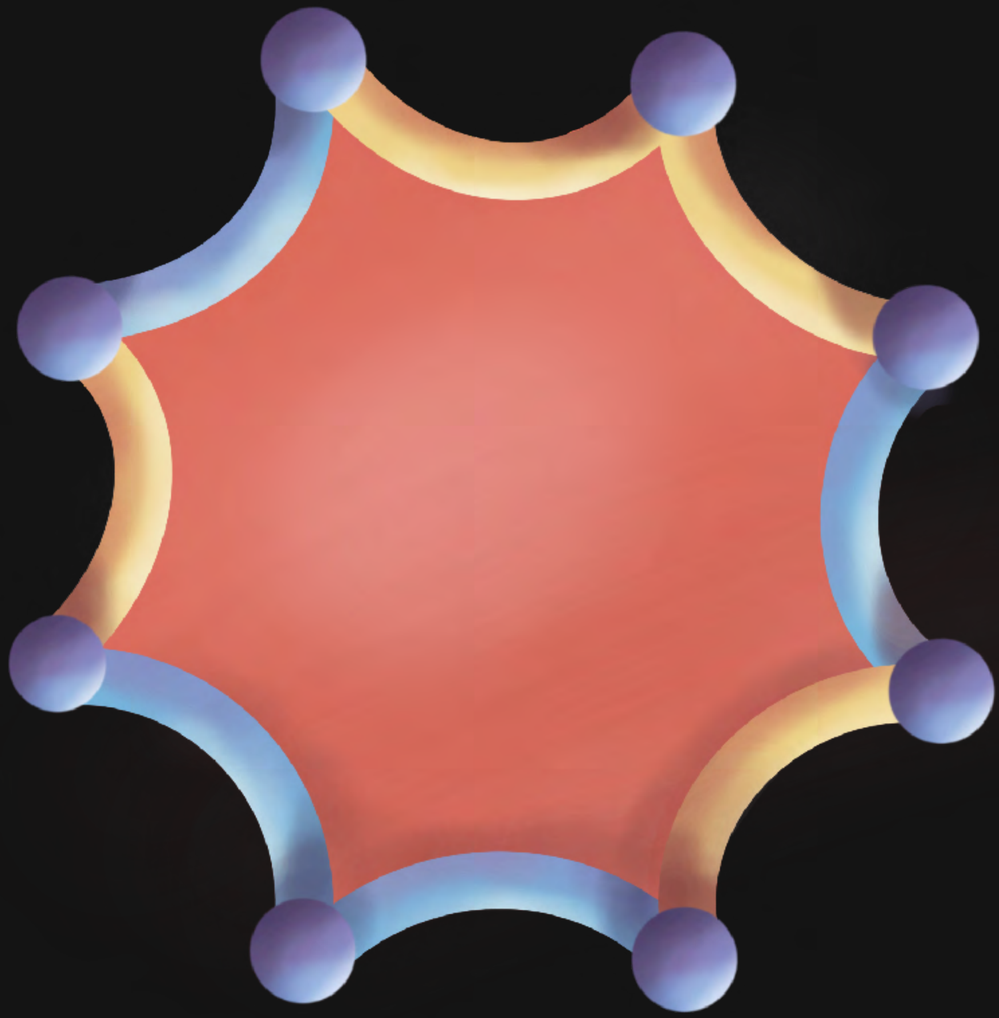


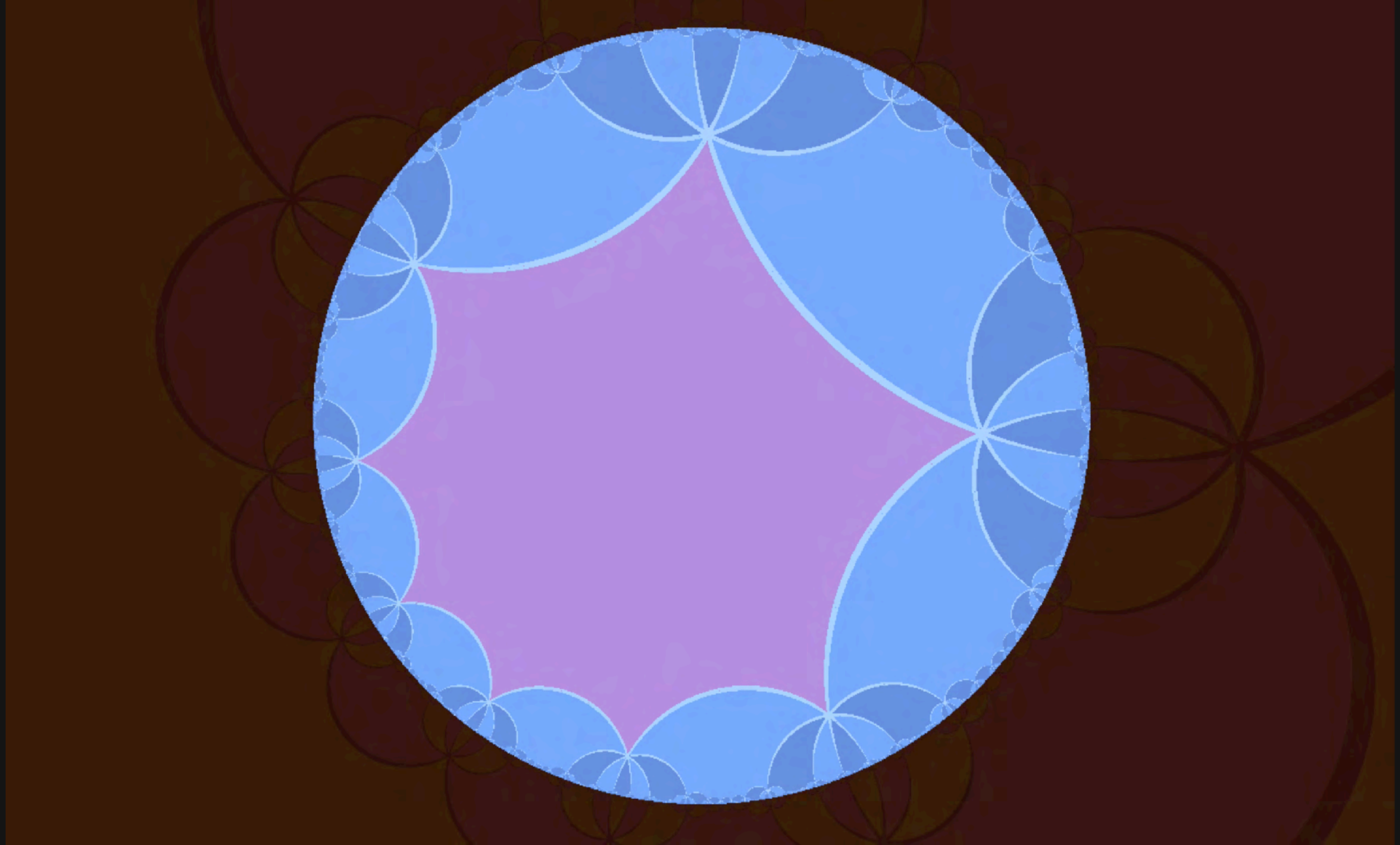
Unrolling along all loops should produce ***a tiling by octagons***

How many octagons need to meet at each vertex?

Fitting eight octagons about a vertex requires ***hyperbolic geometry!***

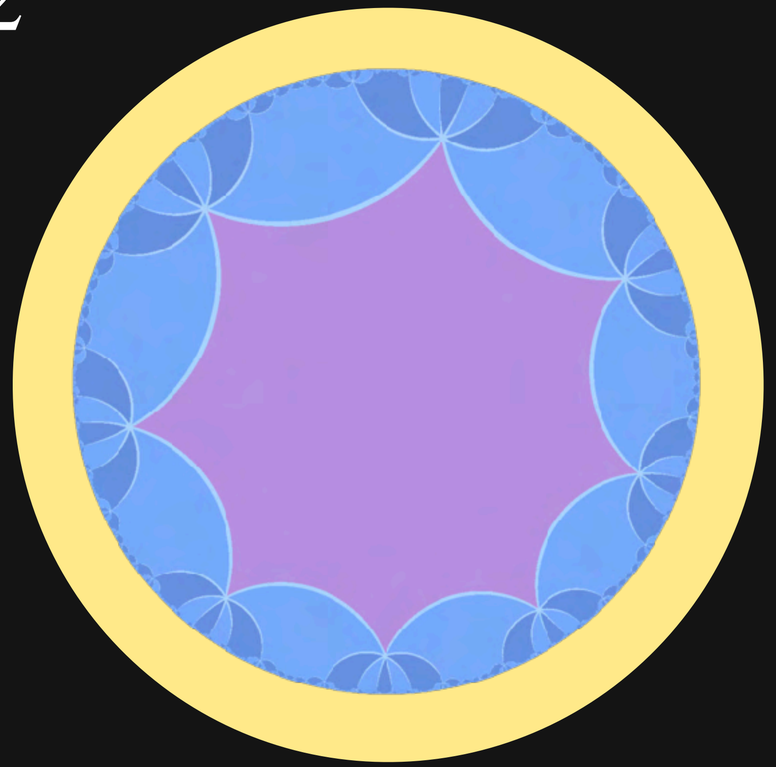






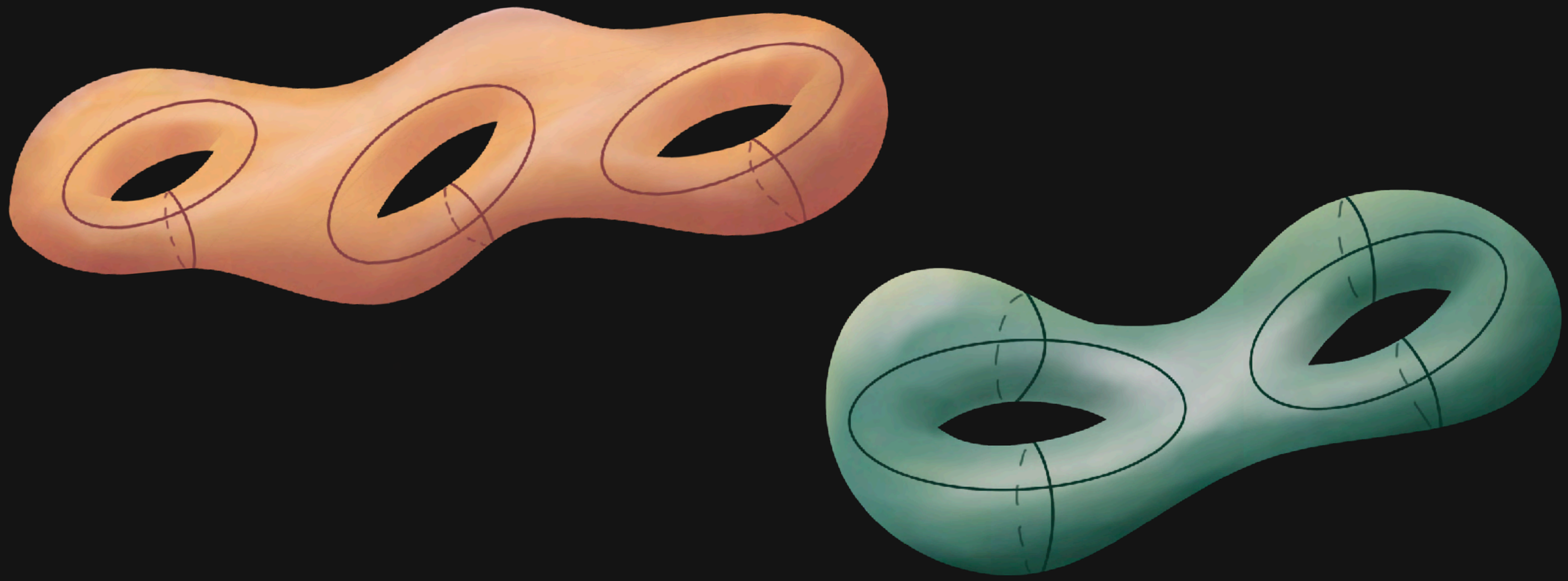
$$\pi_1\Sigma_2 \cong \langle a, b, c, d \mid [a, b][c, d] \rangle$$

acts geometrically on $\mathbb{H}^2$

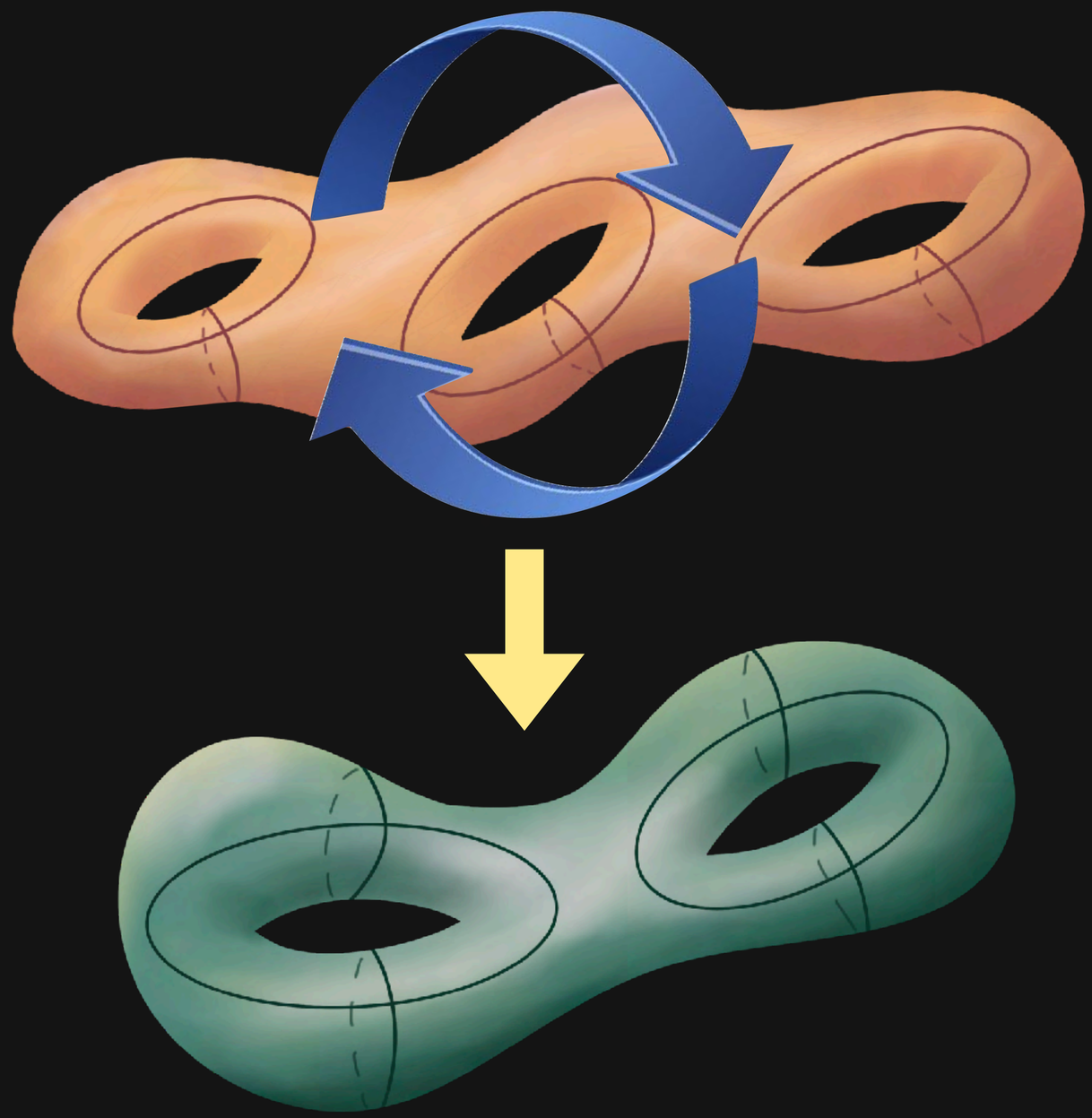


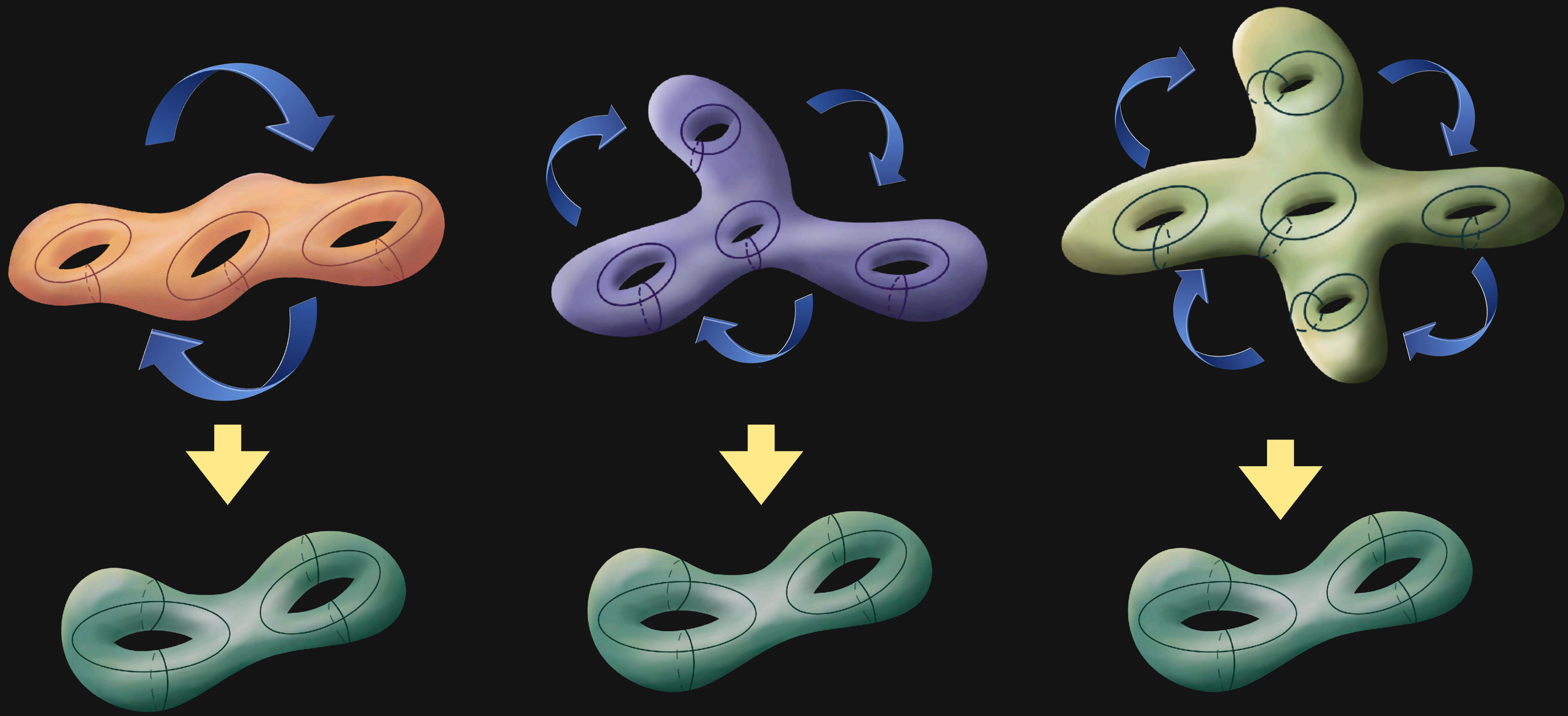
So $\pi_1\Sigma_2$ is QI to $\mathbb{H}^2$, and its boundary is homeomorphic to $\partial\mathbb{H}^2 \cong \mathbb{S}^1$

What about Higher Genus?

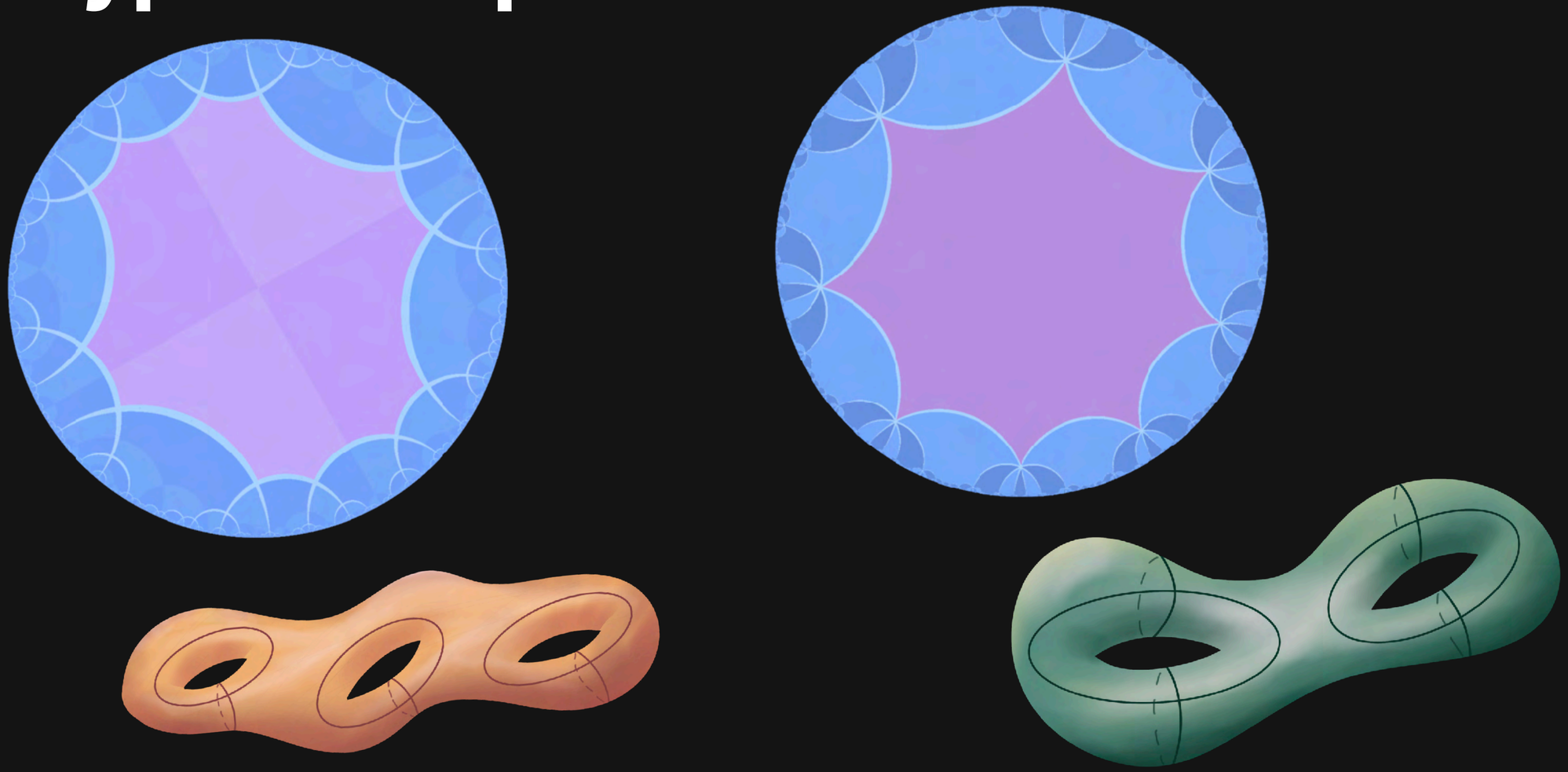


Genus g Covers Genus 2

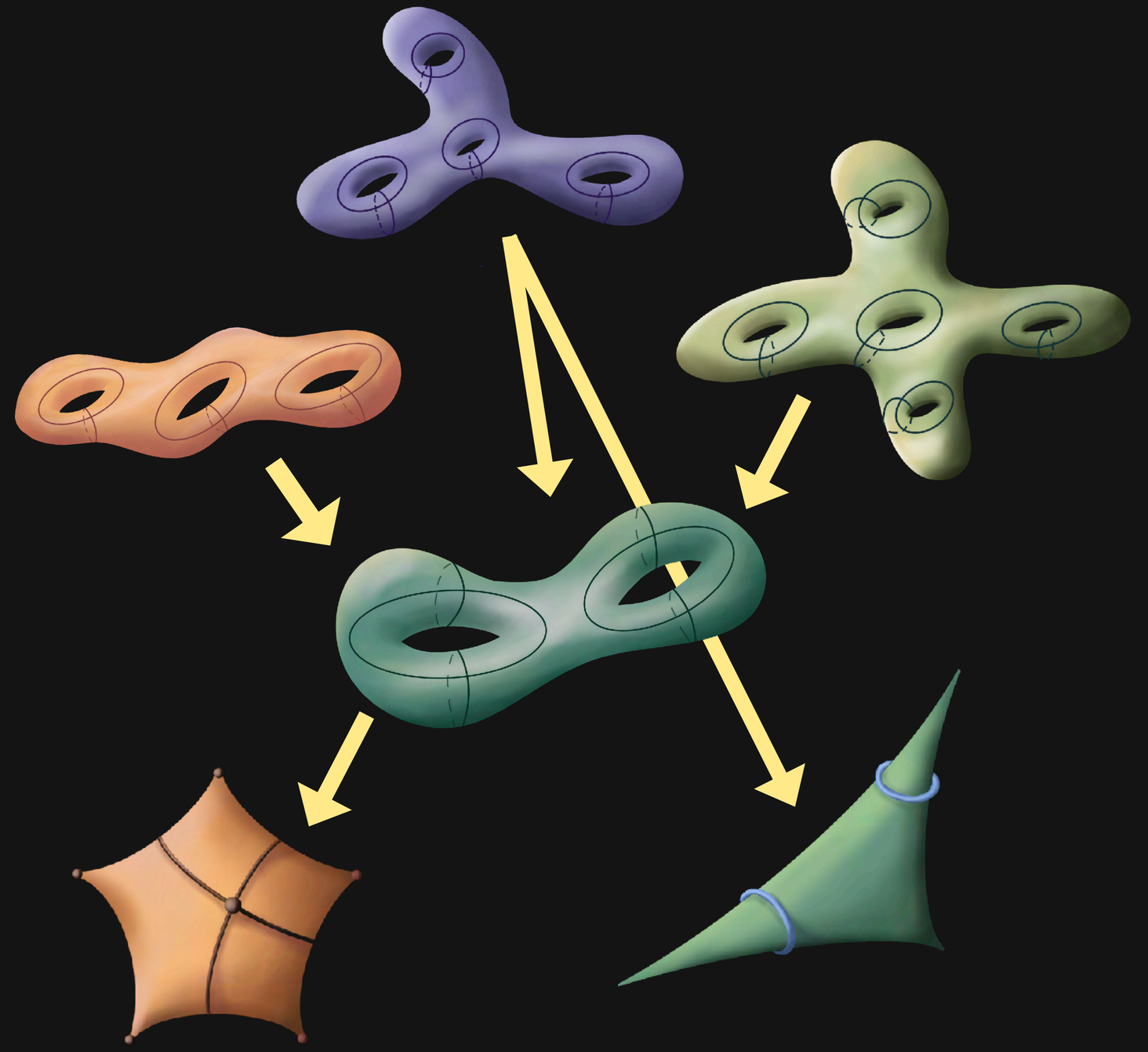




All surface groups of genus g>1 are QI to the hyperbolic plane

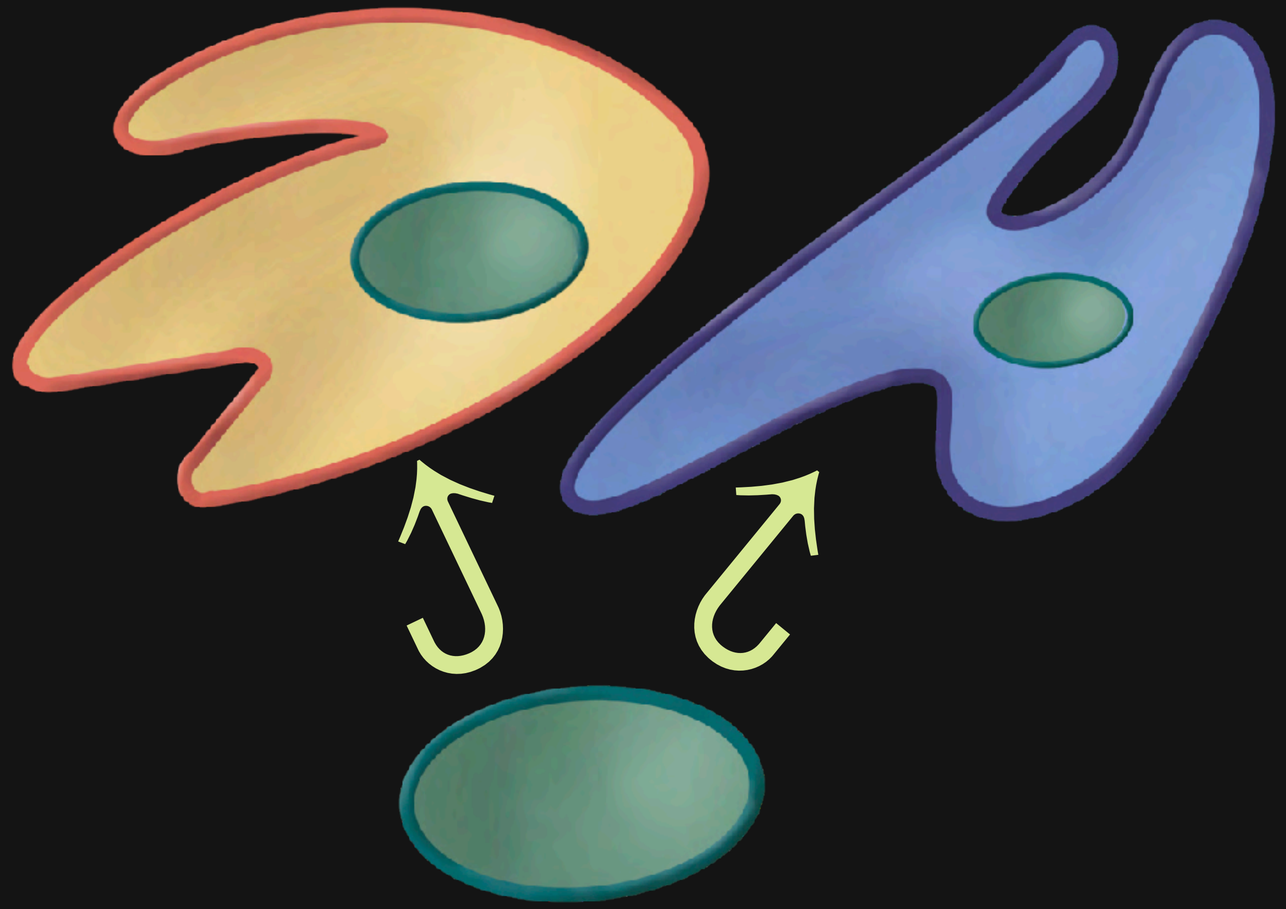


This is really one example:



Commensurability

Groups are (abstractly) commensurable if they share a finite index subgroup



Commensurability

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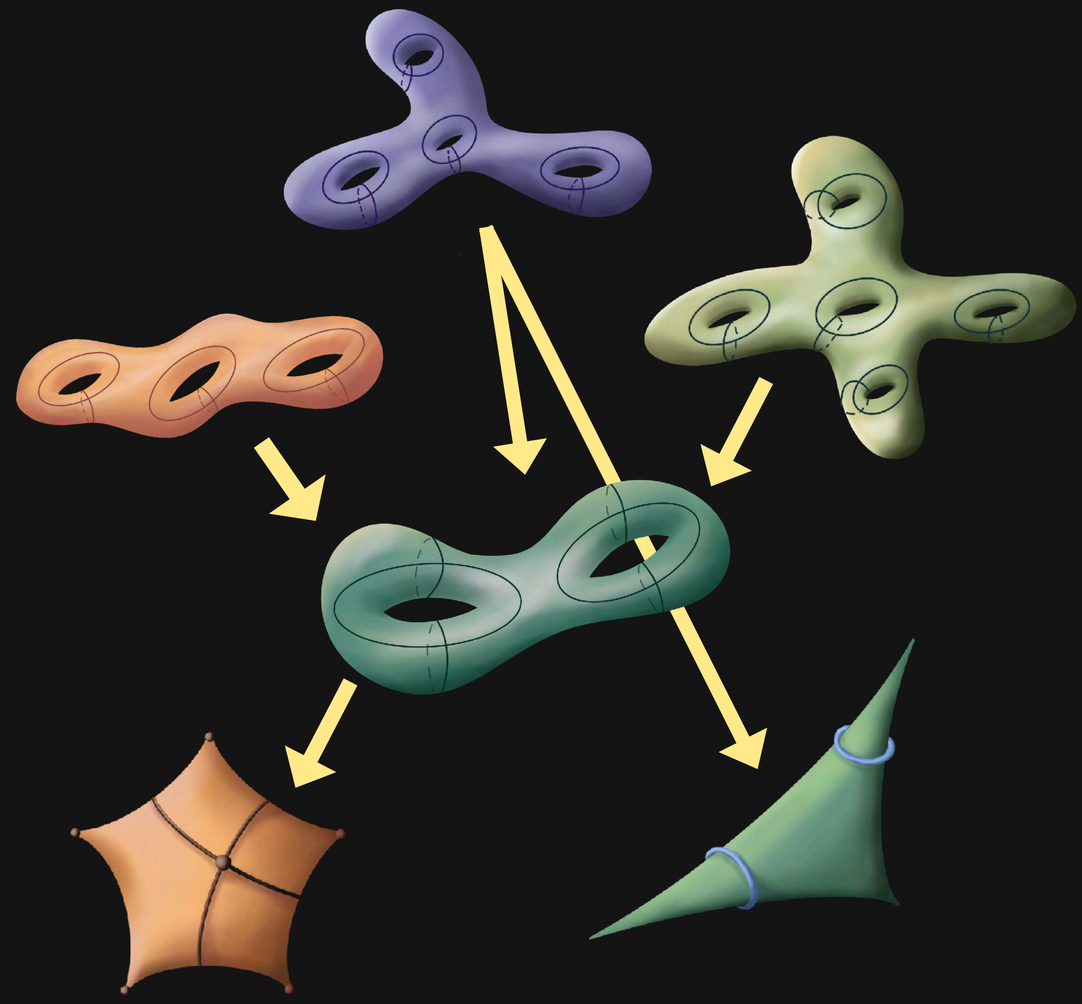


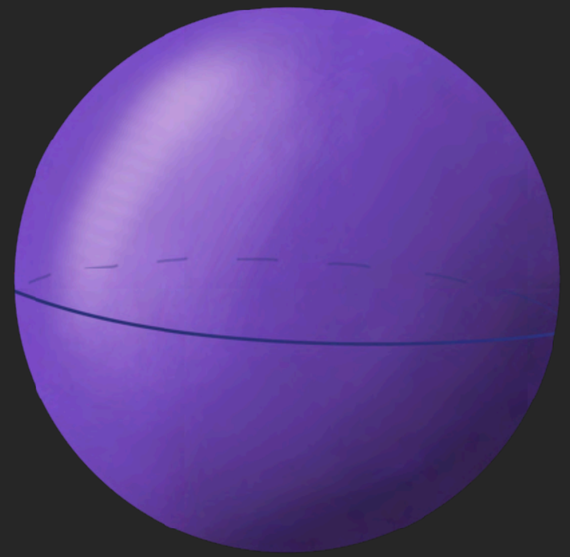
This is the ONLY example!

Tukia Gabai Freden Casson Jungreis, 1988

A hyperbolic group has boundary $\mathbb{S}^1$ if and only if it's one of these.

Groups with boundary

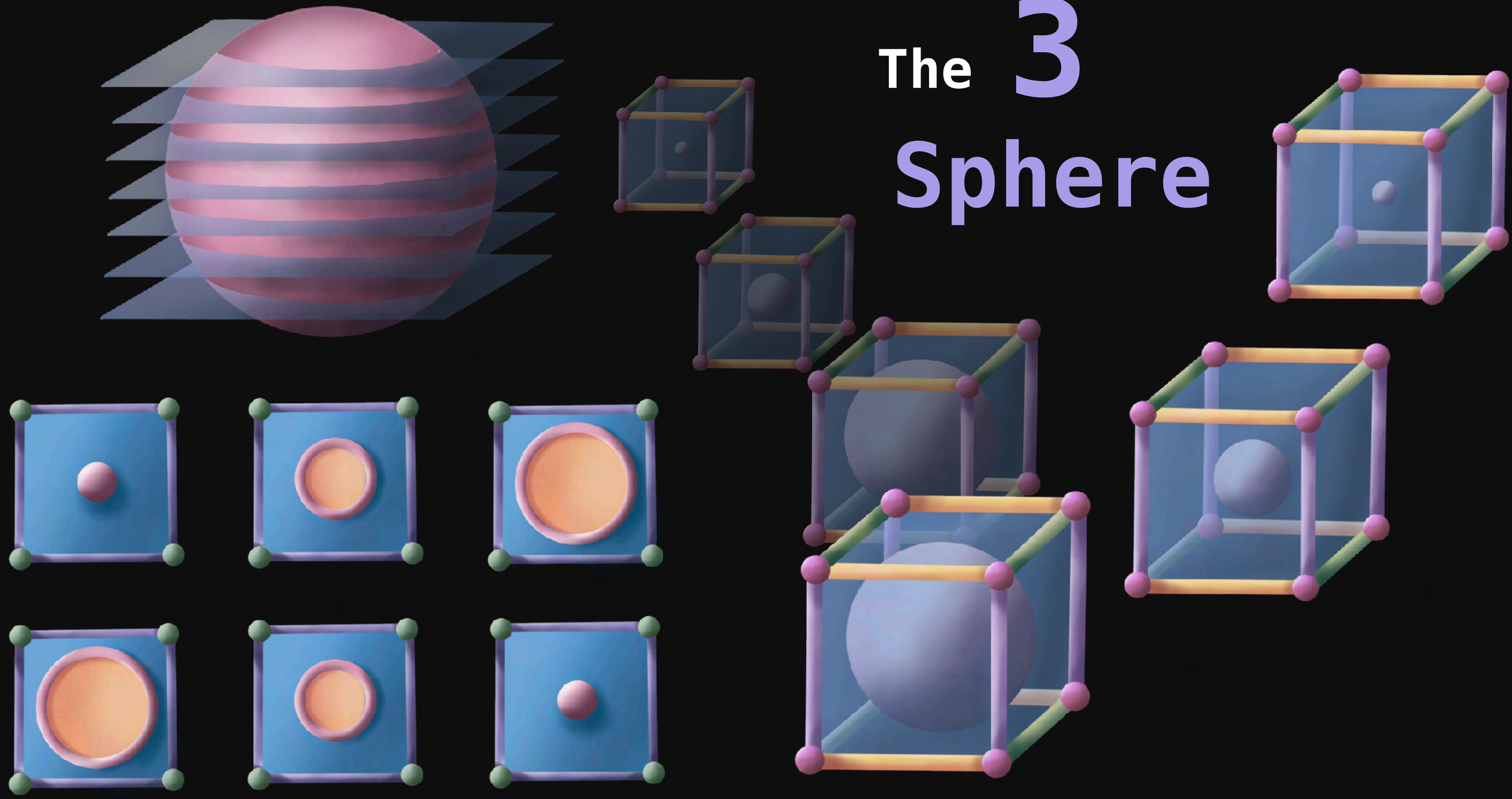
The 2-Dimensional Sphere



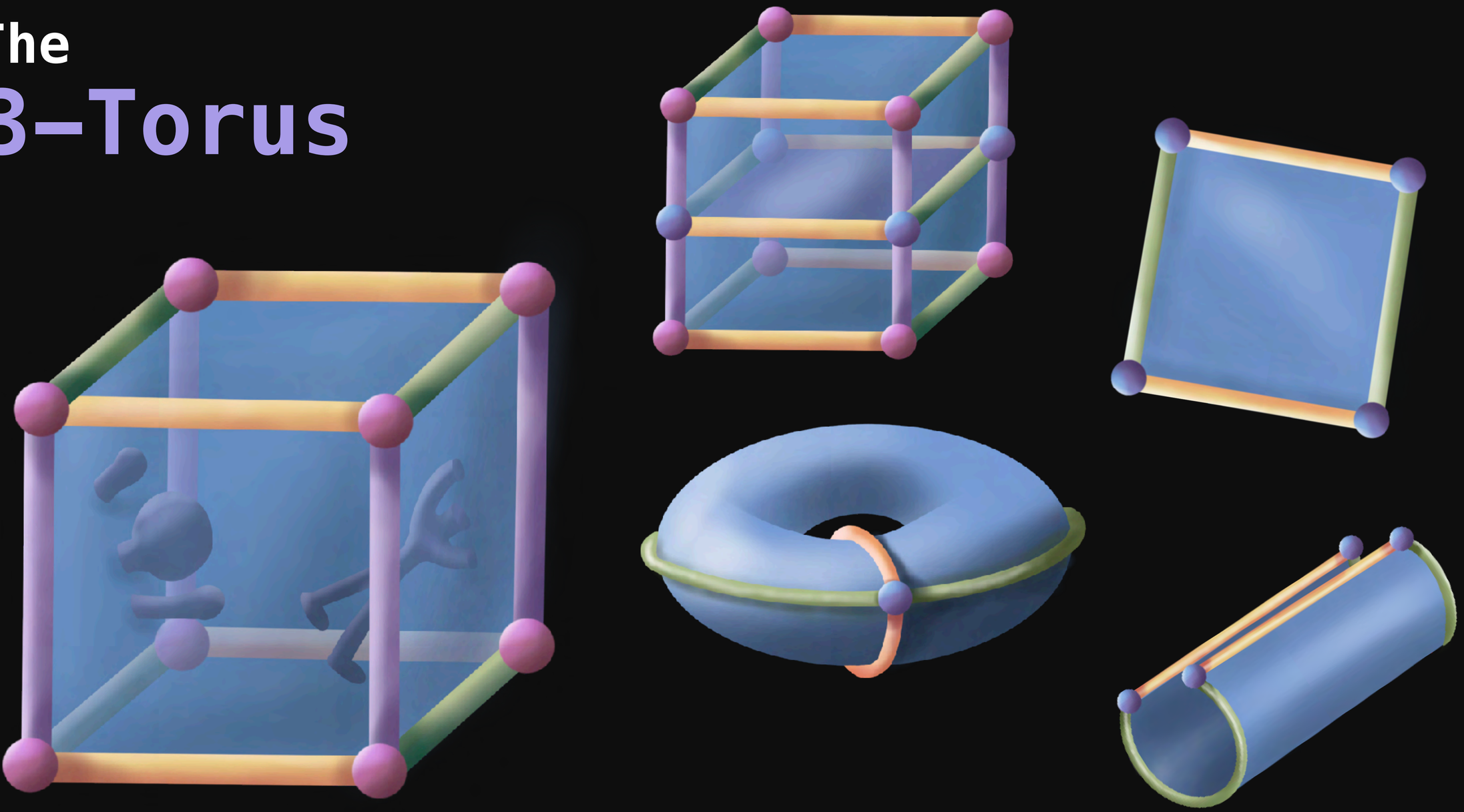
Where do we start?!?!

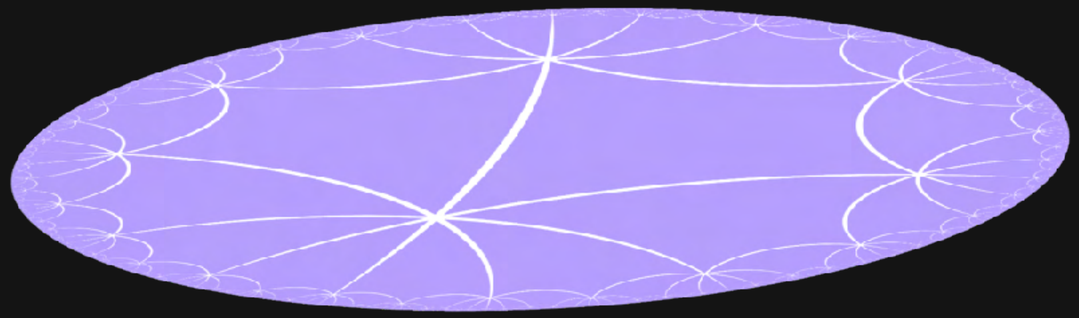
In 2D we started with a nice shape we knew....do we know nice shapes here?

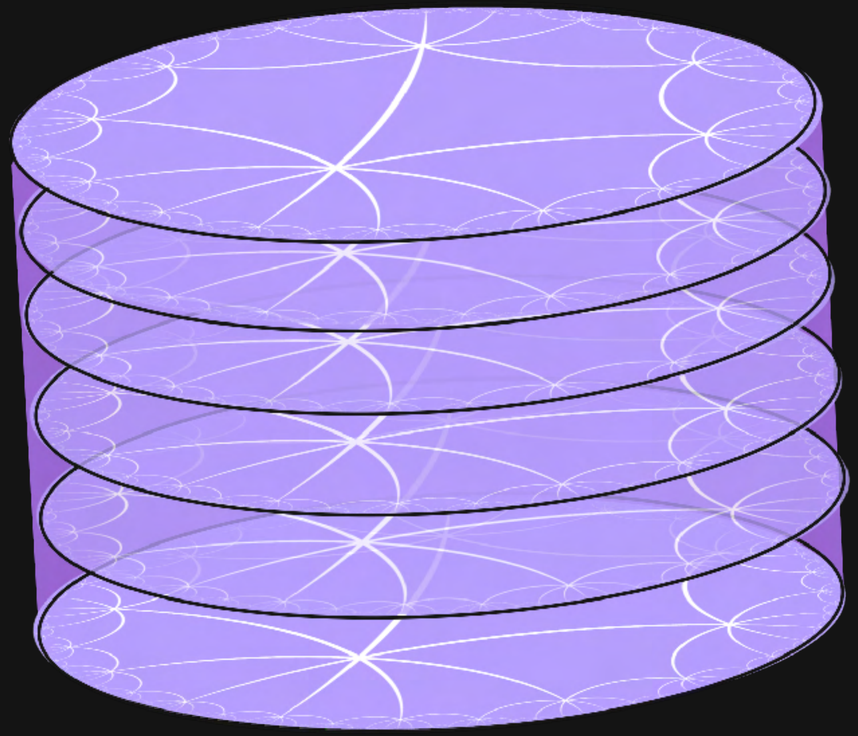
The 3 Sphere



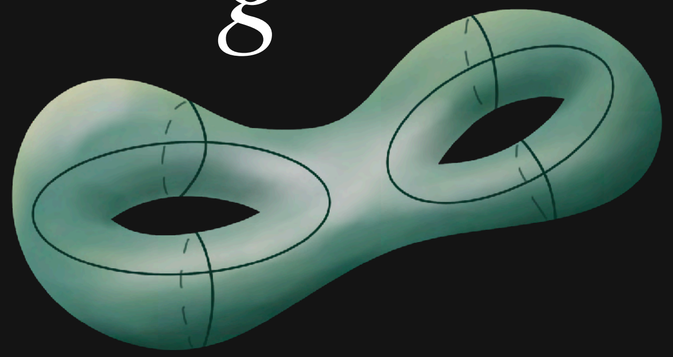
The 3-Torus



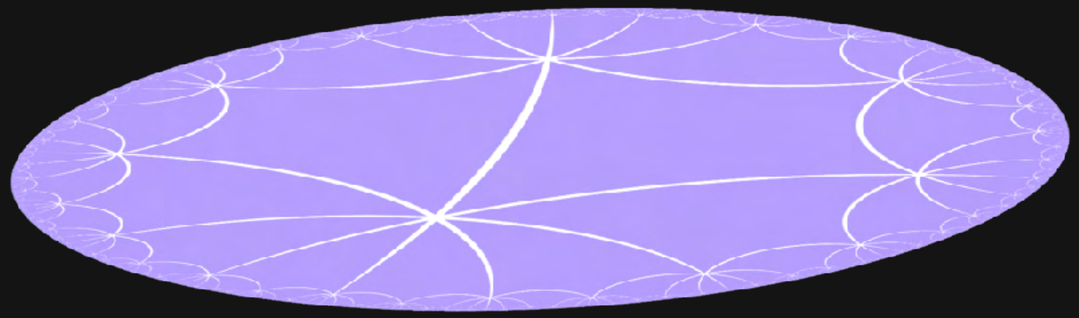


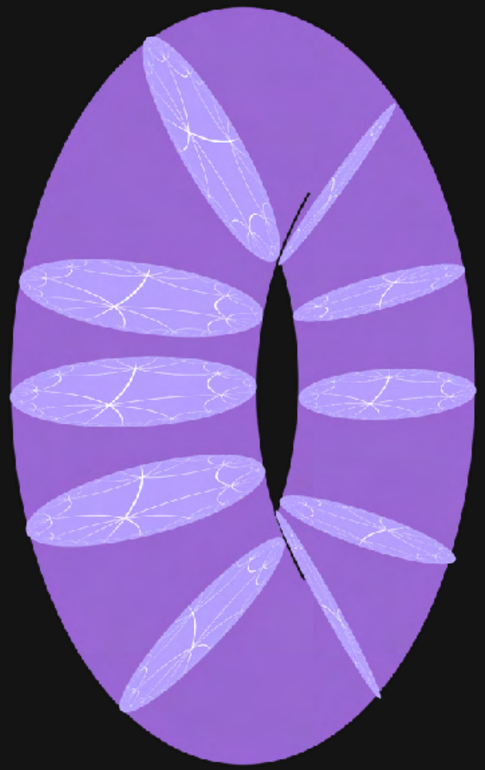


Σ_g

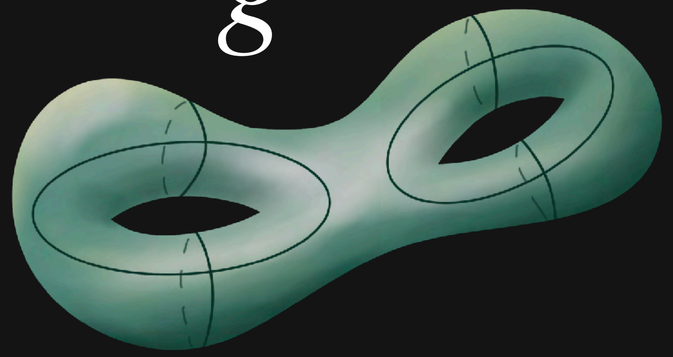


$\Sigma_g \times S^1$





Σ_g



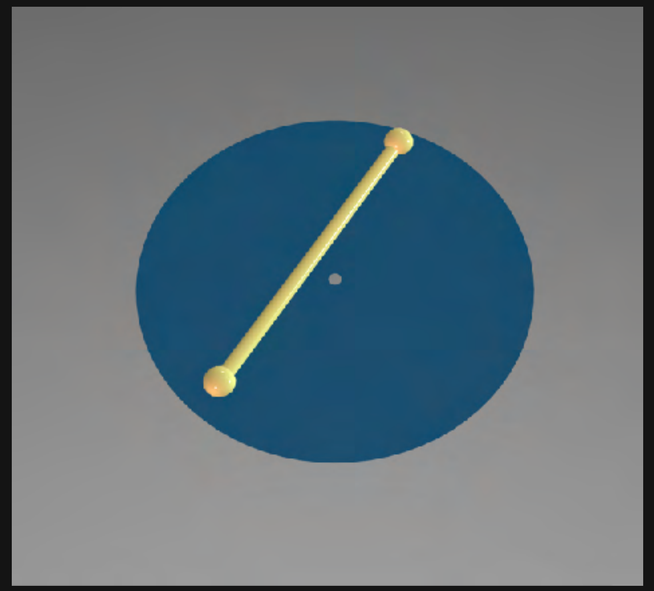
$UT(\Sigma_g)$

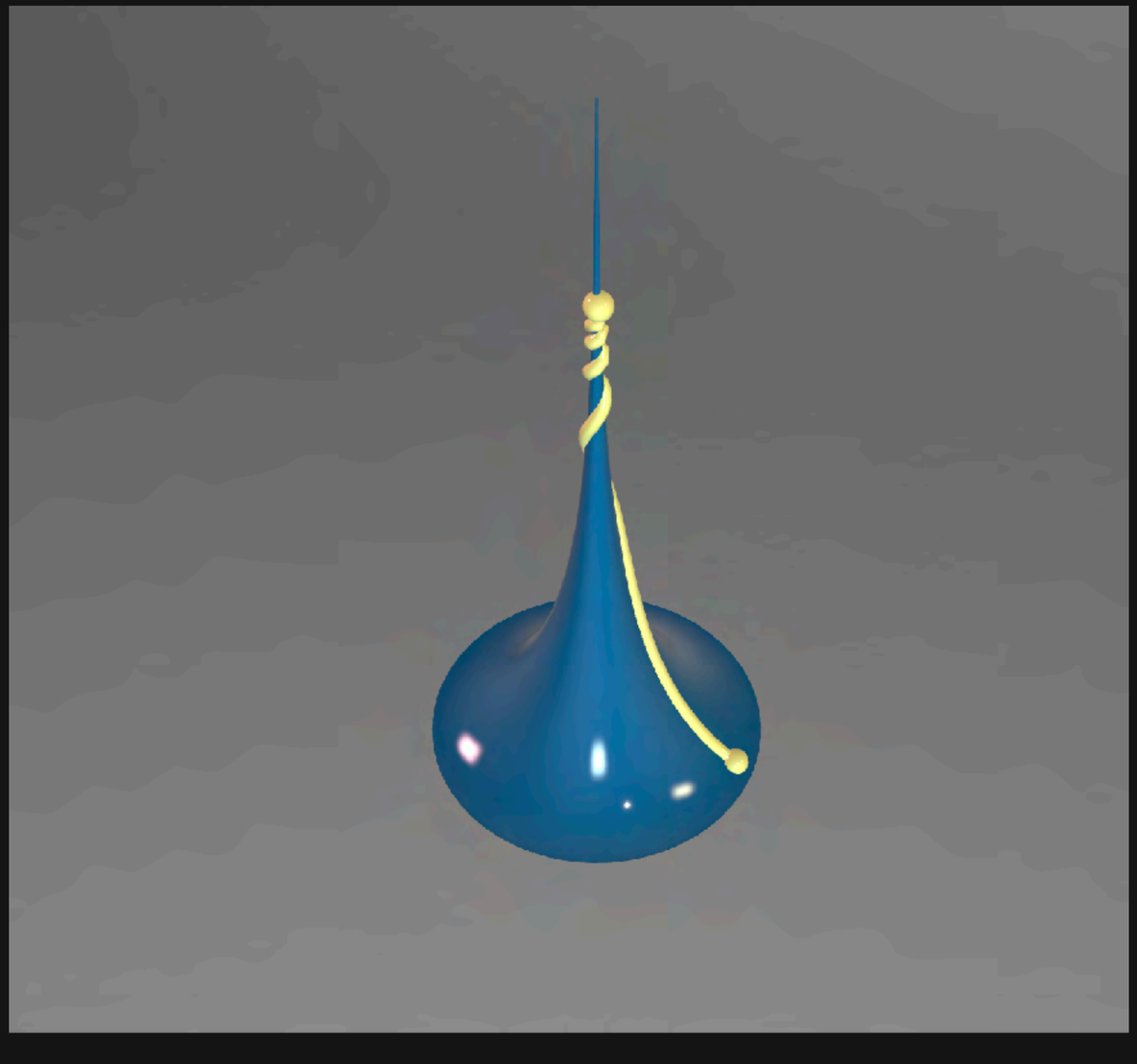
How do we find even a single closed hyperbolic 3-manifold?

Step 1: Ask for less

If we allow the freedom to work with open manifolds as well, can we succeed?

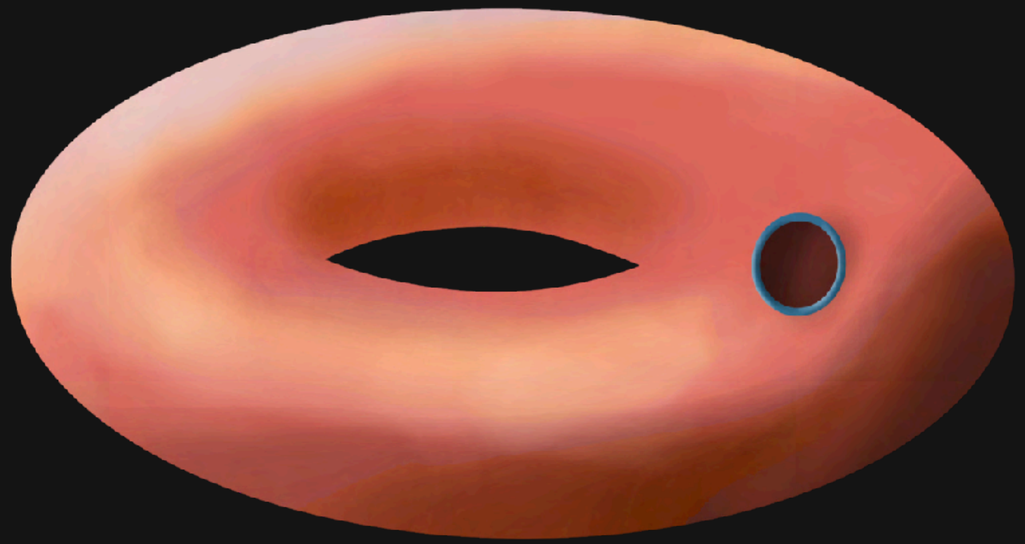
Geometrzation of the punctured disk:





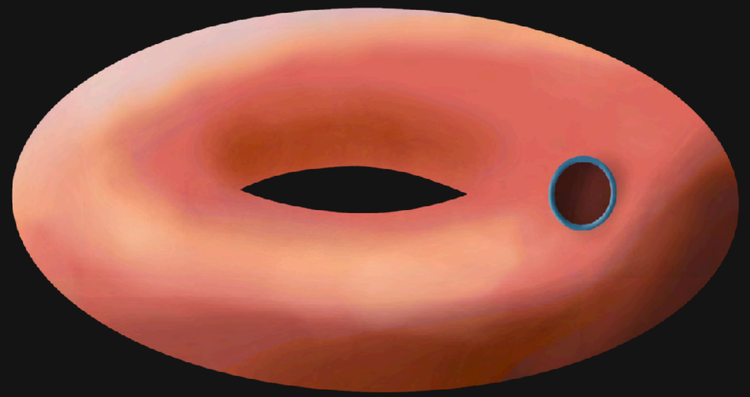
(Open Animation)

Putting a complete metric on a punctured torus.



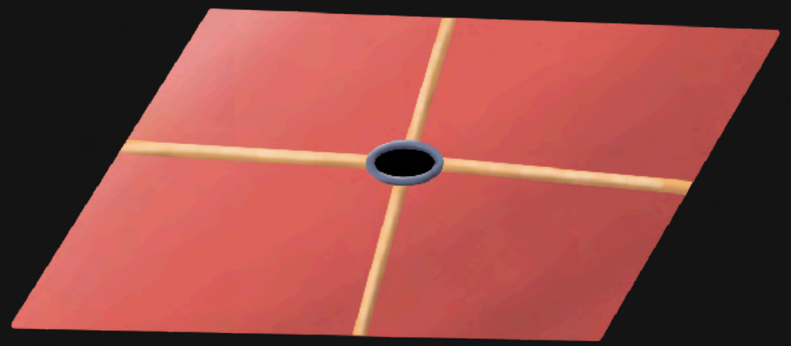
Complete
Infinite Area

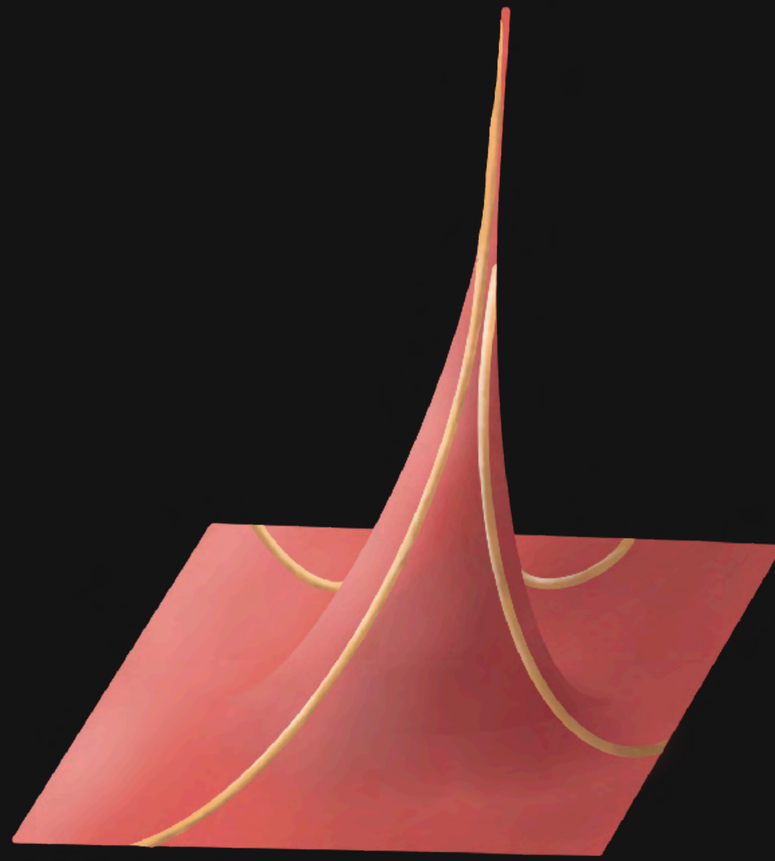
Incomplete



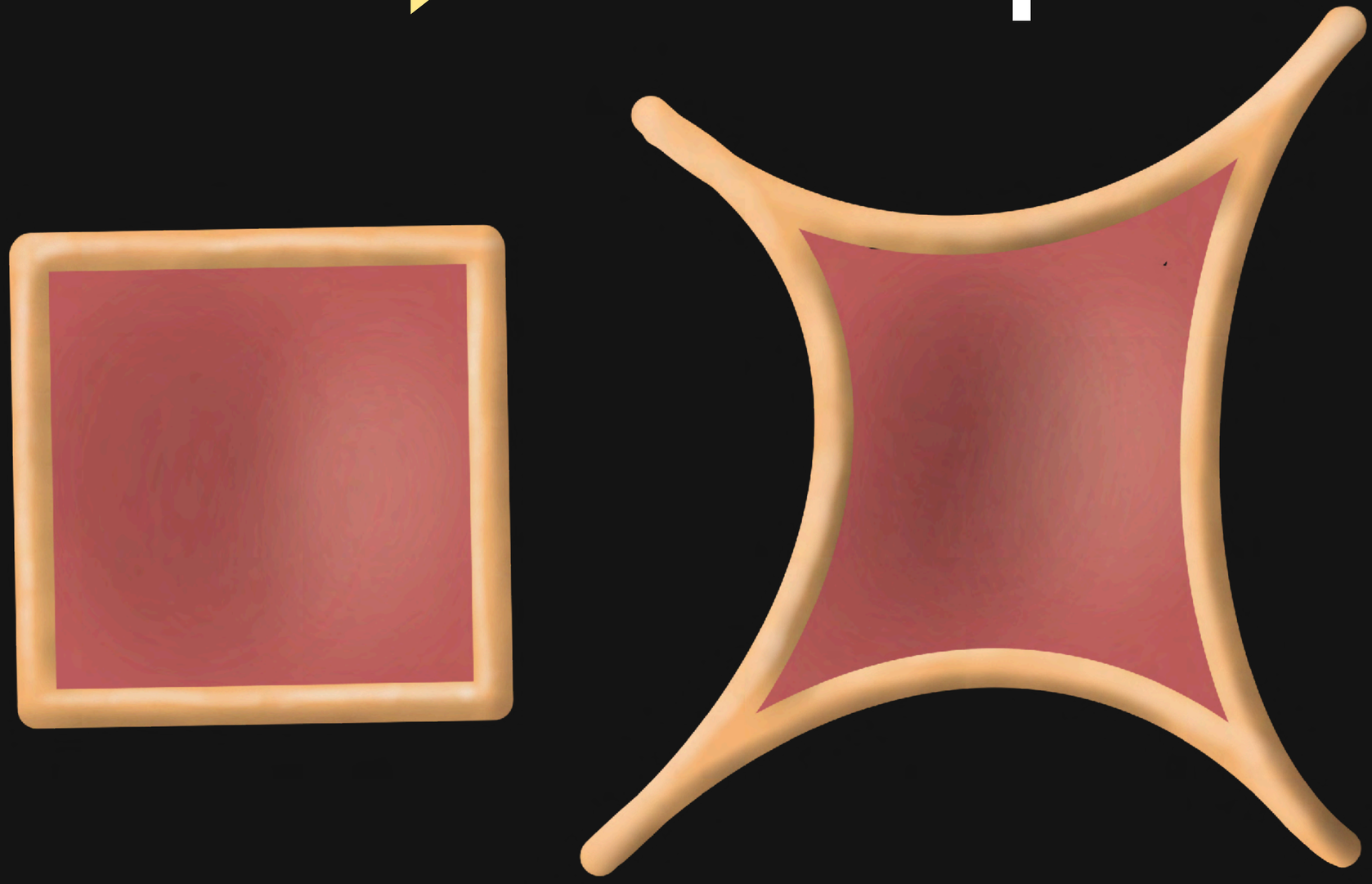
Complete
Finite Area

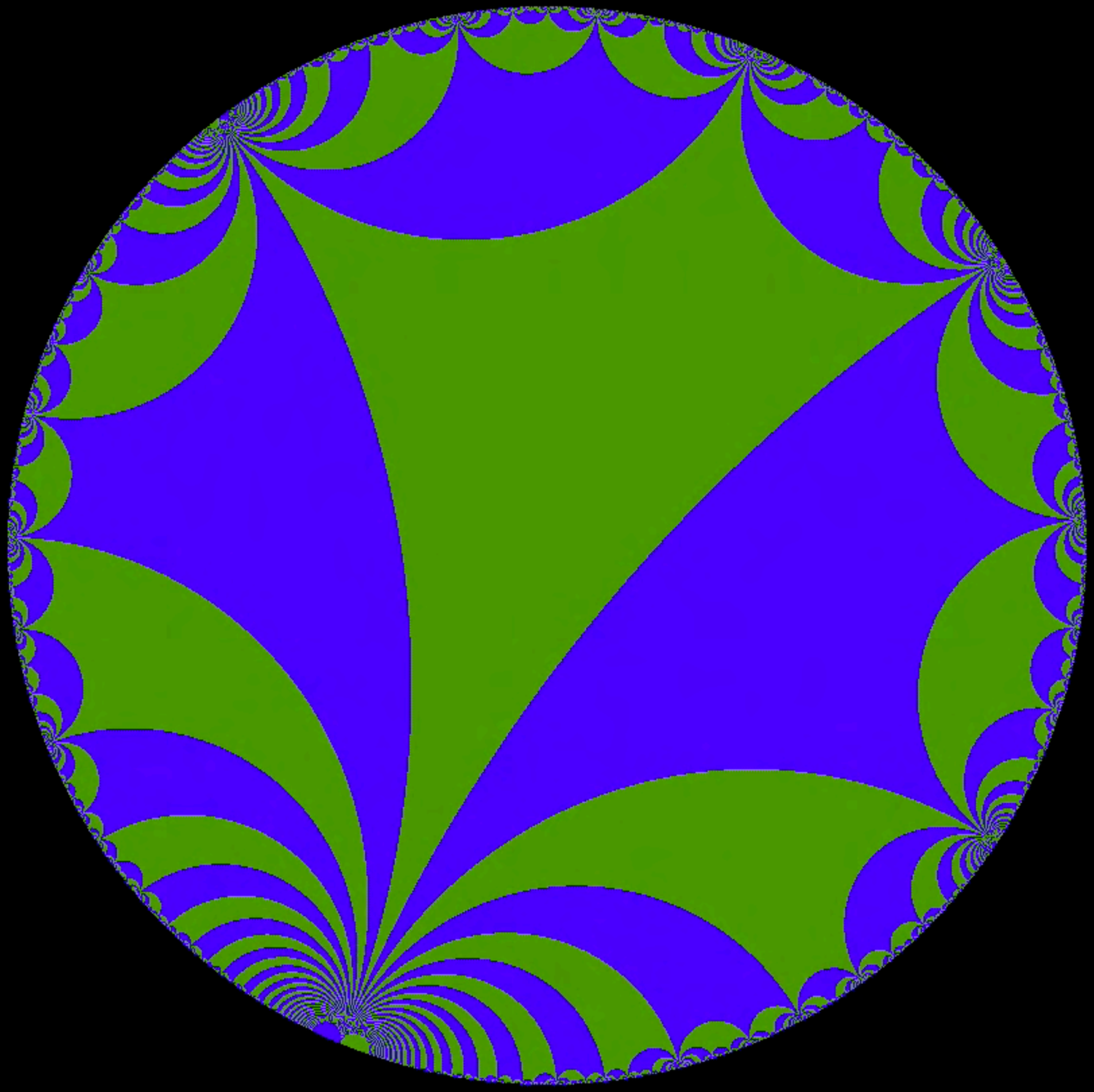
What happens to the metric near the puncture?





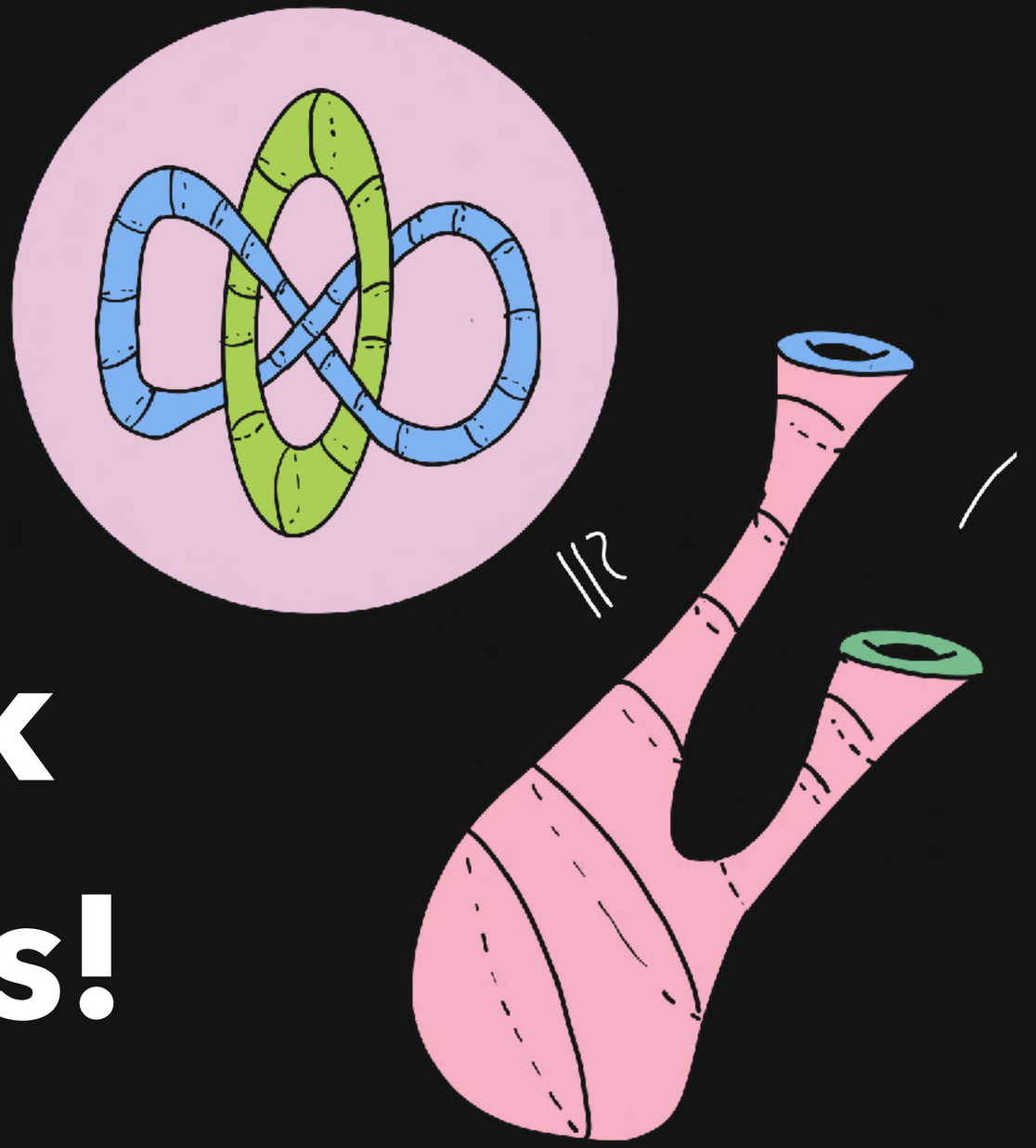
Squares → Ideal Squares





Is it easier to think of interesting 3D open manifolds?

YES



Knot and Link Complements!

Make the disks orthogonal to the knot....hyperbolic!

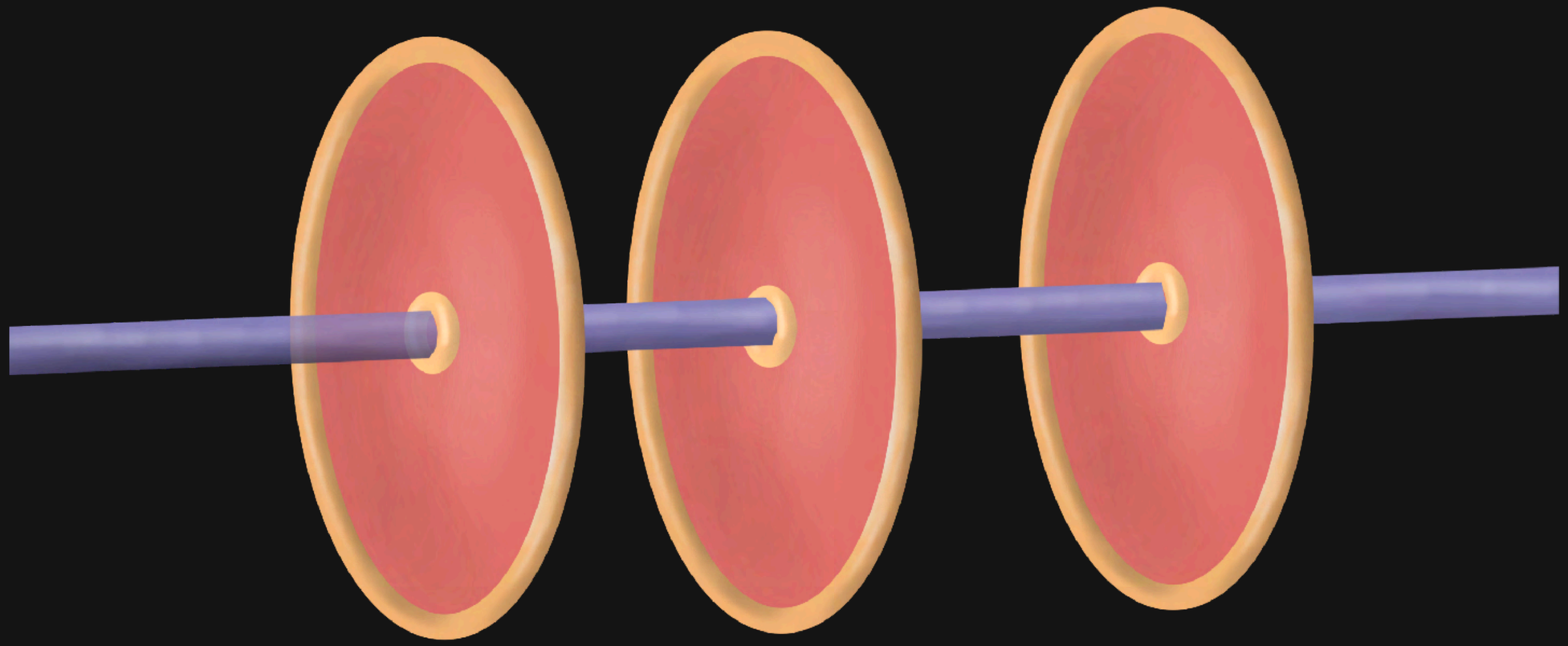
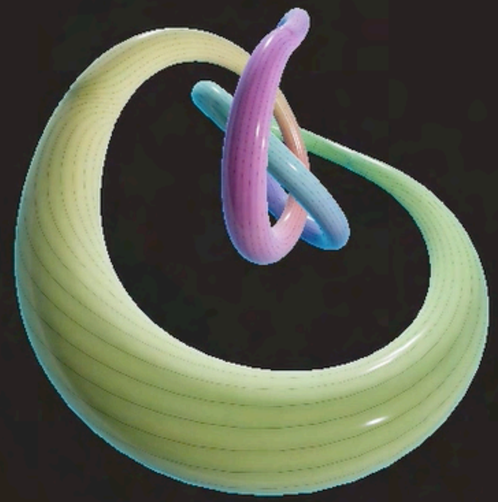
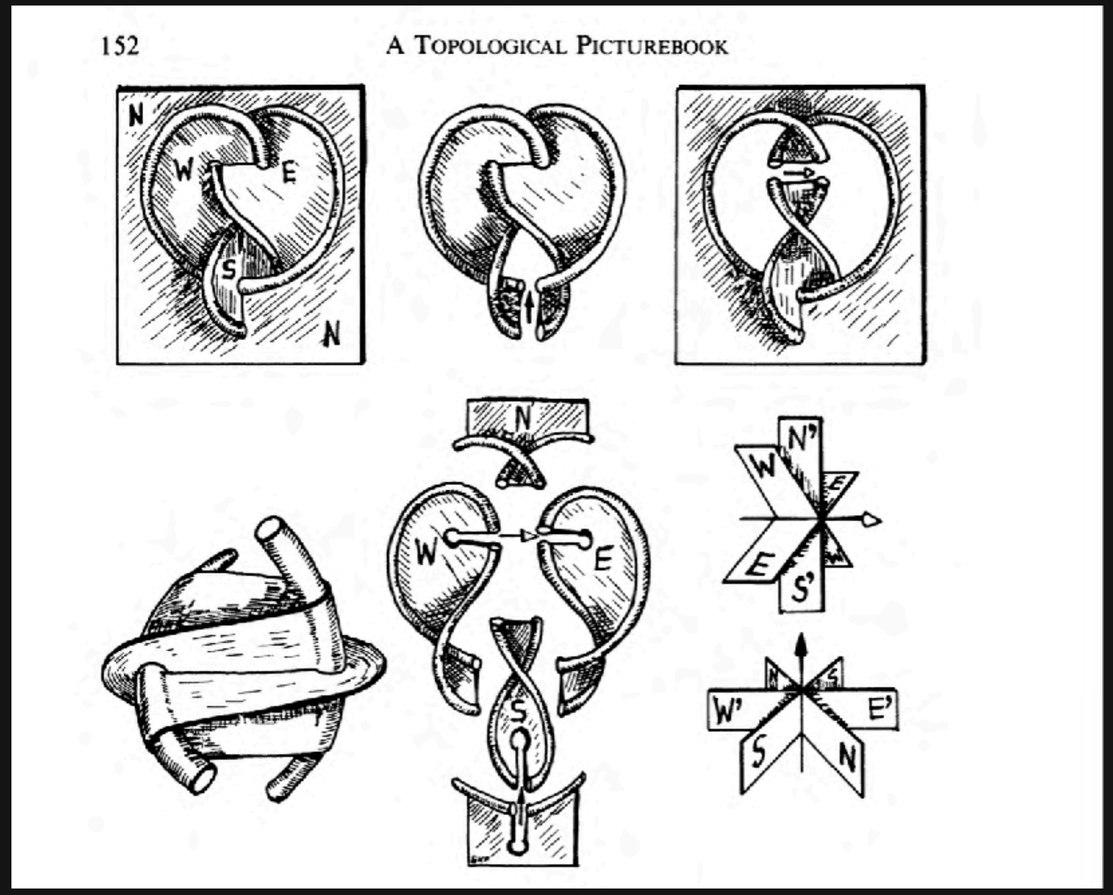
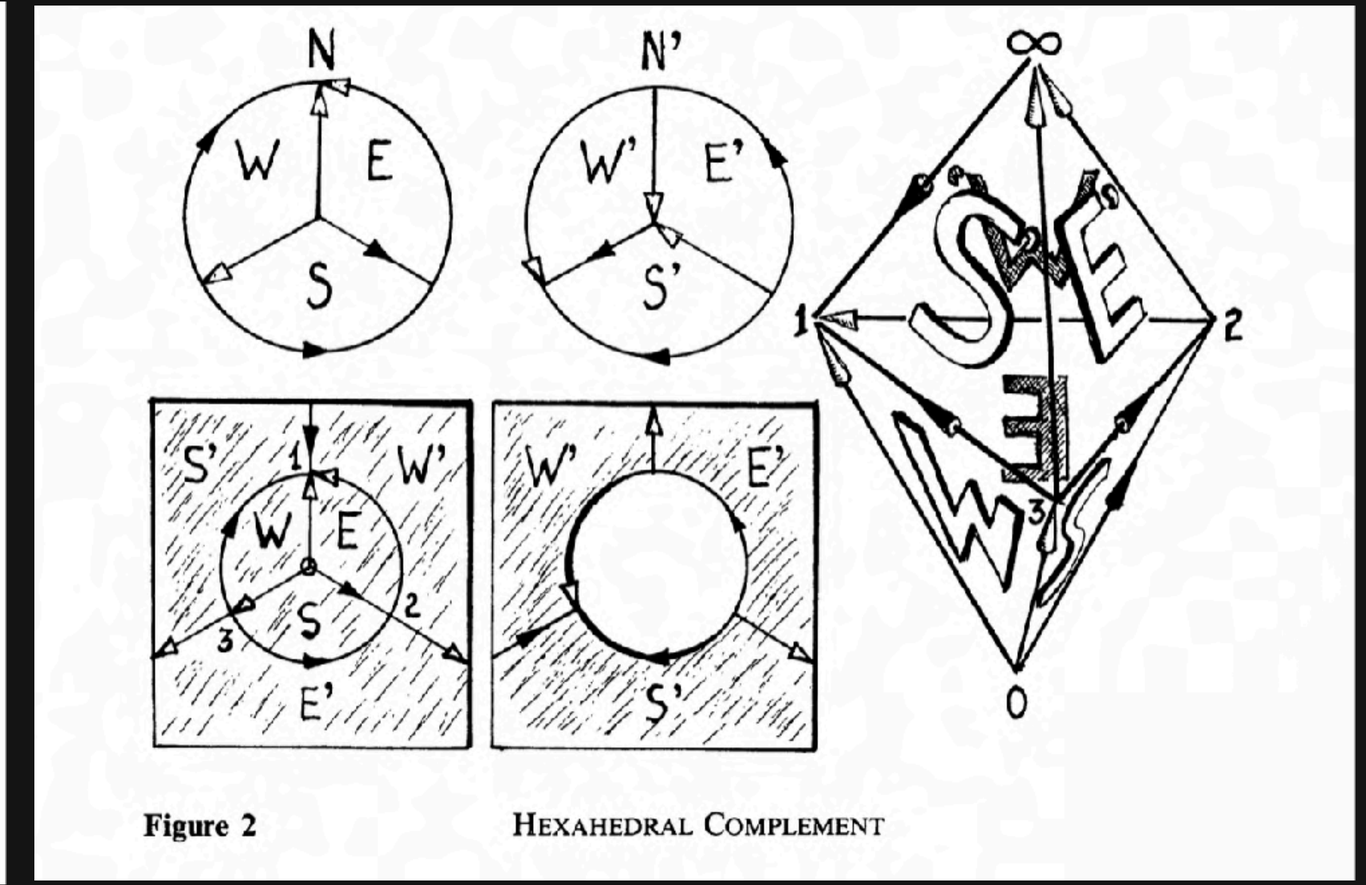


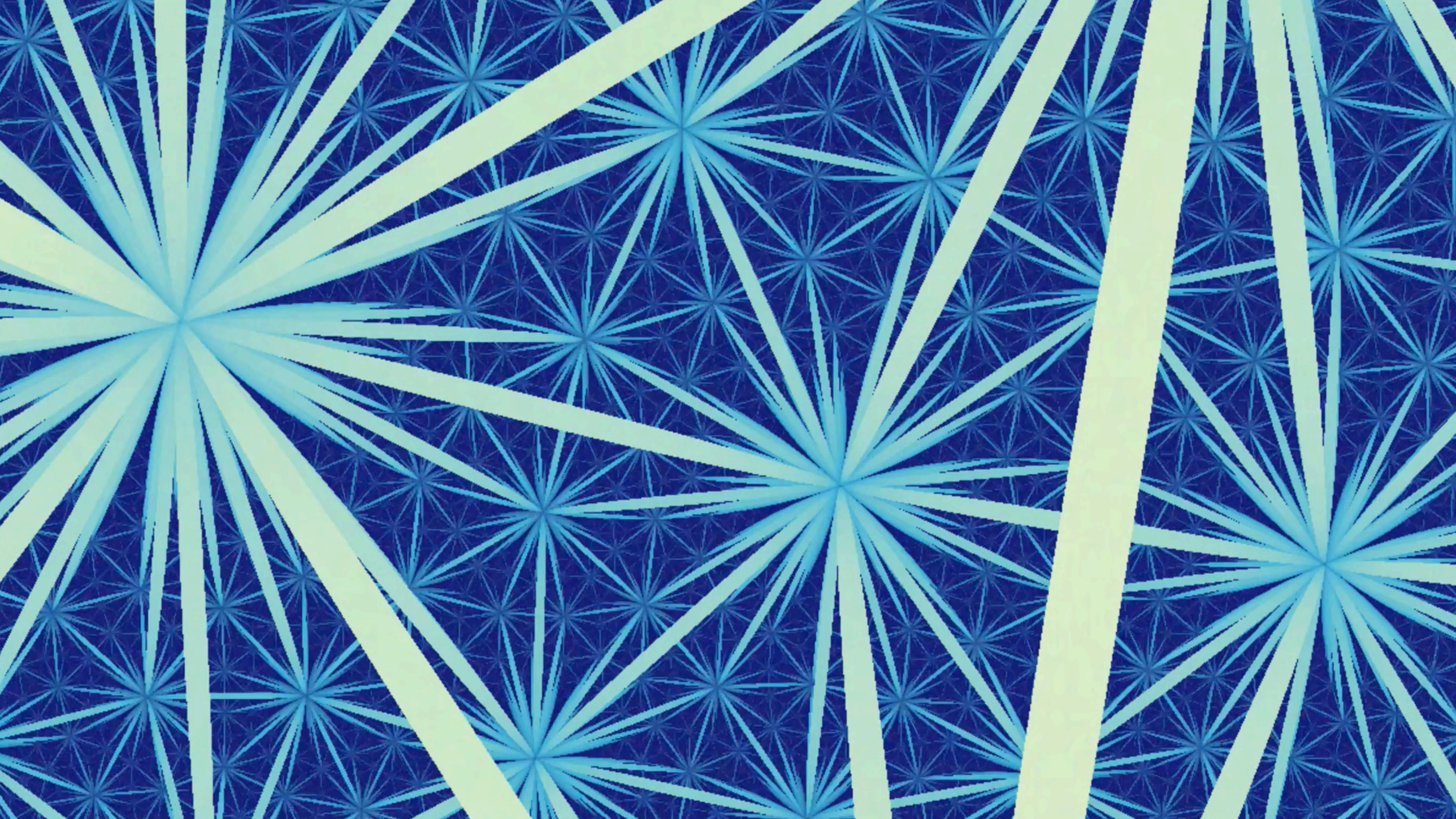
Figure 8 Knot



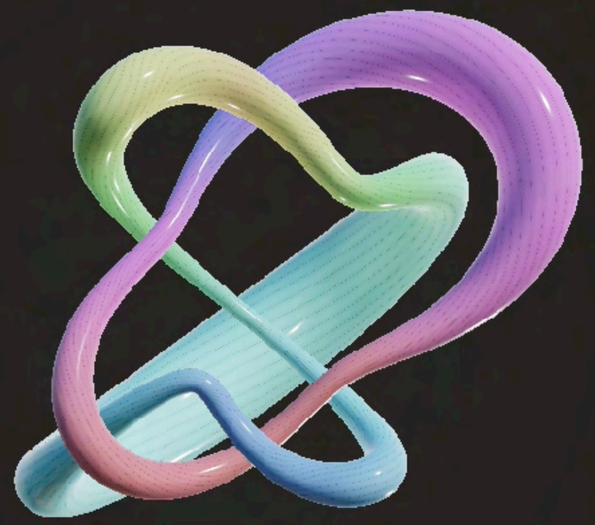
Building the Hyperbolic Structure

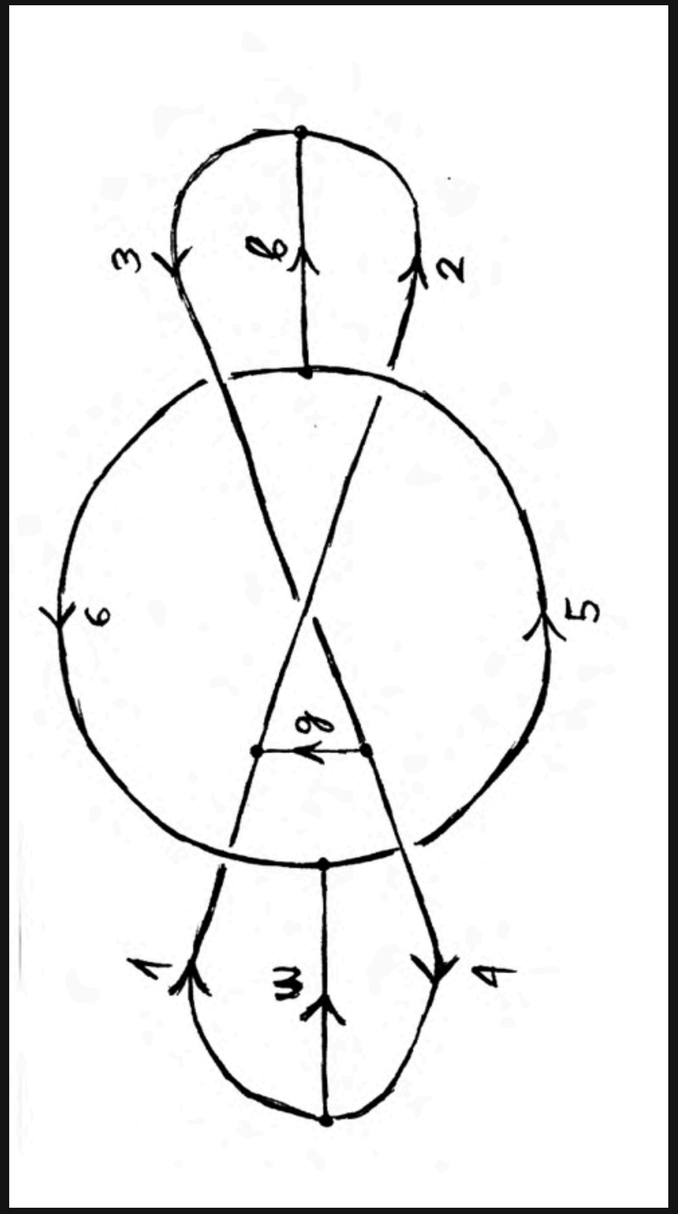


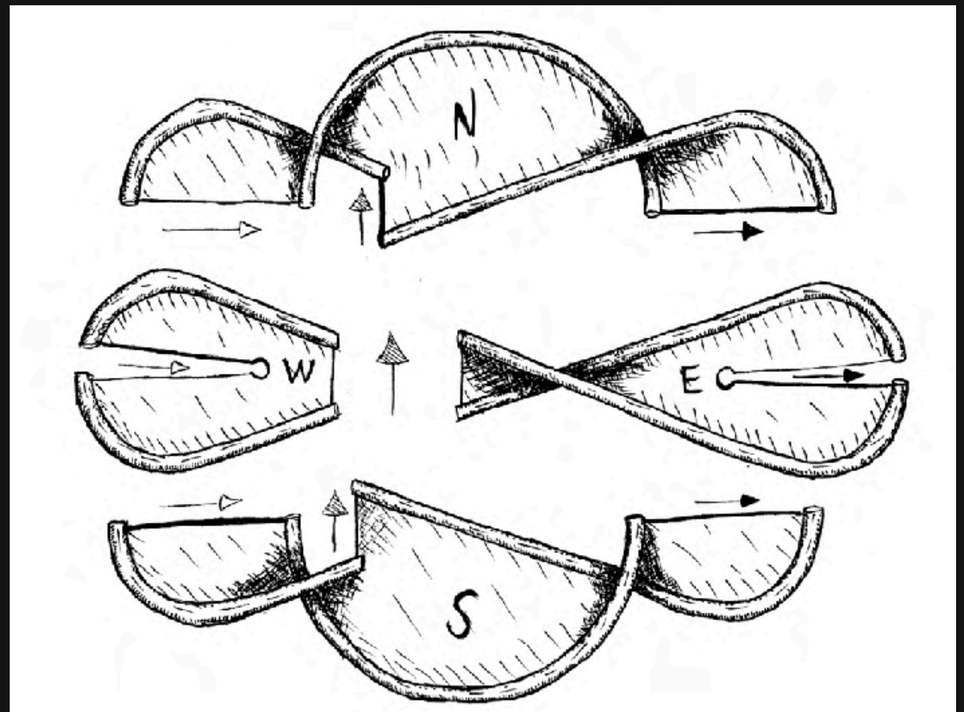


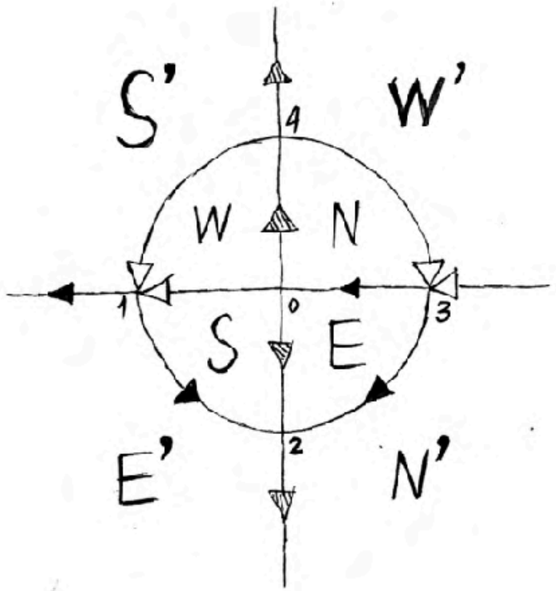


The Whitehead Link

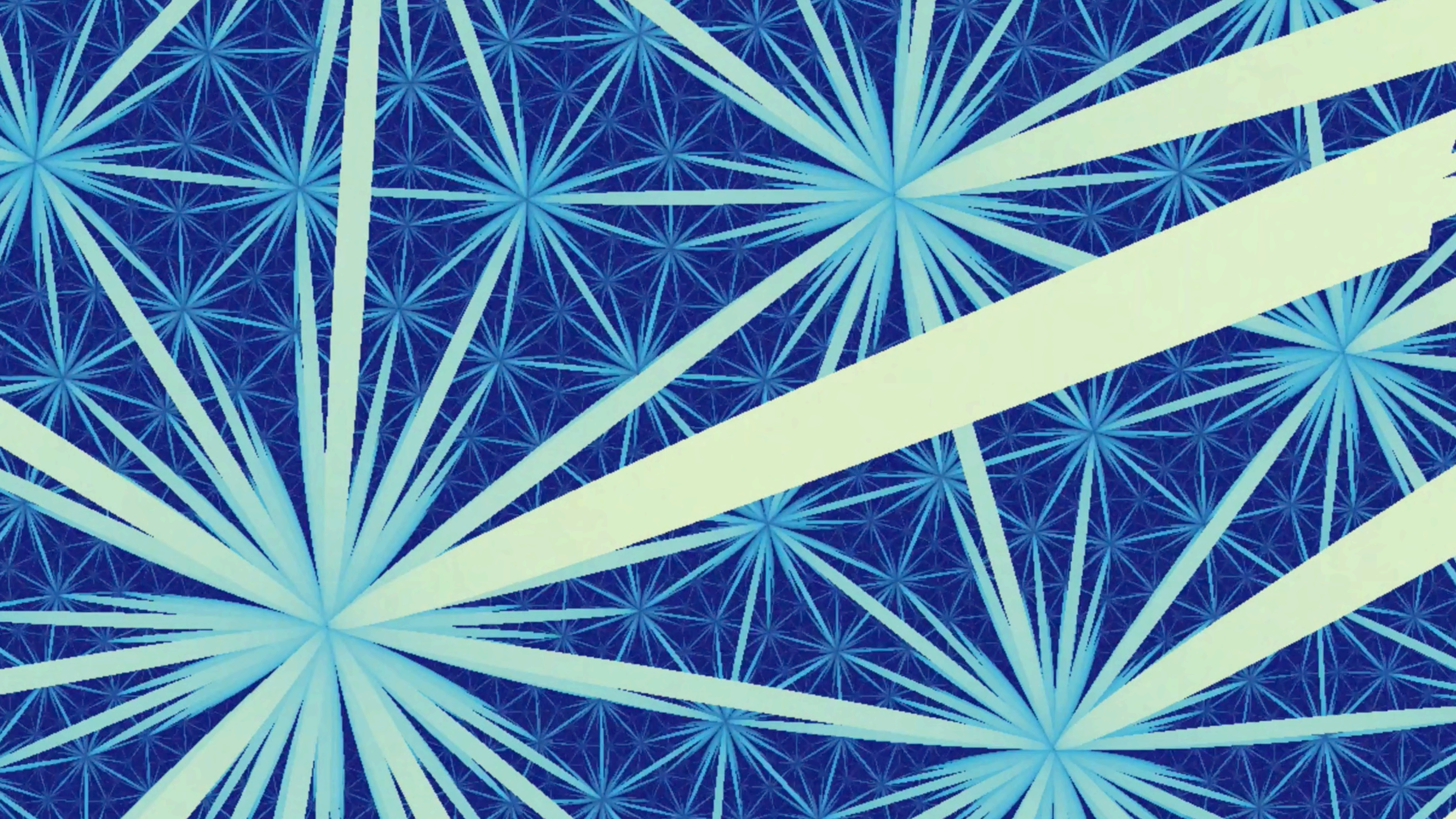



3
8
2
6
5
9
1
7
4


N
W
E
S


S'
W'
4
W
N
1
0
3
S
E
2
E'
N'

0
W
N
S
E
1
3
2
∞



When does this Actually work?

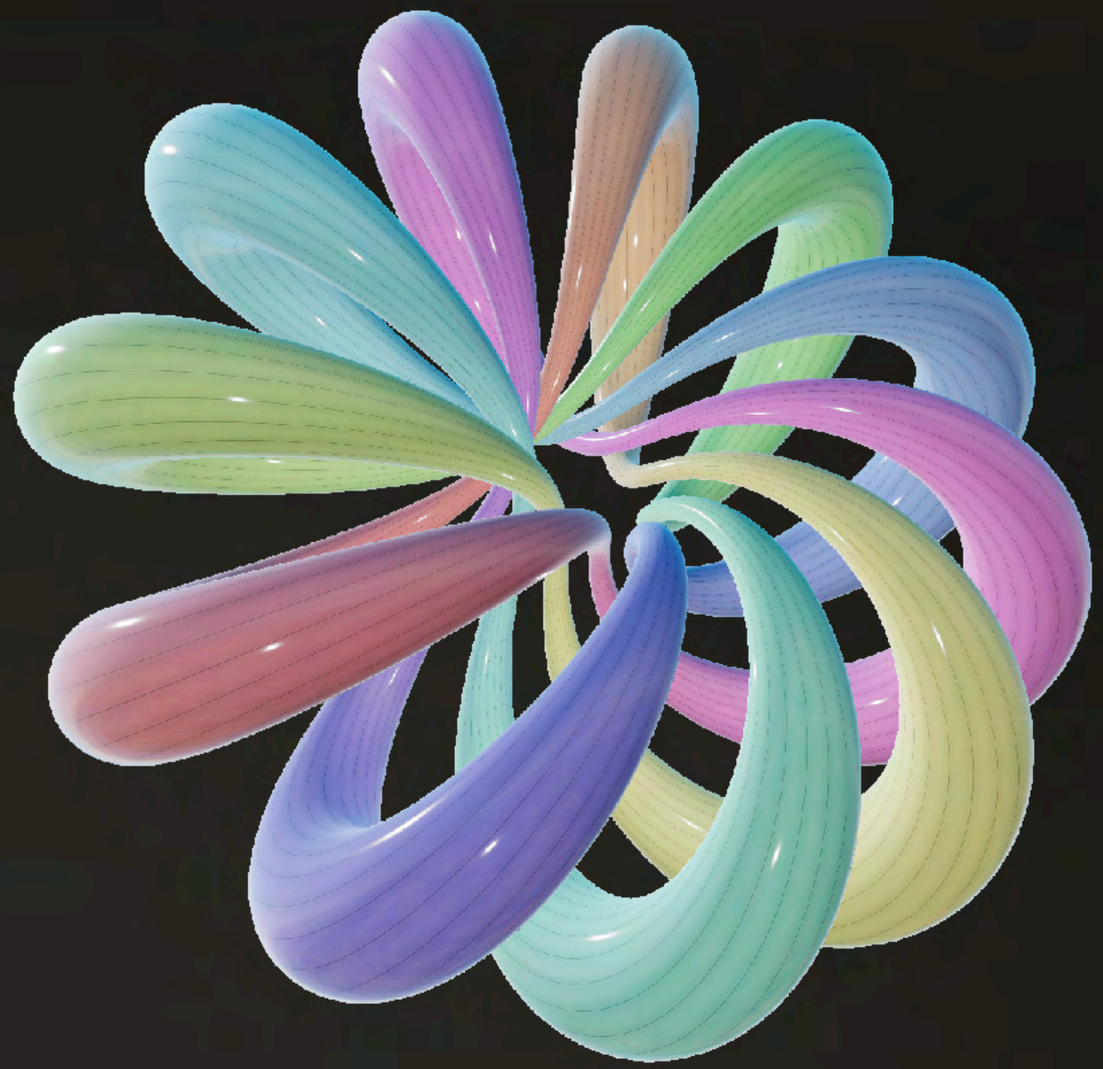
How many knot complements admit hyperbolic metrics?

THURSTON

Every knot, with the exception of torus and satellite knots, is hyperbolic.

Torus Knots

Lie on the surface of a torus

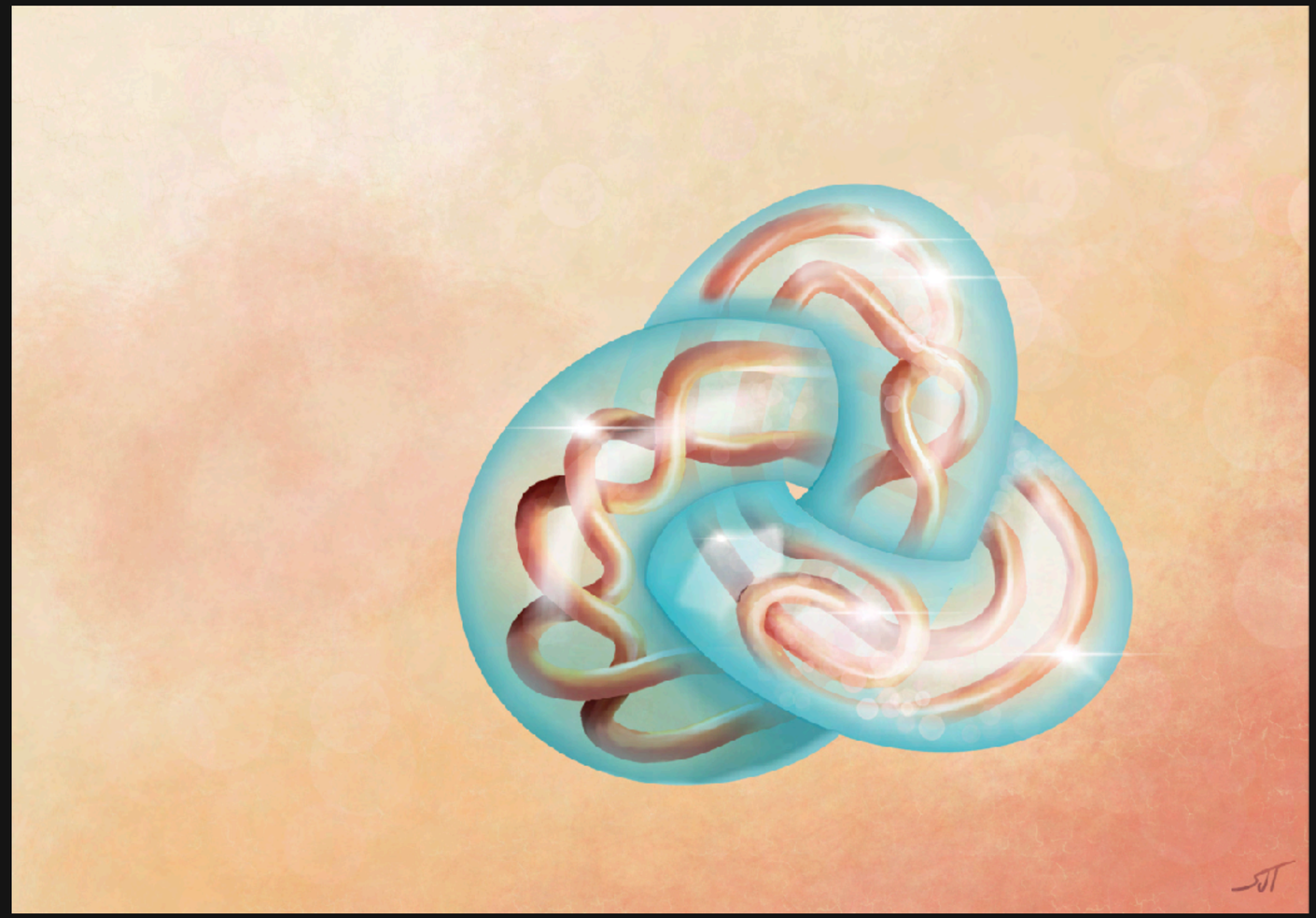


Satellite Knots

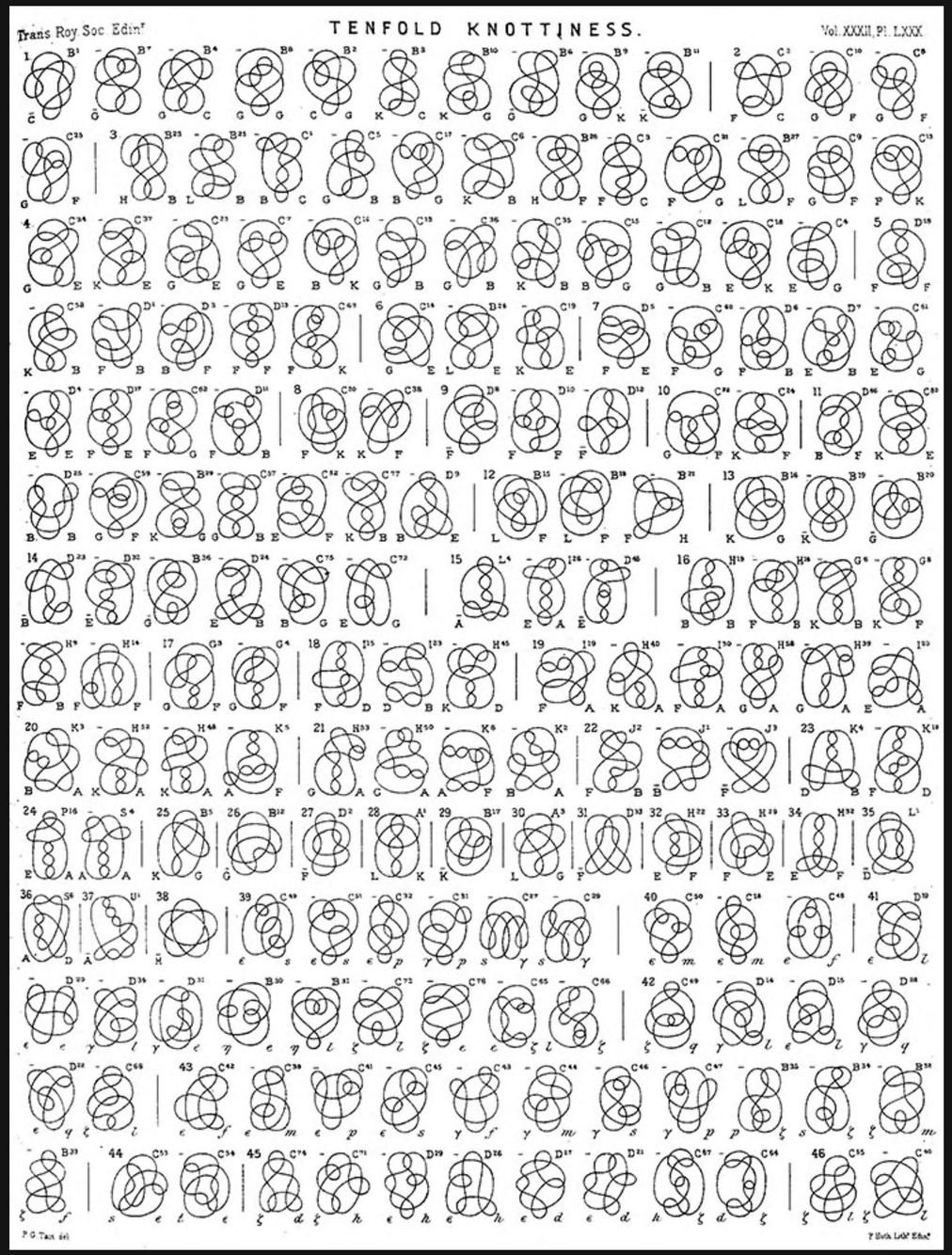
"Knots within knots"

Satellite Knots

"Knots within knots"



Most knots are hyperbolic!



How Does this Help?

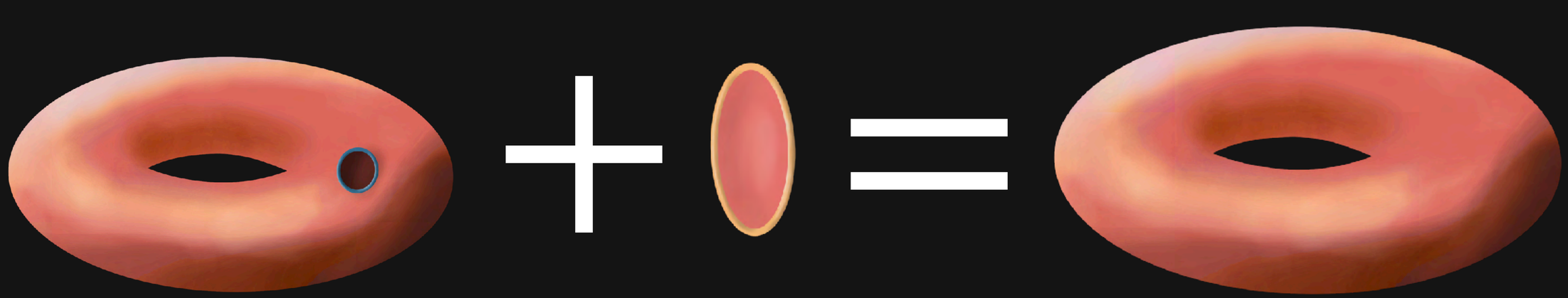
We need a **CLOSED MANIFOLD!**

Idea!

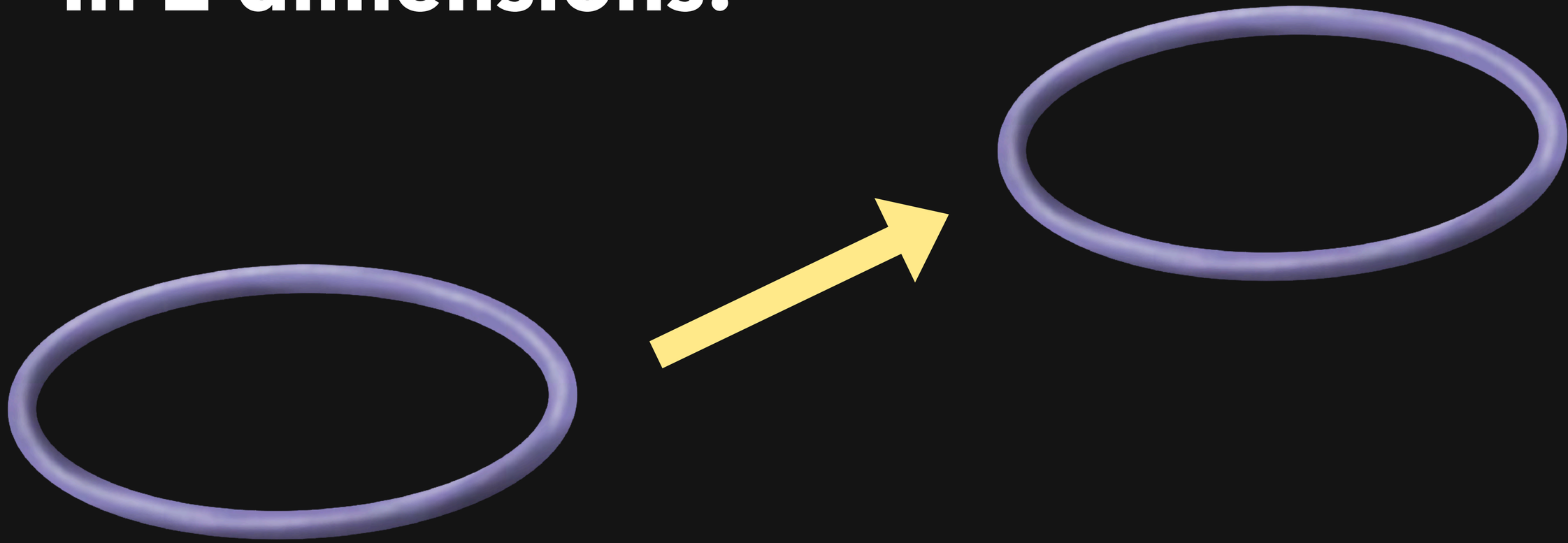
“Plug” the (torus shaped) “hole”

This is called Dehn Filling

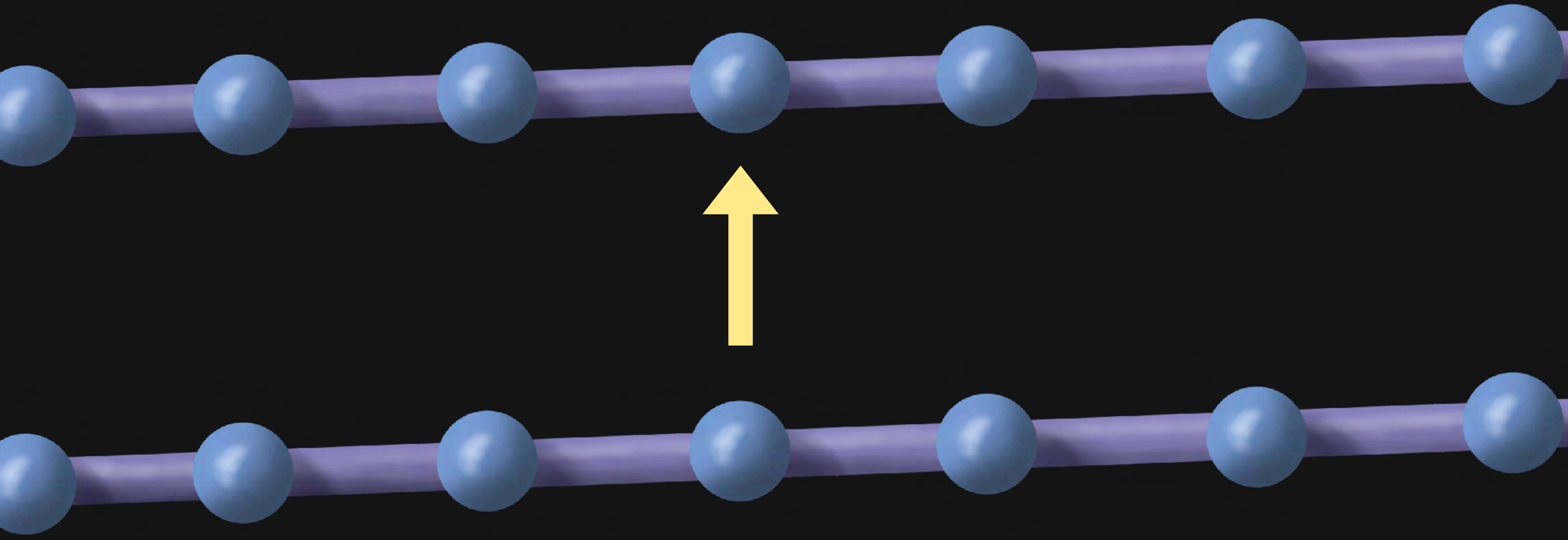
Nothing Interesting Happens in 2 dimensions:



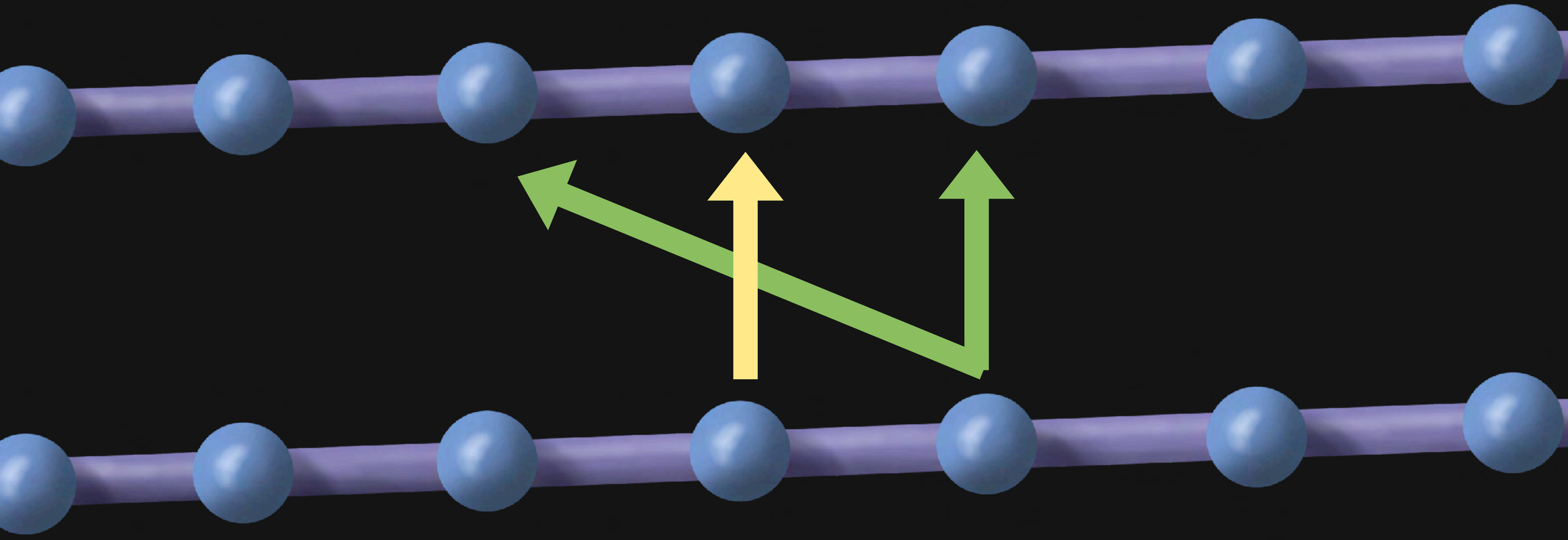
Nothing Interesting Happens in 2 dimensions:



Nothing Interesting Happens in 2 dimensions:

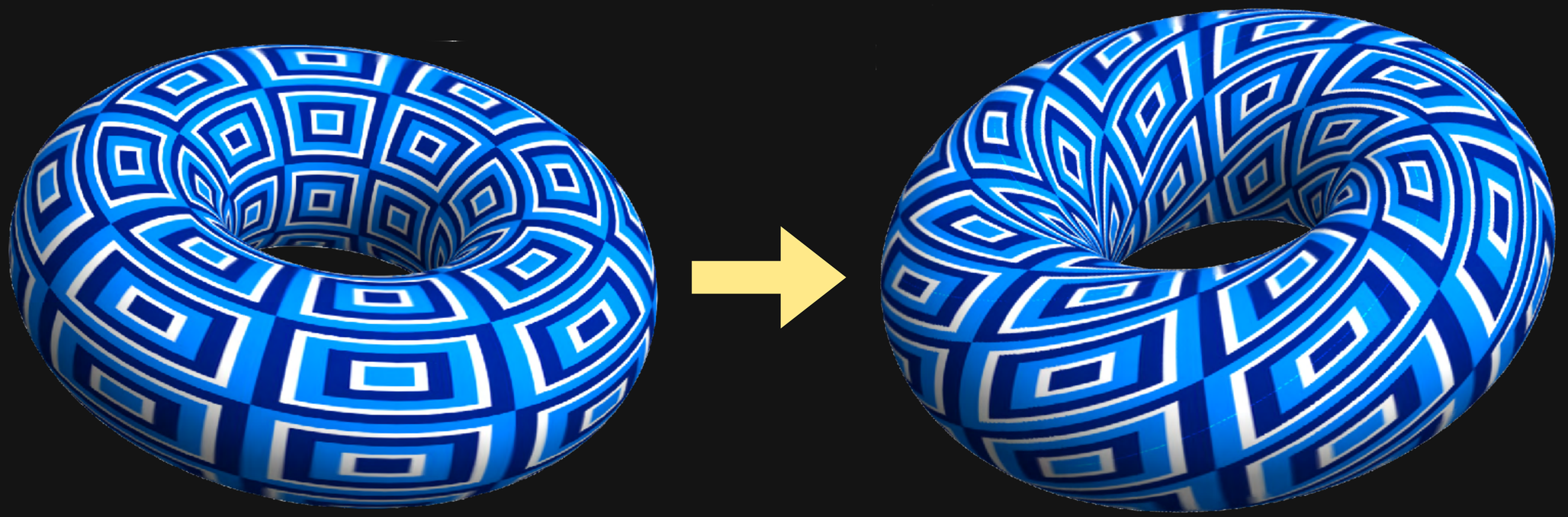


Nothing Interesting Happens in 2 dimensions:

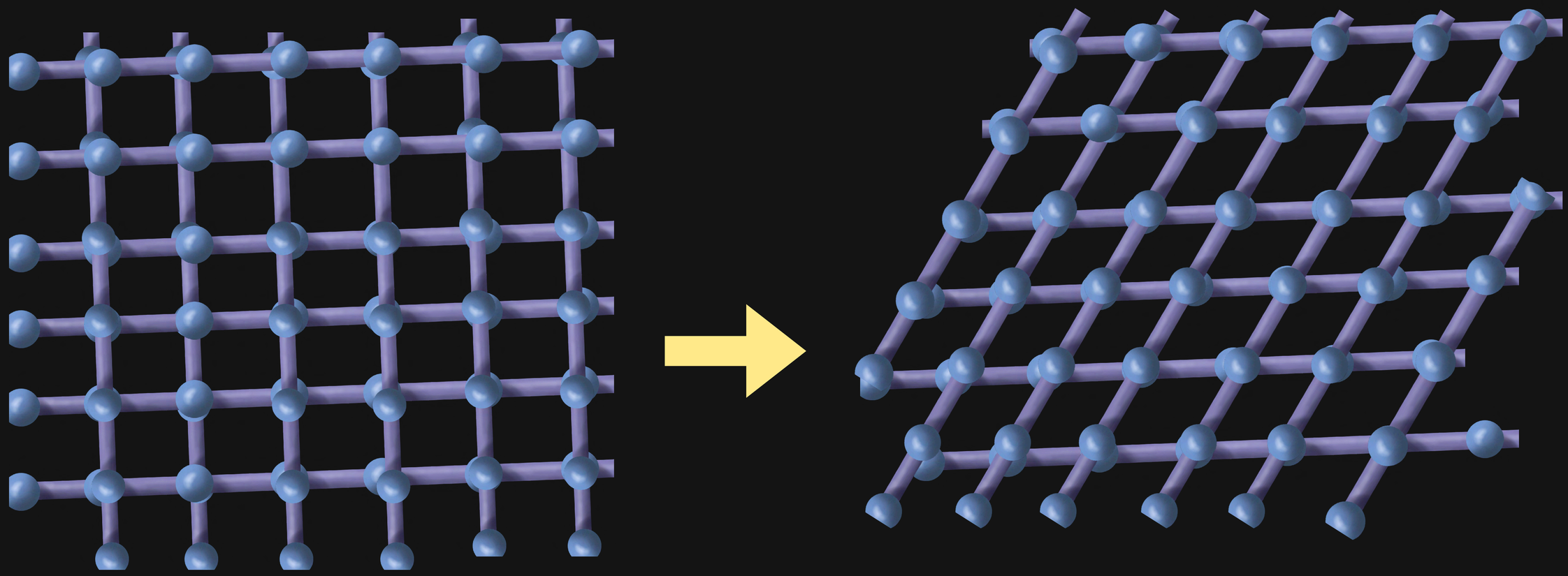


What about in 3D?

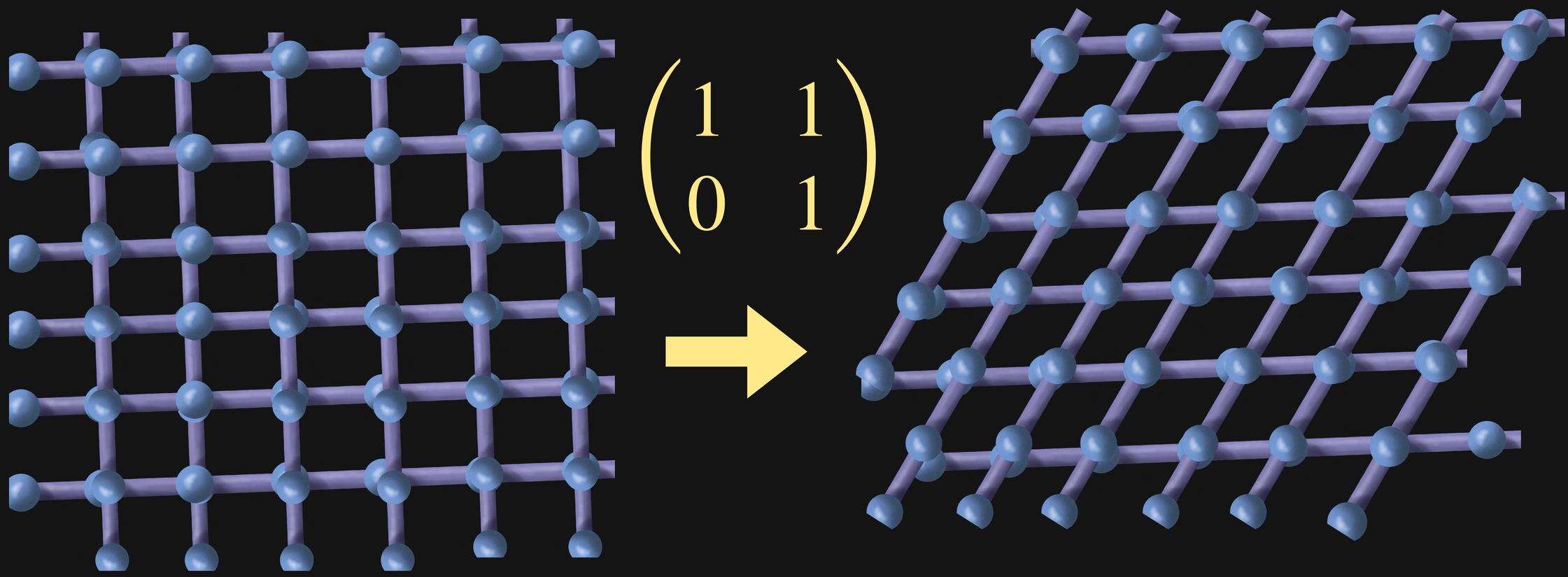
Self-maps of a torus



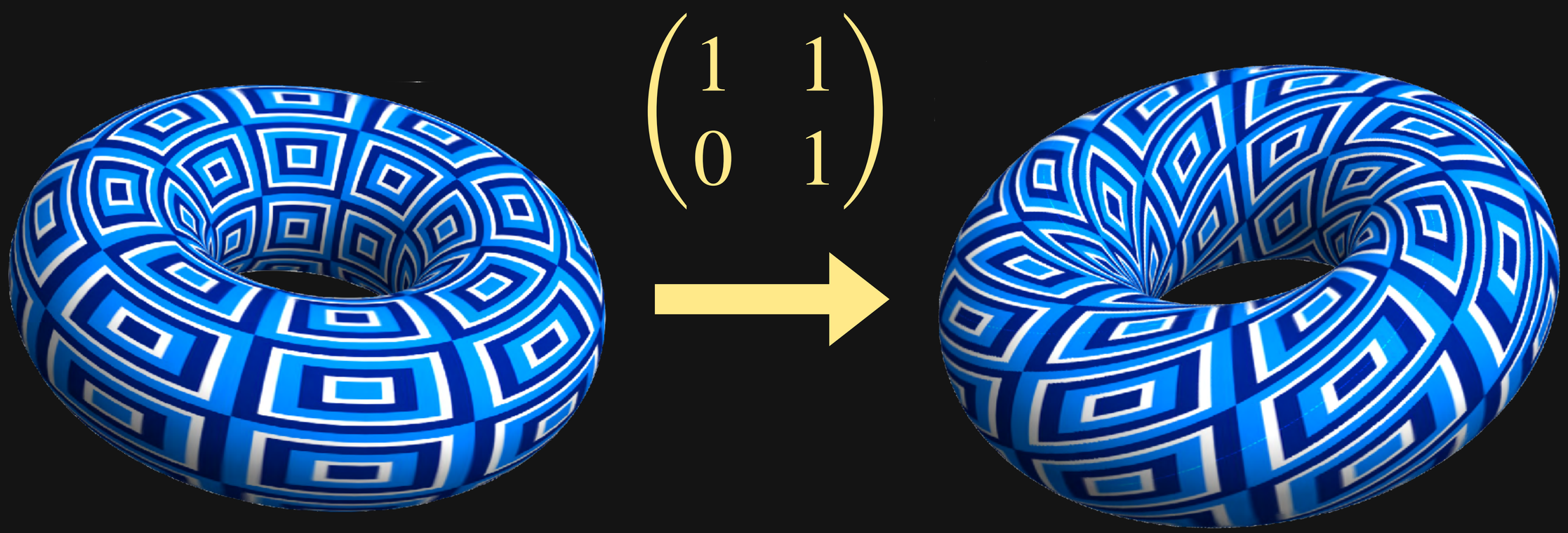
Self-maps of a torus



Self-maps of a torus



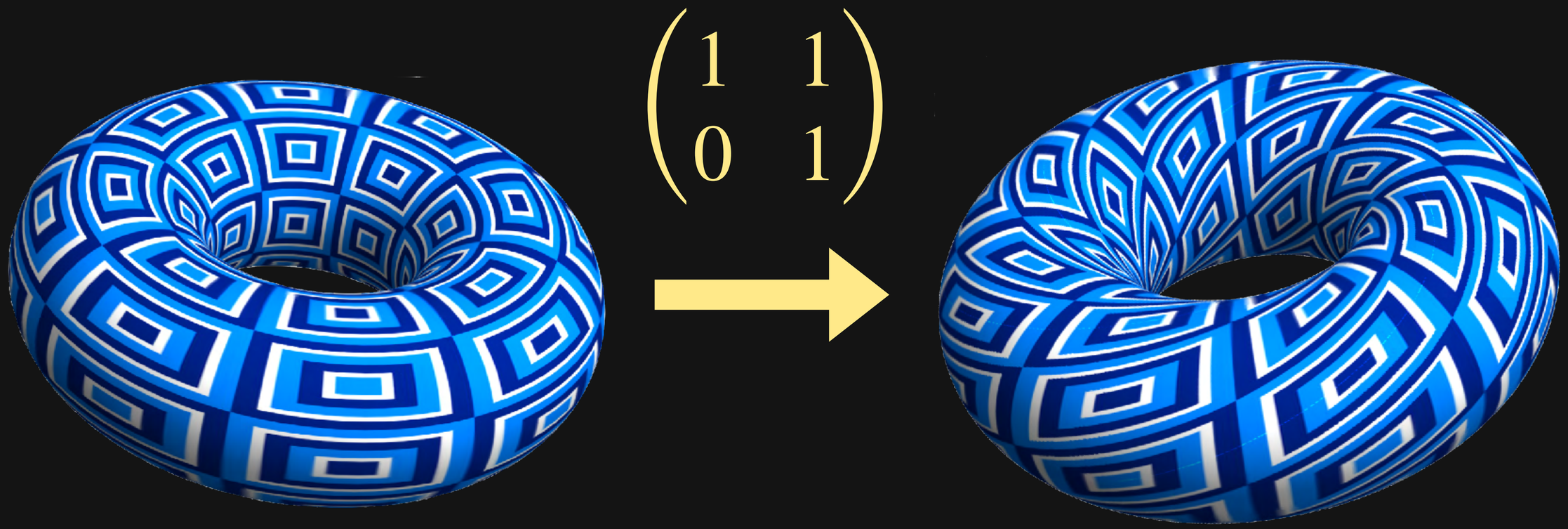
Self-maps of a torus

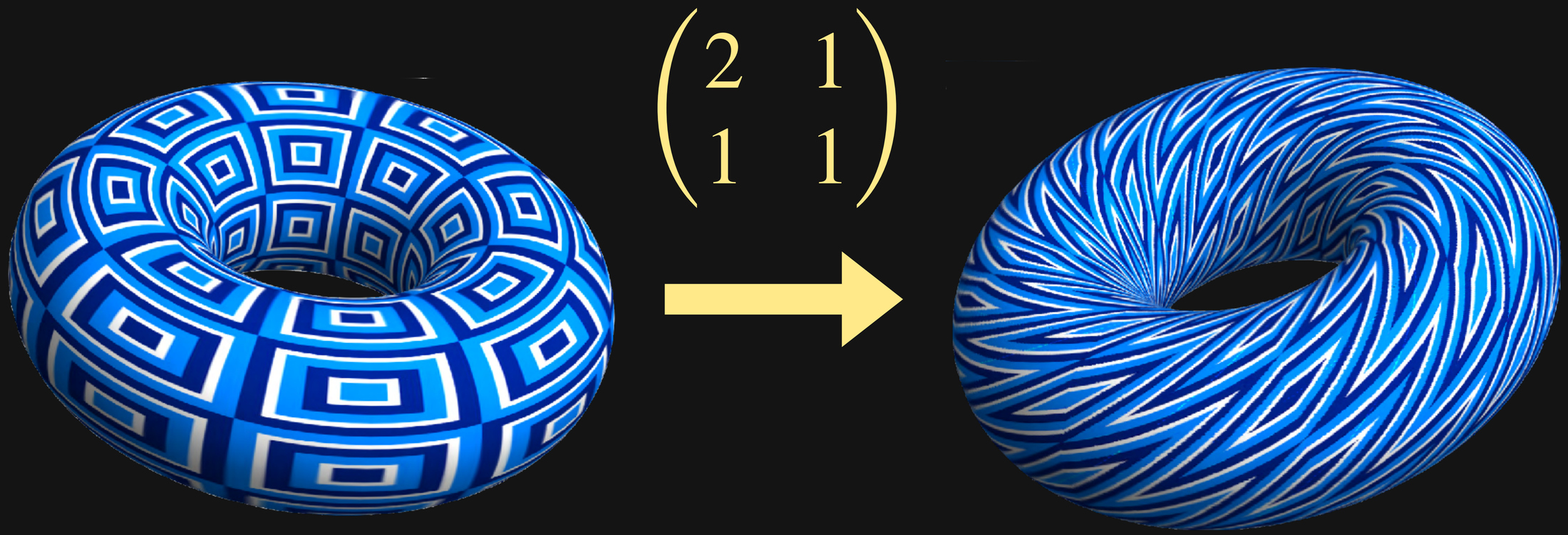


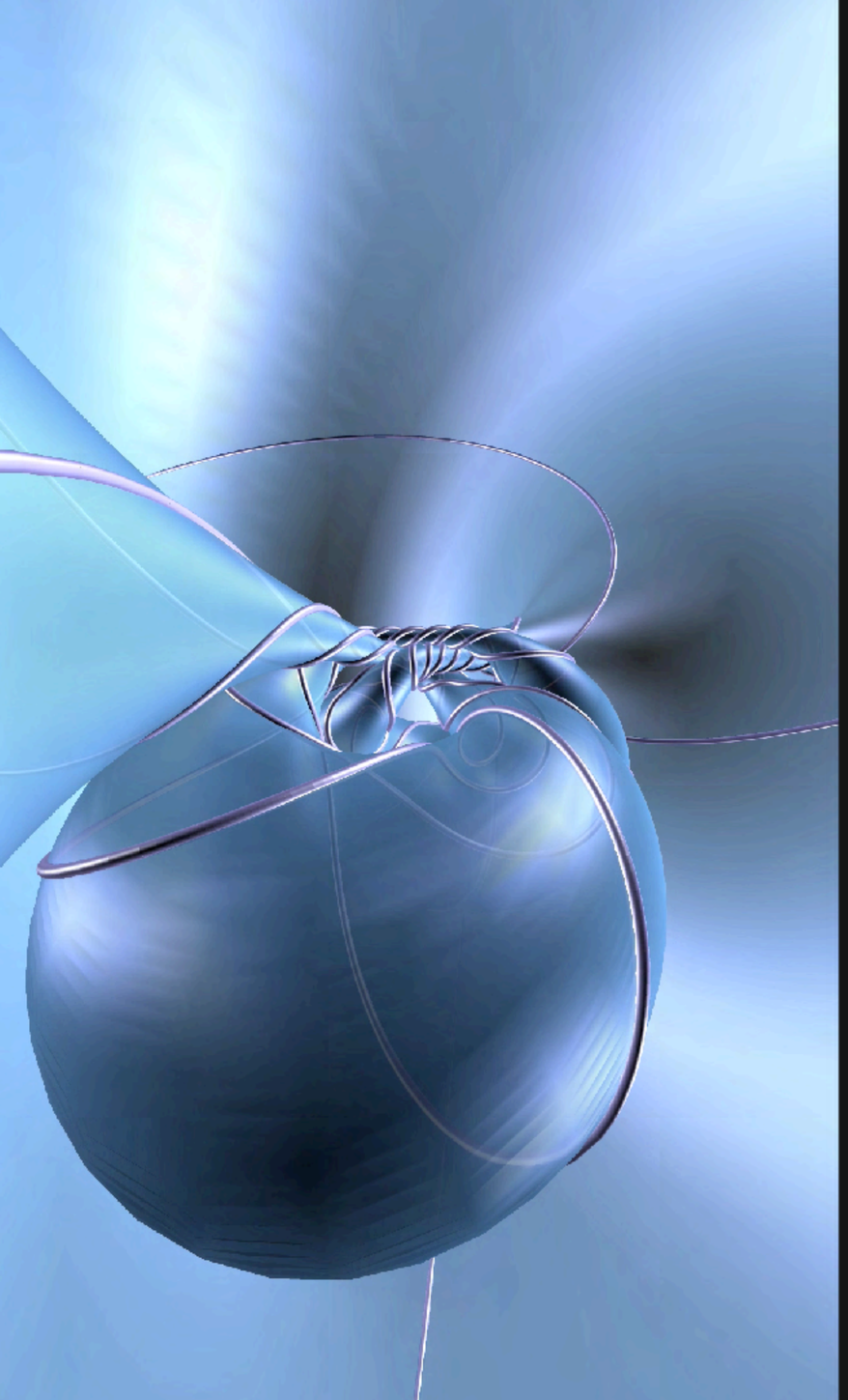
Self-maps of a torus

The Mapping Class Group of the torus (self homeomorphisms modulo isotopy) is

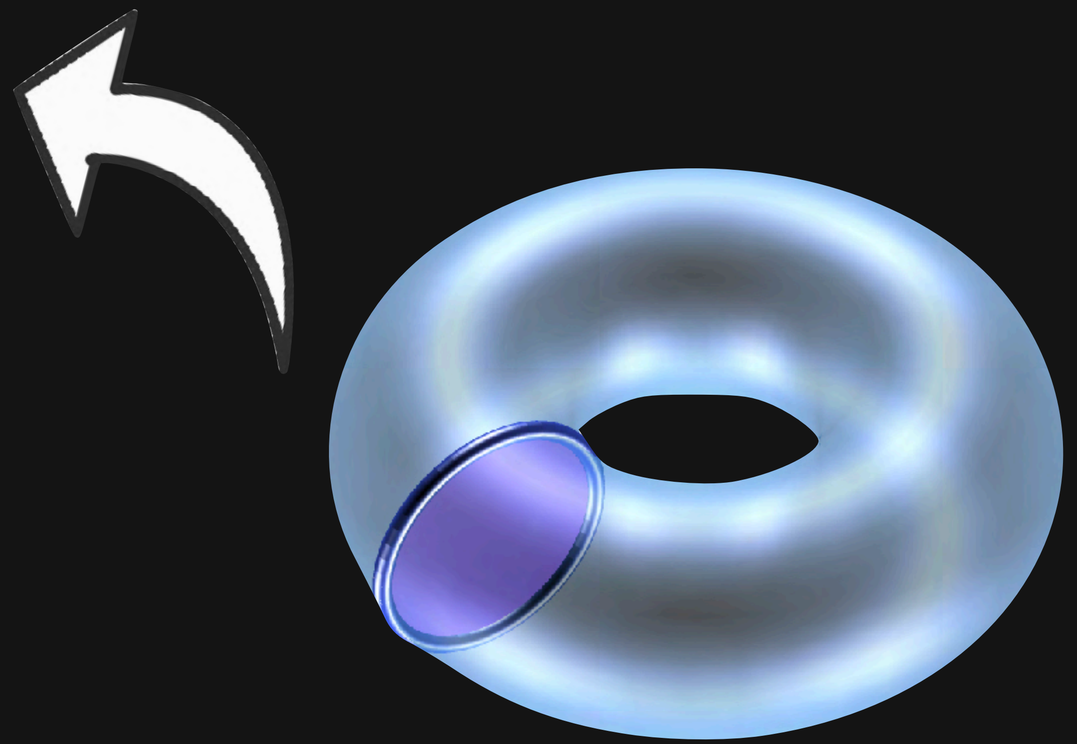
$$MCG(Torus) = SL(2,Z)$$

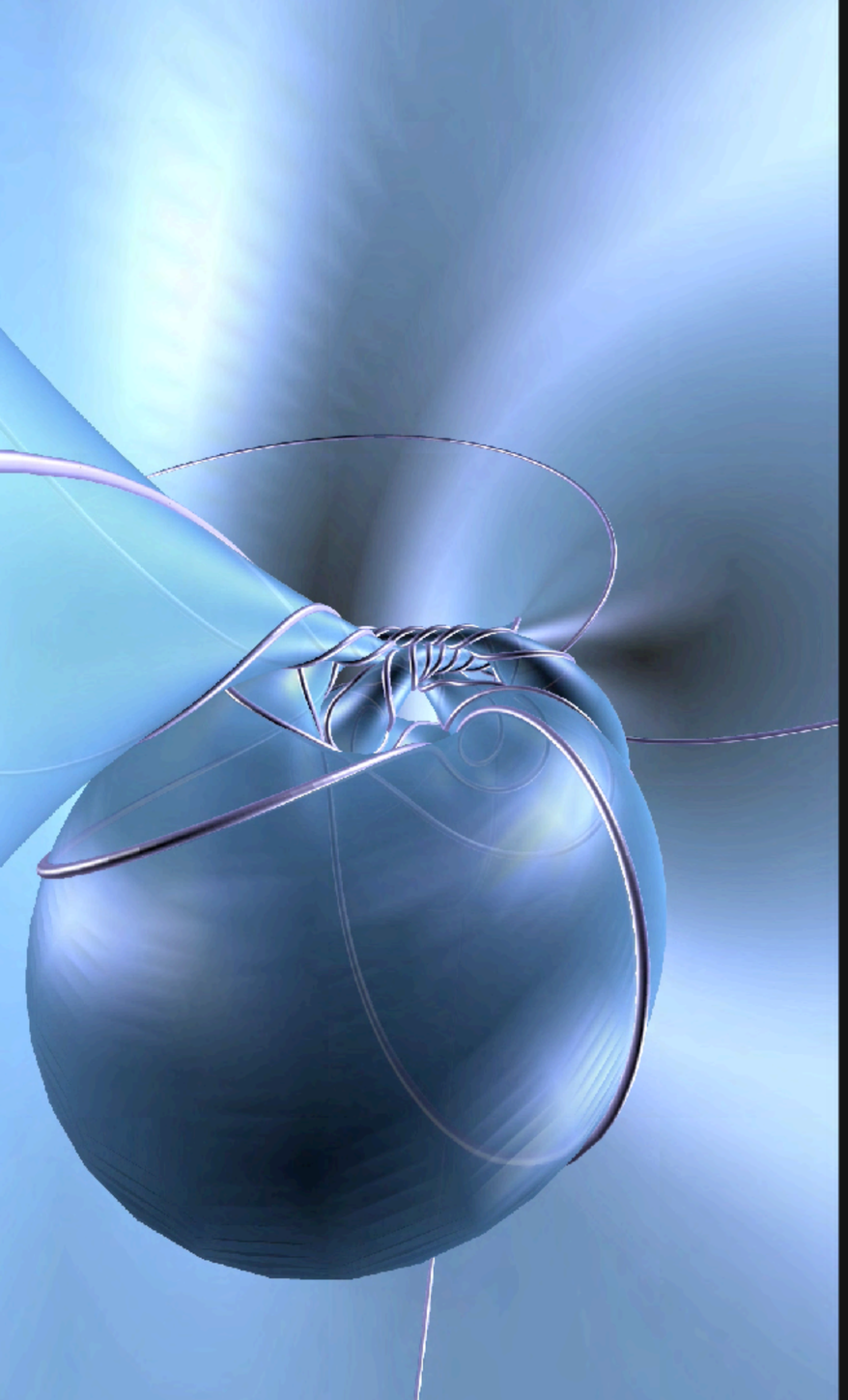

$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$


$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$



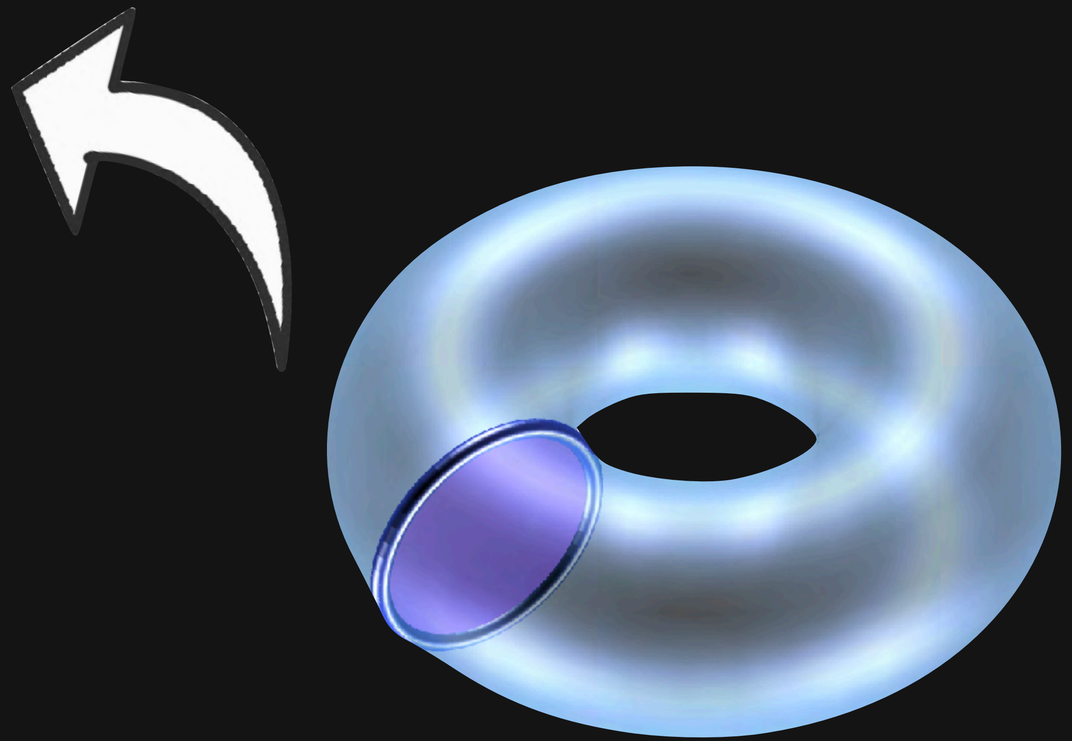
What does this look like?



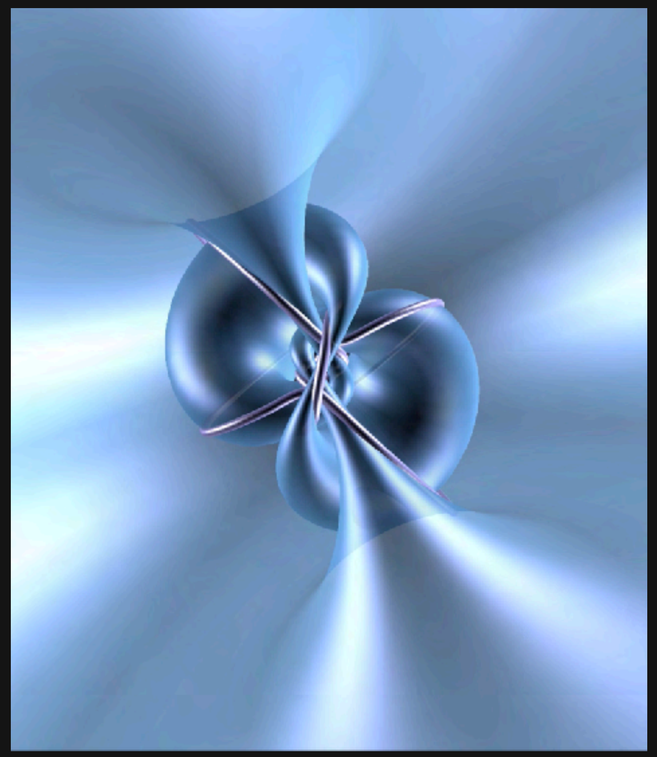


Effect on Fundamental group:

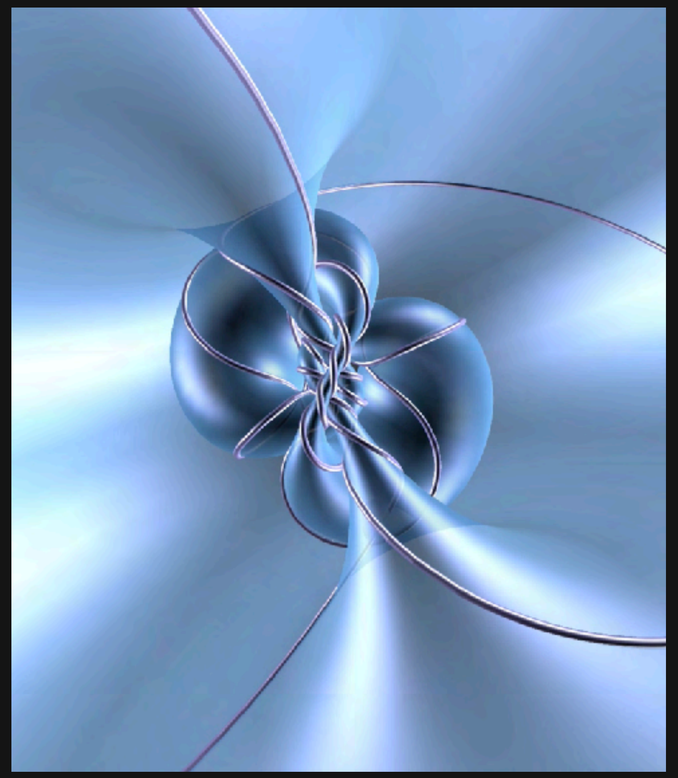
The loop on the left hand side becomes contractible!



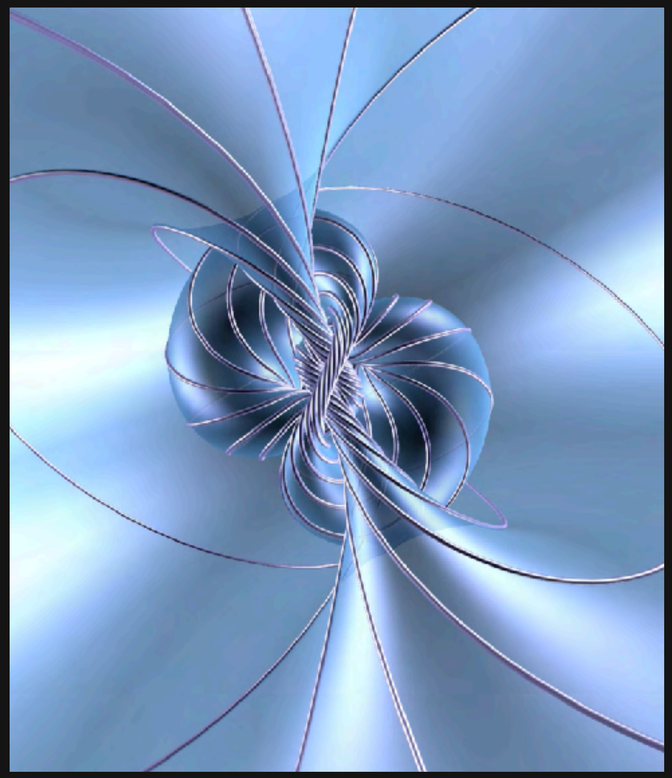
For each knot, this gives us an infinity of different closed manifolds!



5/1

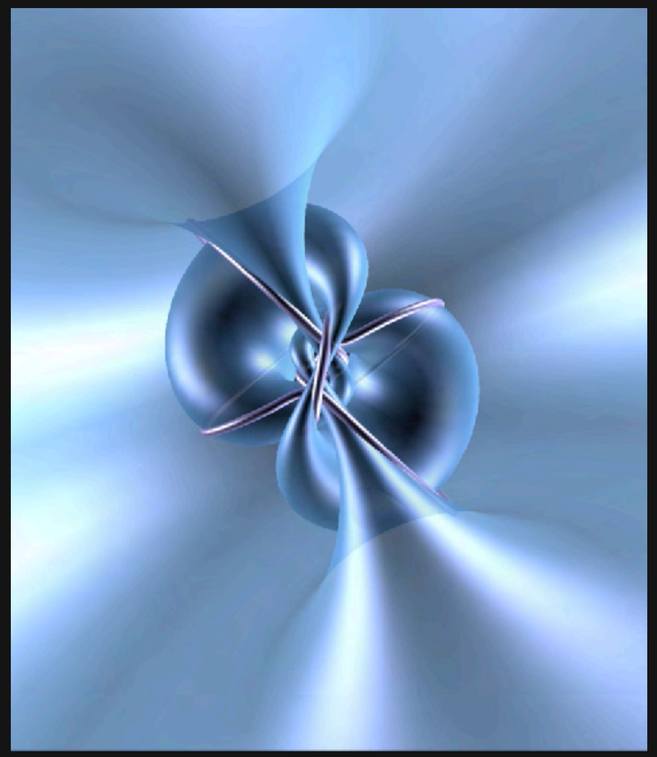


21/2



55/7

These are parameterized by a rational number, the “slope” of the gluing map



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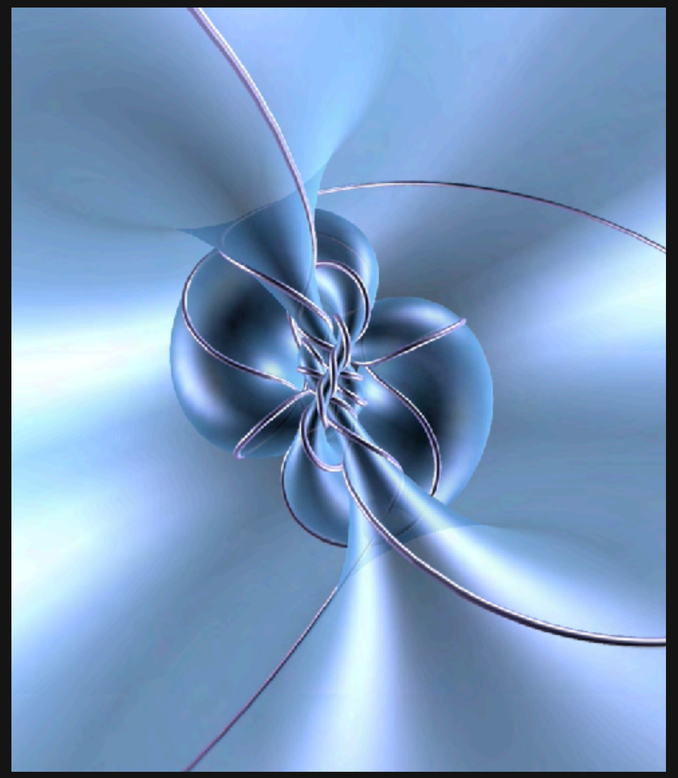

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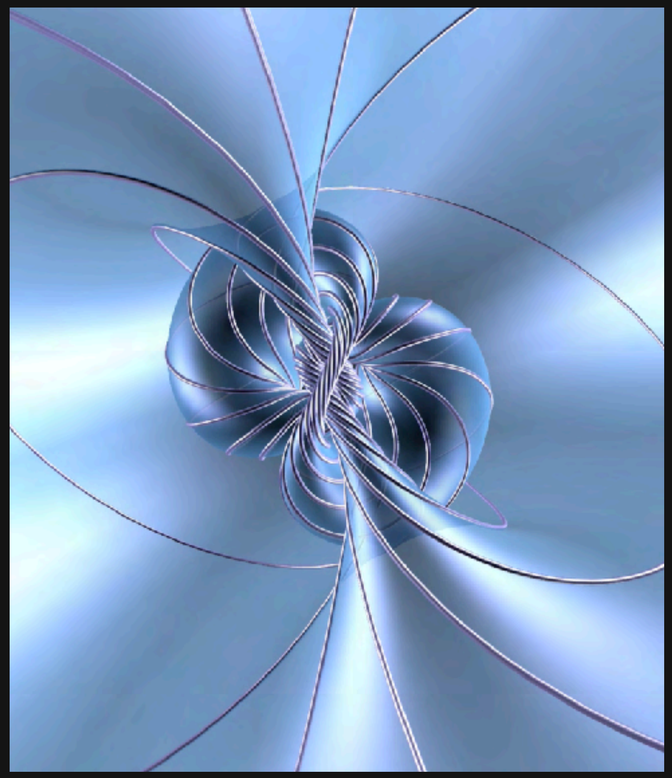


55/7

What does this do to the geometry?!?

Dimension 2: filling the puncture drastically changes the geometry

Dehn filling doesn't even KNOW about geometry!

Why should it matter that the manifold we started with was hyperbolic?

Using a computer (Snappy) we can now just be brave and give it a try!

dehn_fill(*filling_data, which_cusp=None*)

Set the Dehn filling coefficients of the cusps. This can be specified in the following ways, where the cusps are numbered by 0,1,...,(num_cusps - 1).

- Fill cusp 2:

```
>>> M = Manifold('8^4_1')
>>> M.dehn_fill((2,3), 2)
>>> M
8^4_1(0,0)(0,0)(2,3)(0,0)
```

- Fill the last cusp:

```
>>> M.dehn_fill((1,5), -1)
>>> M
8^4_1(0,0)(0,0)(2,3)(1,5)
```

- Fill the first two cusps:

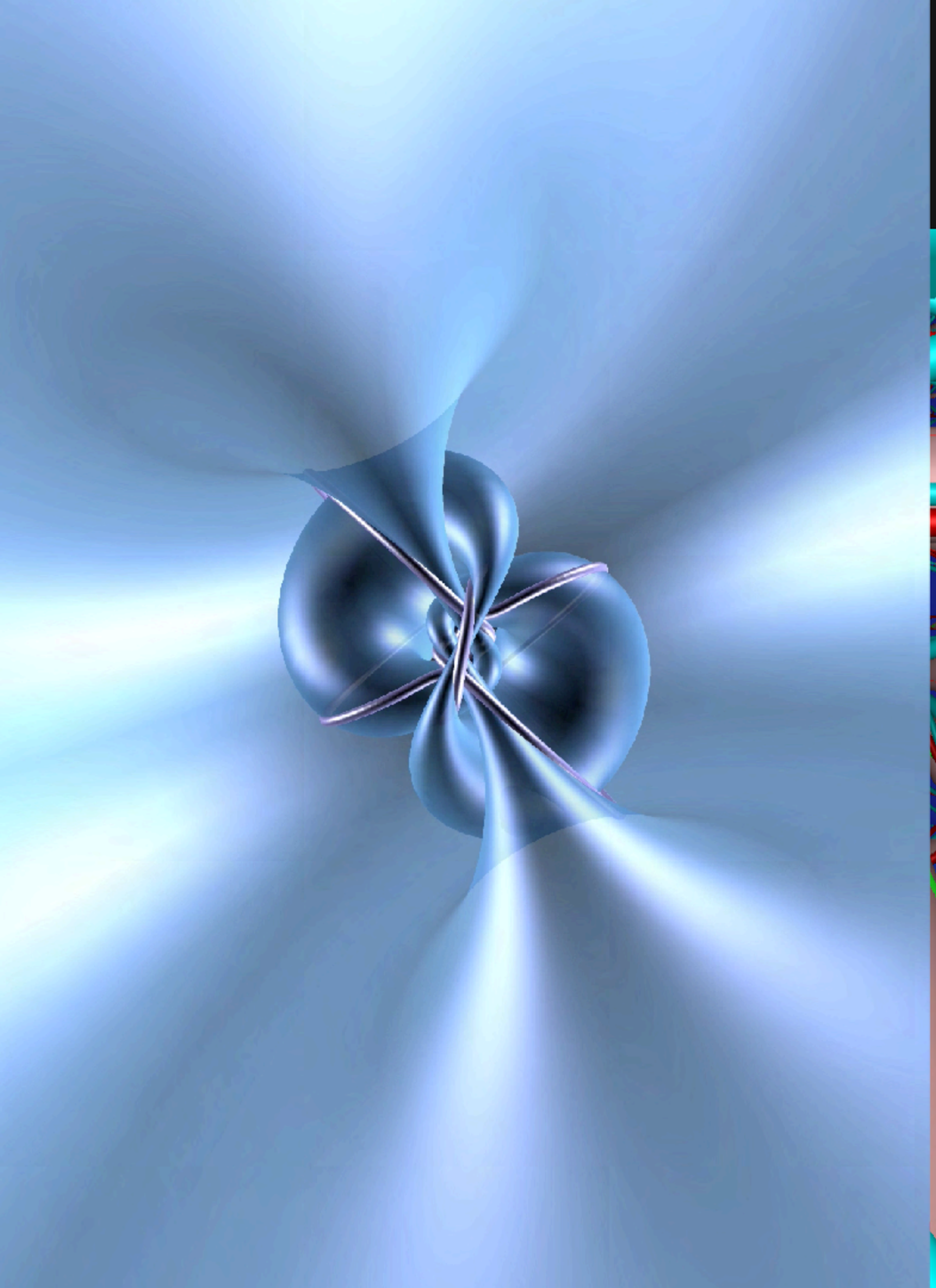
```
>>> M.dehn_fill( [ (3,0), (1, -4) ])
>>> M
8^4_1(3,0)(1,-4)(2,3)(1,5)
```

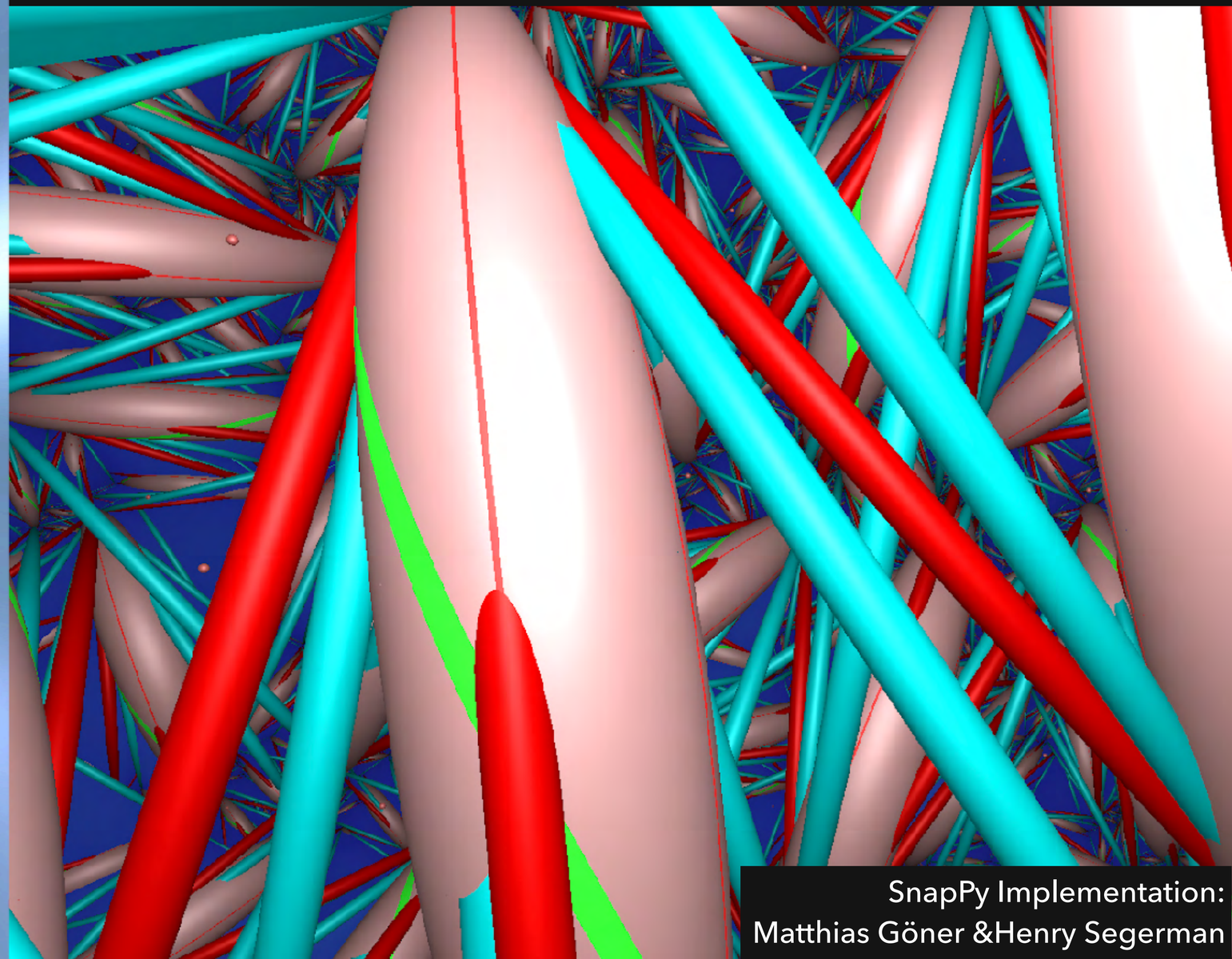
- When there is only one cusp, there's a shortcut

```
>>> N = Manifold('m004')
>>> N.dehn_fill( (-3,4) )
>>> N
m004(-3,4)
```

Does not return a new Manifold.

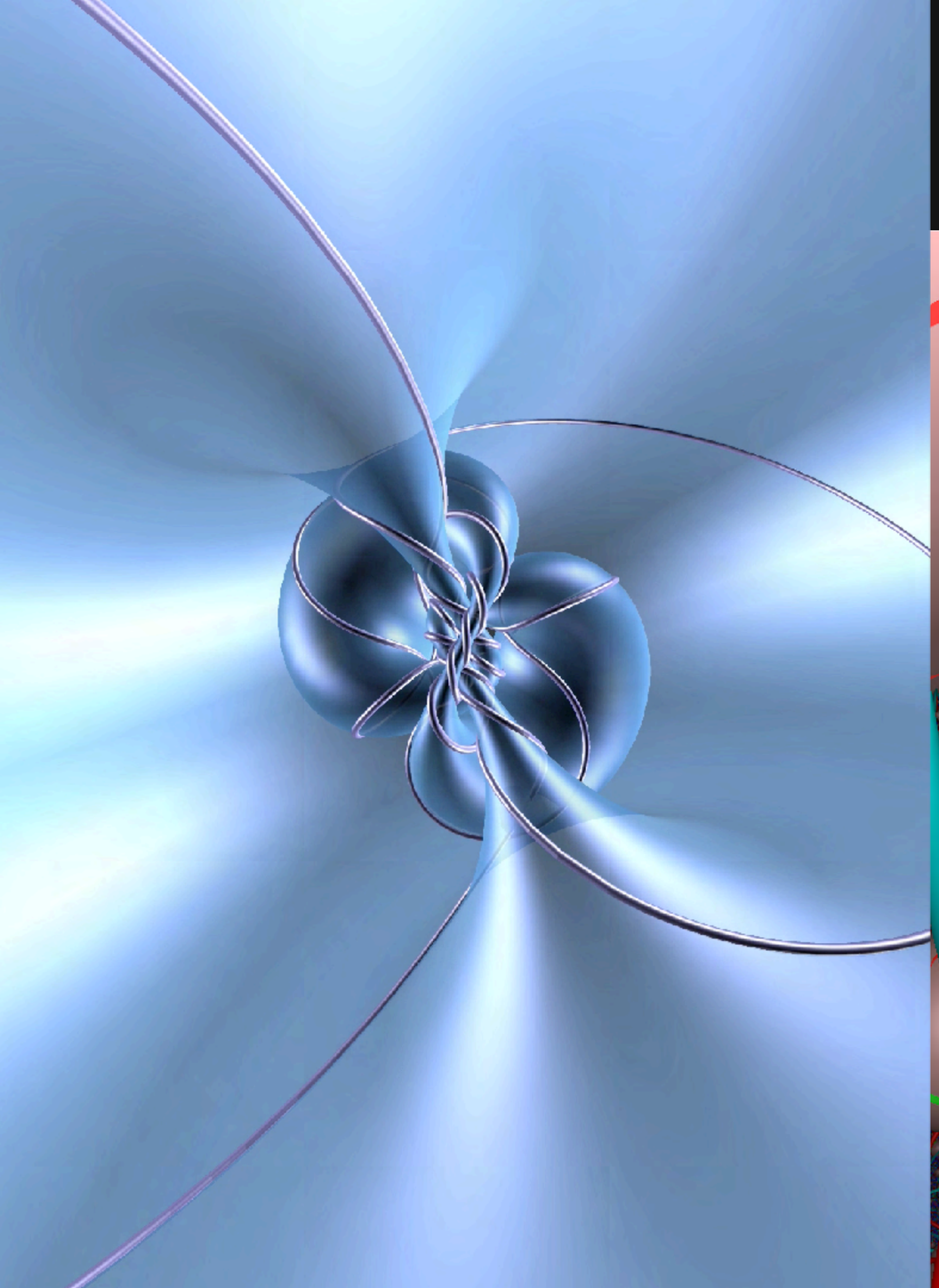
5/1

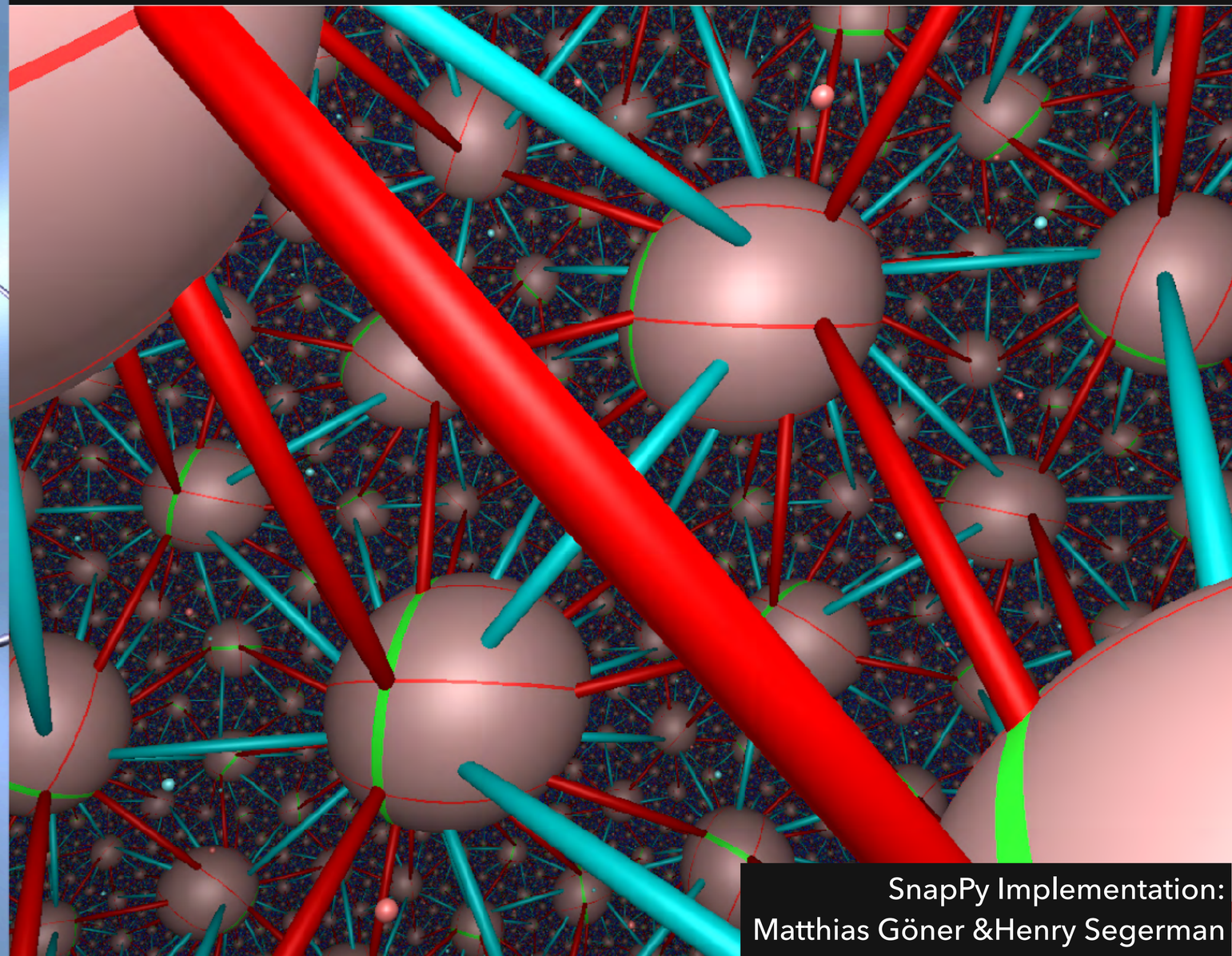




SnapPy Implementation:
Matthias Göner &Henry Segerman

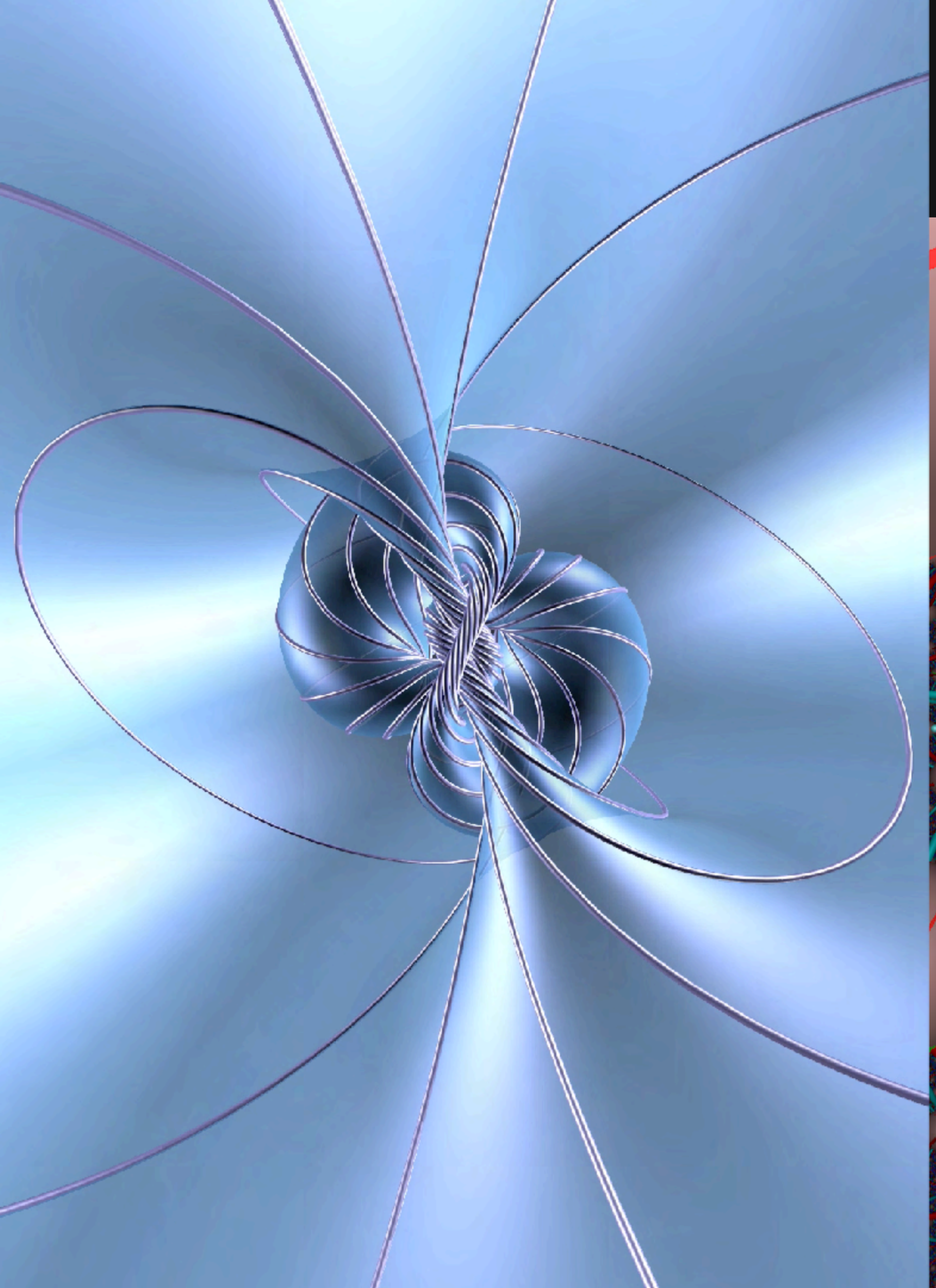
21/2

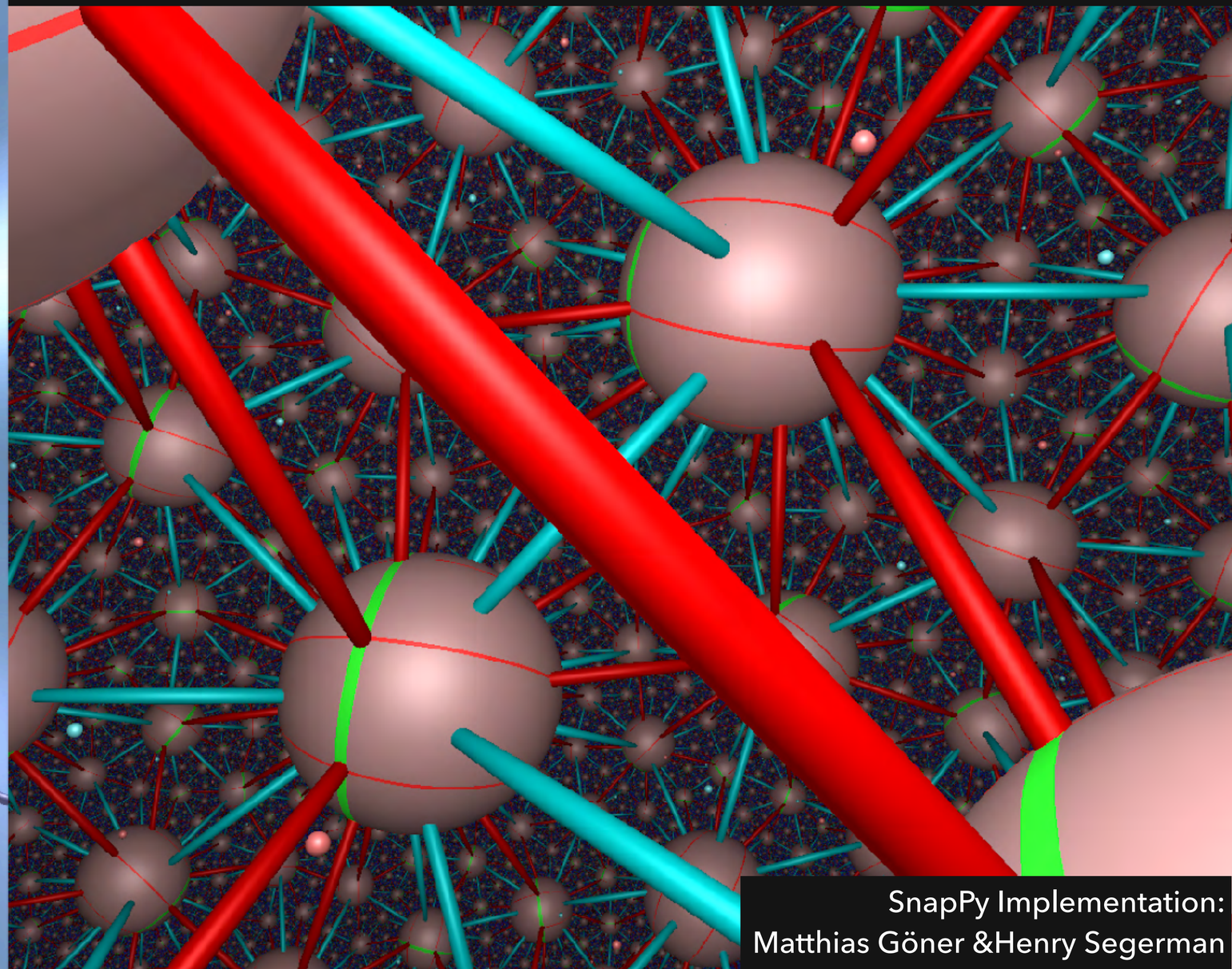




SnapPy Implementation:
Matthias Göner &Henry Segerman

55/7





SnapPy Implementation:
Matthias Göner &Henry Segerman

THURSTON 1980

For all but finitely many exceptional slopes, this results in a closed hyperbolic manifold.

If the slope curve has length 2π, the filling is hyperbolic

Idea of the proof:

Dehn fillings are parameterized by pairs of integers = elements of $\mathrm{H}_1(\partial M, \mathbb{Z})$

What if instead, we attempted to ‘fill’ using elements of $\mathrm{H}_1(\partial M, \mathbb{R})$**?**

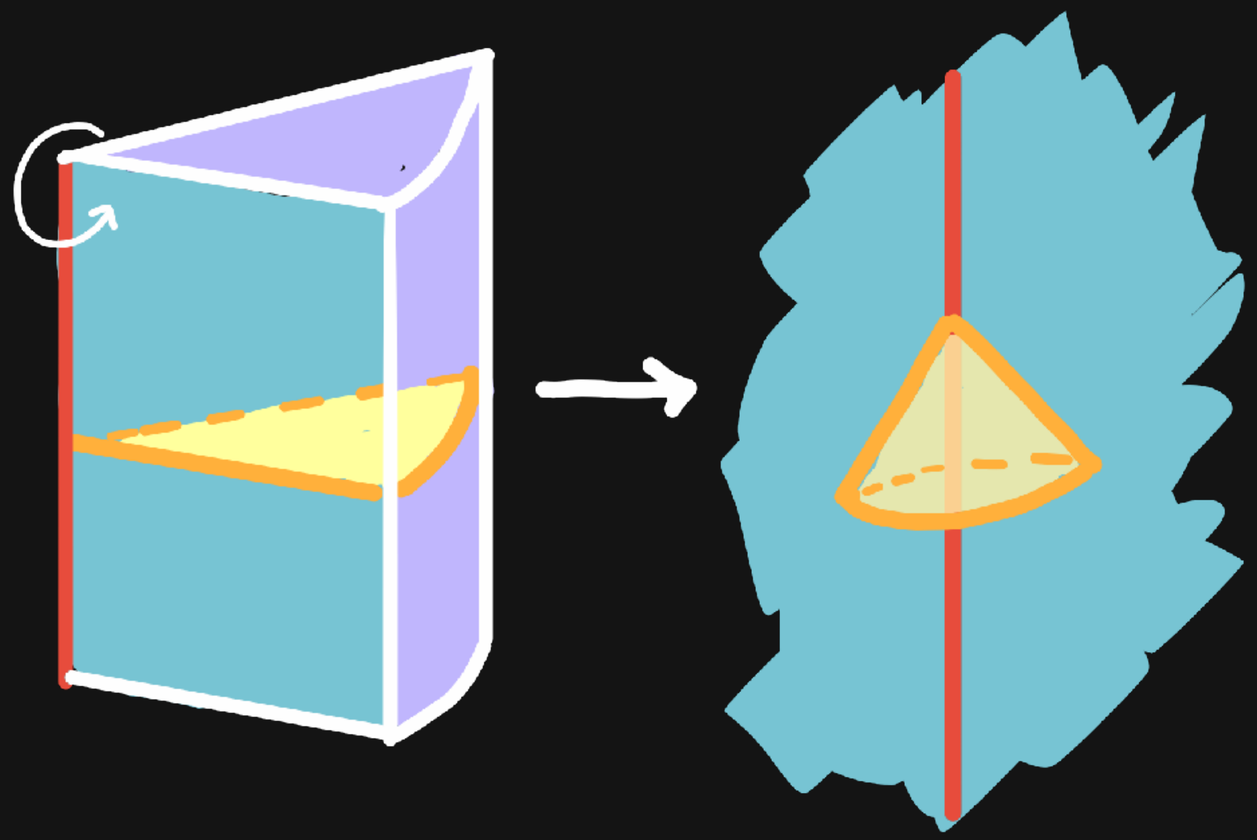
What do these points with real number coordinates **mean**?

The result: cone manifolds

Like orbifolds with arbitrary cone angle.

Complement of singular locus is an incomplete hyperbolic manifold

Can deform complete manifolds to conemanifolds.



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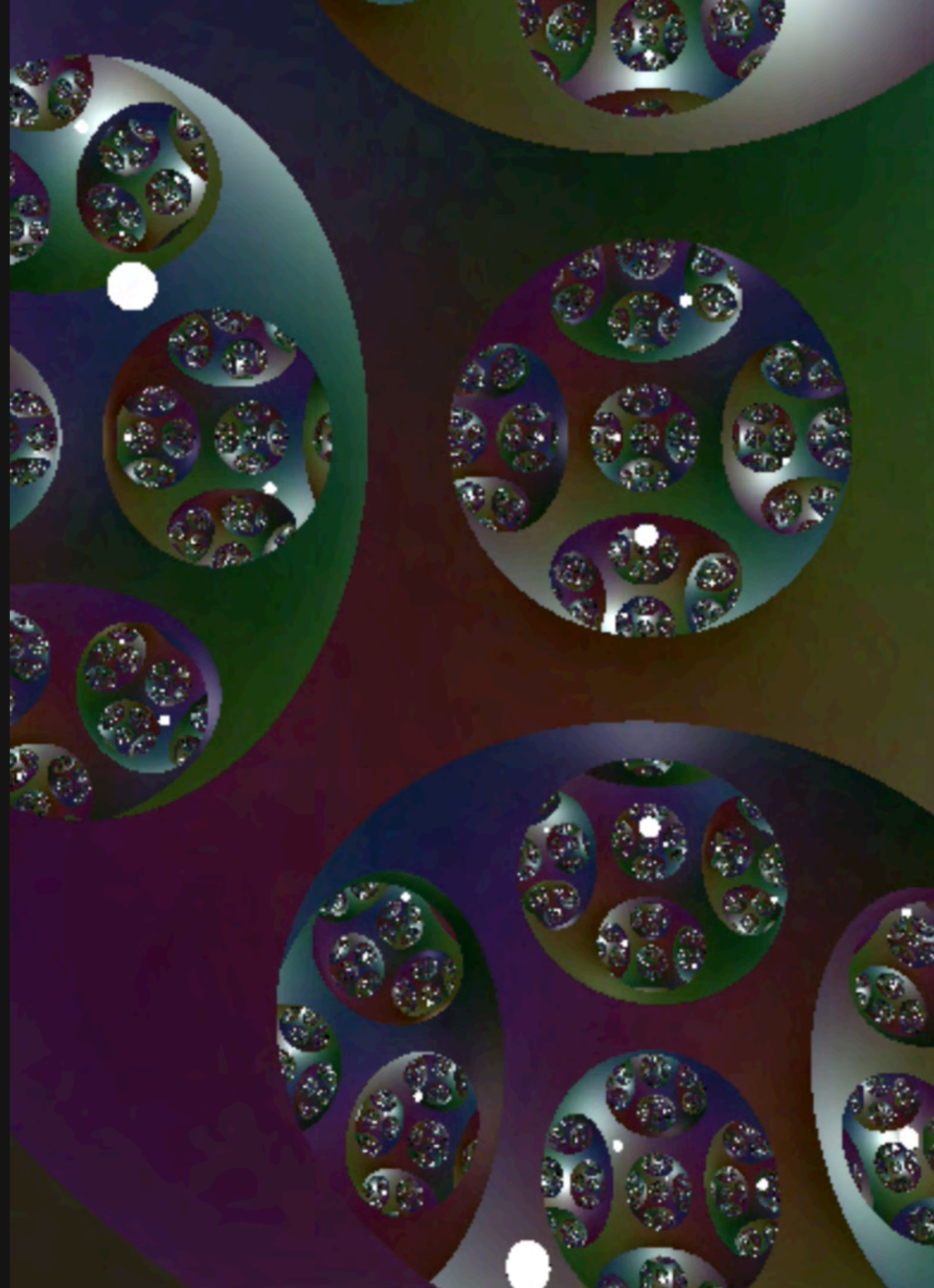
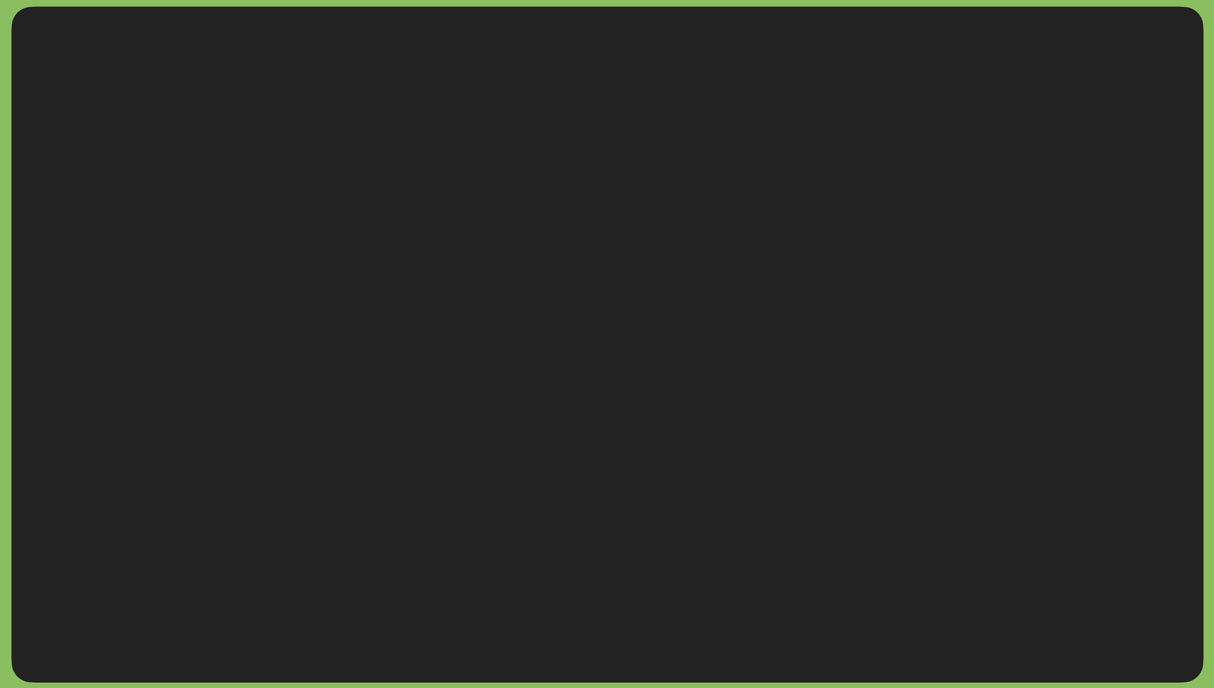
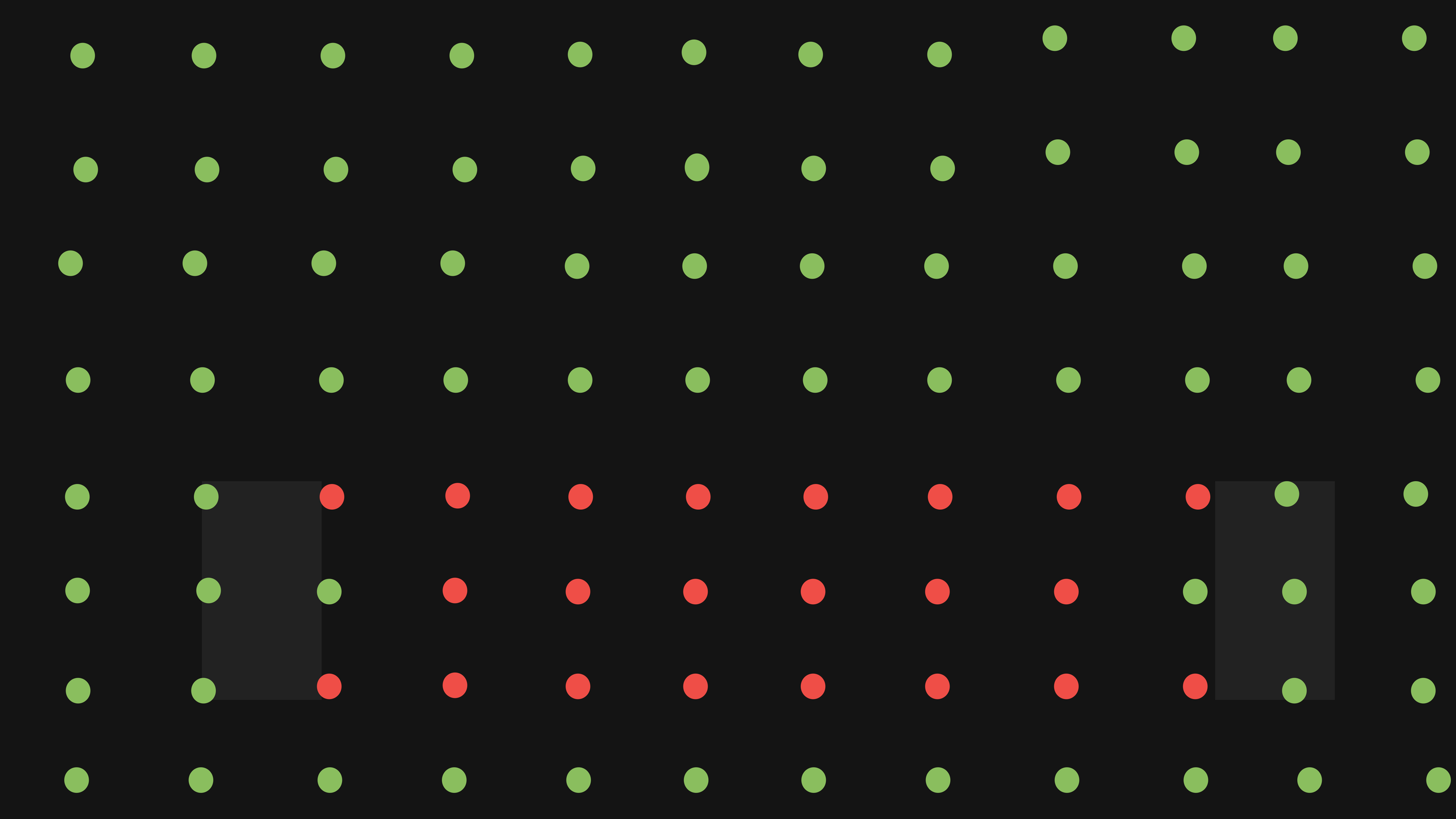


Hyperbolic Dehn Surgery Space:

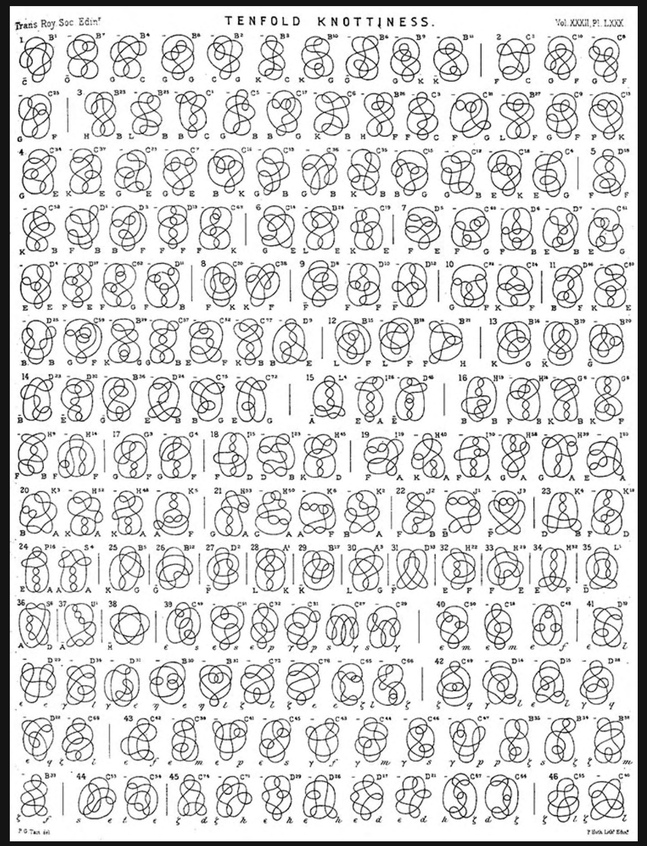
Subset of $\mathrm{H}_1(\partial M, \mathbb{R})$ **corresponding to hyperbolic conemanifold structures on M.**

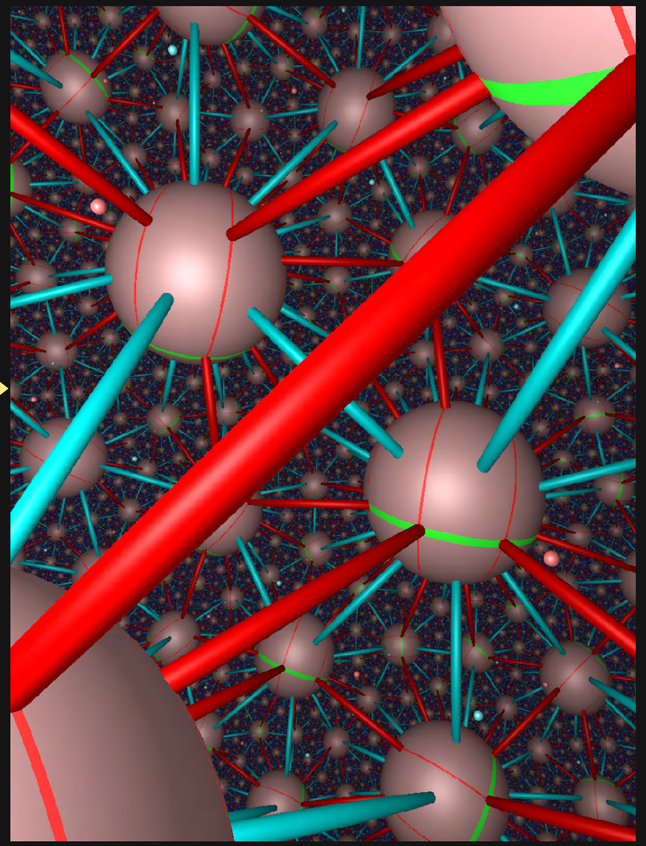
Theorem: This contains a neighborhood of infinity.





TONS of Examples!





Almost any Knot **Almost any Slope** **Group with $\partial \cong \mathbb{S}^2$**

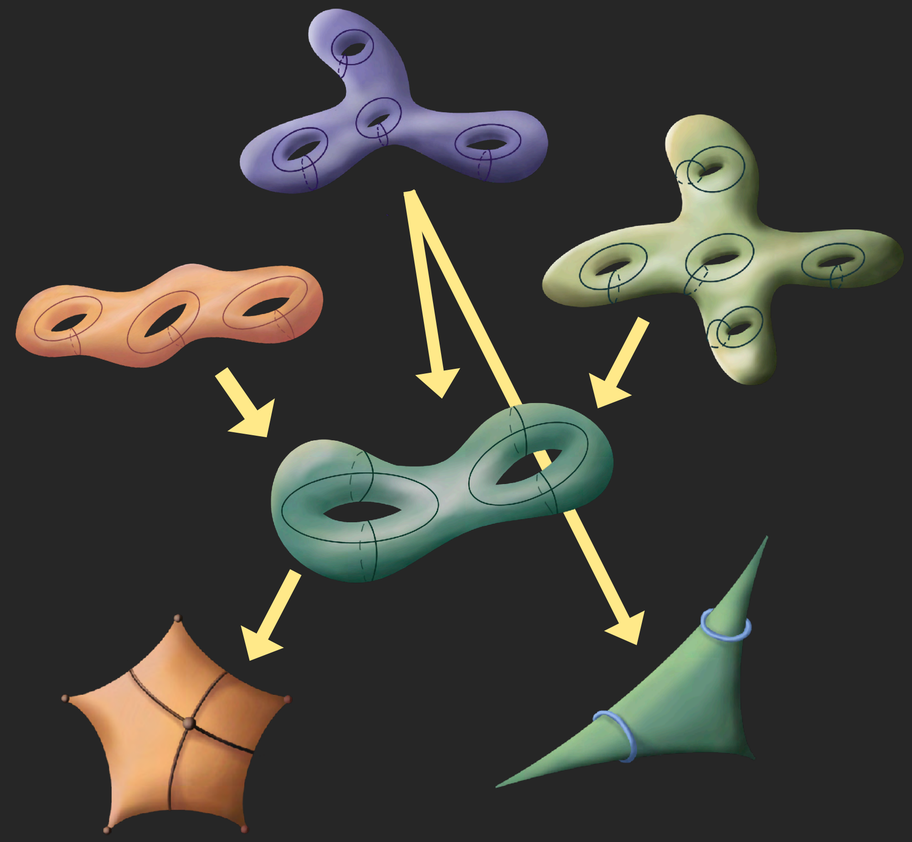
Explicit Examples!

$$\pi_1\left(\mathbb{S}^3 - K_8\right) \cong \langle a, b, t \,|\, tat^{-1} = ab, tbt^{-1} = bab \rangle$$

$$G_{p,q} = \left\langle a, b, t \;\middle|\; \begin{array}{c} tat^{-1} = ab, \\ tbt^{-1} = bab, \\ \lambda^p \mu^q = e \end{array} \right\rangle$$

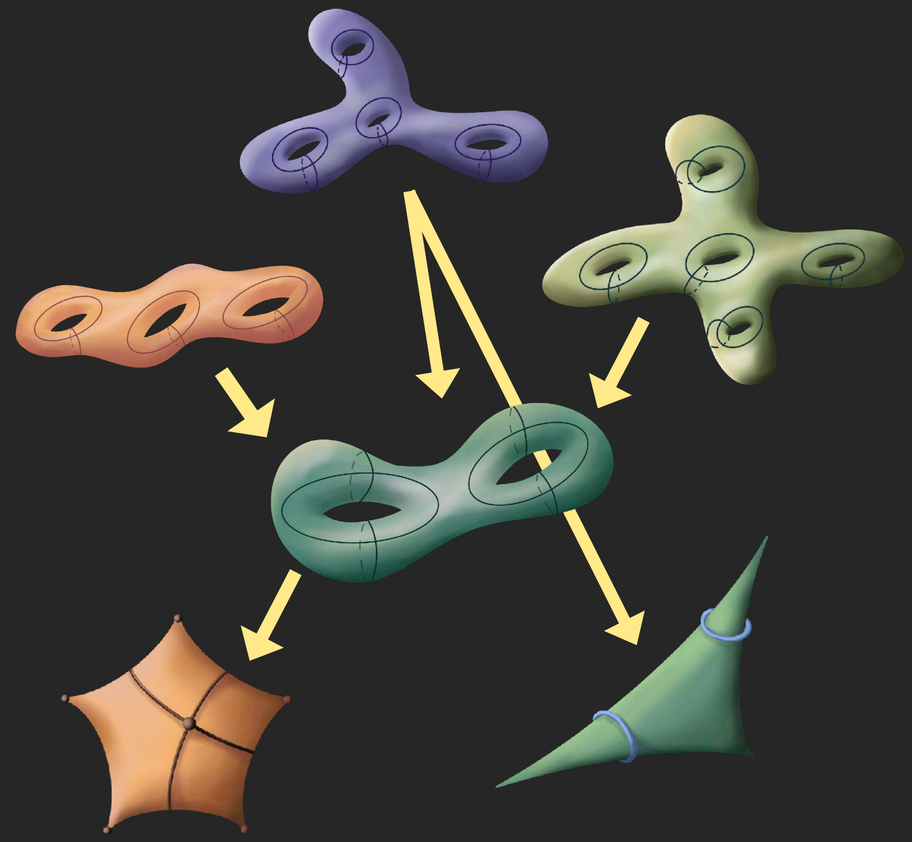
How different are all these examples?

If two manifolds are commensurable, their volumes are rationally related.



What are the volumes of hyperbolic 3-manifolds?

If two manifolds are commensurable, their volumes are rationally related.



What are the volumes of hyperbolic 3-manifolds?

THURSTON

As $(p, q) \to (\infty, \infty)$ the volume of $(\mathbb{S}^3 \setminus K)_{p/q}$ limits to the volume of $(\mathbb{S}^3 \setminus K)$

Fig 8 and Whitehead link not rationally related.

Thm: Kerhalls 2015

Whitehead Link

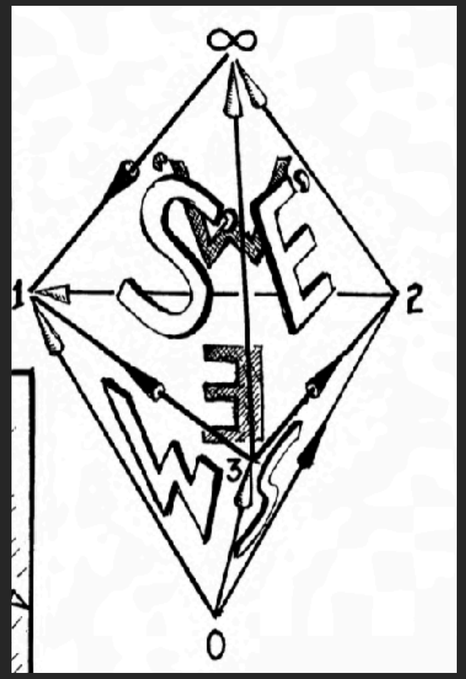


Figure 8 Knot

Volumes are given by summing volumes of hyperbolic tetrahedra.

Fig 8 and Whitehead link not rationally related.

Thm: Kerhalls 2015

$$V_8 = 3\Lambda\left(\frac{\pi}{3}\right) \qquad V_K = 8\Lambda\left(\frac{\pi}{4}\right)$$

Whitehead Link **Figure 8 Knot**

$$\Lambda(\theta) = -\int_0^\theta \log|2\sin t|\,dt$$

Is this all the examples?

Do all groups which have boundary homeomorphic to $\mathbb{S}^2$ come from hyperbolic 3 manifolds?

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Do all groups which have boundary homeomorphic to $\mathbb{S}^2$ come from hyperbolic 3 manifolds?

Open question! (Cannon Conjecture)

Extra!!!

What else can HDS Teach us?

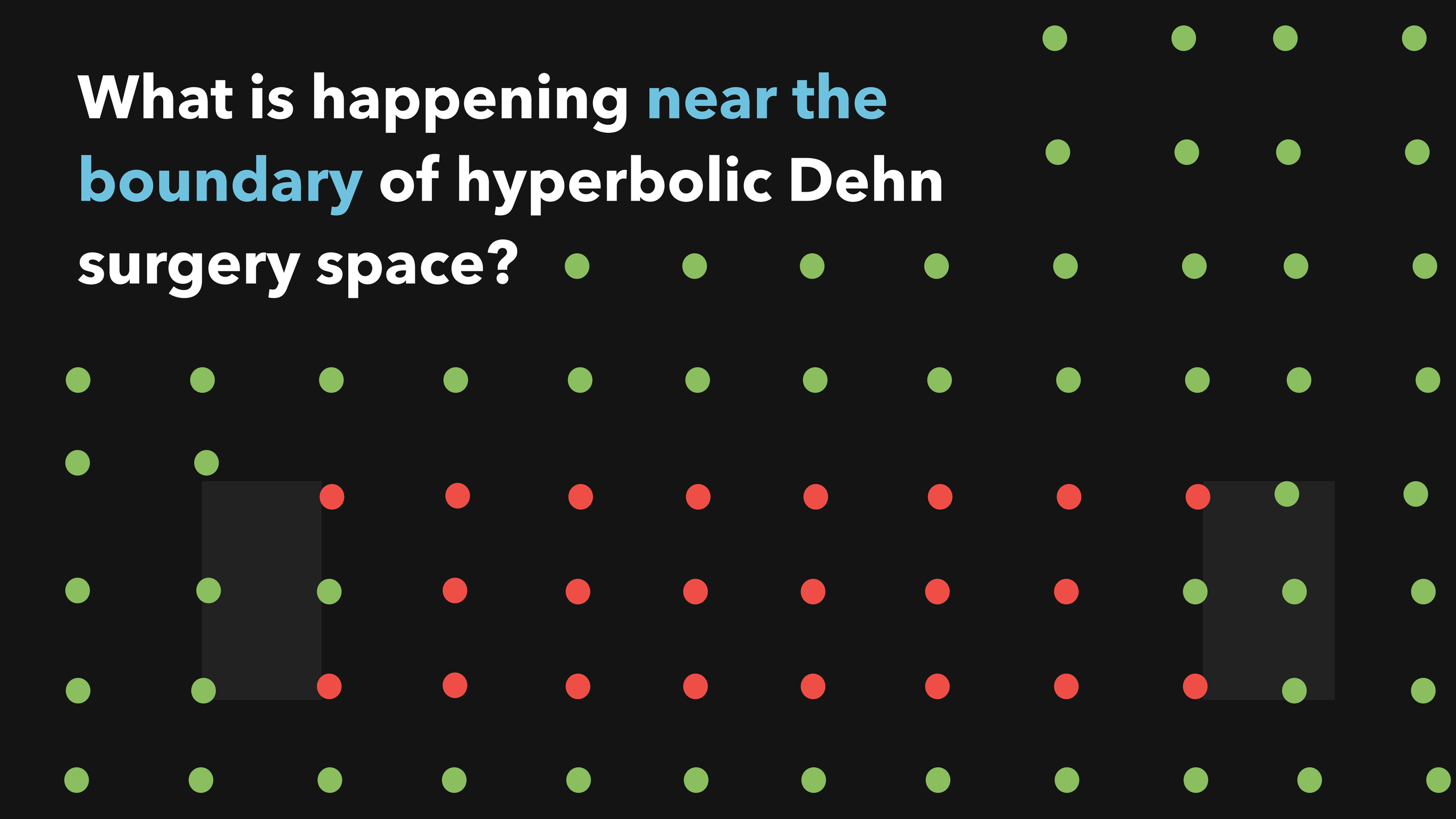
How many slopes are not hyperbolic?

Fig 8 = 10

(-2,3,7) pretzel knot = 7

Conj (Gordon) 10 is max

Thm (Lackenby, Meyerhoff) this is true


What is happening near the boundary of hyperbolic Dehn surgery space?

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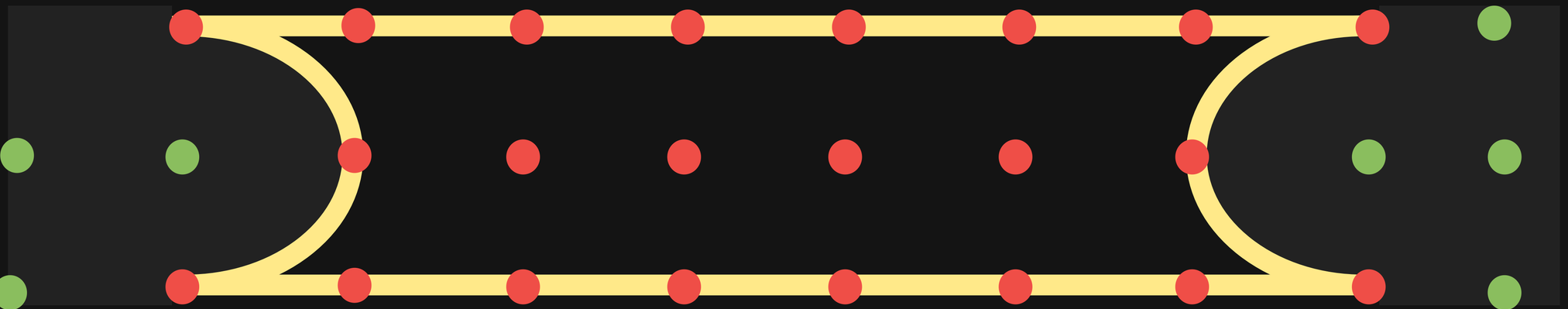
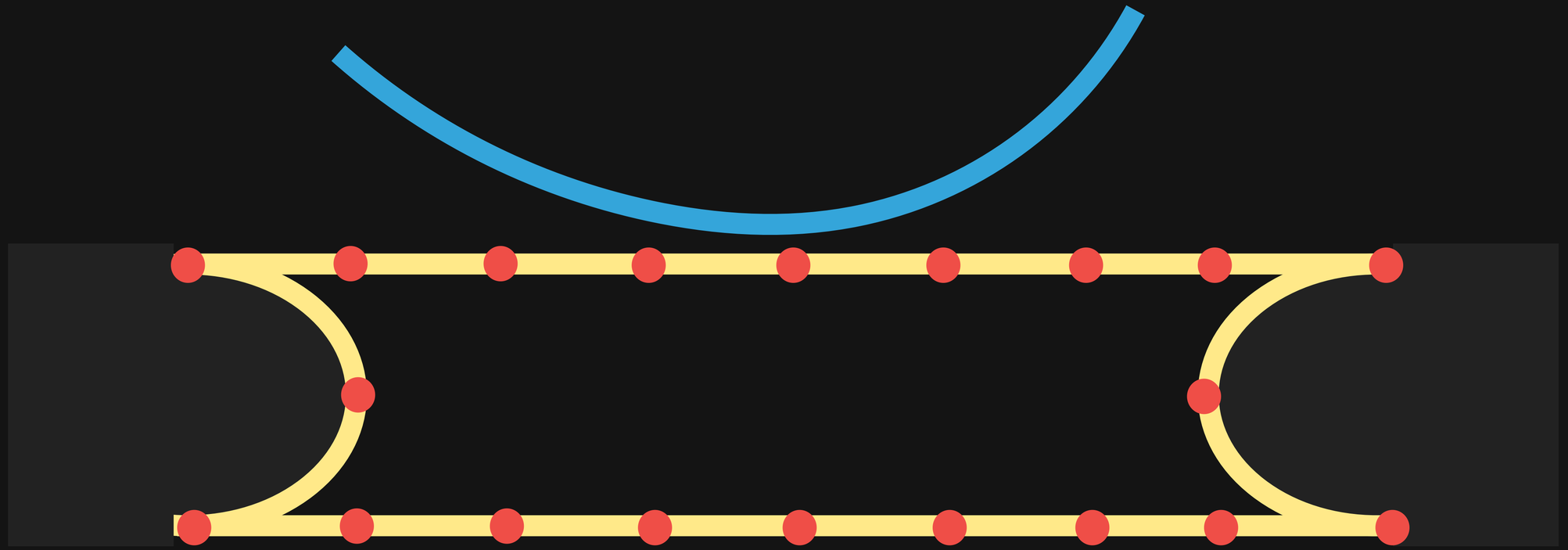
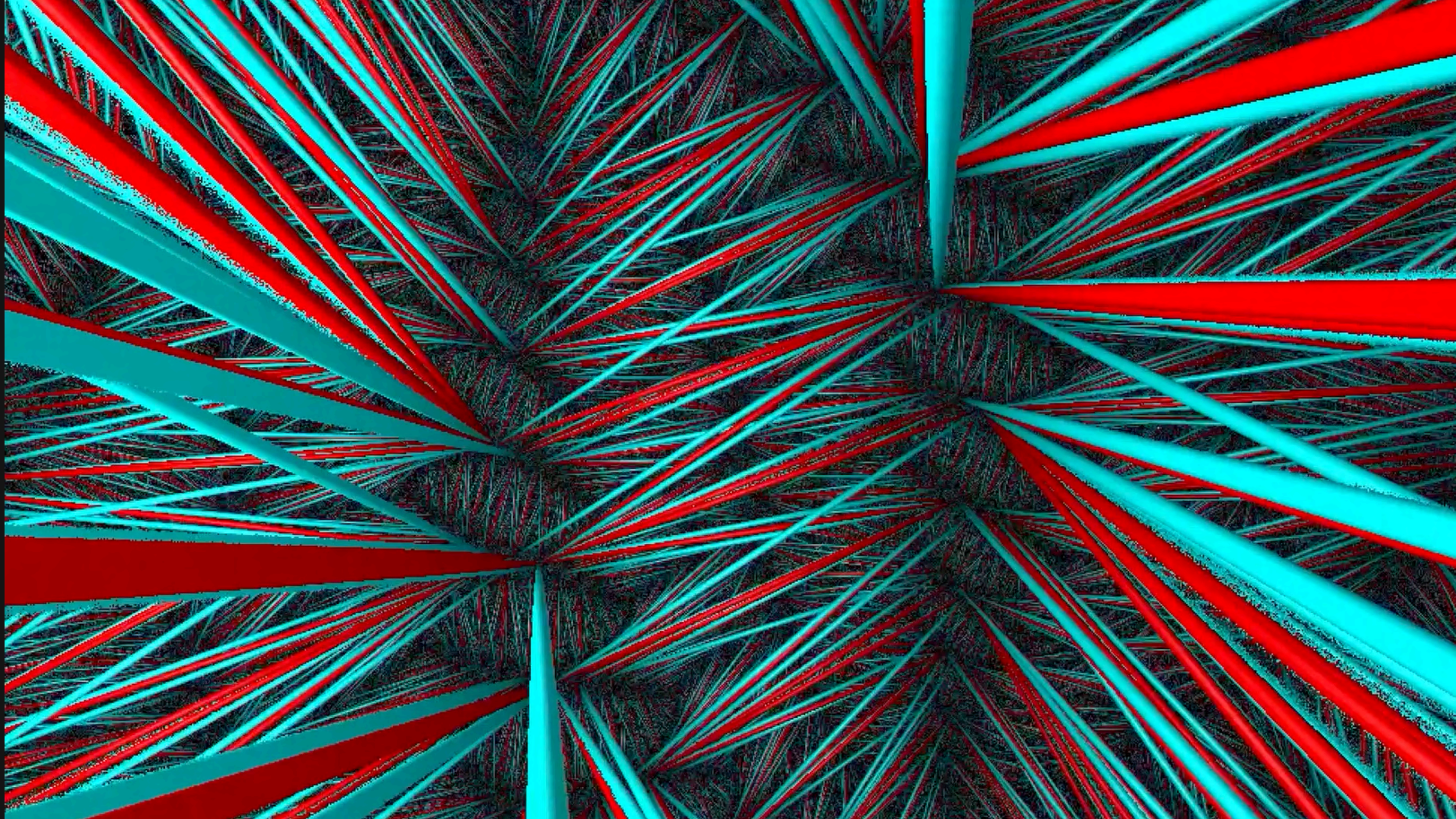


Let’s look at a path of hyperbolic cone manifolds which almost hits the boundary...

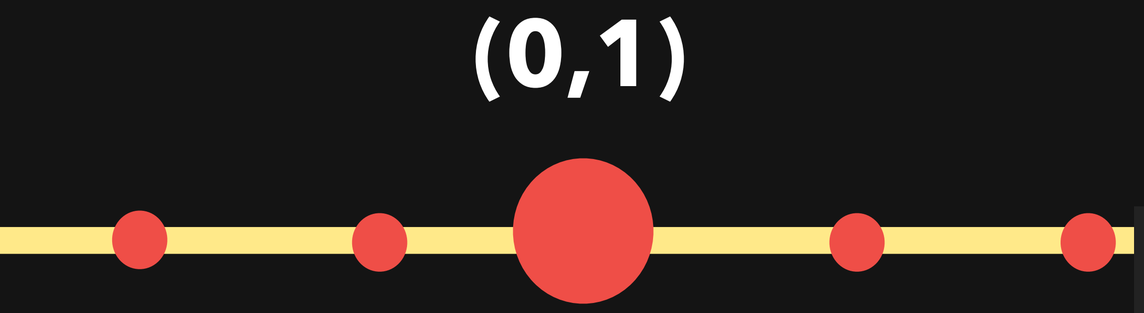


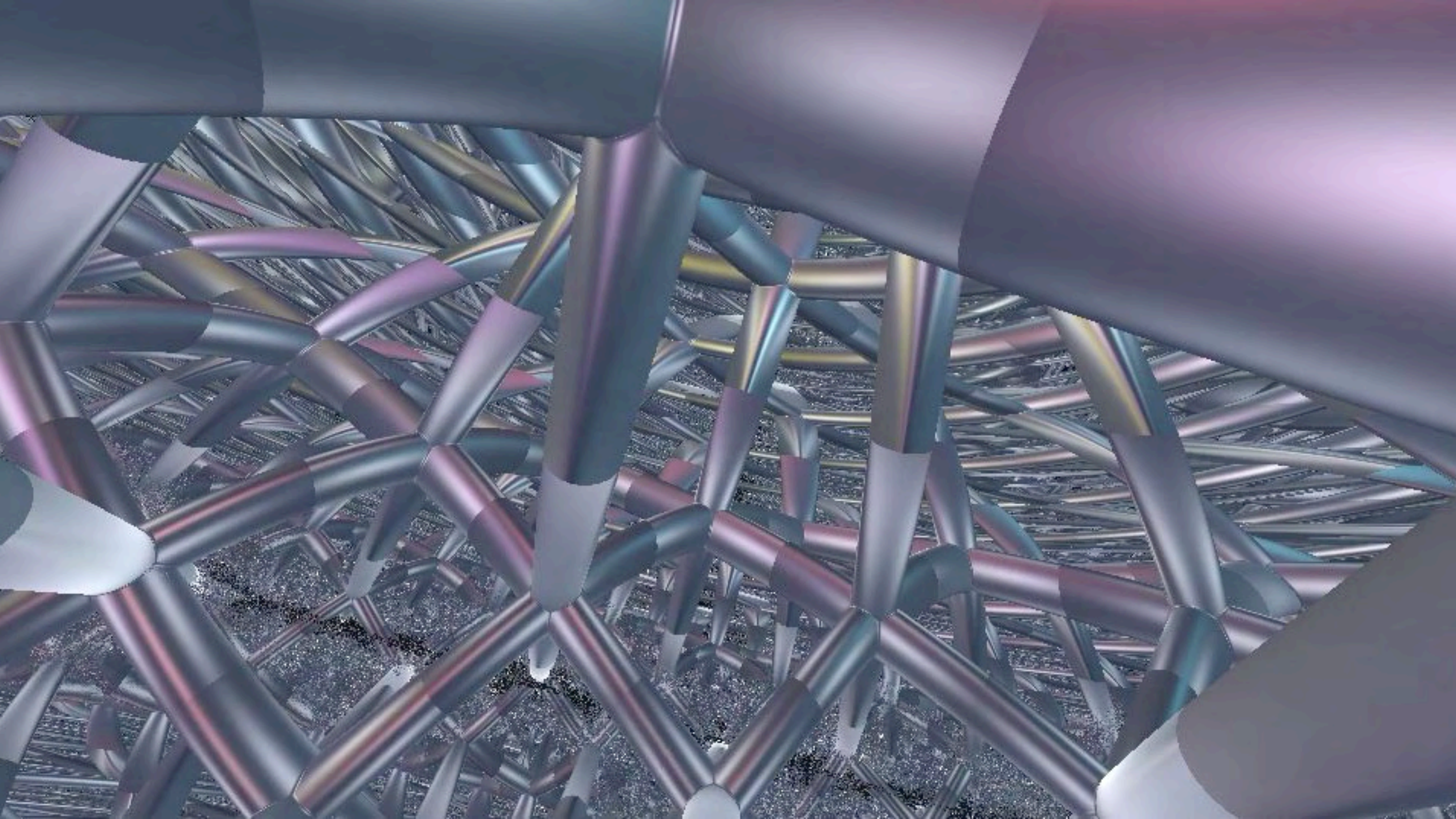


(0,1) SURGERY ON THE FIGURE 8

Result has a Sol structure!

(Torus bundle over the circle with $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ monodromy)



Theorem (Heusener, Porti, Suarez, ’01)

This structure is the limit of collapsing hyperbolic cone manifolds.