Classical Physics in Curved Space

Warnings for willing travelers

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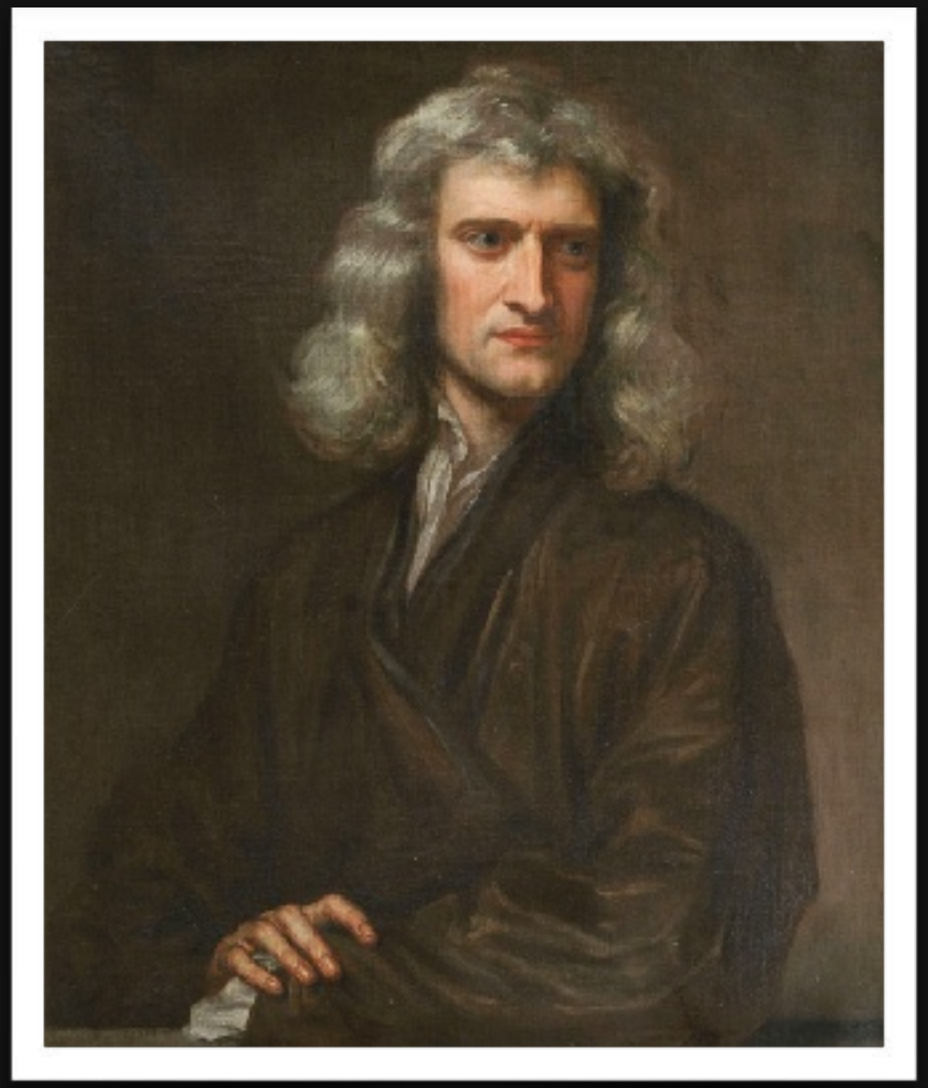
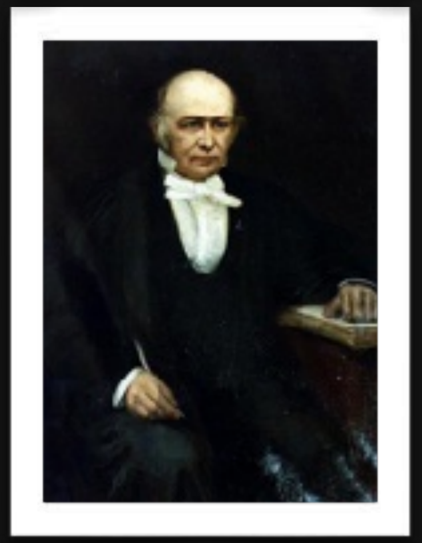


$$\frac{\partial \mathscr{L}}{\partial q} = \frac{d}{dt}\frac{\partial \mathscr{L}}{\partial \dot{q}}$$



$$\frac{d}{dt}(q,p) = X_H$$

$$dH = \iota_{X_H}\omega$$

$$\vec{F} = m\vec{a}$$

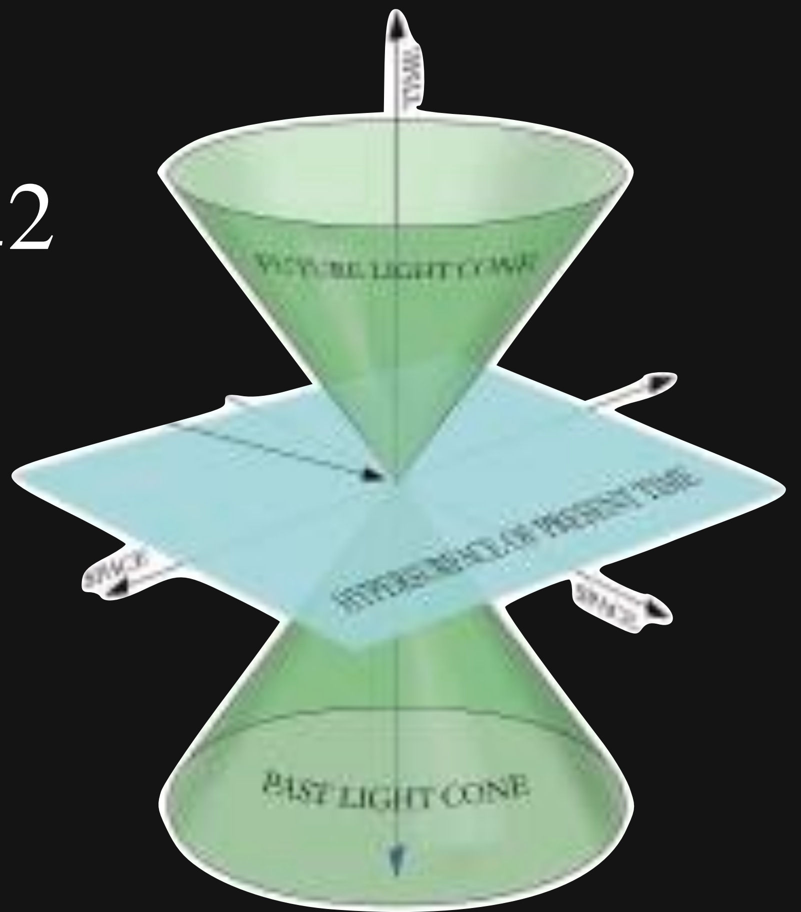
Newtonian Mechanics

(x, y, z)

$\langle v_x, v_y, v_z \rangle$

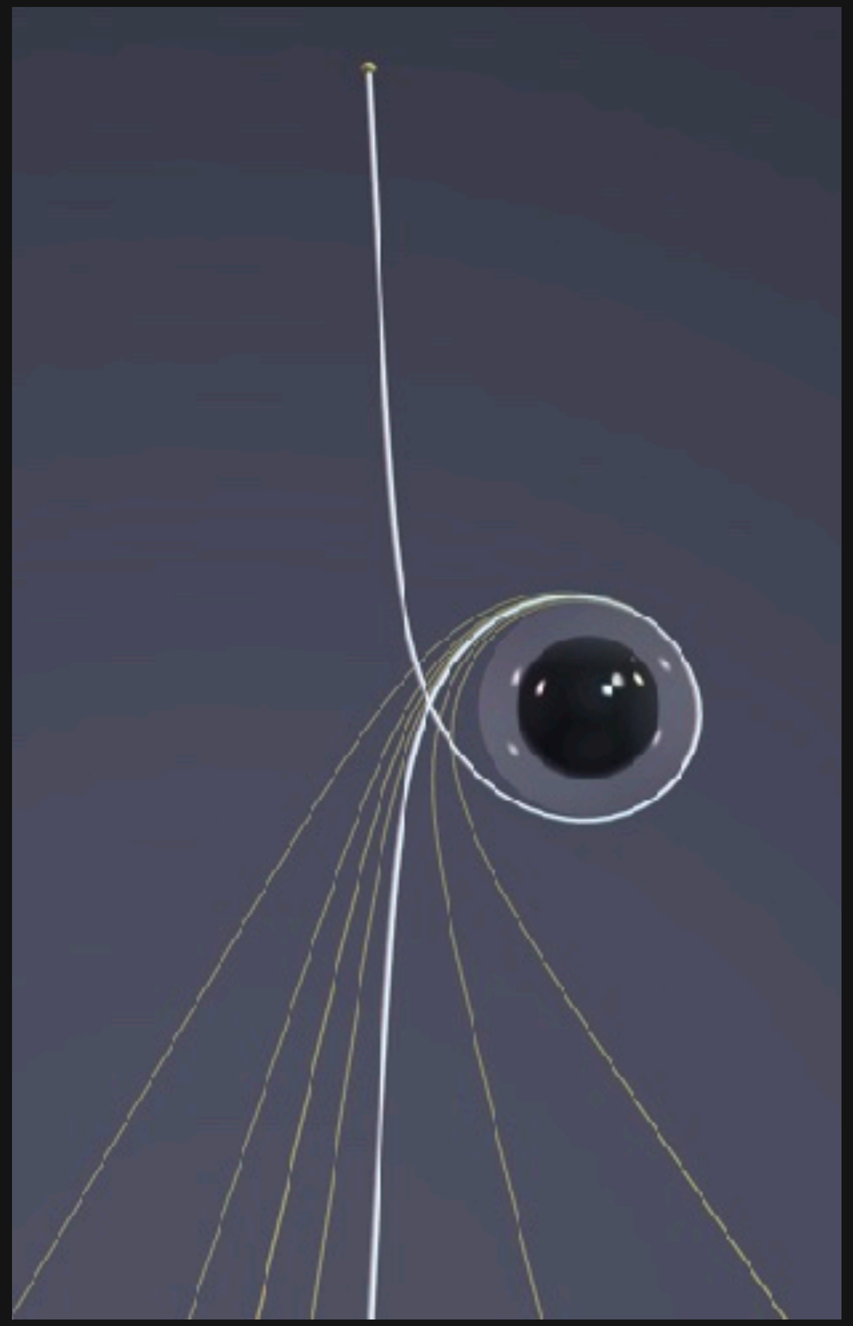
Special Relativity

$$ds^2 = (dx^2 + dy^2 + dz^2) - dt^2$$

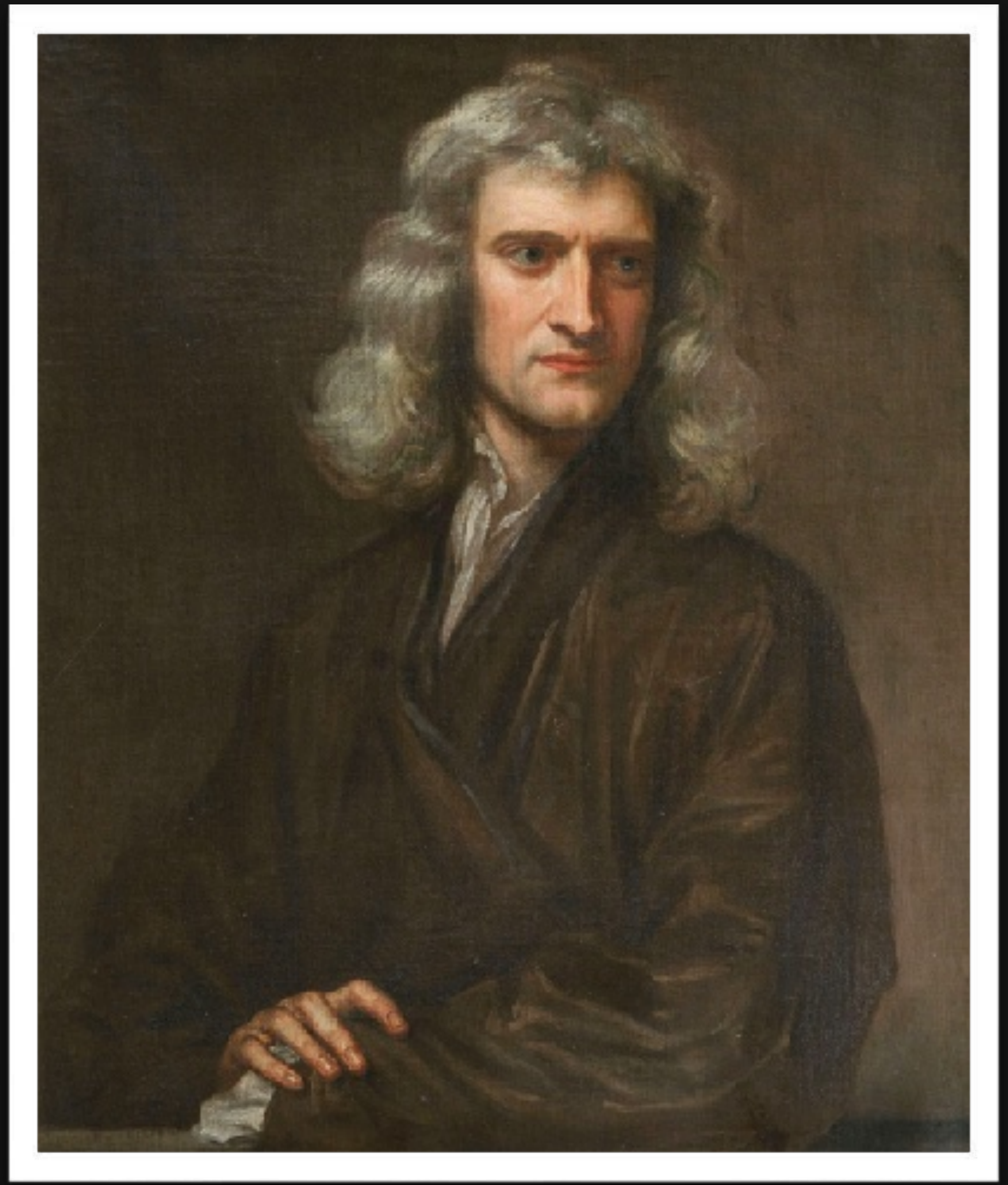


General Relativity

$$\mathrm{Ric} = 8\pi G\left(\mathbf{T} - \frac{1}{2}T\mathbf{g}\right)$$



Flat Space



Flat Spacetime

Curved Space

Curved Spacetime

Everything is a Mirage

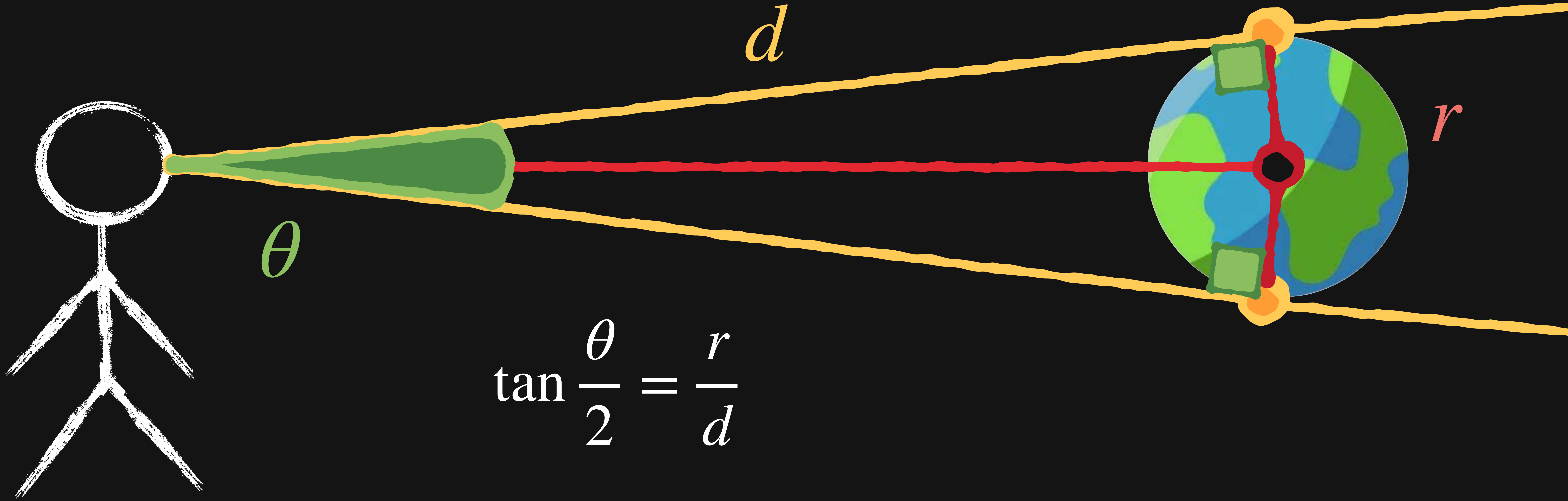
and the distant stars are deadly

A story from joint work with Remi Coulon,
Sabetta Matsumoto, and Henry Segerman

The geometry of vision

The geometry of vision

A real example: Apollo 17 en route to the moon.

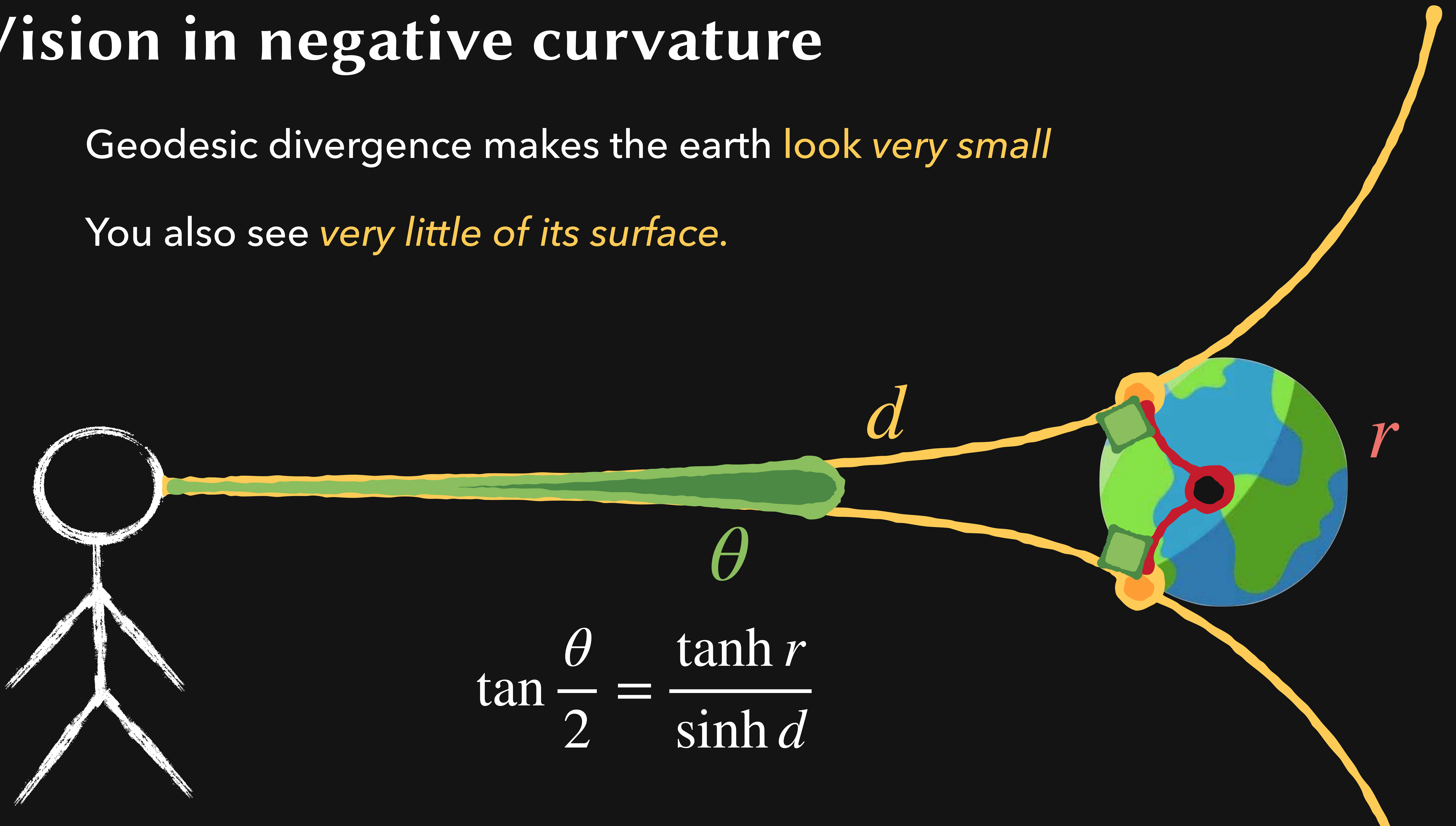


$$\tan\frac{\theta}{2} = \frac{r}{d}$$

Vision in negative curvature

Geodesic divergence makes the earth *look very small*

You also see *very little of its surface.*

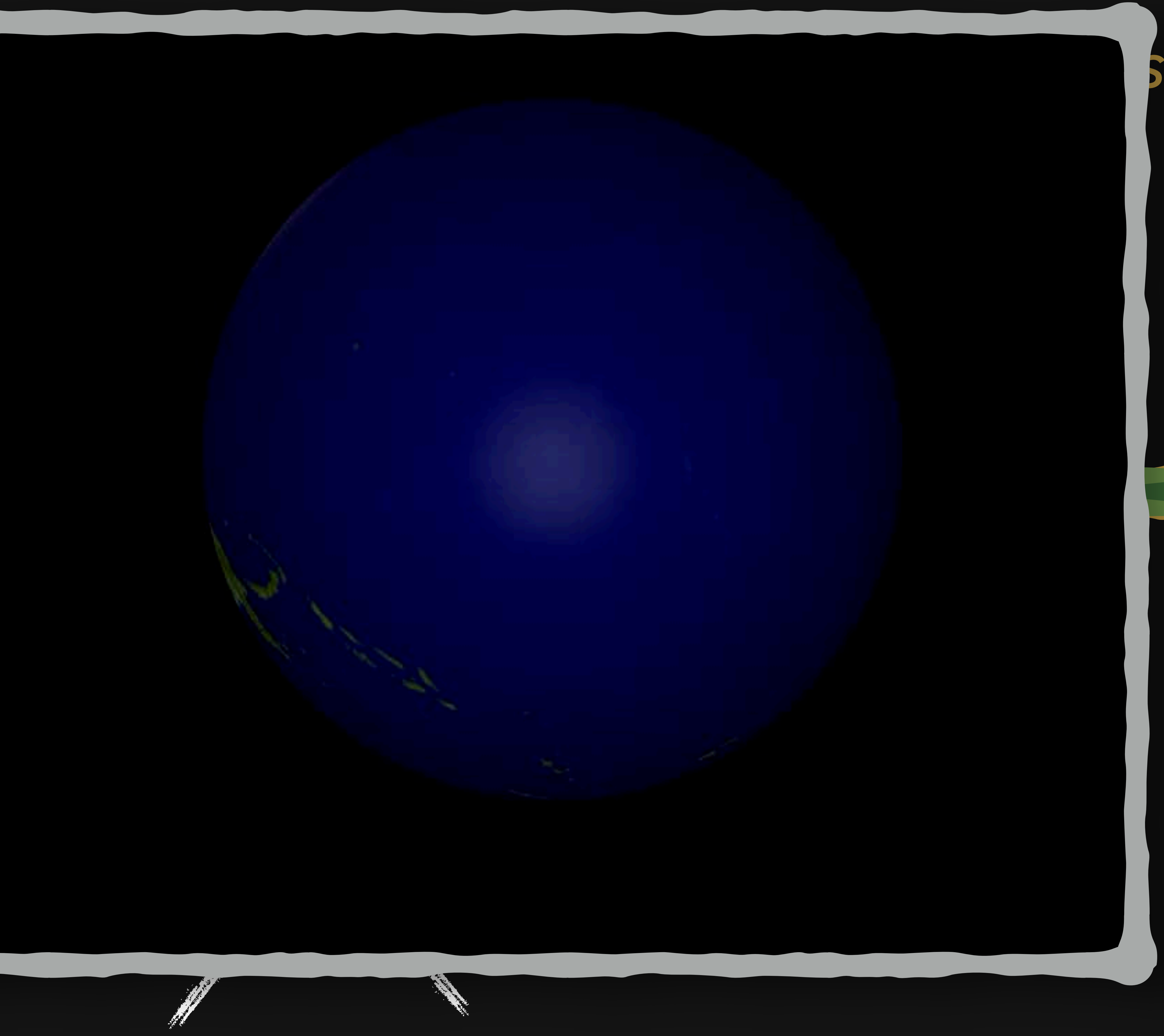


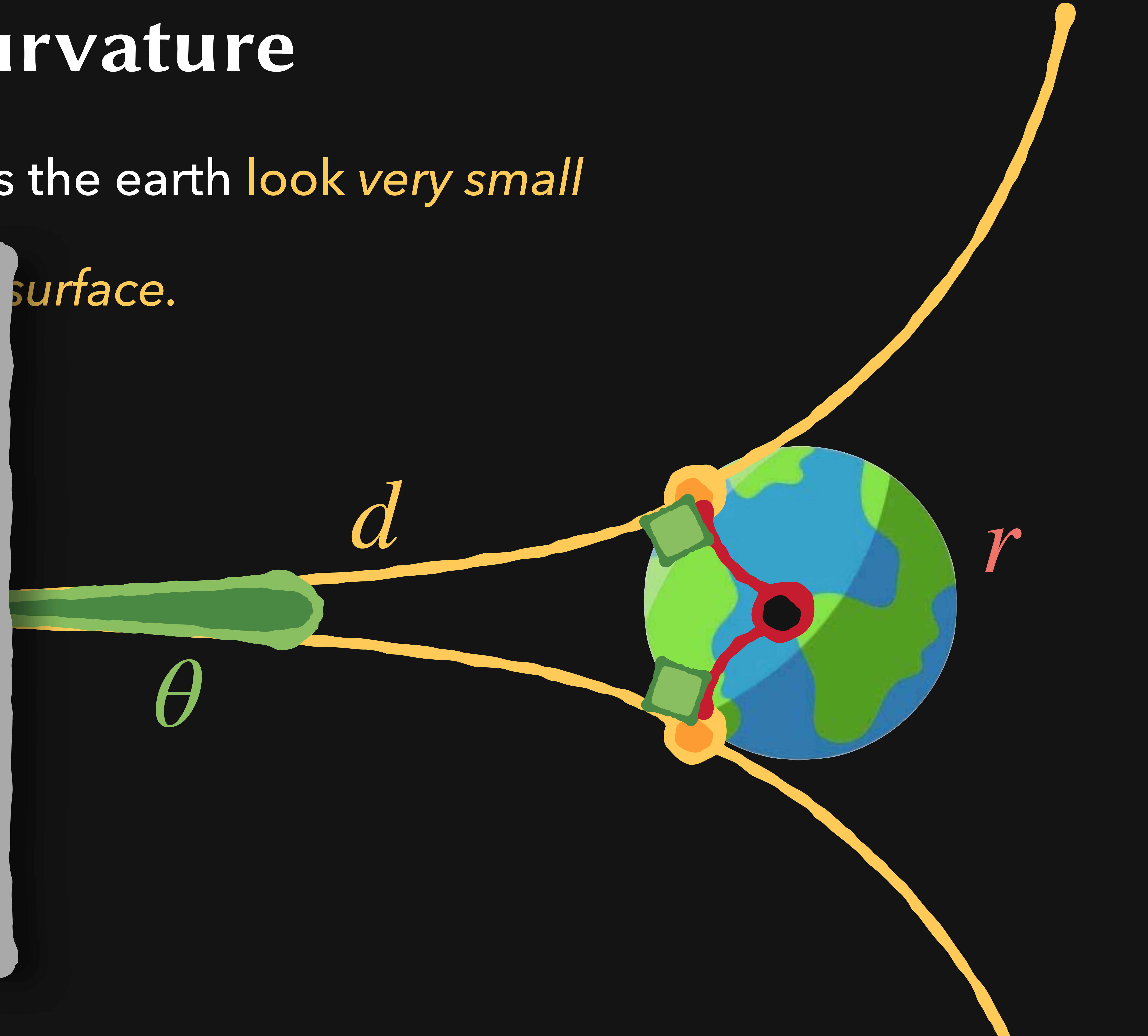
$$\tan\frac{\theta}{2} = \frac{\tanh r}{\sinh d}$$

Vision in negative curvature

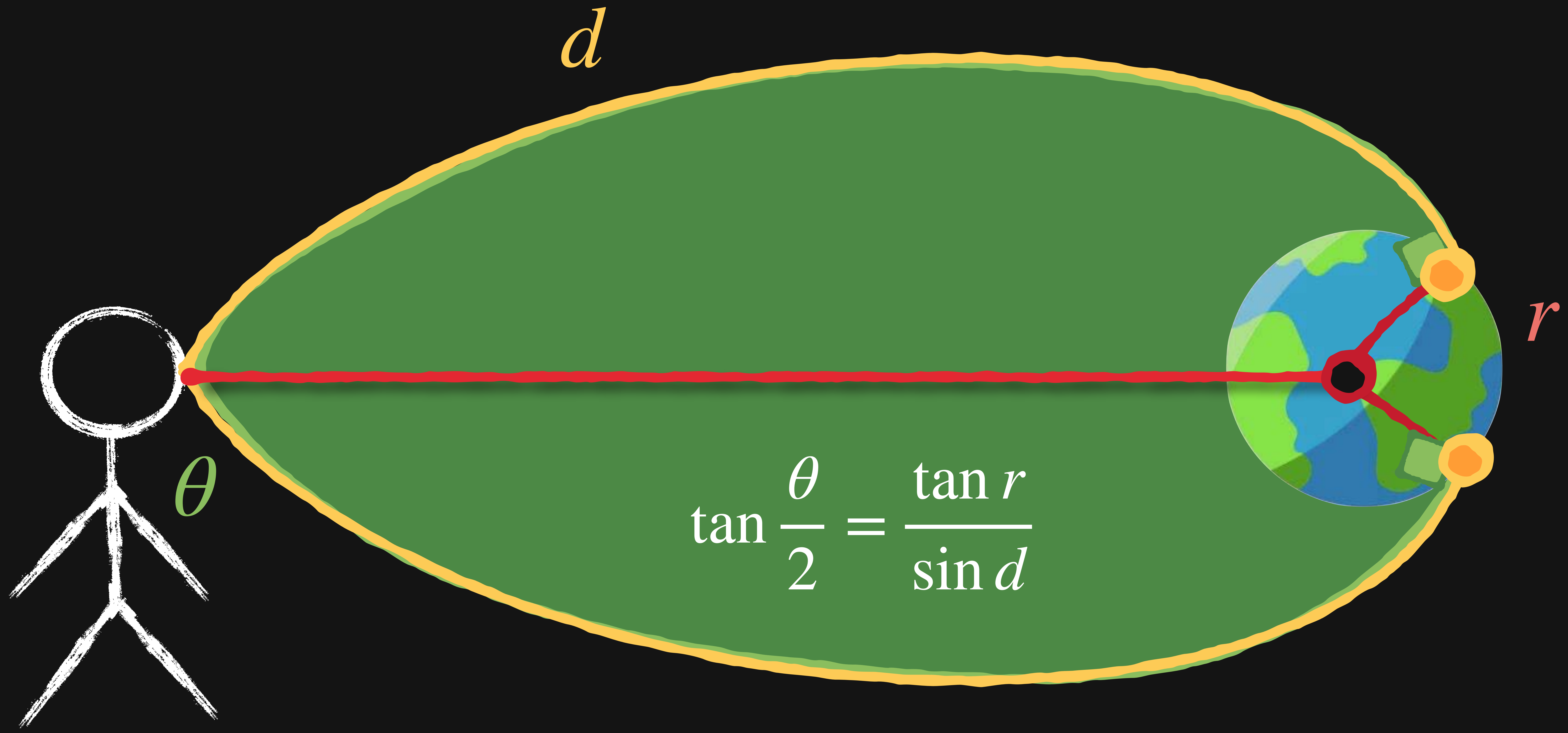
Geodesic divergence makes the earth look *very small*

surface.



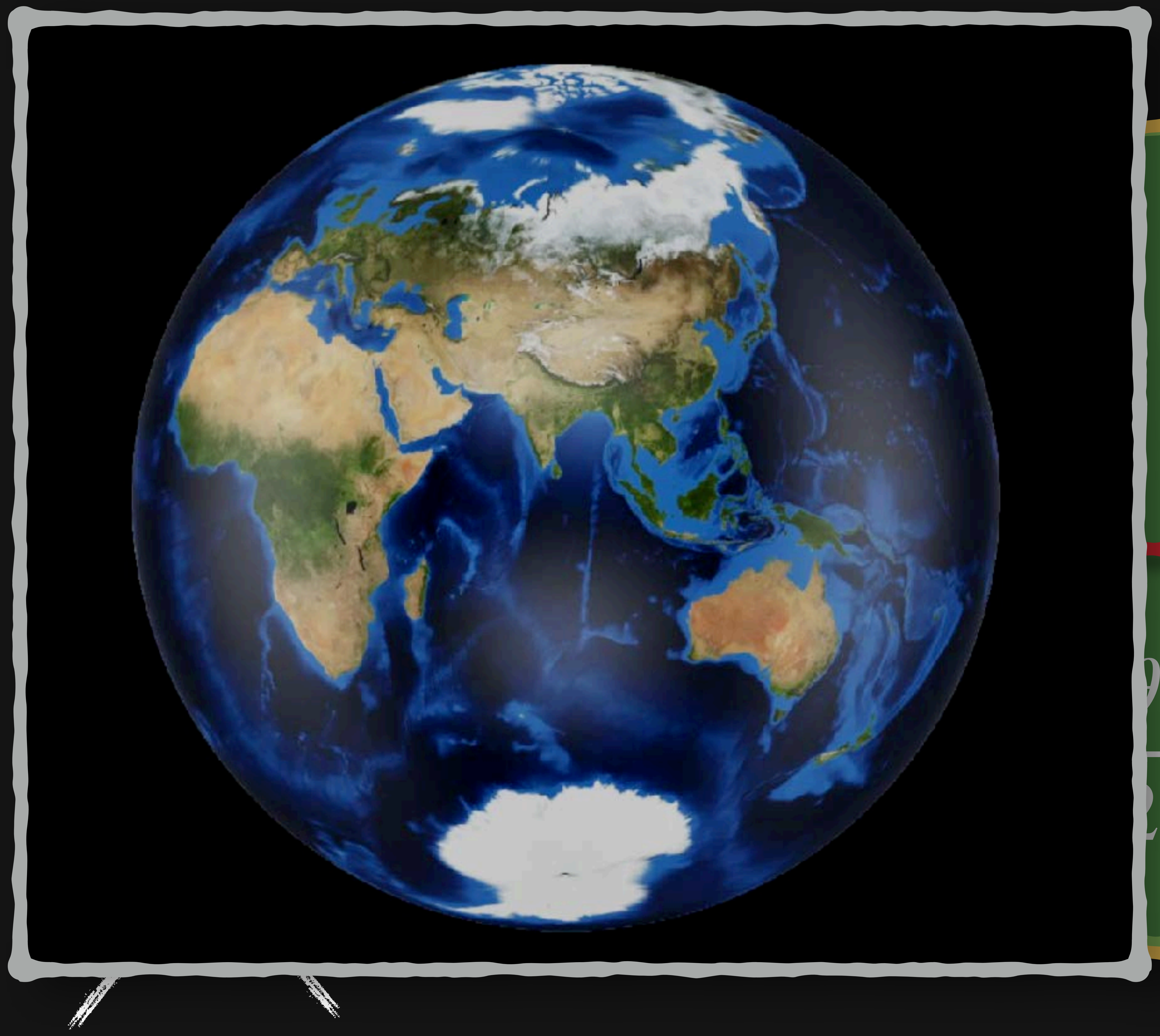


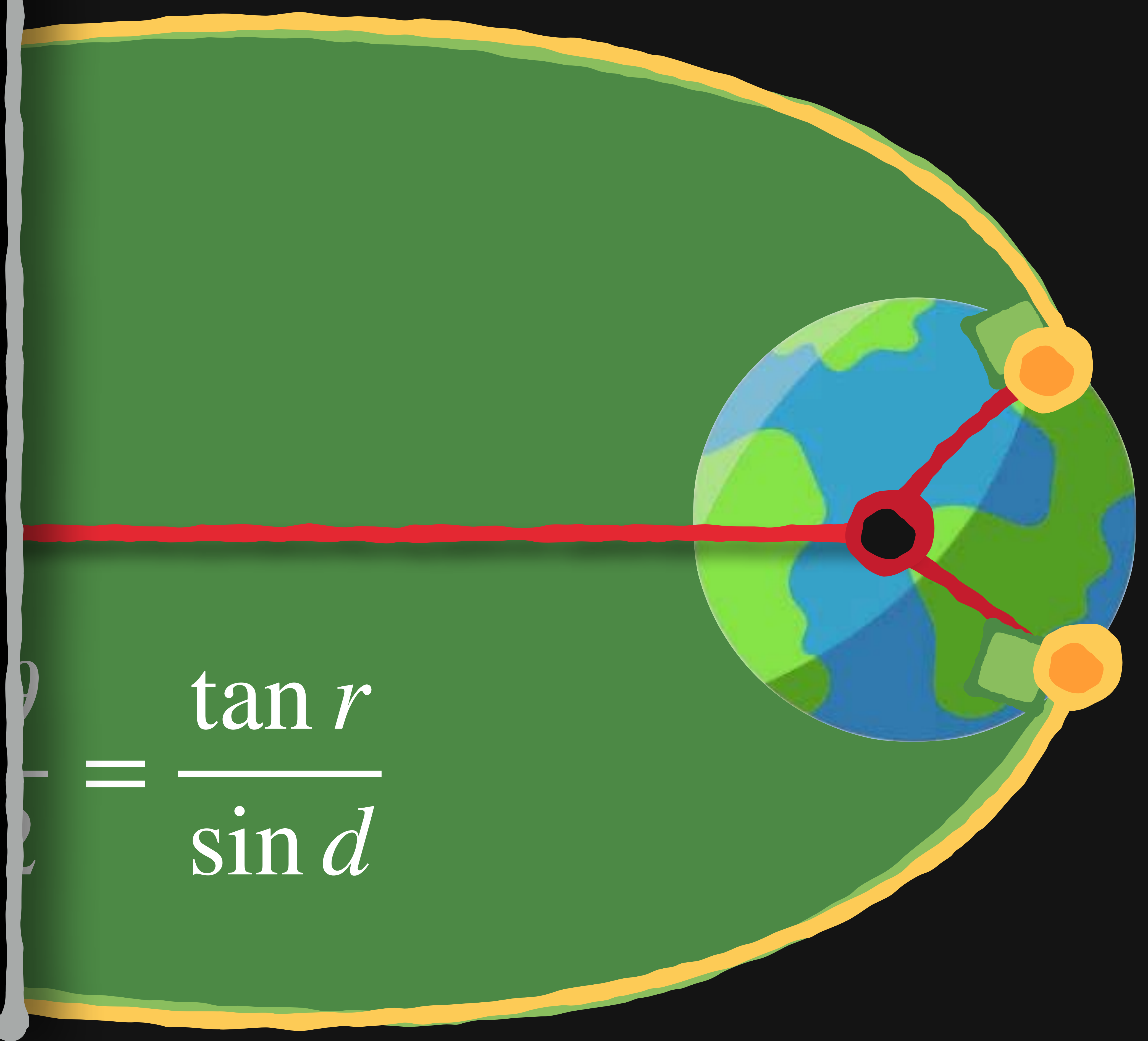
Vision in positive curvature



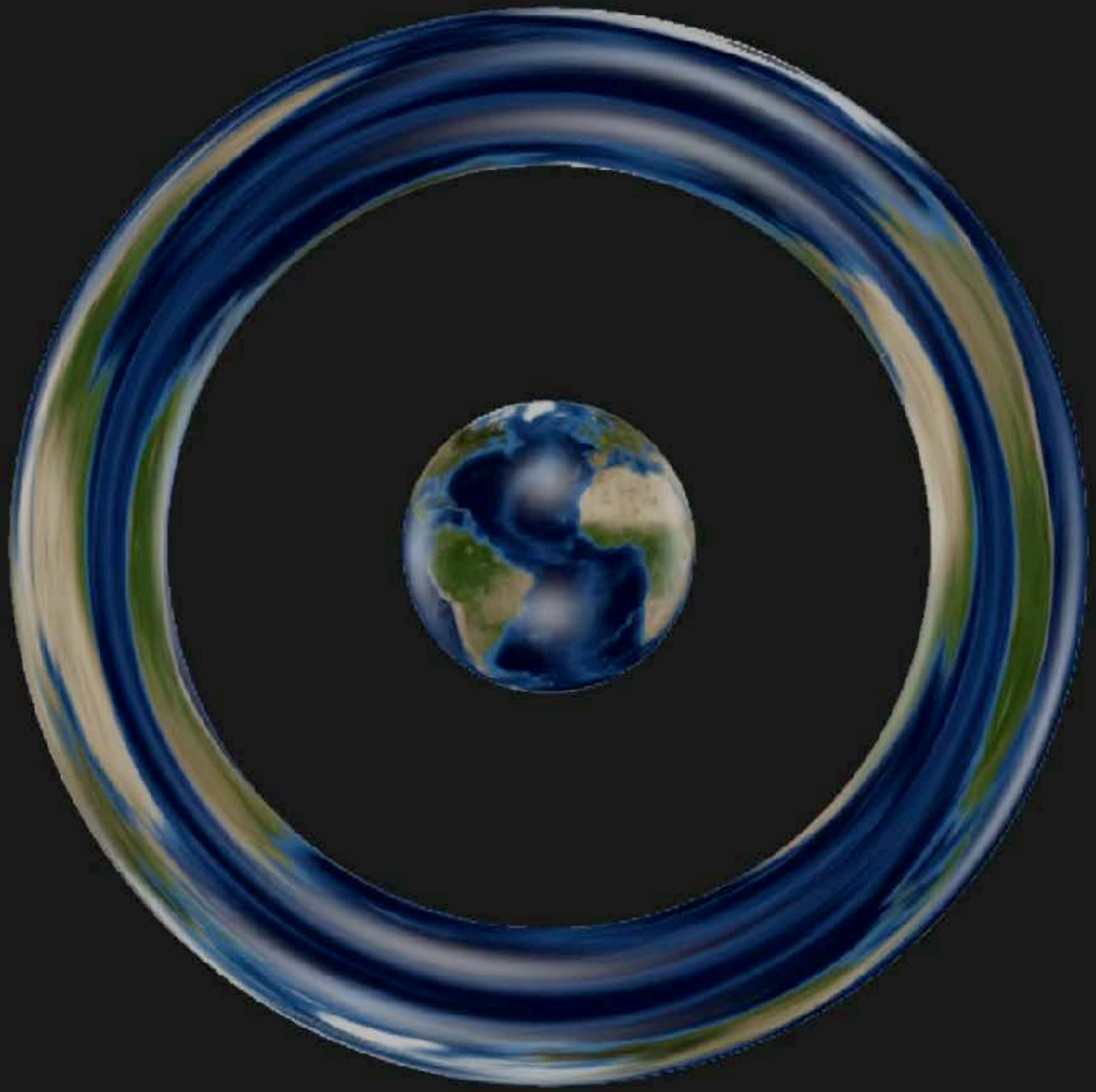
$$\tan\frac{\theta}{2} = \frac{\tan r}{\sin d}$$

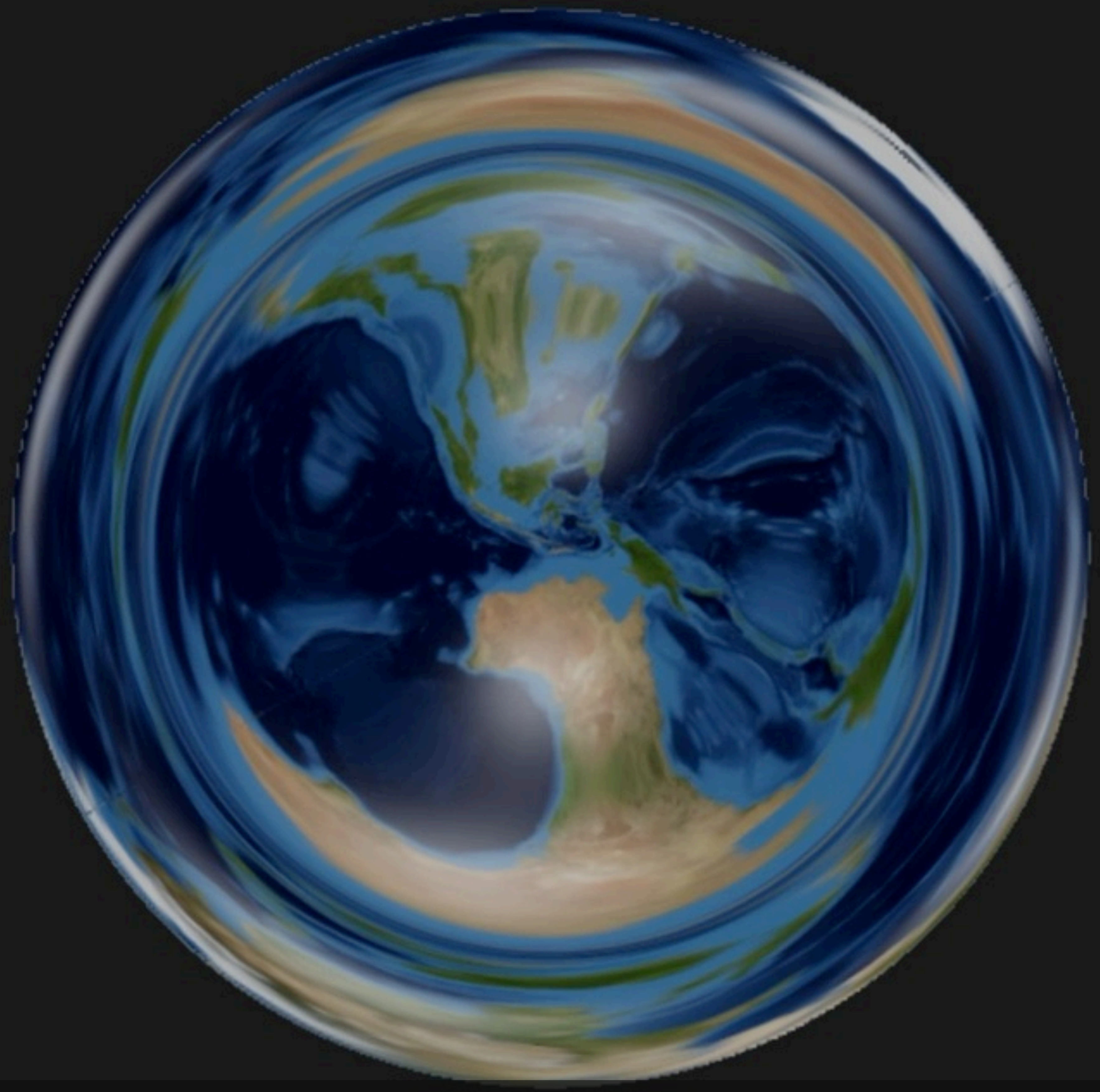
Vision in positive curvature





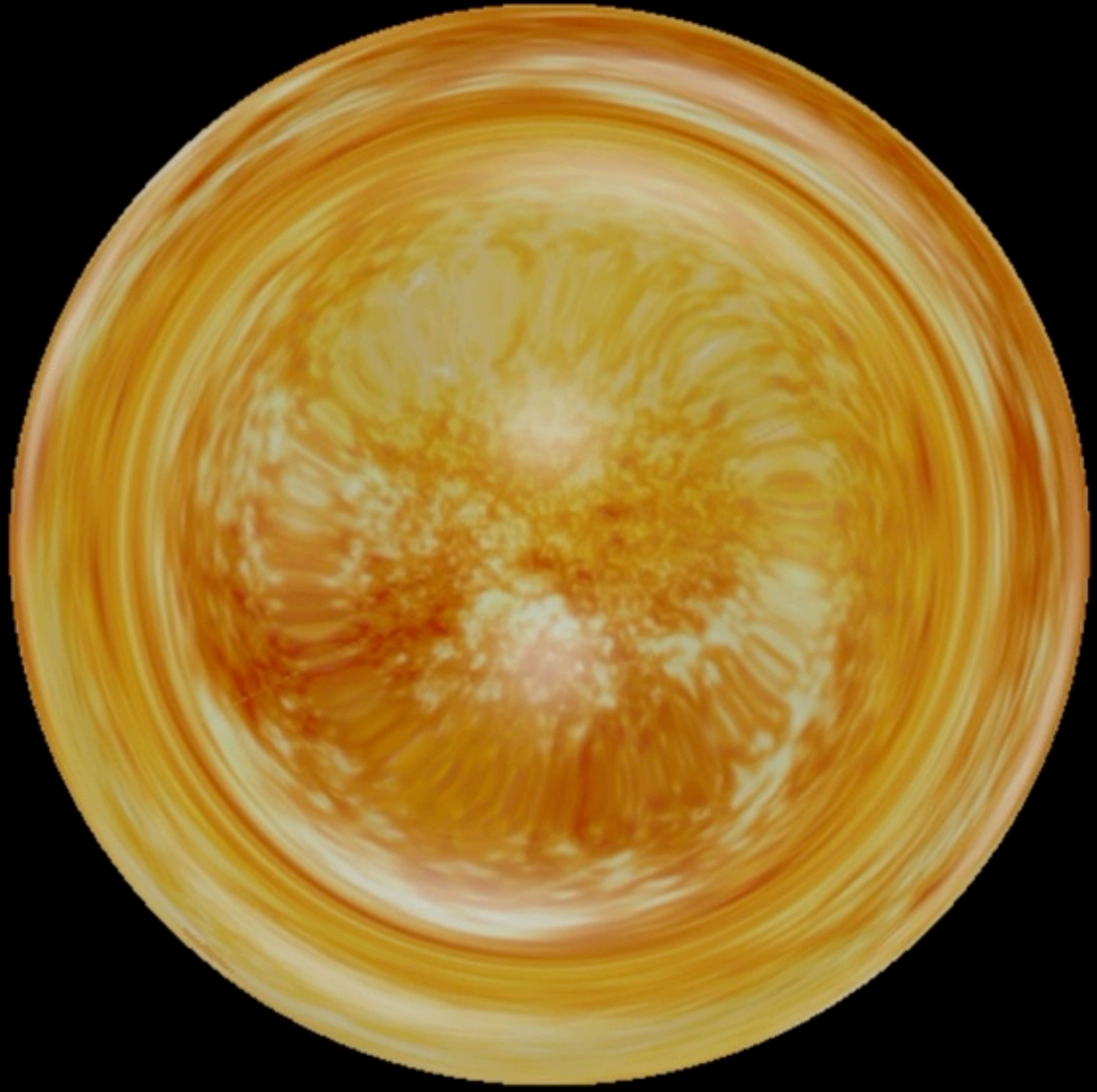
Vision in nil geometry

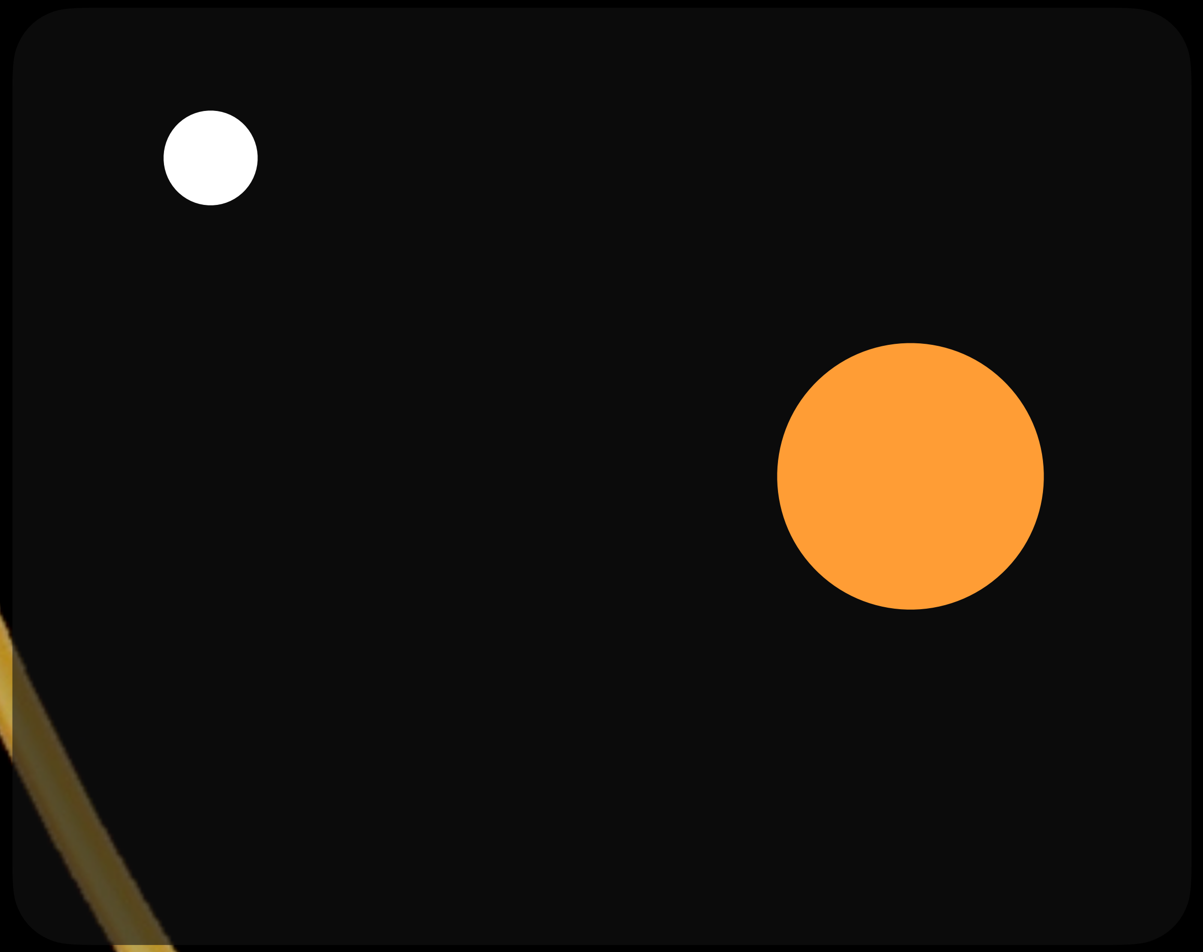




Consequence:

Death by space laser





Moving Hurts

and the earth explodes

We feel **rotation**
but not
translation

Acceleration along a path is proportional to geodesic curvature

$$
\begin{aligned}
\text{acc} &= \frac{D}{dt}\gamma(vt) \\
&= \nabla_{v\dot\gamma} v\dot\gamma \\
&= v\,\nabla_{\dot\gamma} v\dot\gamma \\
&= v\left(\dot v\dot\gamma + v\,\nabla_{\dot\gamma}\dot\gamma\right) \\
&= 0 + v^2\nabla_{\dot\gamma}\dot\gamma \\
&= v^2\kappa_\gamma
\end{aligned}
$$

Constant speed translation

$$\text{acc} = V^2\kappa = (v\cosh d)^2\tanh d = \frac{v^2}{2}\sinh 2d$$

$$\kappa = \tanh d$$

$$V = v\cosh d$$

Consequences:
If the earth moves at all, it explodes

R: 13.6 km

v: 30km/s

$$\text{acc} = \frac{(30)^2}{2}\sinh(2 \cdot 13.6)$$

$$= 146{,}215{,}170{,}000{,}000\frac{\text{km}}{\text{s}^2}$$

$$= 1.5 \times 10^{15} g$$

Consequences:
If the earth moves at all, it explodes

R: 13.6 km

v: 1 mm/s

$$\text{acc} = \frac{(0.00001)^2}{2}\sinh(2 \cdot 13.6)$$

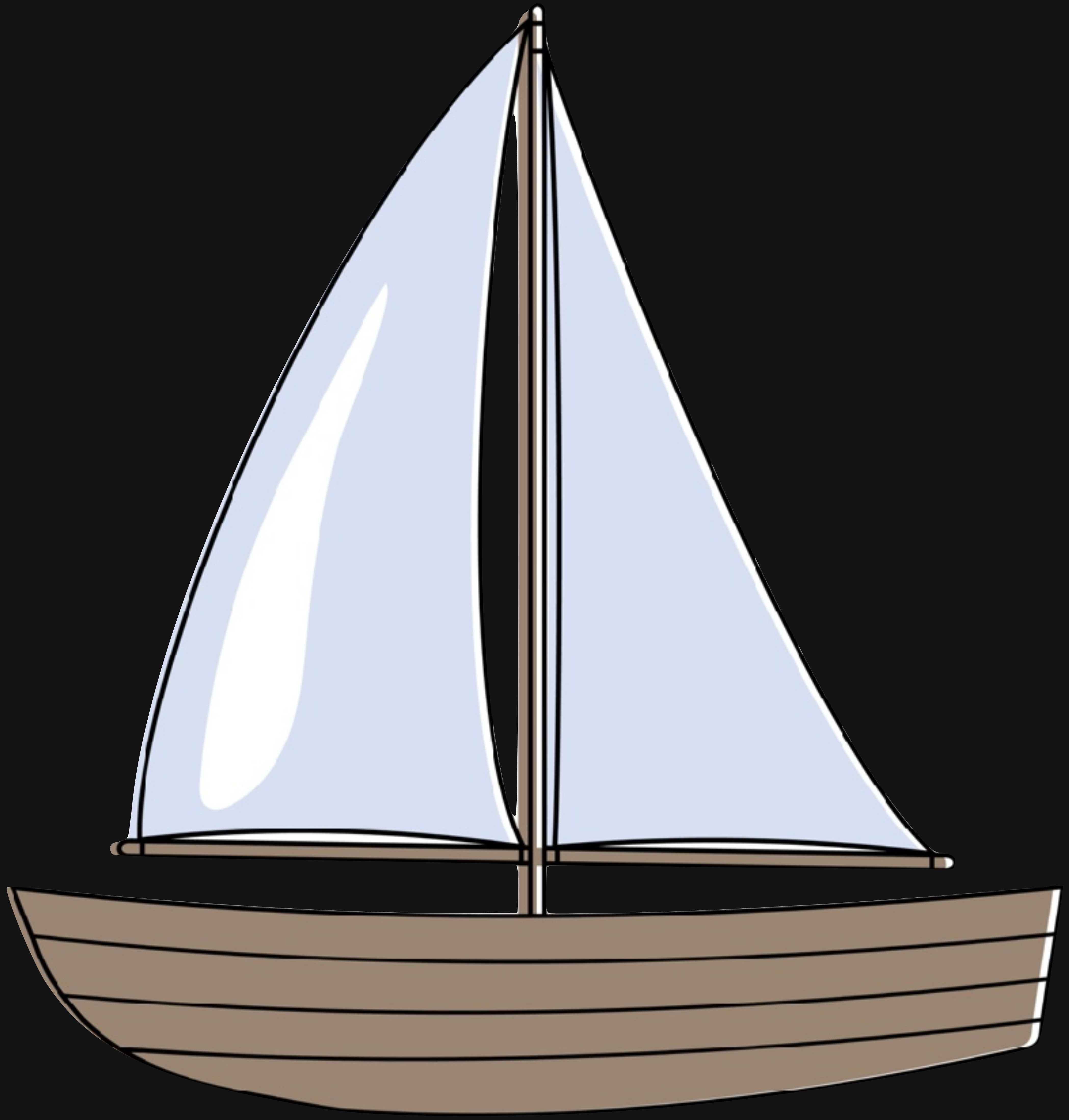
$$= 16.2\frac{\text{km}}{\text{s}^2}$$

$$= 1{,}600g$$

Relativity

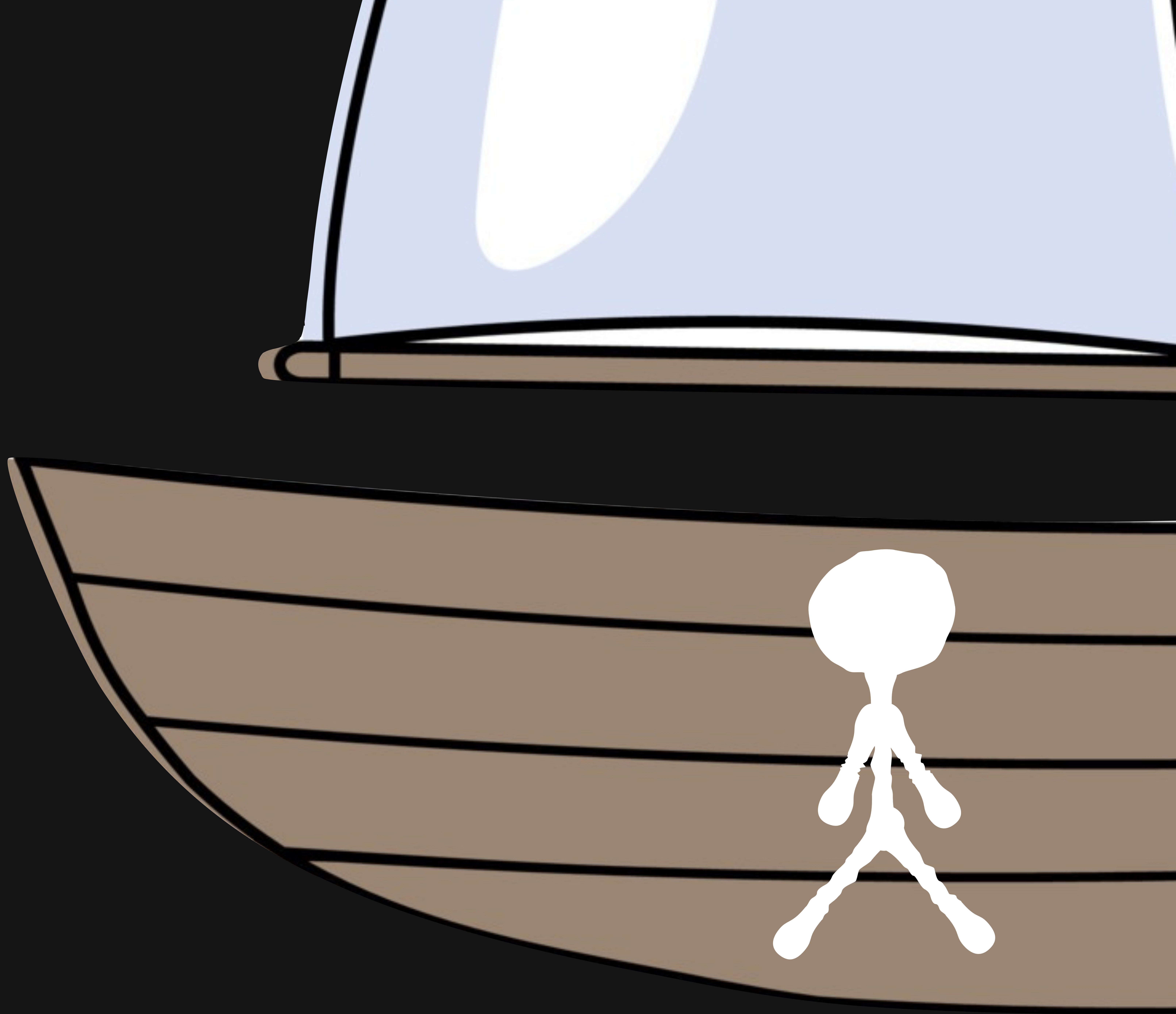
If you are below deck on a ship sailing smooth seas, there is no way to tell if you're moving.

No isolated experiment can detect the experimenter's velocity of motion through space.



Relativity

This is **false** in hyperbolic space!



Relativity

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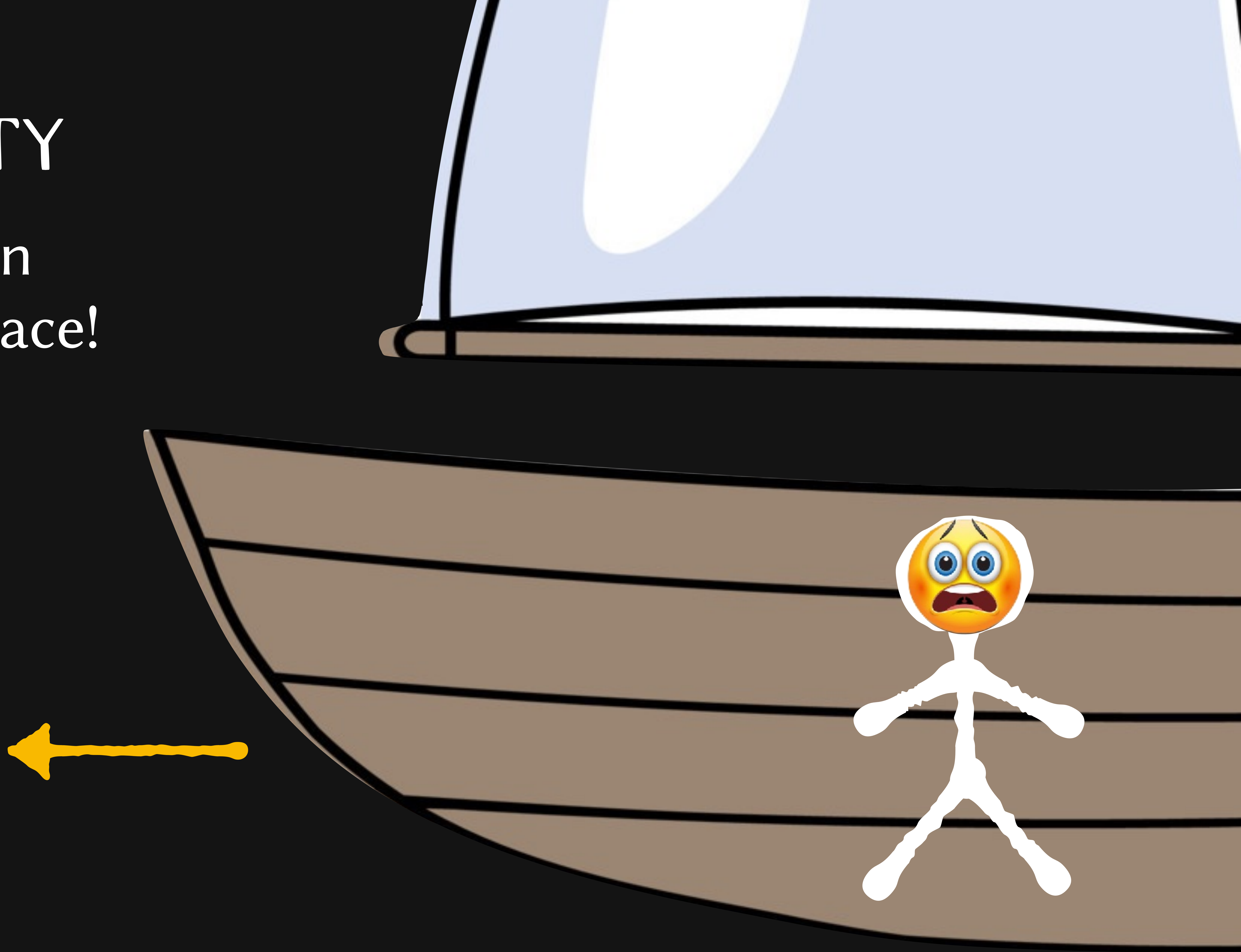
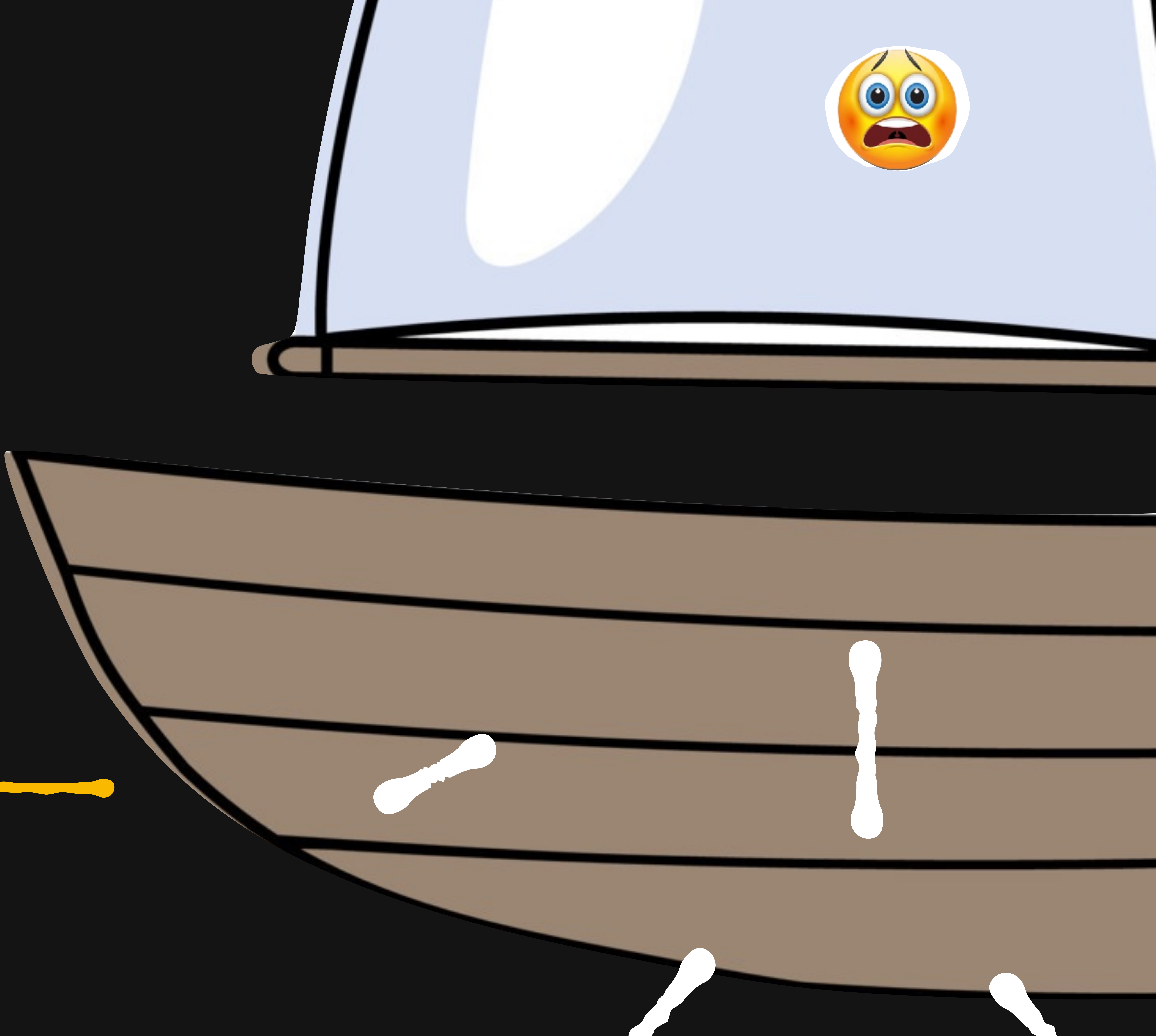


Relativity

This is **false** in hyperbolic space!

Indeed, its possible to build a device that can measure your exact velocity vector



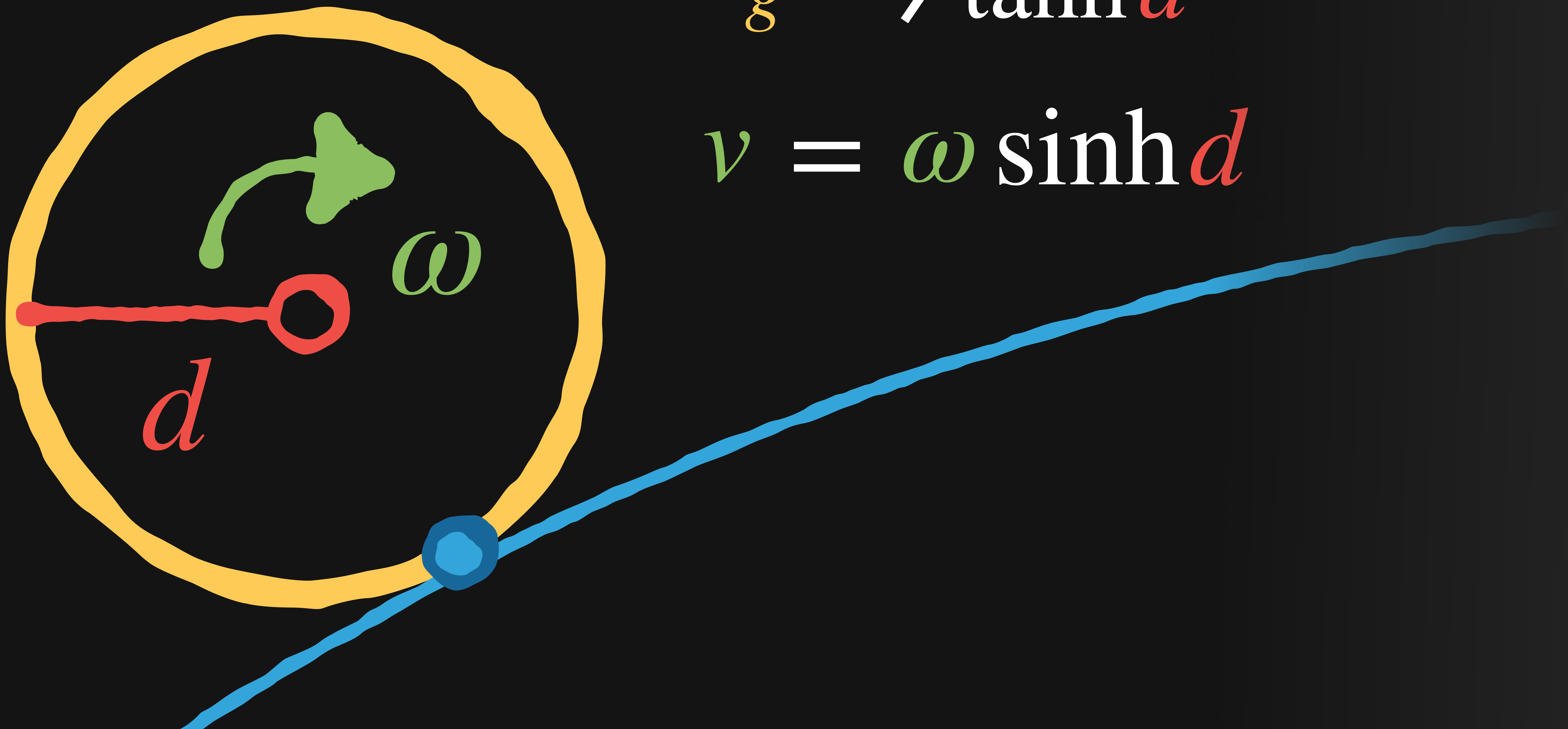
Relativity

But....a new form of relativity emerges between translational and rotational motion.

$$\kappa_g = 1/\tanh d$$

$$v = \omega \sinh d$$

$$\text{acc} = v^2 \kappa_g = \frac{(\omega \sinh d)^2}{\tanh d} = \frac{\omega^2}{2} \sinh 2d$$



Consequences:

Spin the earth and it explodes

R: 13.6 km

ω: 1/24hrs

$$\text{acc} = \frac{\sinh(2 \cdot 13.6)}{2(24 \cdot 3600)^2}$$

$$522.3 \frac{\text{km}}{\text{s}^2}$$

$$522{,}316 \frac{\text{m}}{\text{s}^2}$$

$$53{,}000g$$

Consequences:

You sit upside down on trains

(Not to Scale)

R: 13.6 km

v: 430 kph
= 0.12 km/s

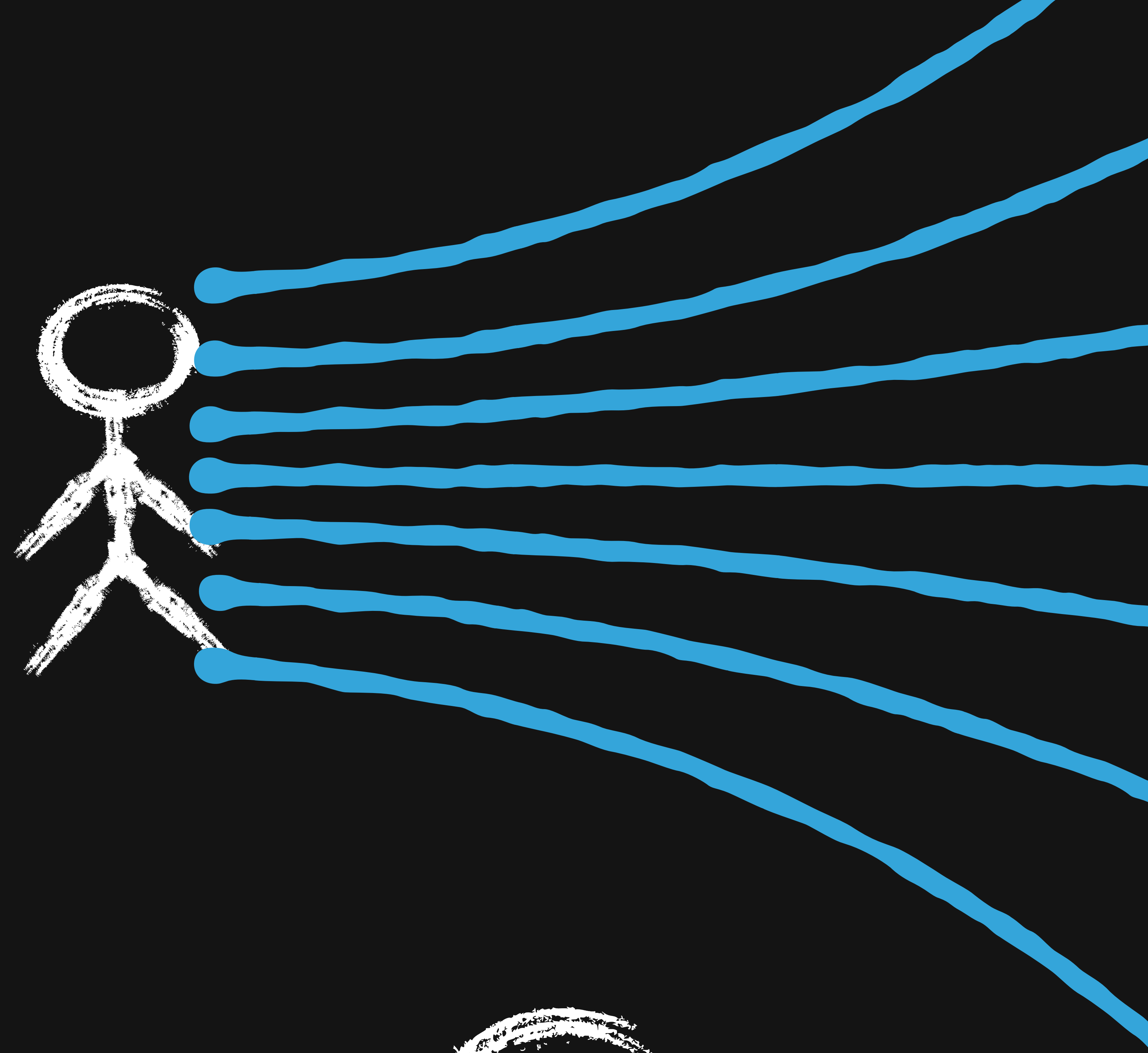
$$\text{acc} = \frac{(0.12)^2}{\tanh 14.6}$$

$$0.014\frac{\text{km}}{\text{s}^2}$$

$$14\frac{\text{m}}{\text{s}^2}$$

$$1.5g$$

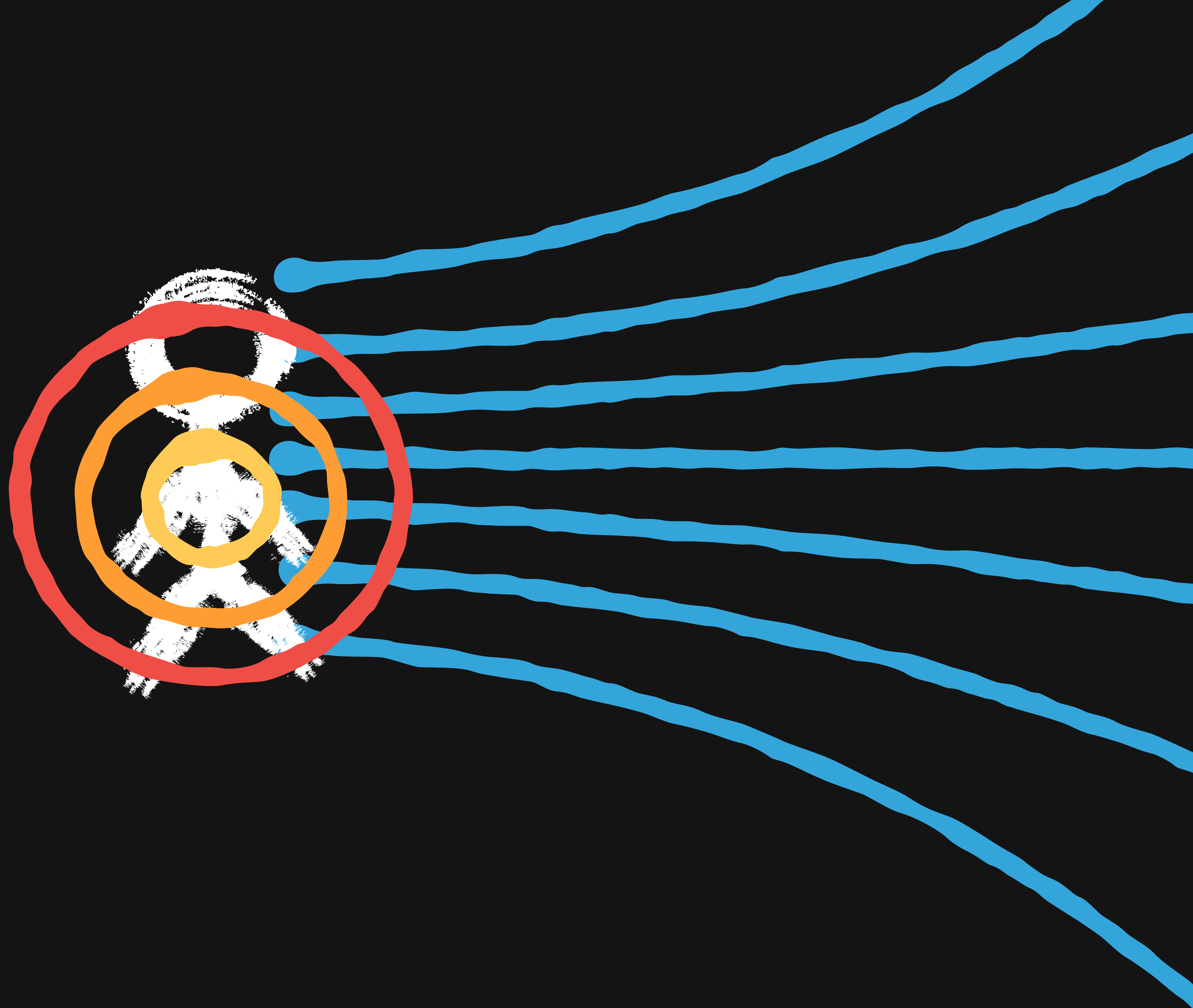
$$\text{acc} = \frac{v^2}{2} \sinh 2d$$



Spinning and translating feel qualitatively the same!

$$\text{acc} = \frac{\omega^2}{2} \sinh 2d$$

$$\text{acc} = \frac{v^2+\omega^2}{2} \sinh 2d$$

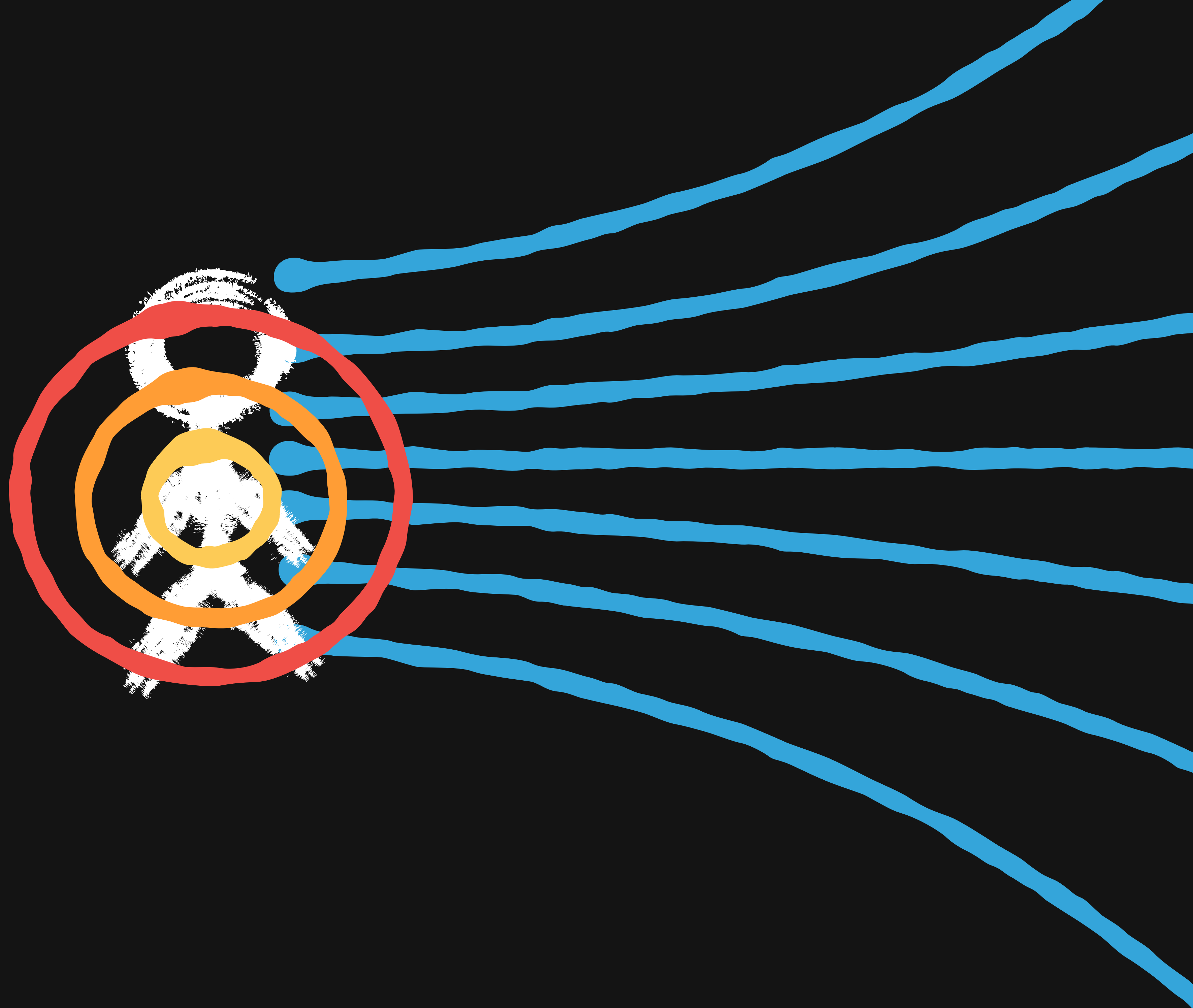


New Relativity:

Define the *total velocity* of a rigid object moving along a geodesic as $V = \sqrt{\omega^2 + v^2}$

Experiments can measure total velocity, but not angular or linear velocity separately.

$$\text{acc} = \frac{v^2+\omega^2}{2}\sinh 2d$$

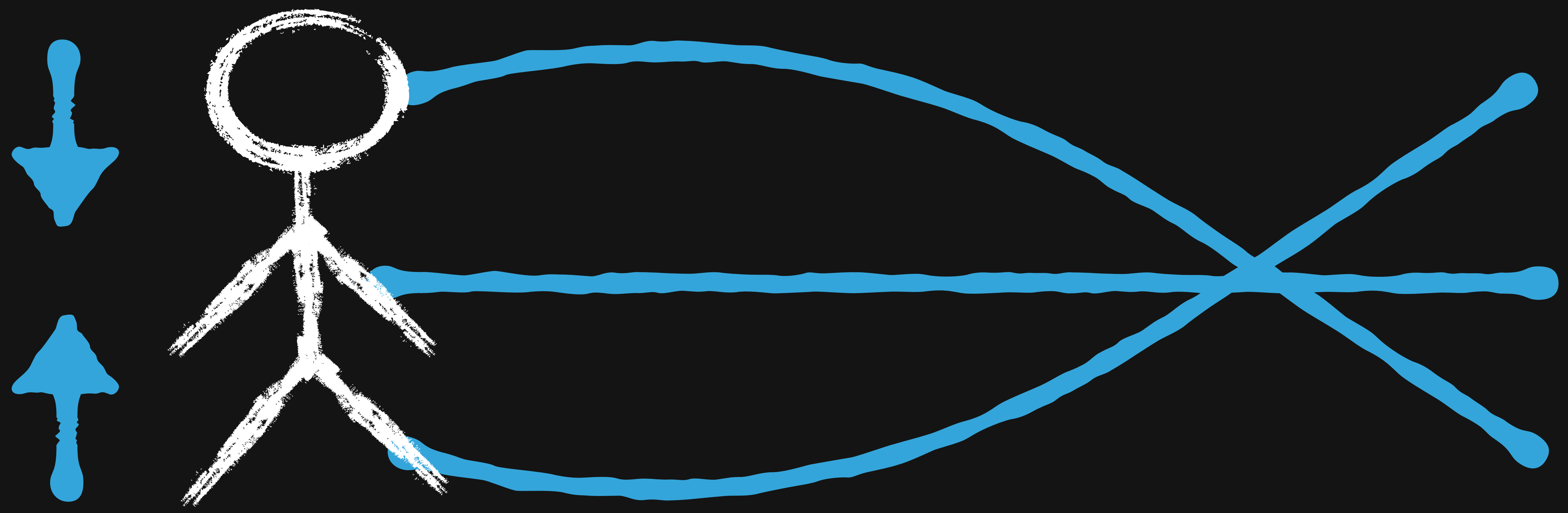


New Relativity:

Let G be the group of isometries preserving a chosen geodesic, and $H_t = \exp(tX)$ be a 1-parameter subgroup of G.

Then one can only measure $\|X\|$, not X itself.

Spherical Geometry



$\oplus \ \frac{\omega^2}{2}\sin(2d)$

$\ominus \ \frac{v^2}{2}\sin(2d)$

Spherical Geometry

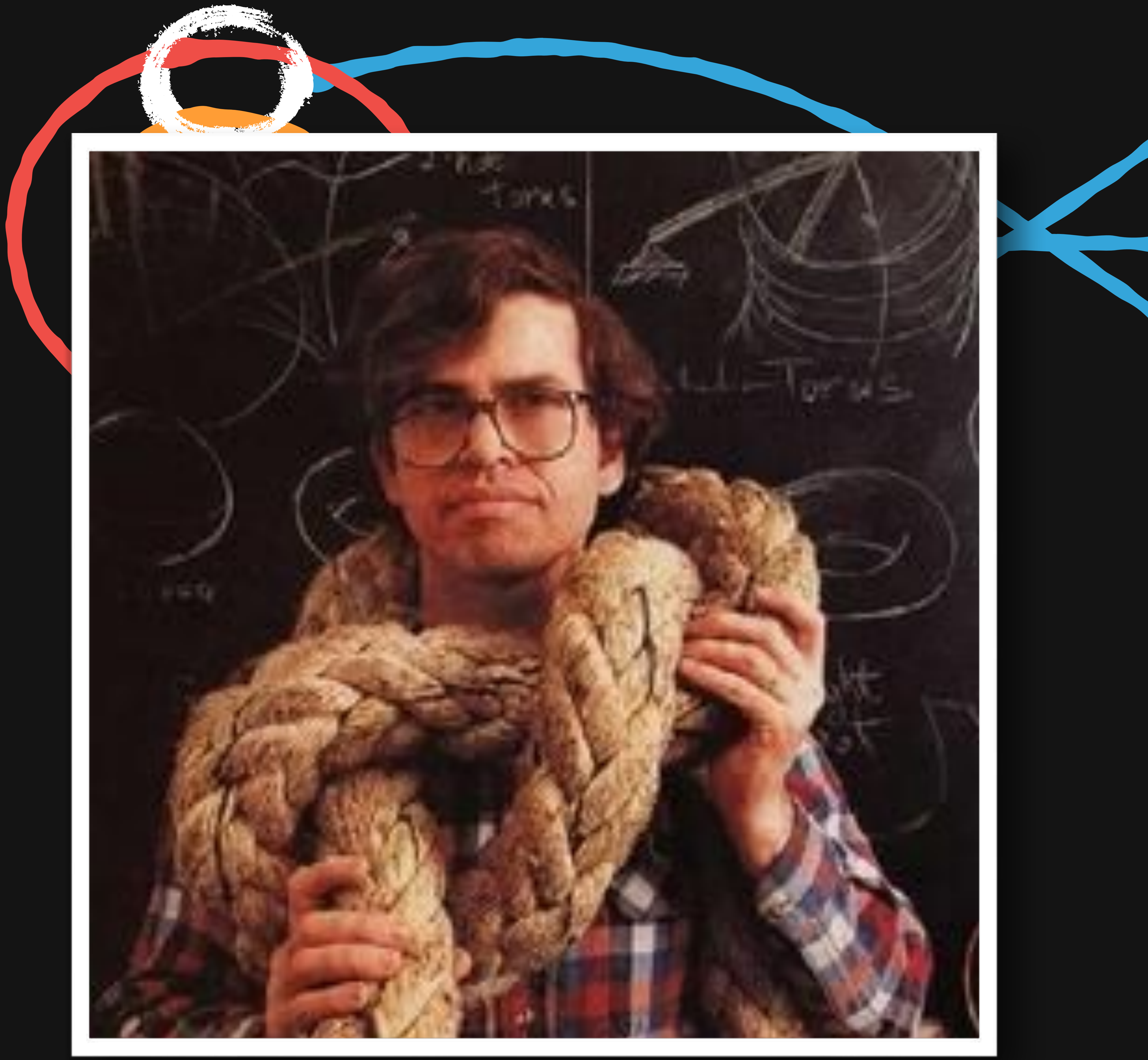
$$\mathrm{acc} = \frac{\omega^2 - v^2}{2} \sin 2d$$

The relativity in positive curvature
is in their *difference*.

If angular velocity equals linear velocity,
our explorer *feels no force at all.*

What paths are these?

The Hopf Fibration!

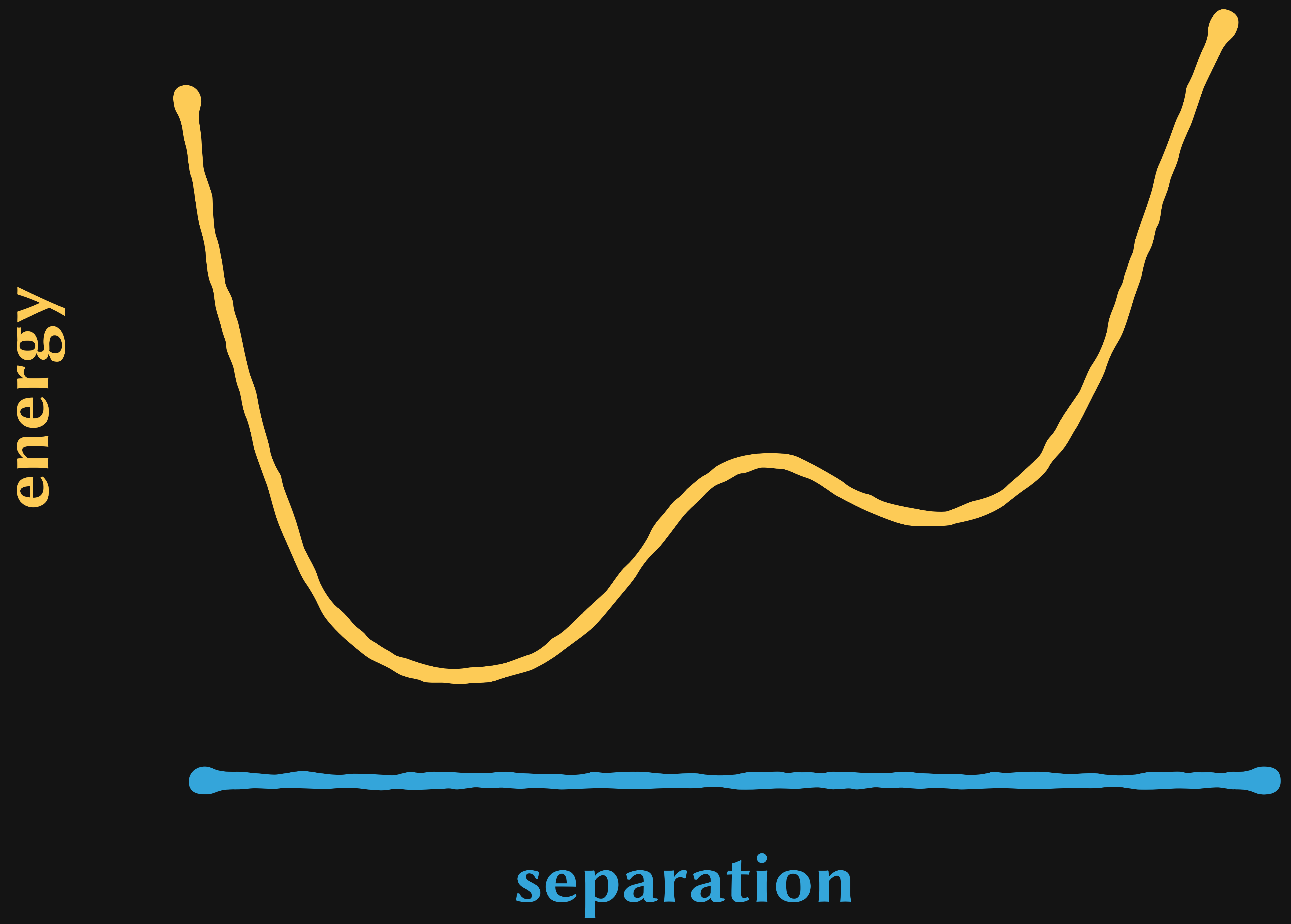


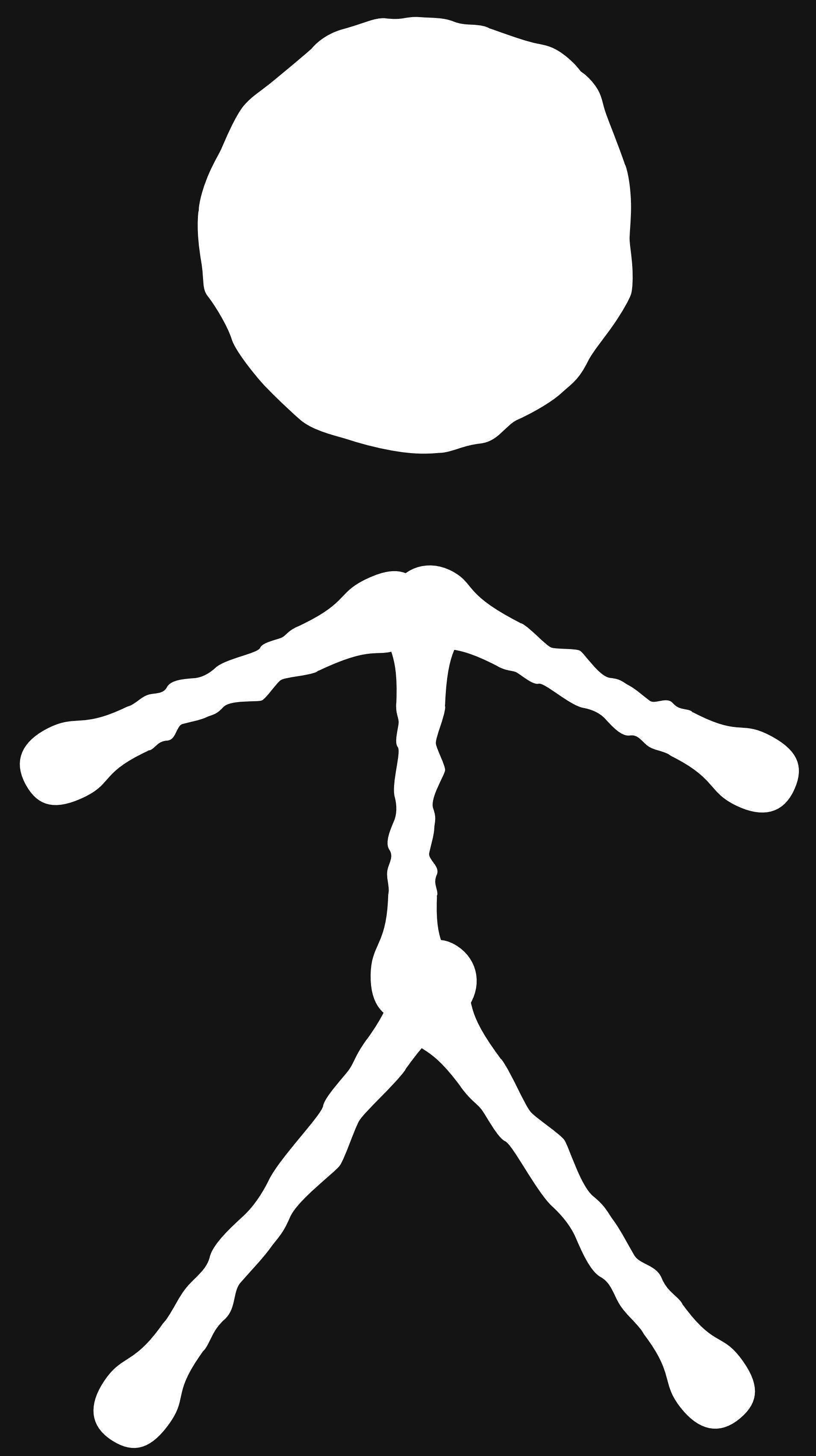
Thurston must have known all of this...

Space has…Friction?

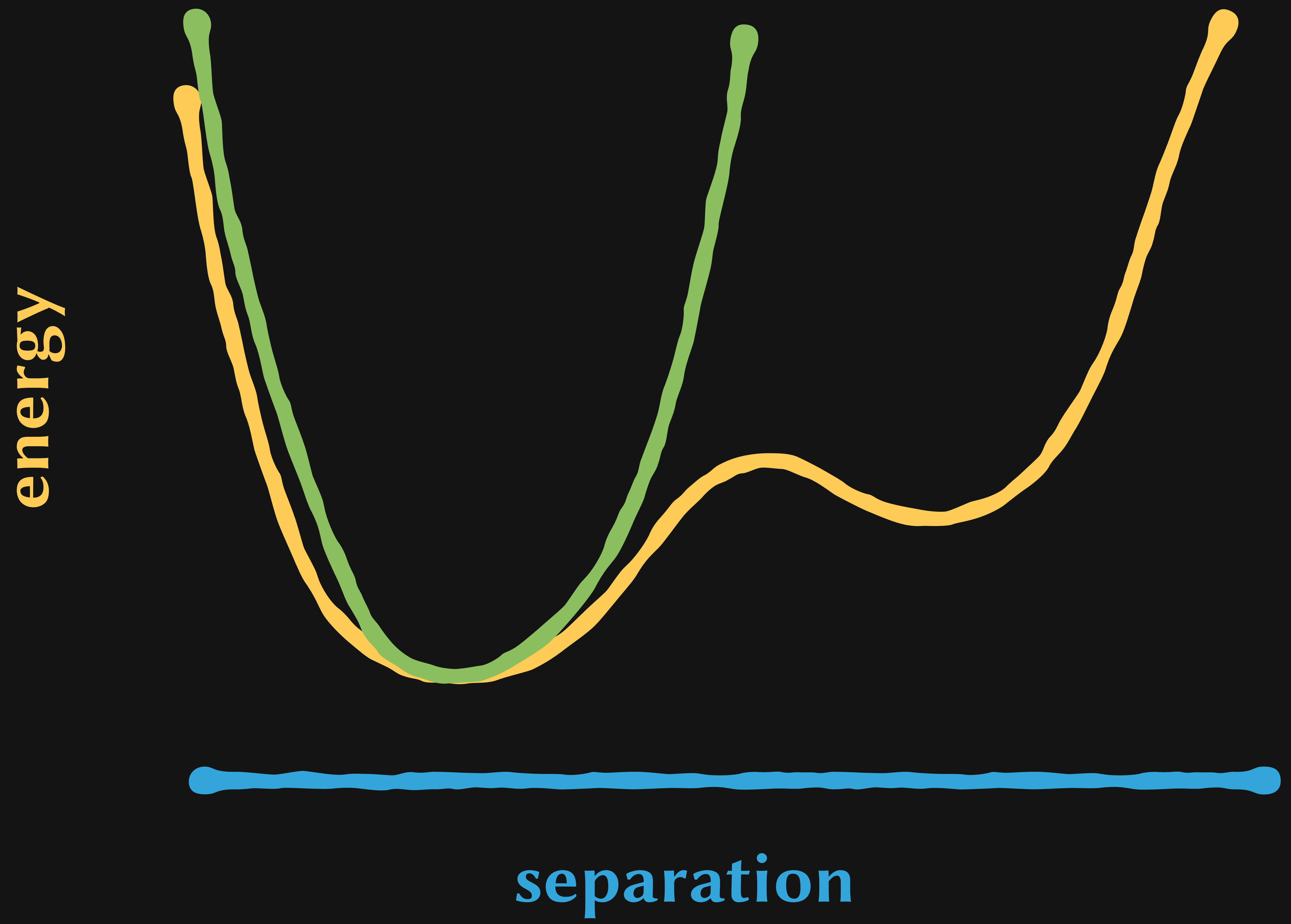
because everything jiggles

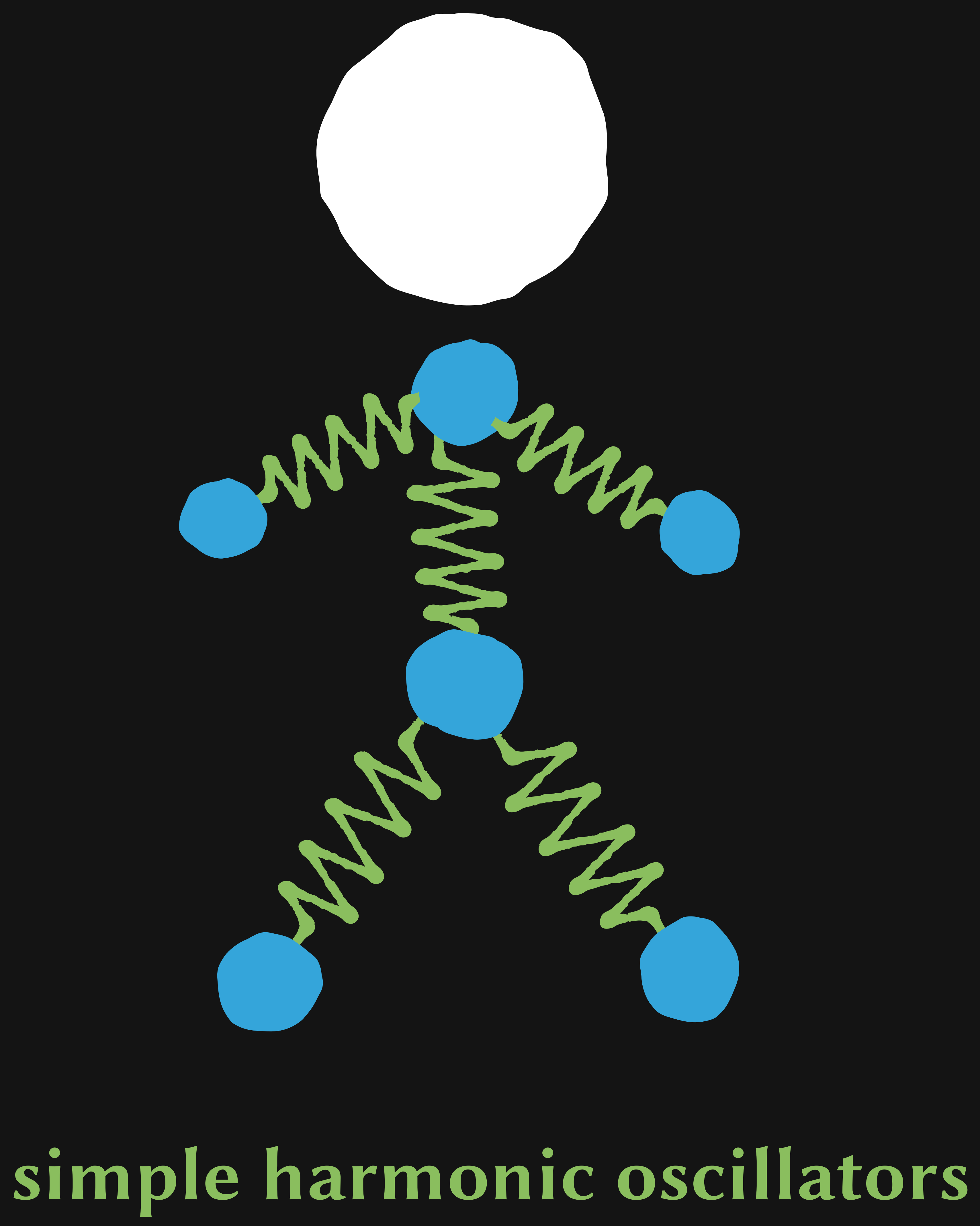
Objects are made of matter held together by inter-molecular forces





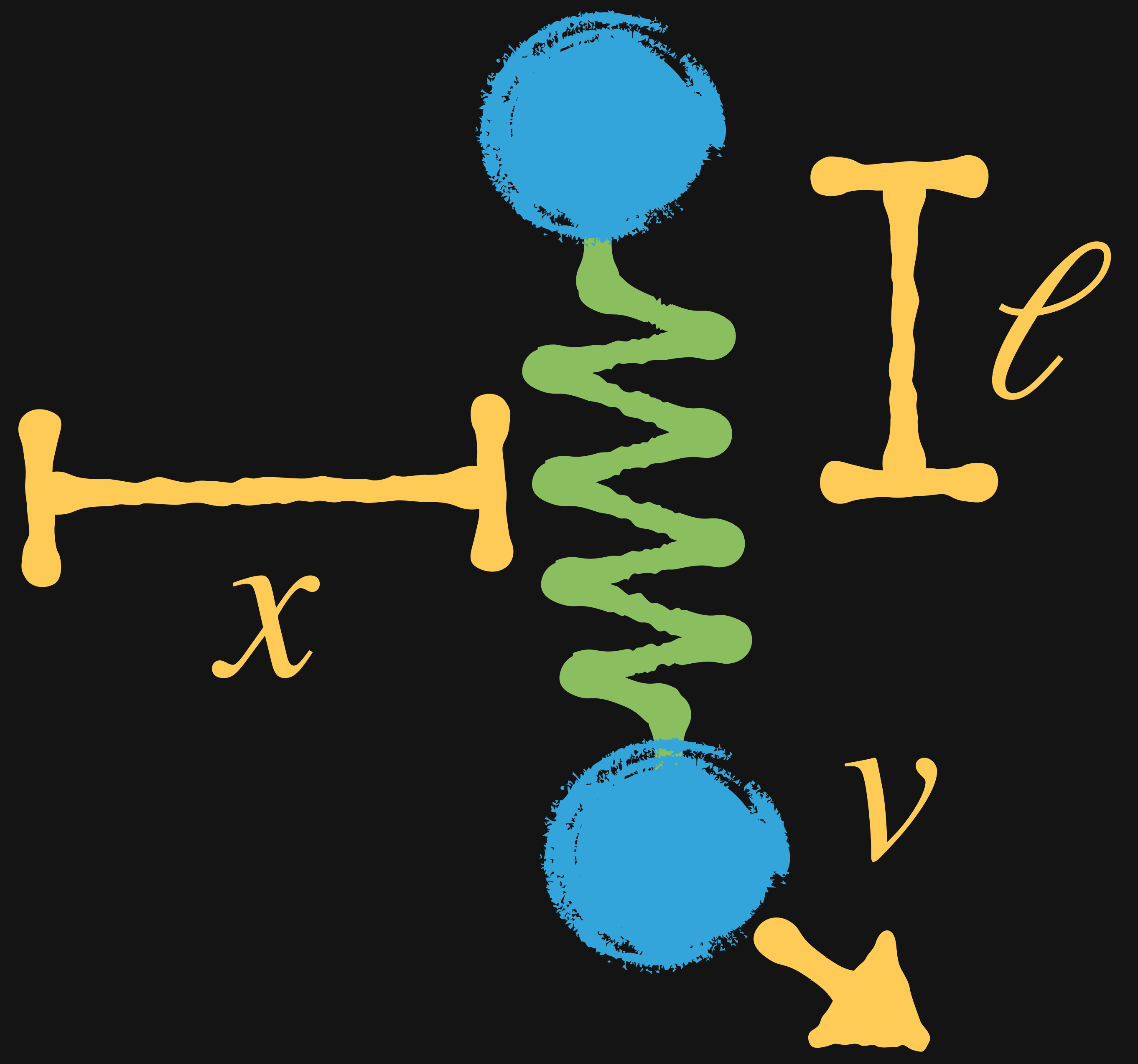
Objects are made of matter held together by inter-molecular forces





Lagrangian for a Spring

$$\mathscr{L} = \frac{1}{2} m \|v\|^2 - \frac{1}{2} k \left(\Delta \ell\right)^2$$



Calculus of Variations in $\mathbb{H}^3$

$$\left\{ \dot{x} = \frac{v_o \cosh \ell_o}{\cosh^2 \ell} \qquad \ddot{\ell} = \frac{v_o^2 \sinh \ell}{\cosh^3 \ell} - \frac{2k}{m}\left(\ell - \ell_0\right) \right\}$$

Determined by ℓ

Bounded

Unbounded

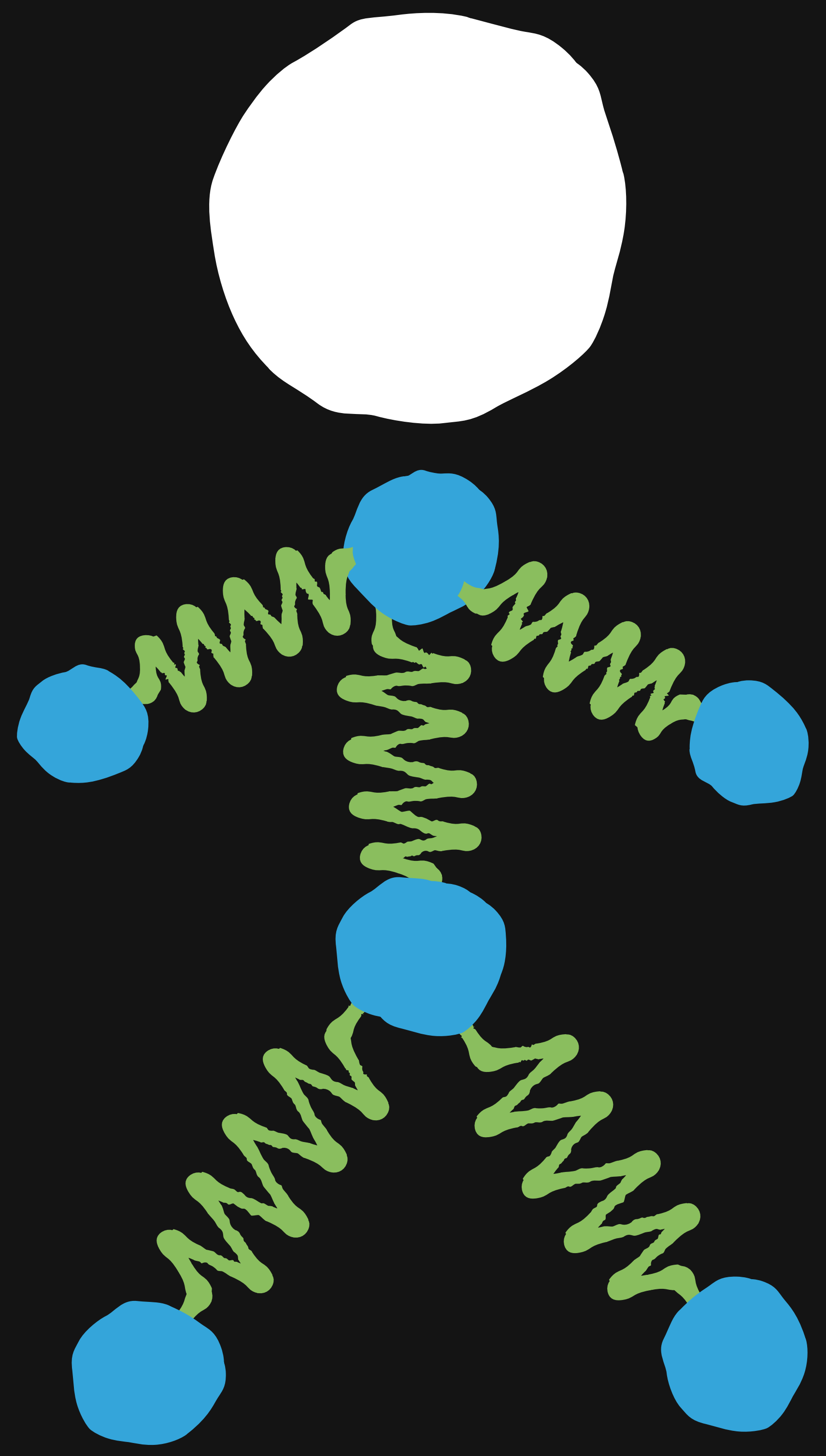
Velocity induced oscillations

Velocity induced oscillations

Magnitude dependent on initial speed

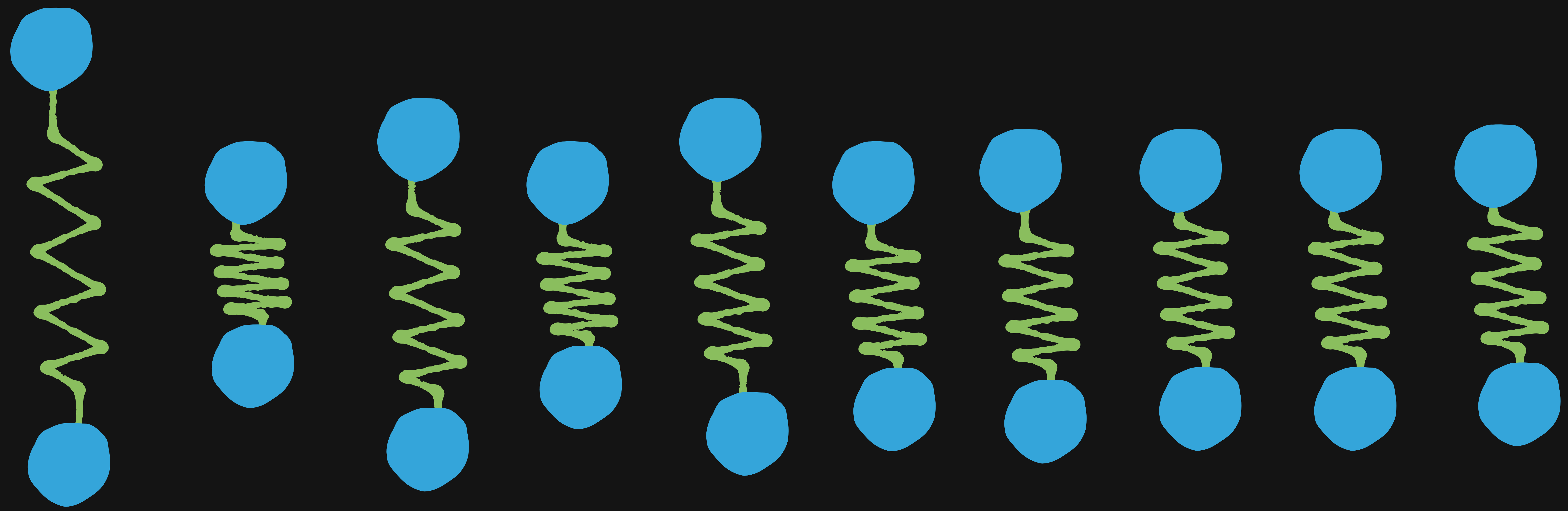
Consequences:

Everything vibrates!



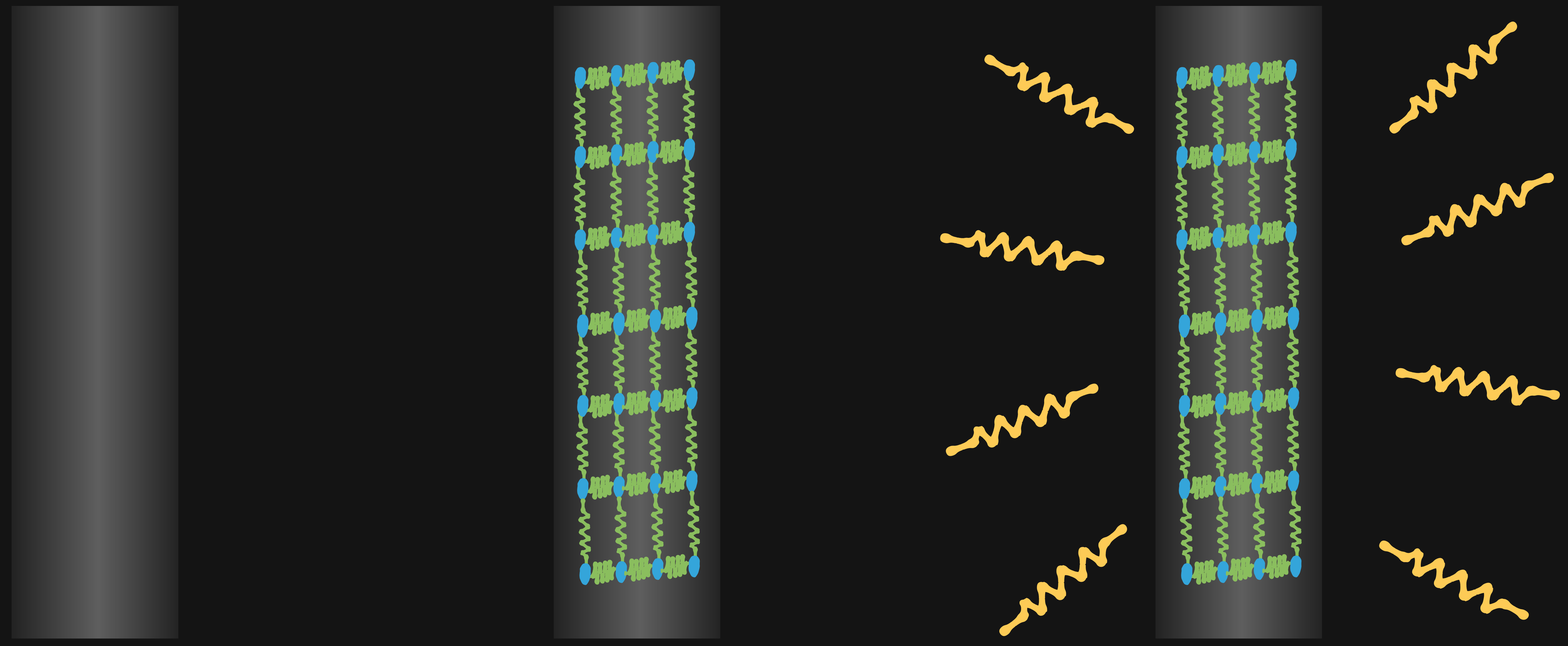
If your planet doesn't explode, it suffers constant earthquakes

Real world objects loose energy

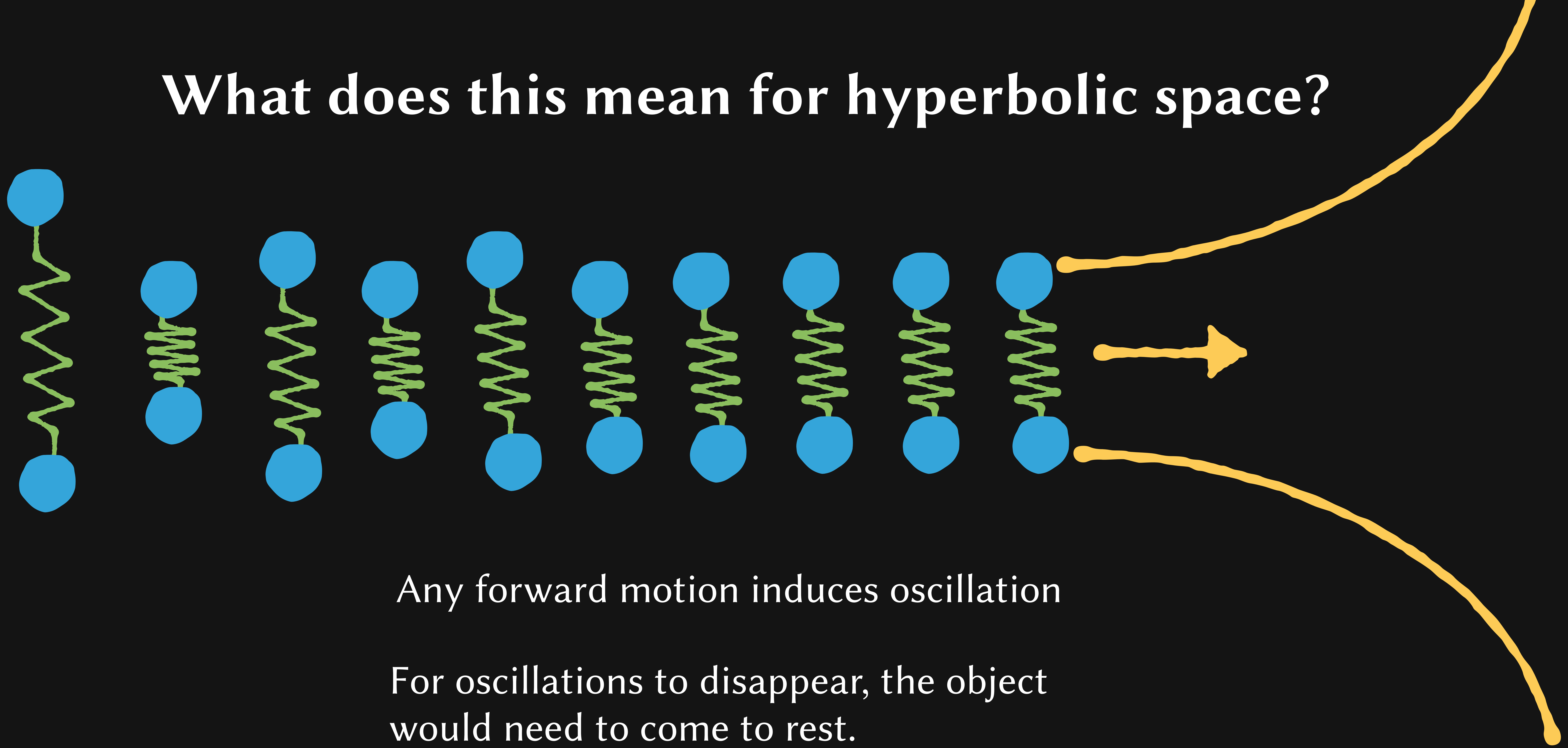


Transferred from macroscopic to microscopic degrees of freedom.

Eventually radiated as heat.



What does this mean for hyperbolic space?



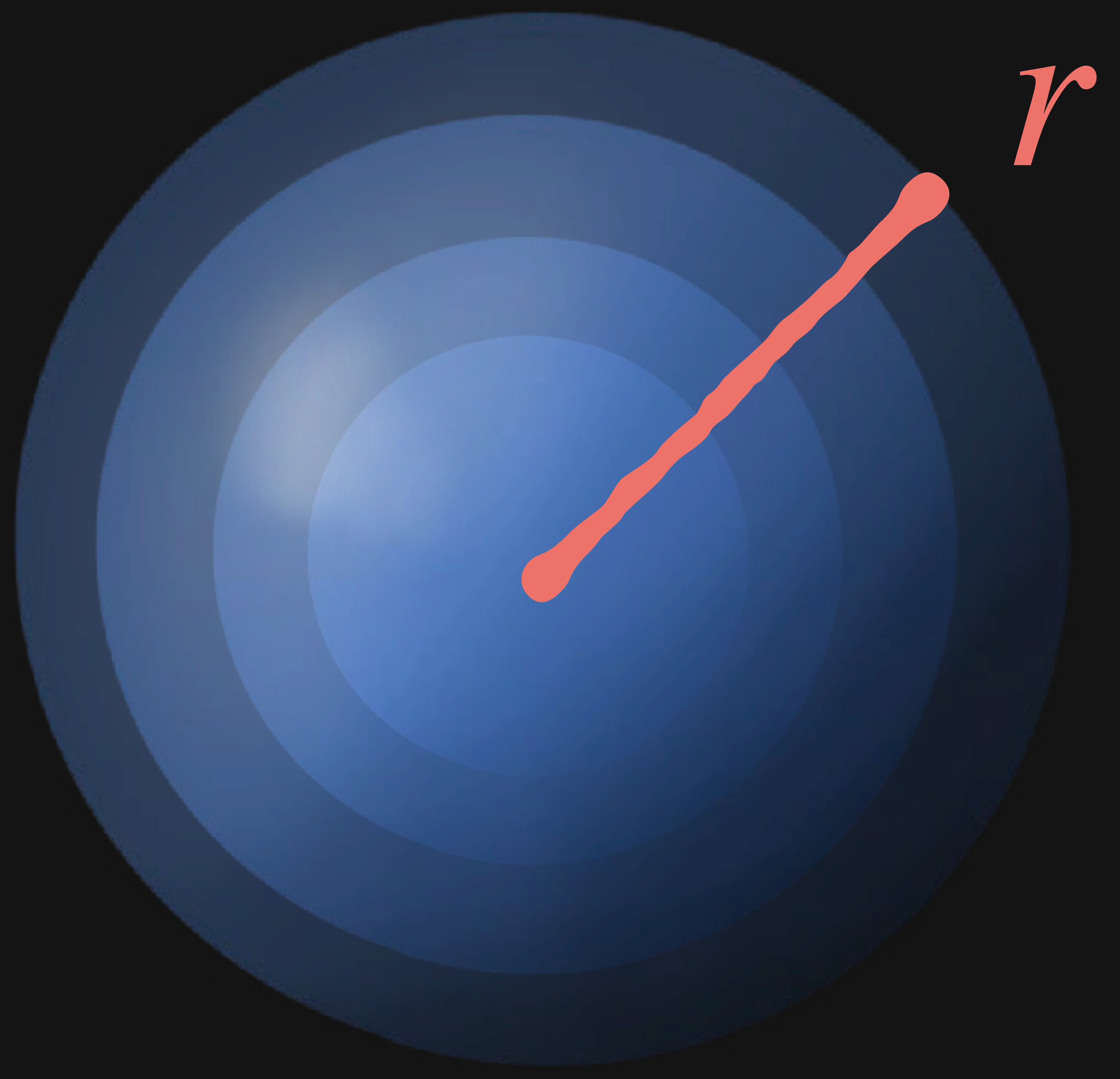
Any forward motion induces oscillation

For oscillations to disappear, the object would need to come to rest.

Do objects slow down in empty space?!

Locked in Space

drinking coffee through a straw



Euclidean

$$\text{Area} = 4\pi r^2$$

$$\text{Volume} = 4\pi r^3/3$$

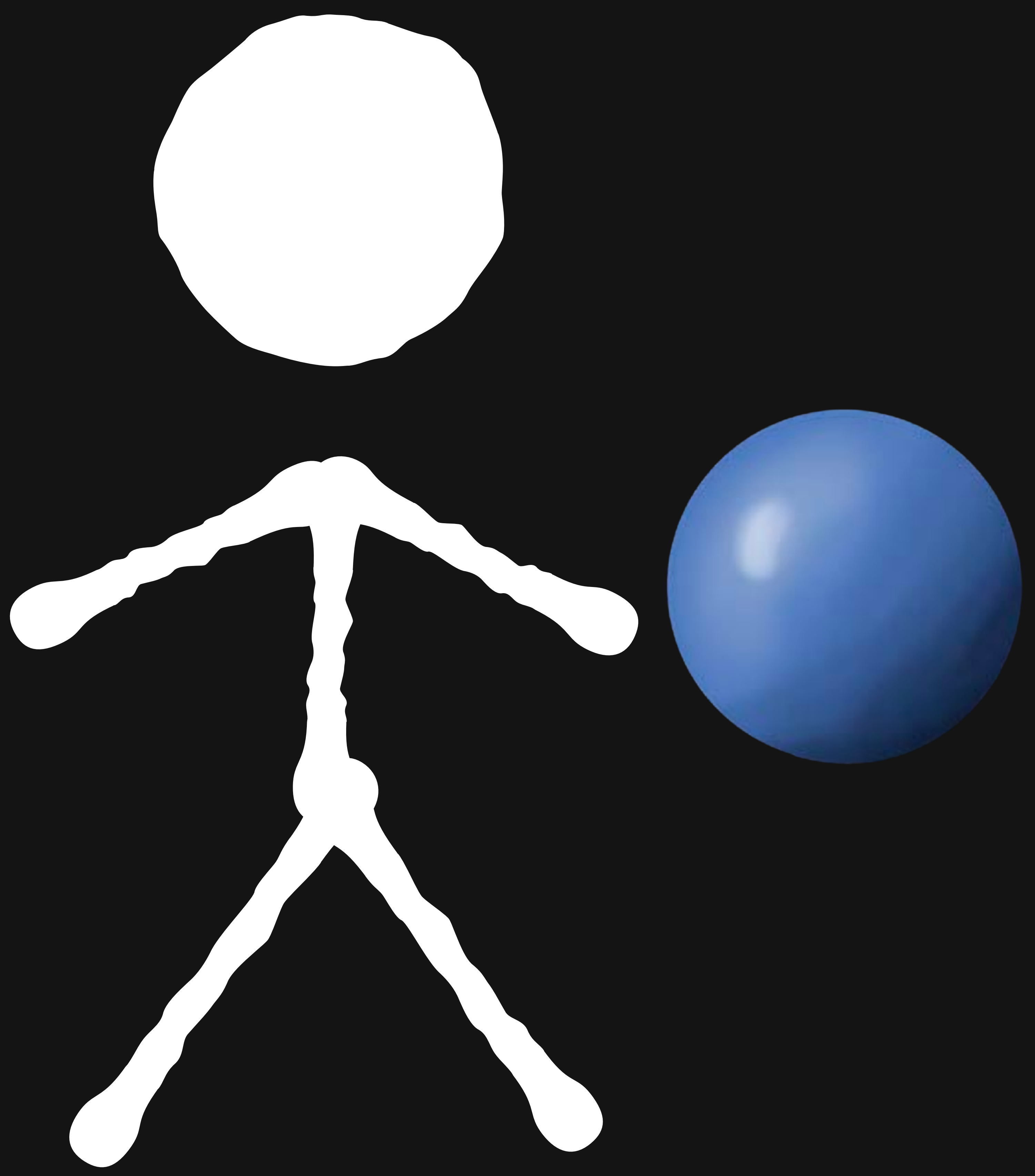
Hyperbolic

$$\text{Area} = 4\pi \sinh^2(r)$$
$$\approx \pi e^{2r}$$

$$\text{Volume} = \pi \left(\sinh(2r) - 2r\right)$$
$$\approx \pi e^{2r}/2$$

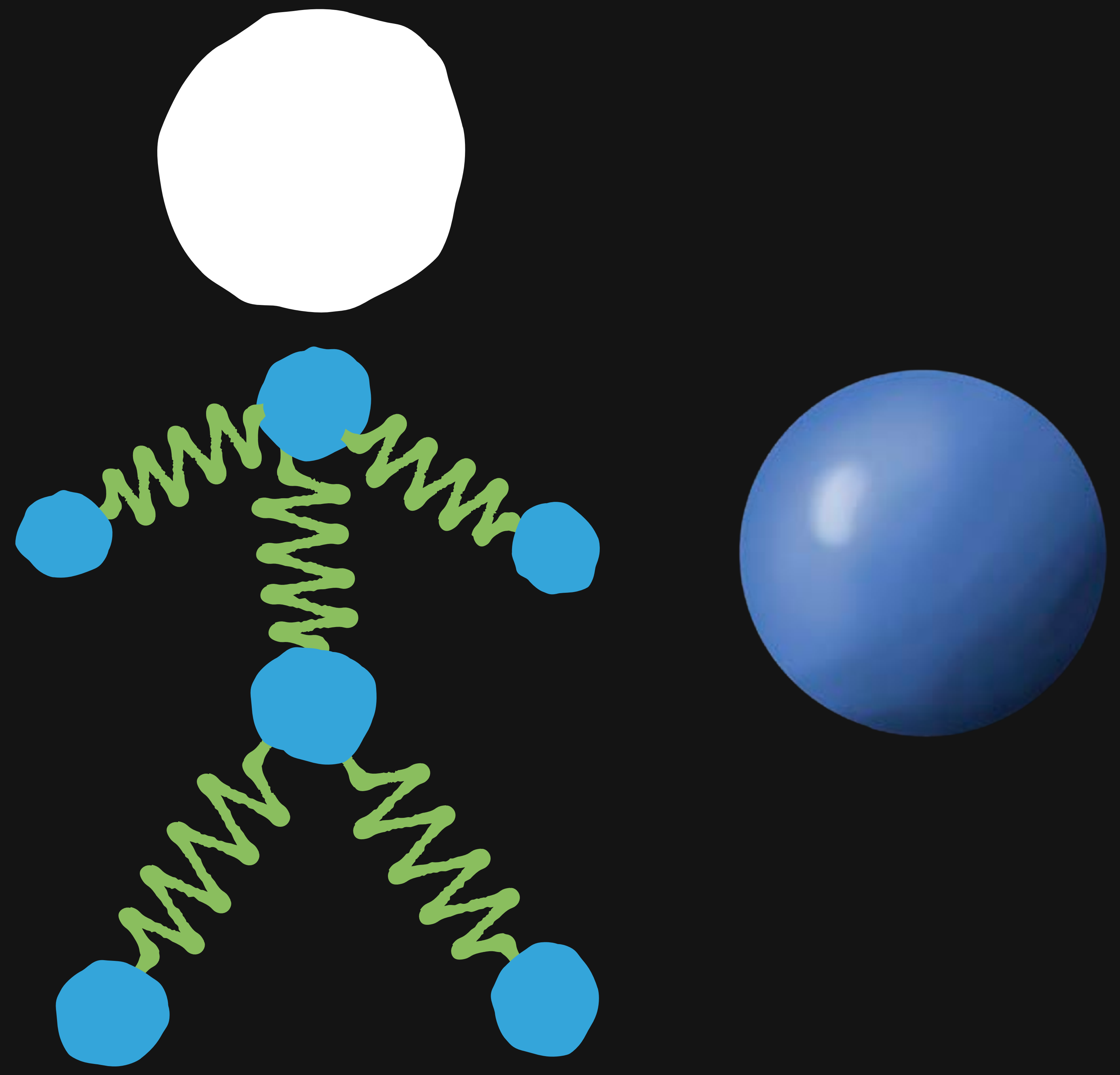
Moving along a curvature gradient

Euclidean **Hyperbolic**



Moving along a curvature gradient

Euclidean **Hyperbolic**

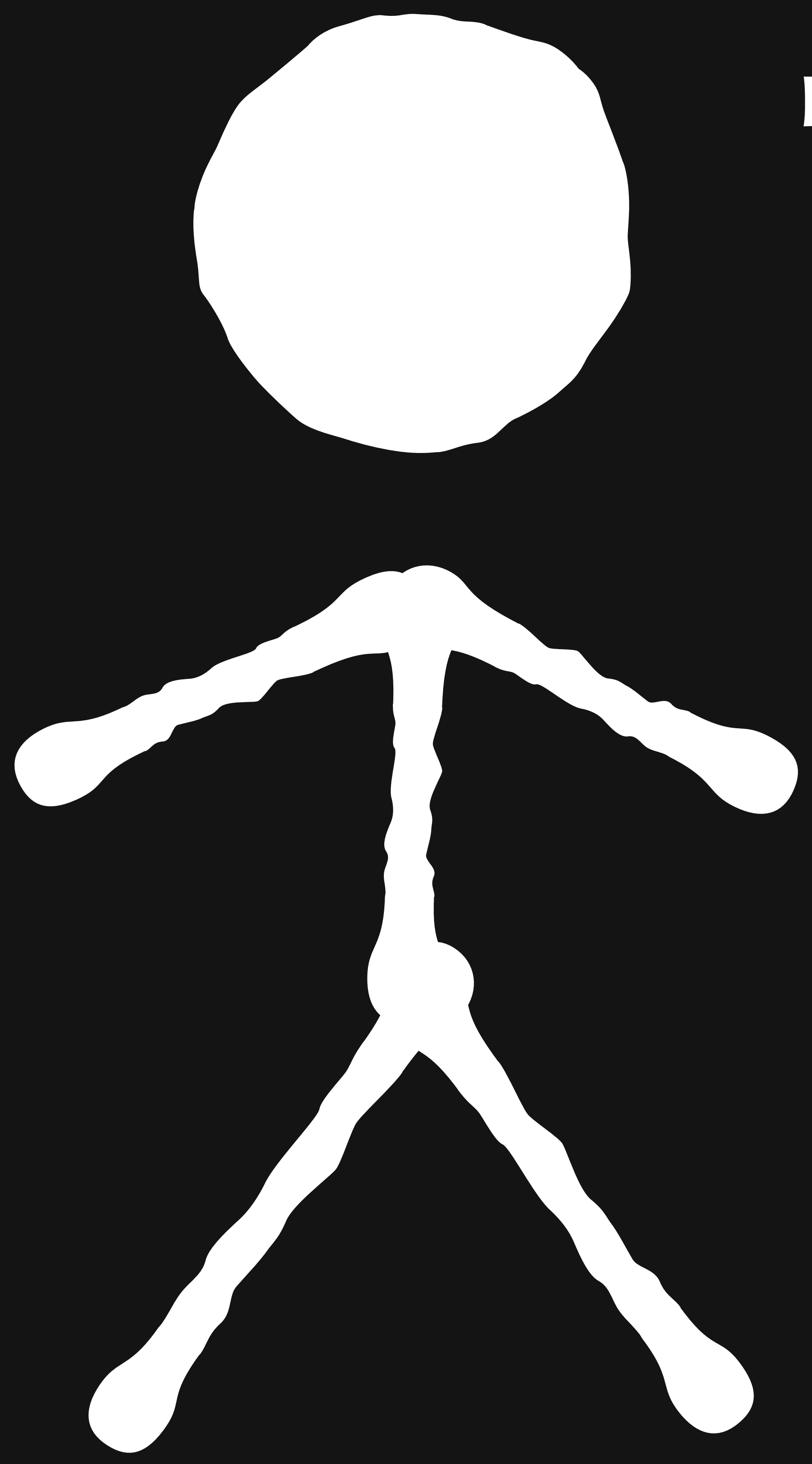


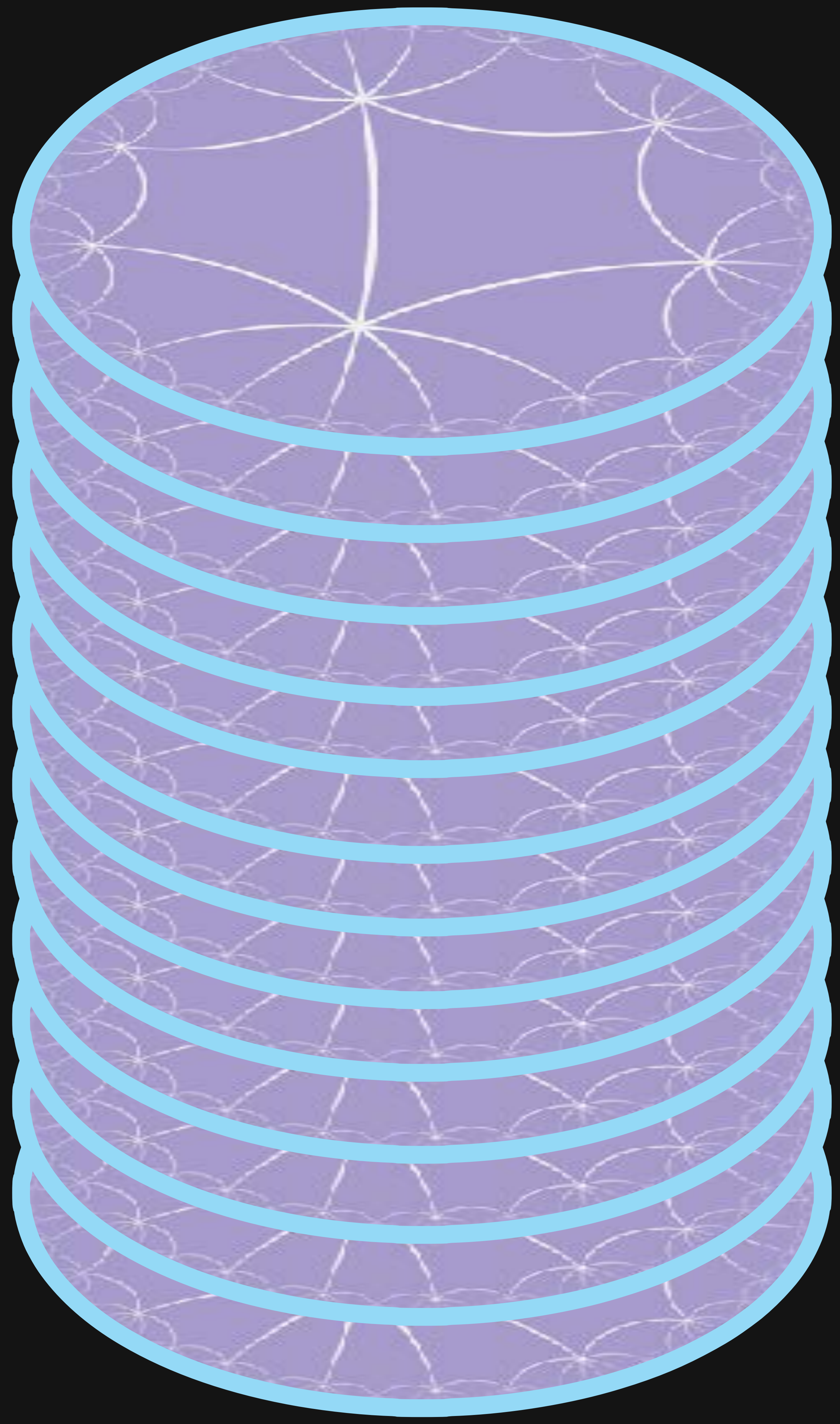
It takes a lot of force to rearrange solid matter

You can pick a table up and move it around, but you can’t flip it over

Same for your coffee: the cup must stay vertical or it shatters

$\mathbb{H}^2 \times \mathbb{R}$





Snakes on the Space Station

and swimming in the vacuum

*A story from joint work with Brian Day
and Sabetta Matsumoto*

Snakes on a Space Station

Should our astronaut be scared?

No, the snake can’t reach them!

The **center of mass** of the snake cannot change without external influence.

$$\text{CoM} = \sum_{p_i \in \text{🐍}} m_i p_i$$

This argument doesn't generalize:

False: curved space is very different.

True, but for different reasons.

This formula is **a weighted sum of points.** It **doesn't make sense** to add points in curved space!

$$\text{CoM} = \sum_{p_i \in} m_i p_i$$

Interlude:

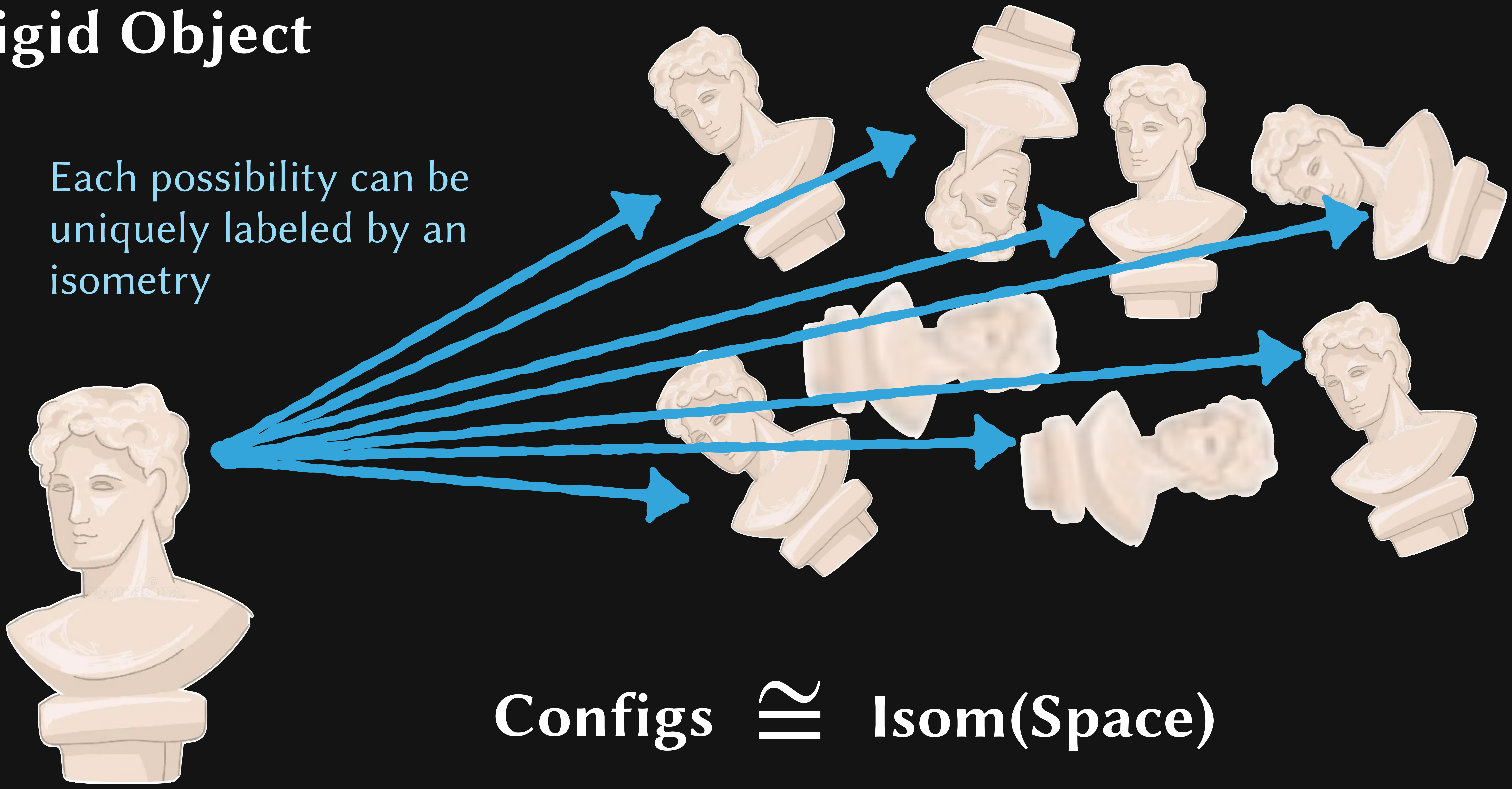
Making sense of deformable objects in classical mechanics

Configurations of a Rigid Object

Configurations of a Rigid Obje

Configurations of a Rigid Object

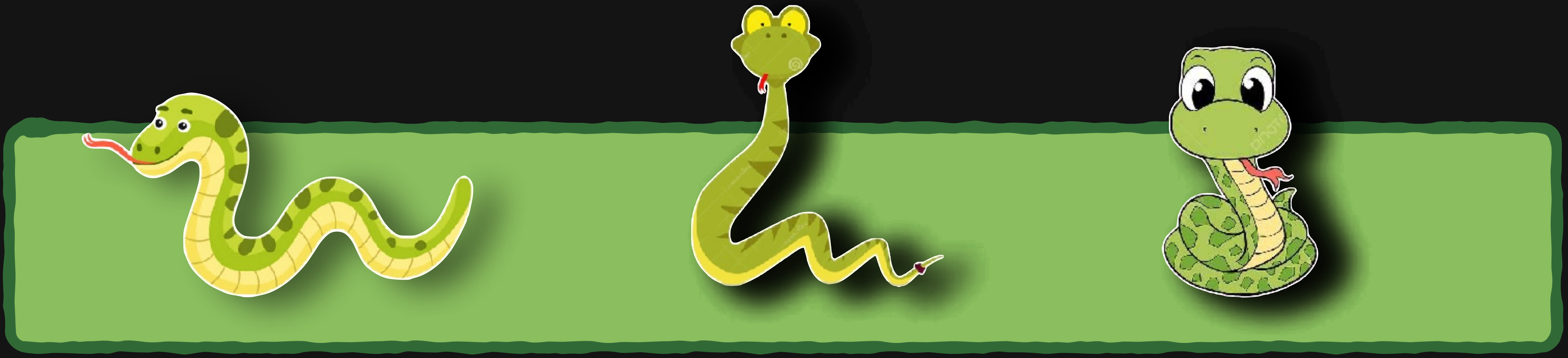
Each possibility can be uniquely labeled by an isometry



Configs $\cong$ Isom(Space)

Configurations of a Deformable Object

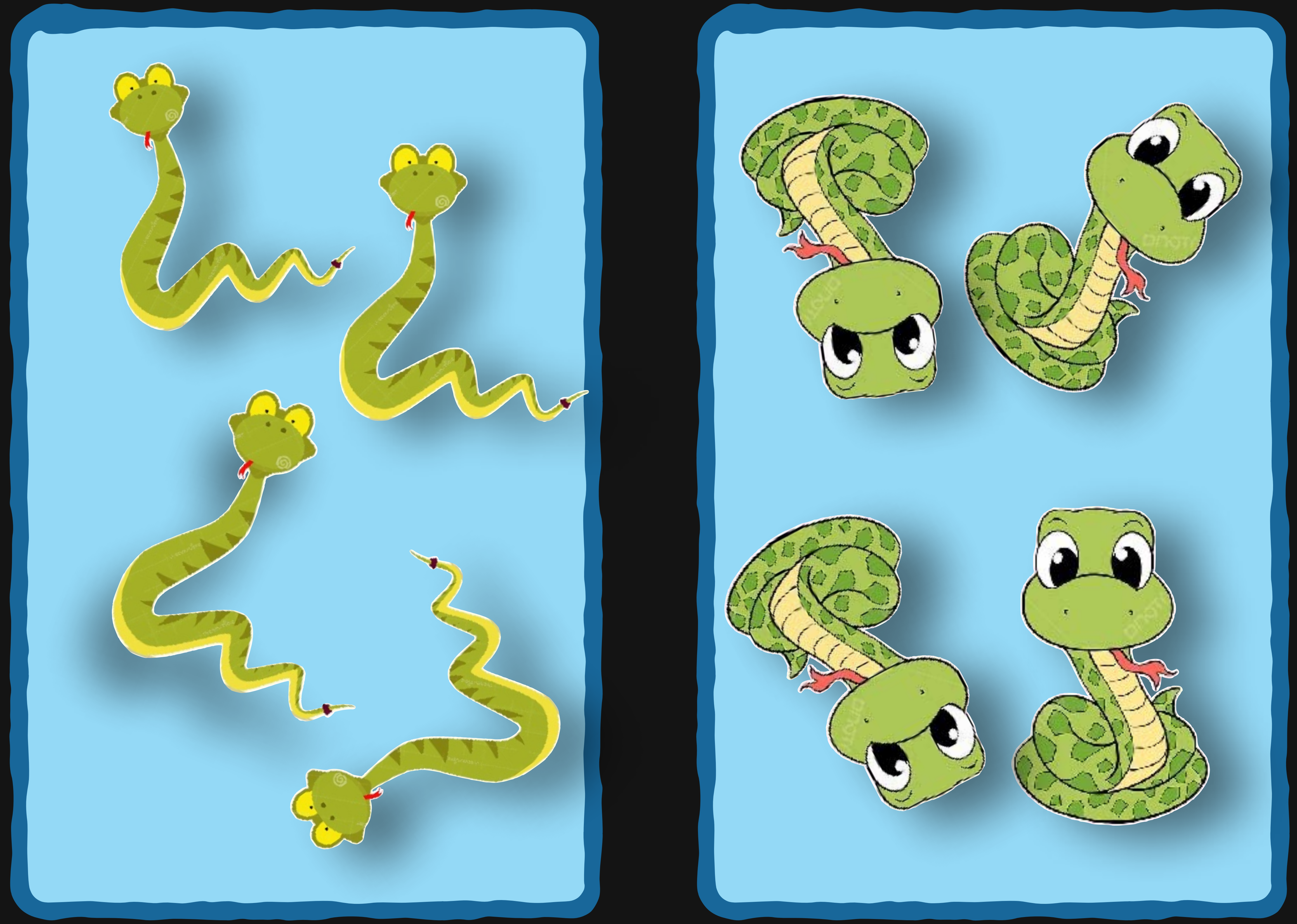
Many different Internal Configurations

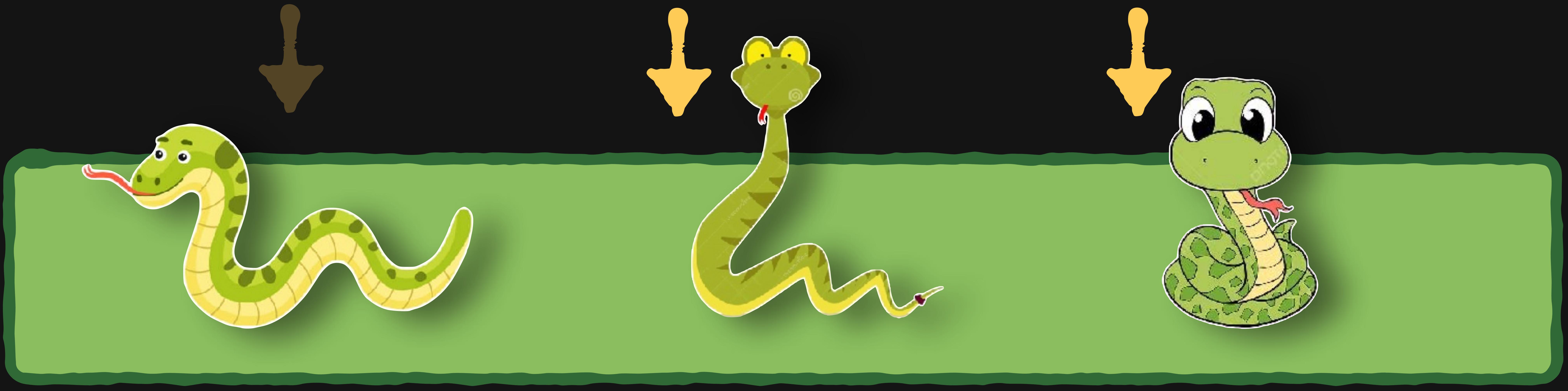


Configurations of a Deformable Object

Many different **Internal Configurations**

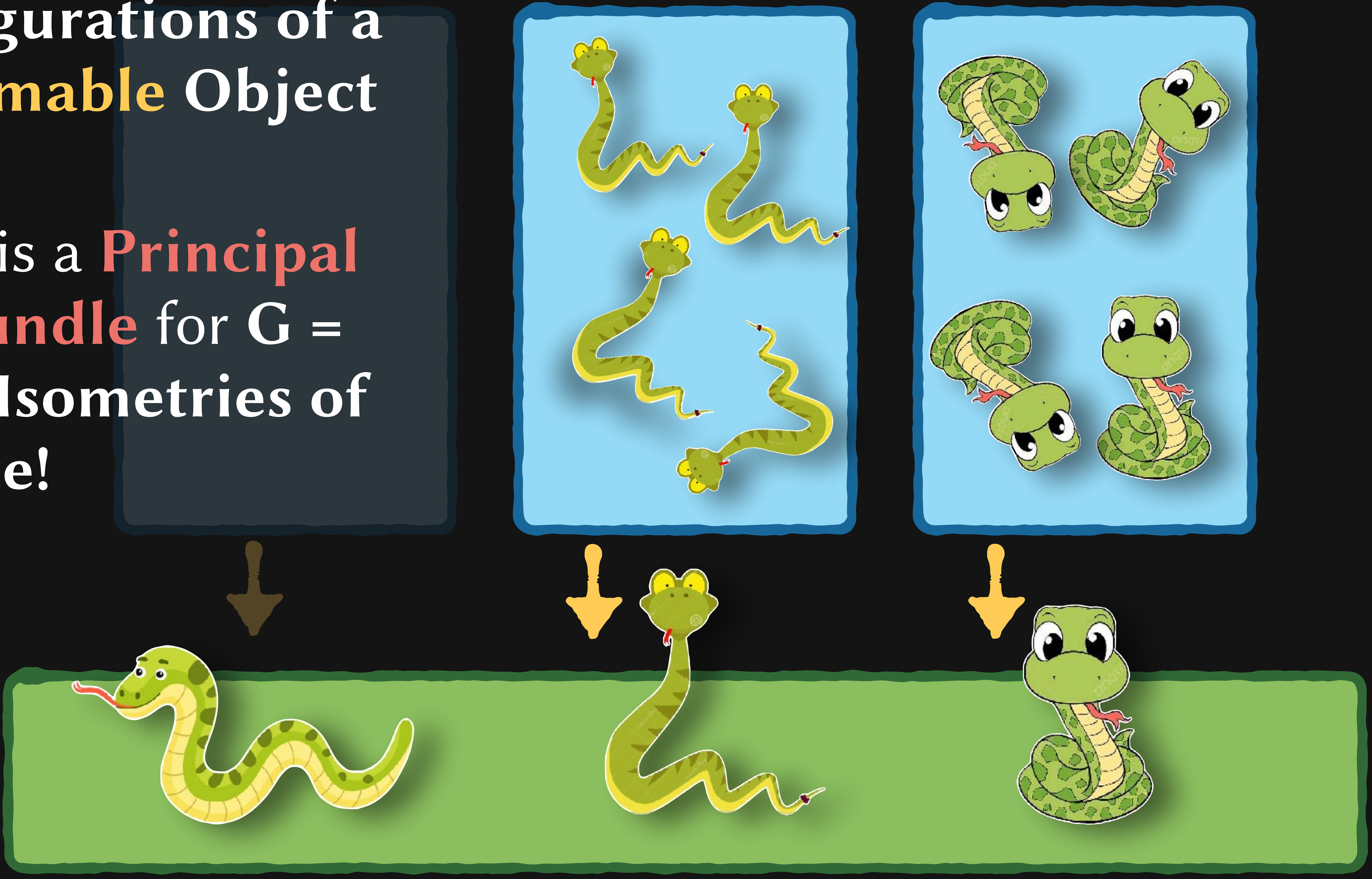
Each internal configuration can be **realized in Isom(Space) many ways**



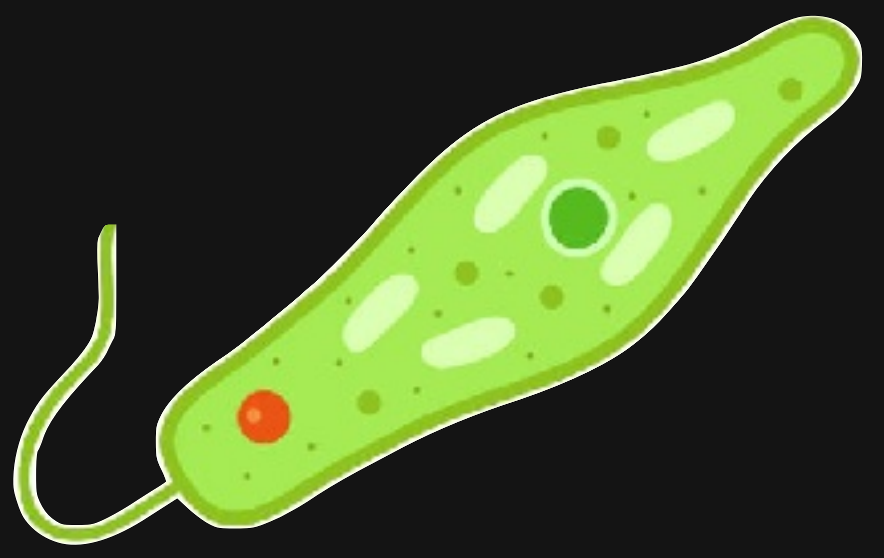


Configurations of a Deformable Object

This is a **Principal G Bundle** for **G = The Isometries of Space!**



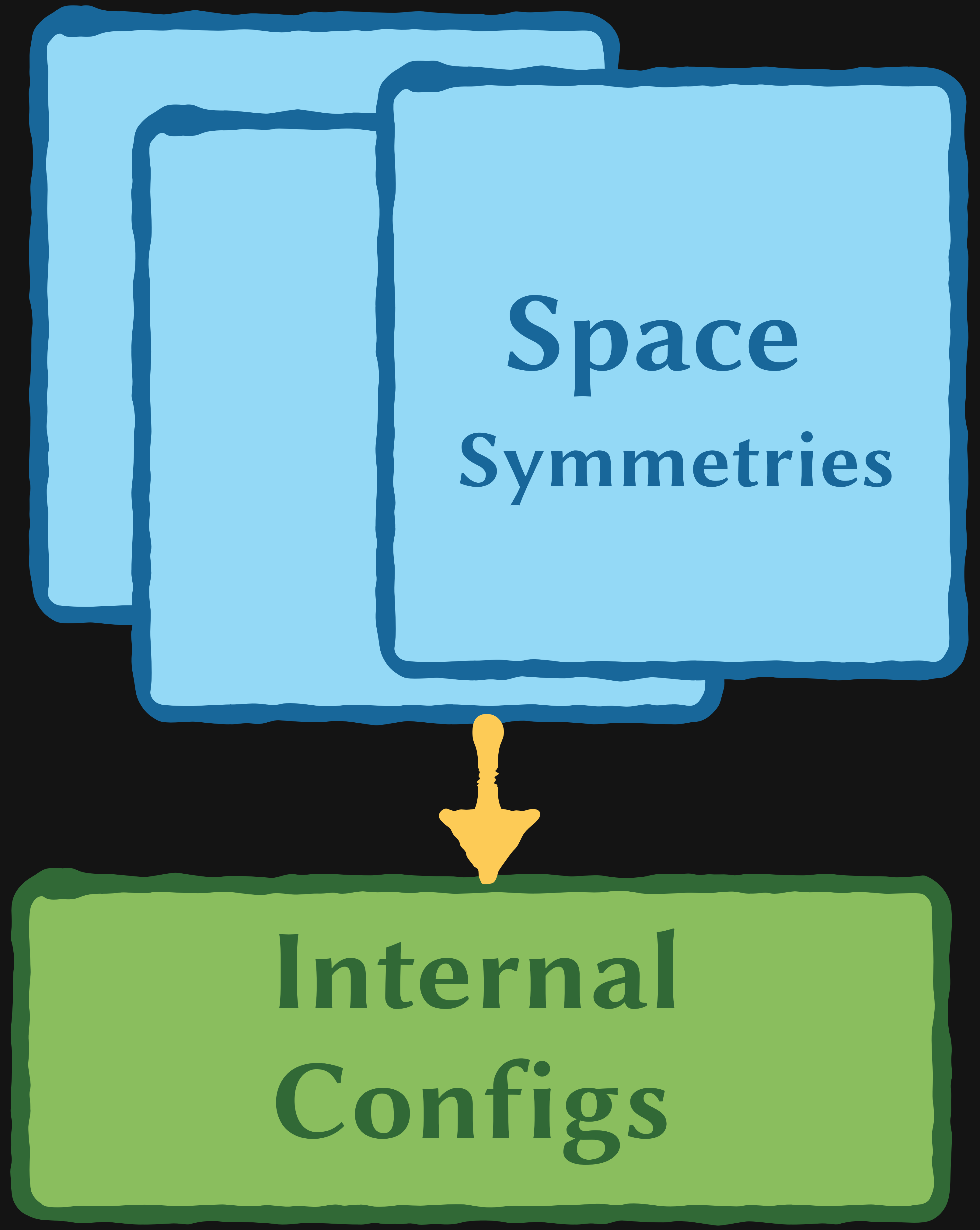
Frank
Wilczek



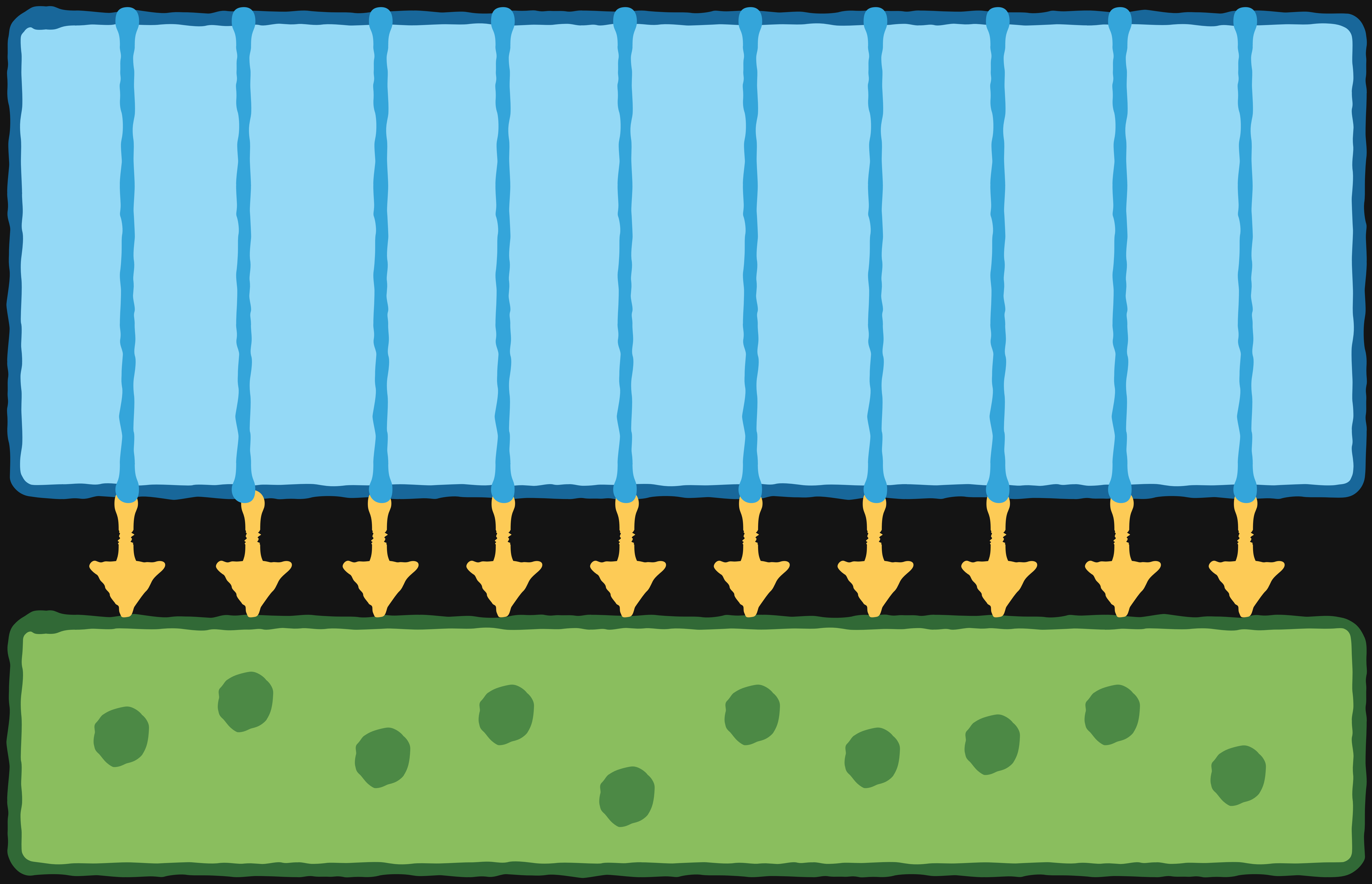
Richard
Montgomery

Jack
Wisdom

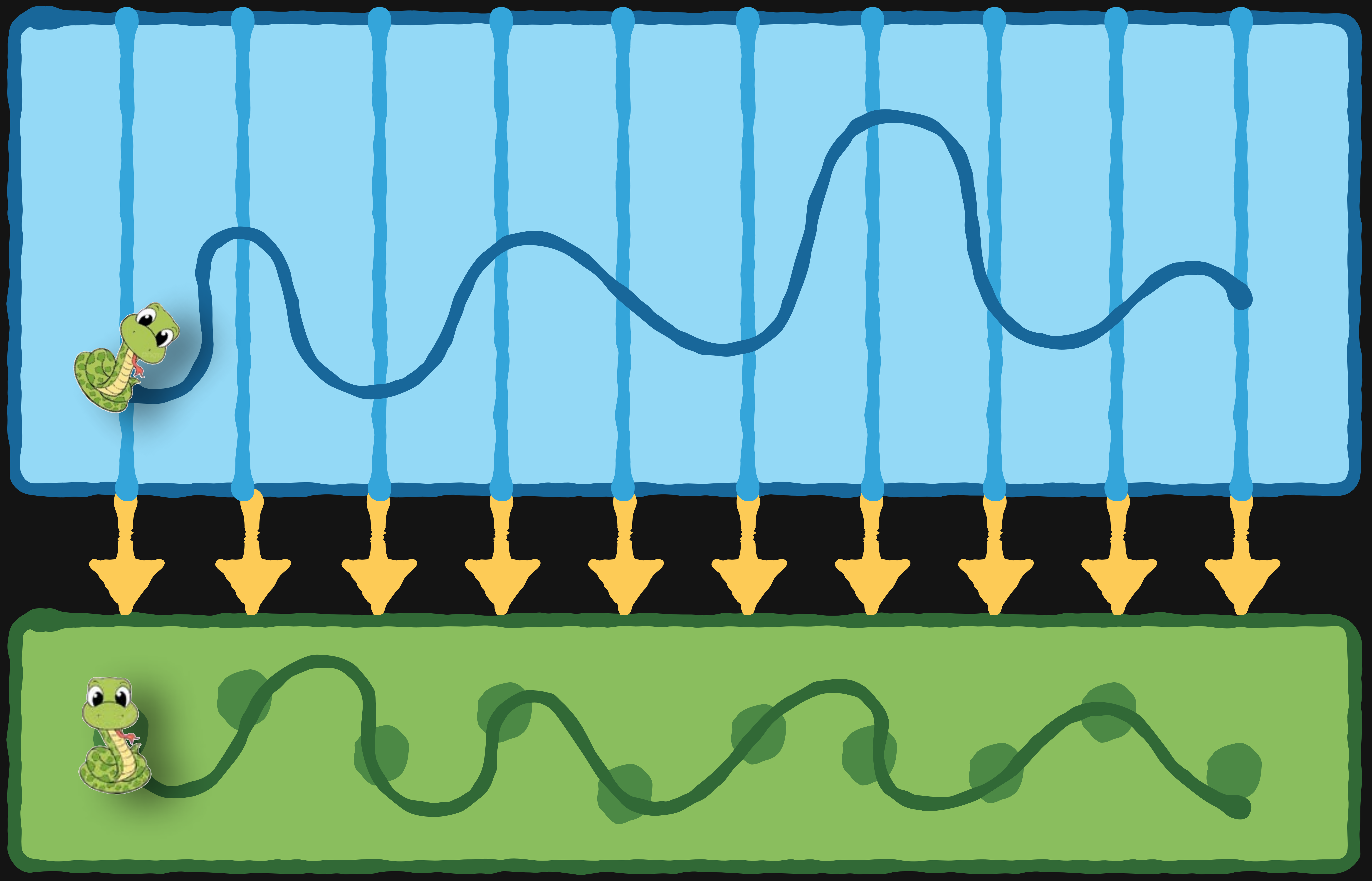
Gauge Theory!



Internal Motion as a path lifting problem



Internal Motion as a path lifting problem



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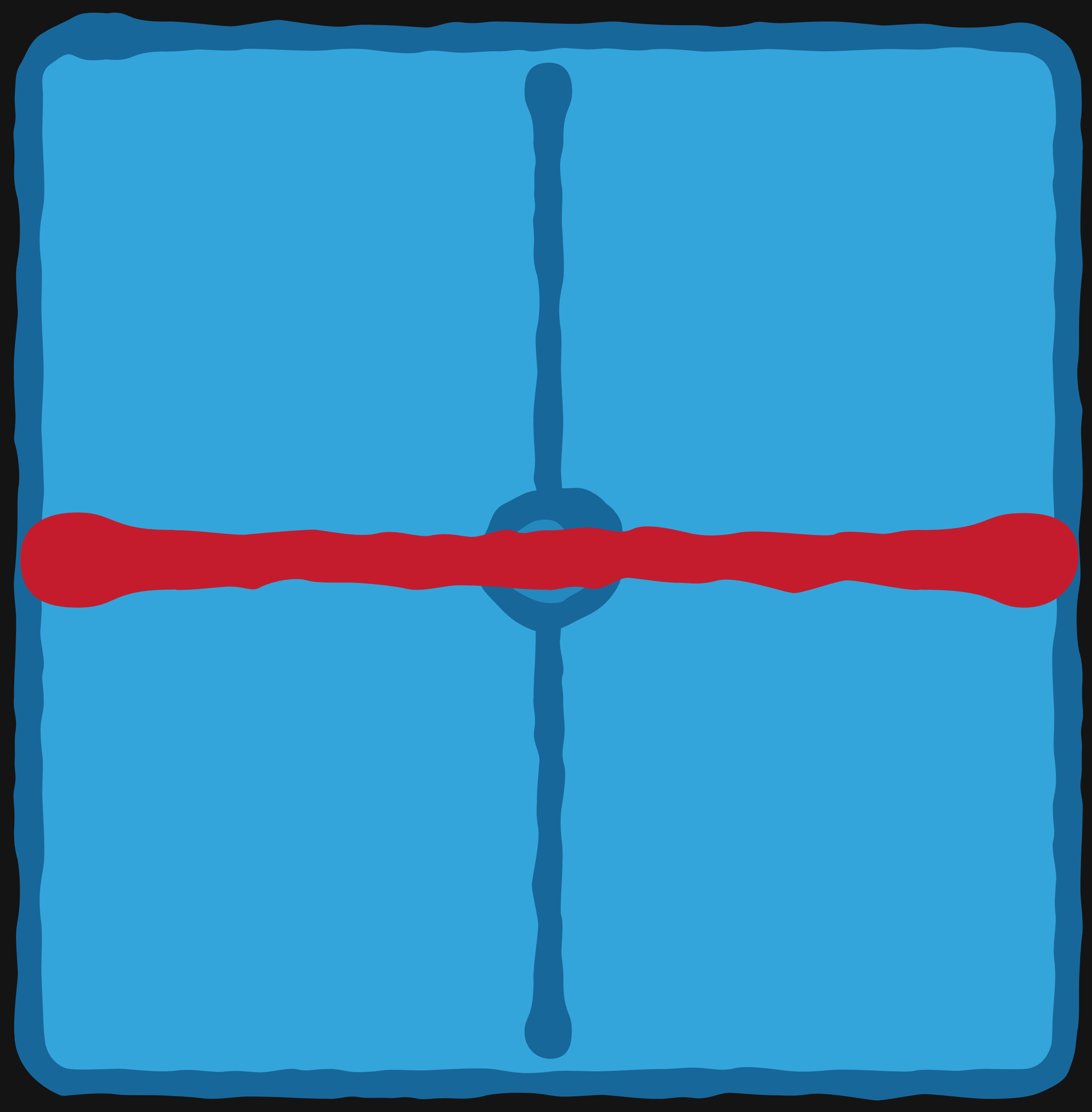
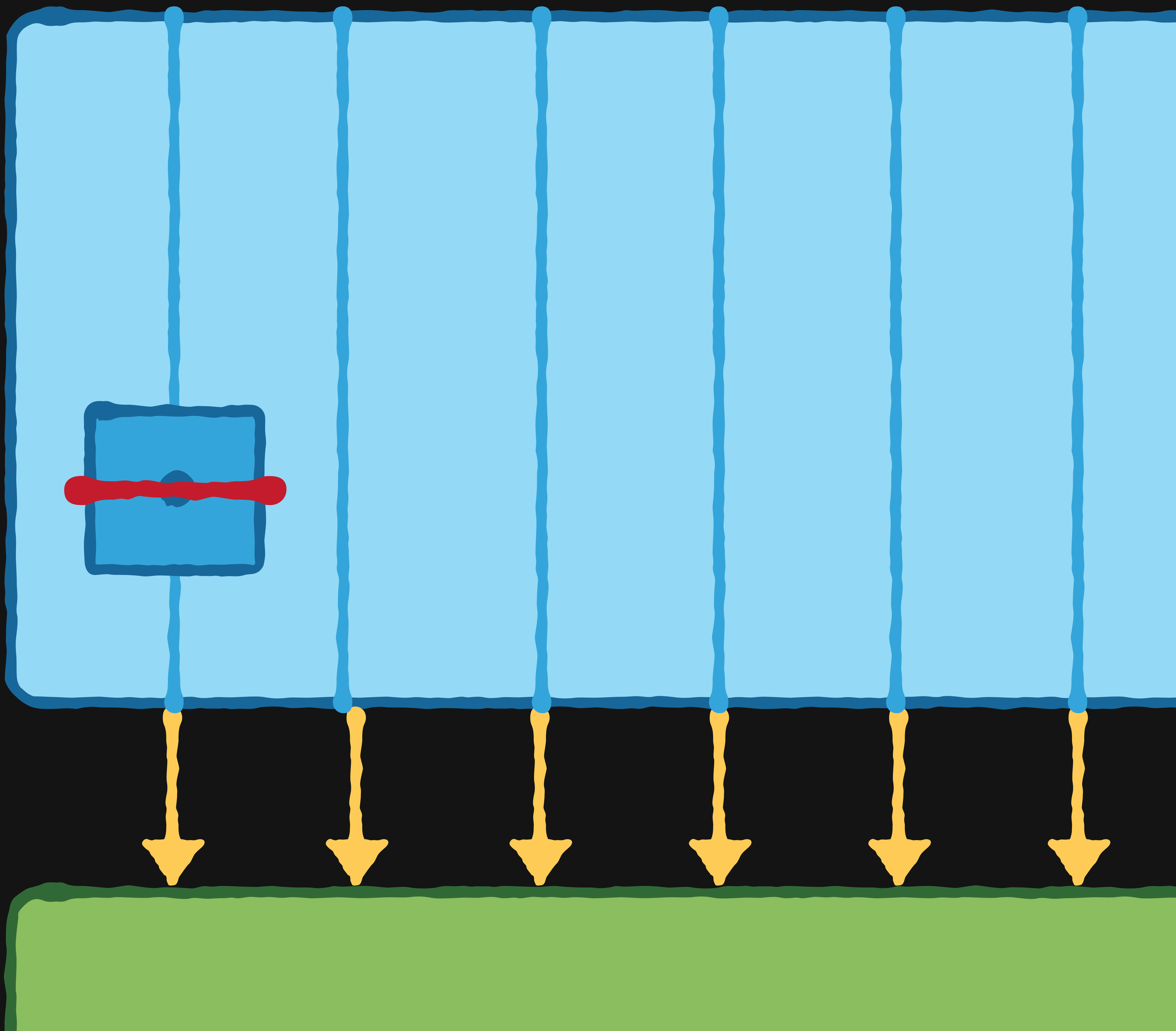
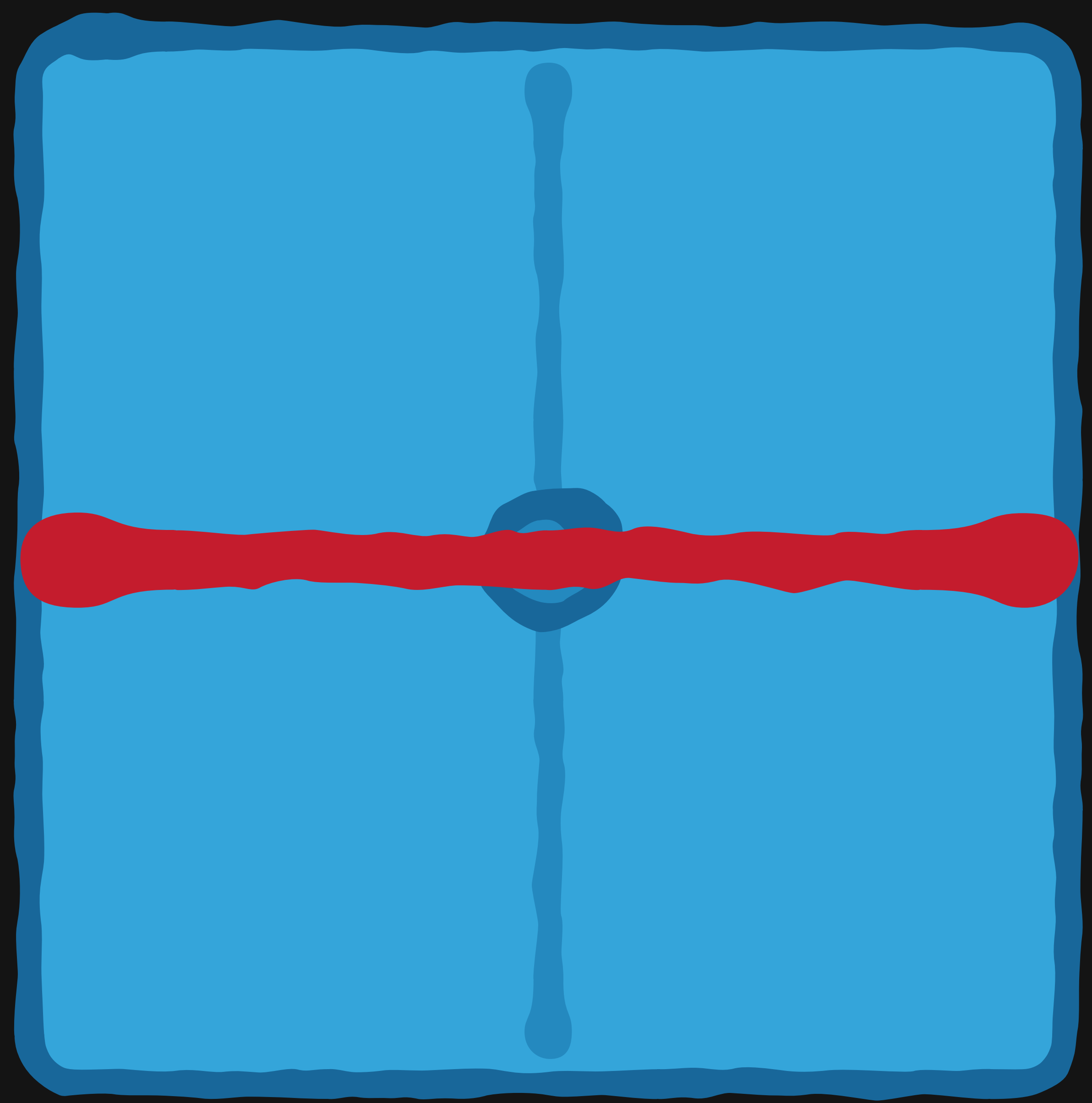
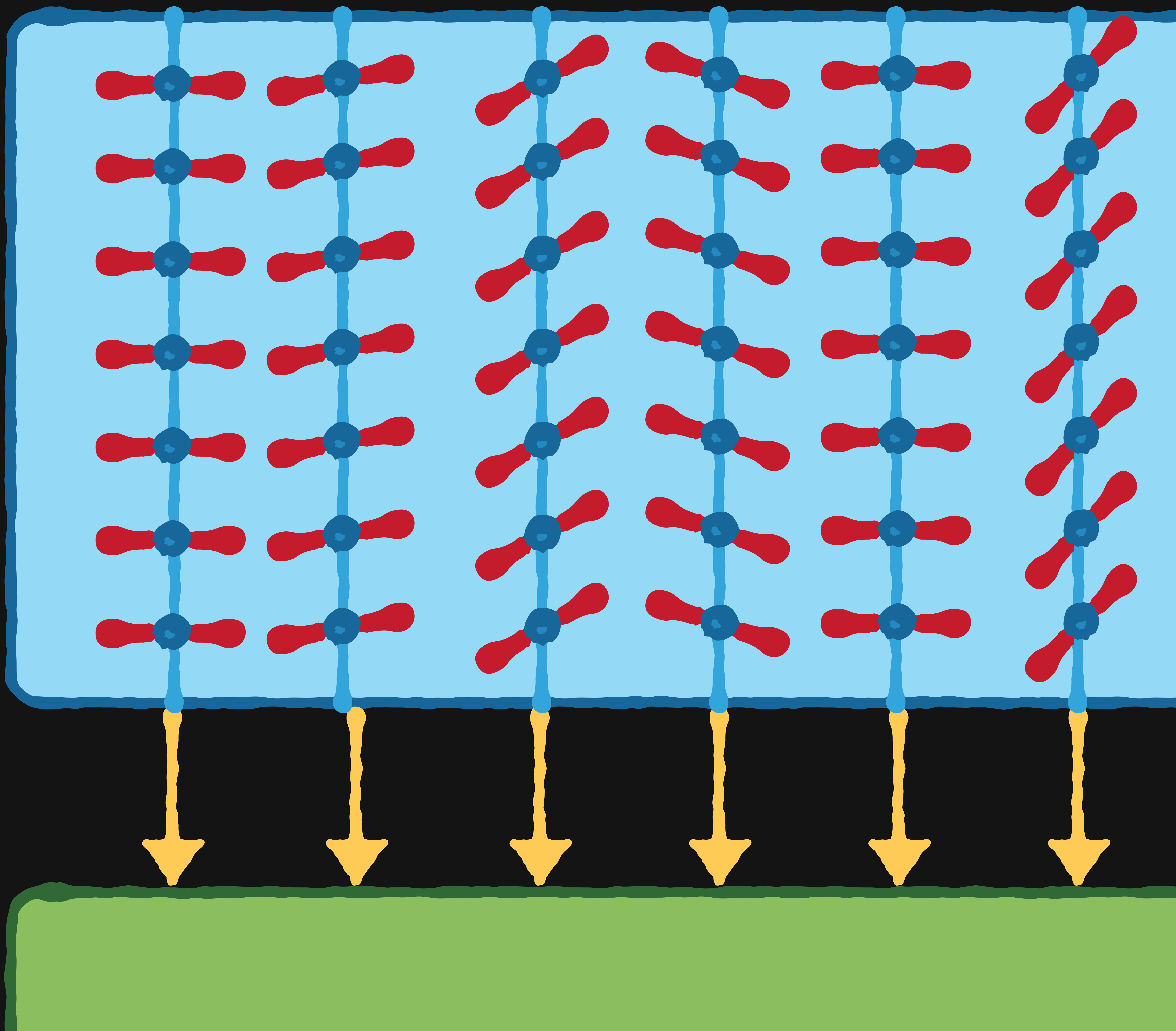




Internal Motion as a path lifting problem

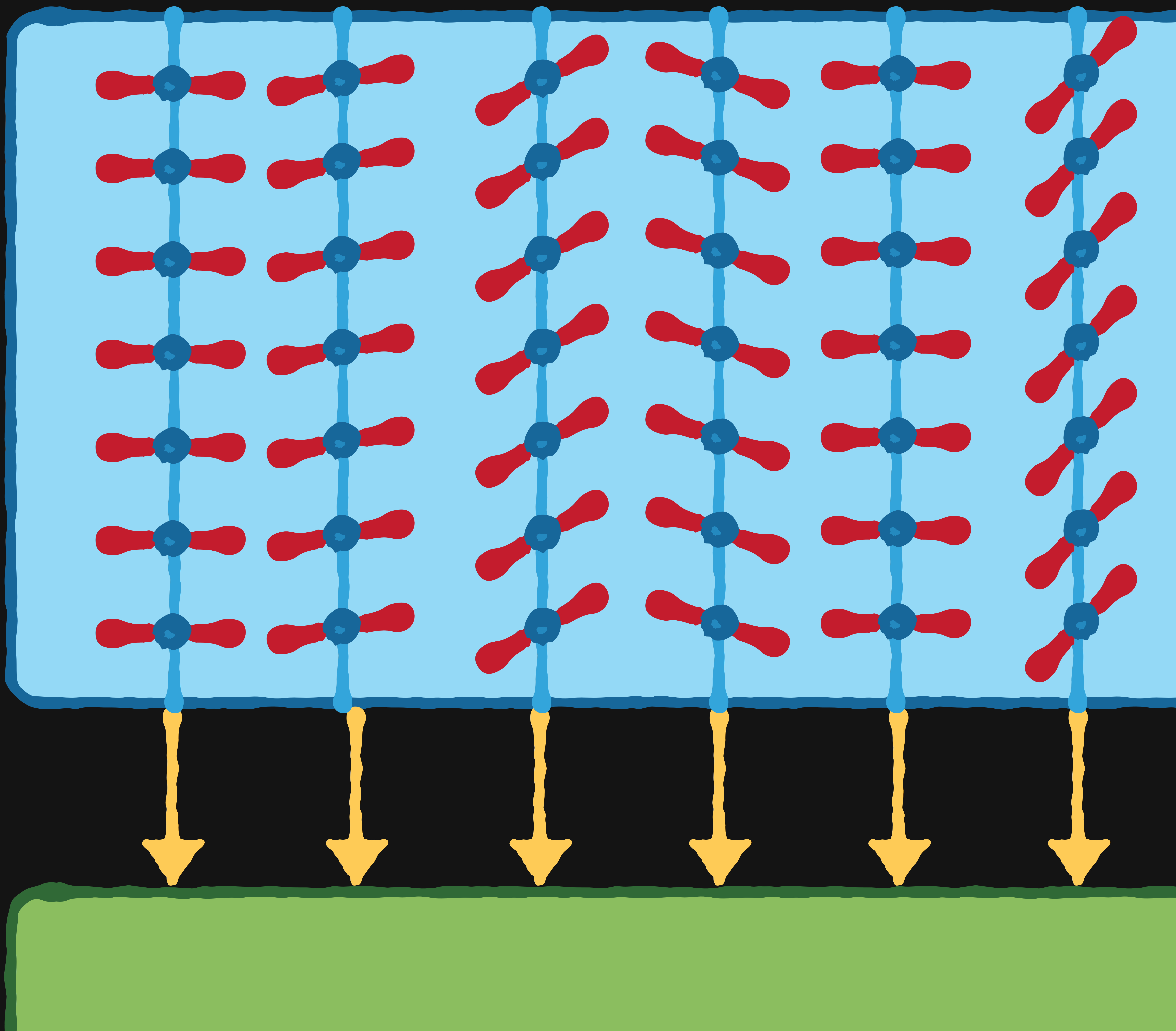




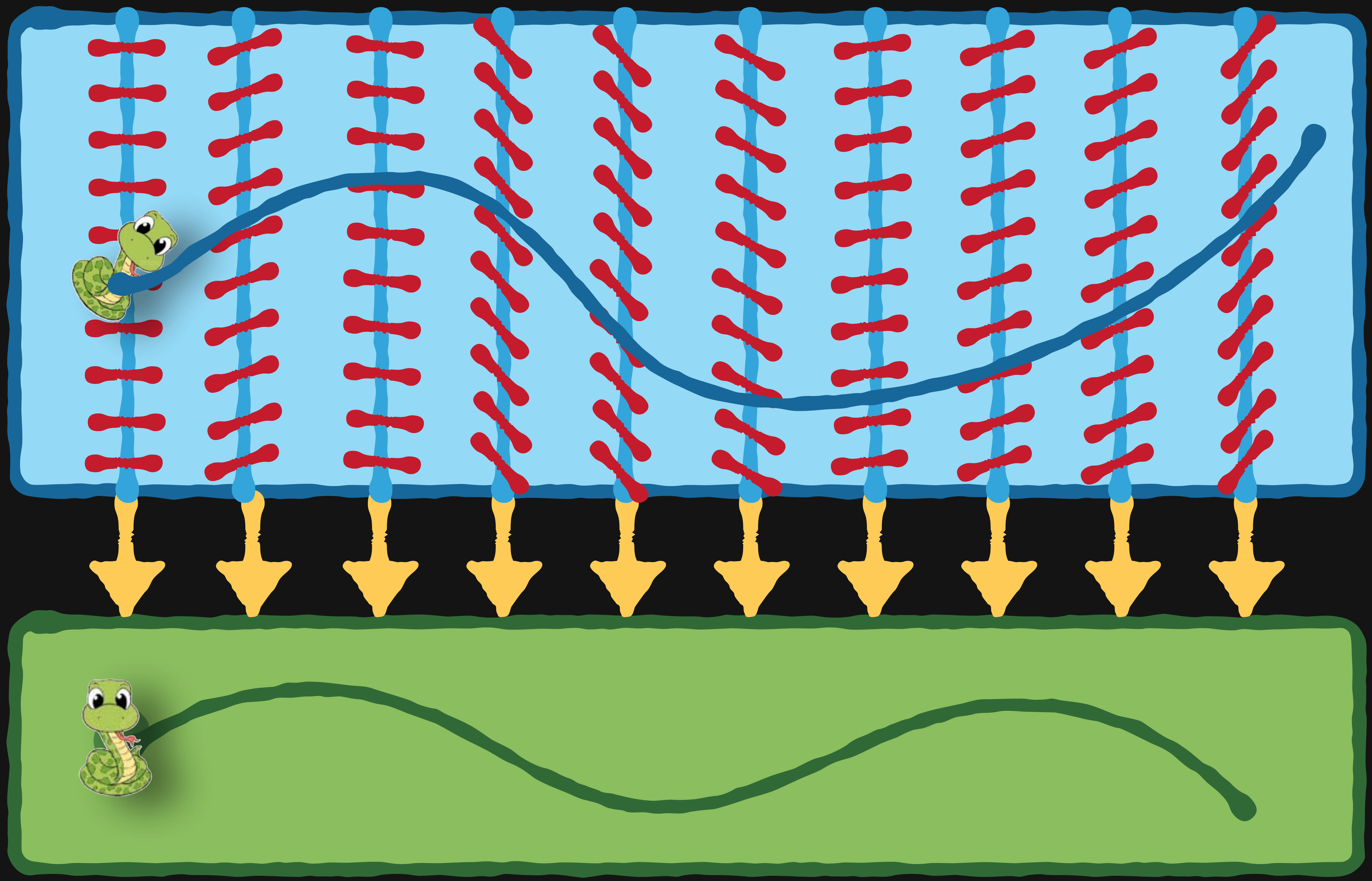
Internal Motion as a path lifting problem

The constraint of force-free motion is a **plane distribution on configuration space**

This distribution defines a **connection**



Internal Motion as a path lifting problem

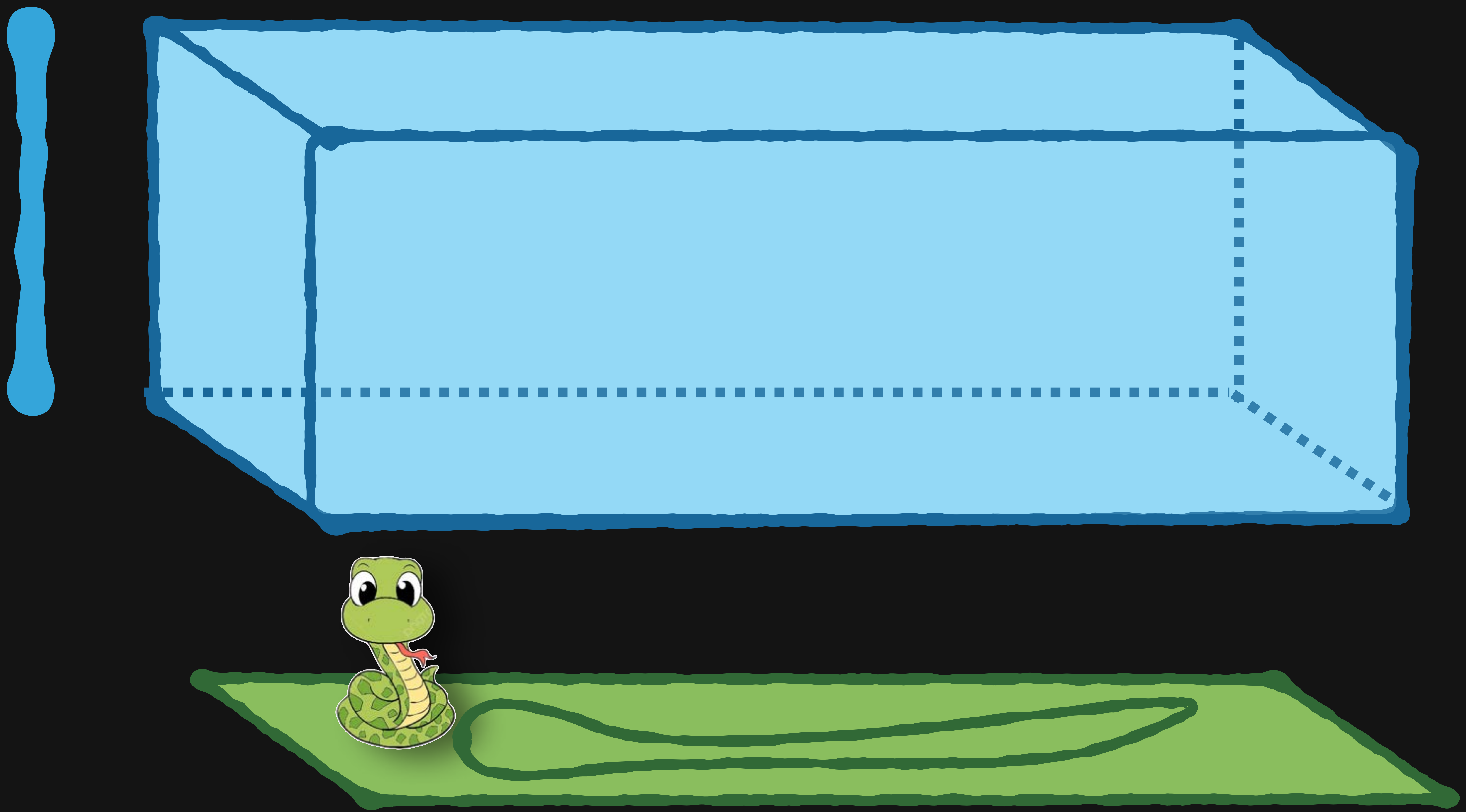


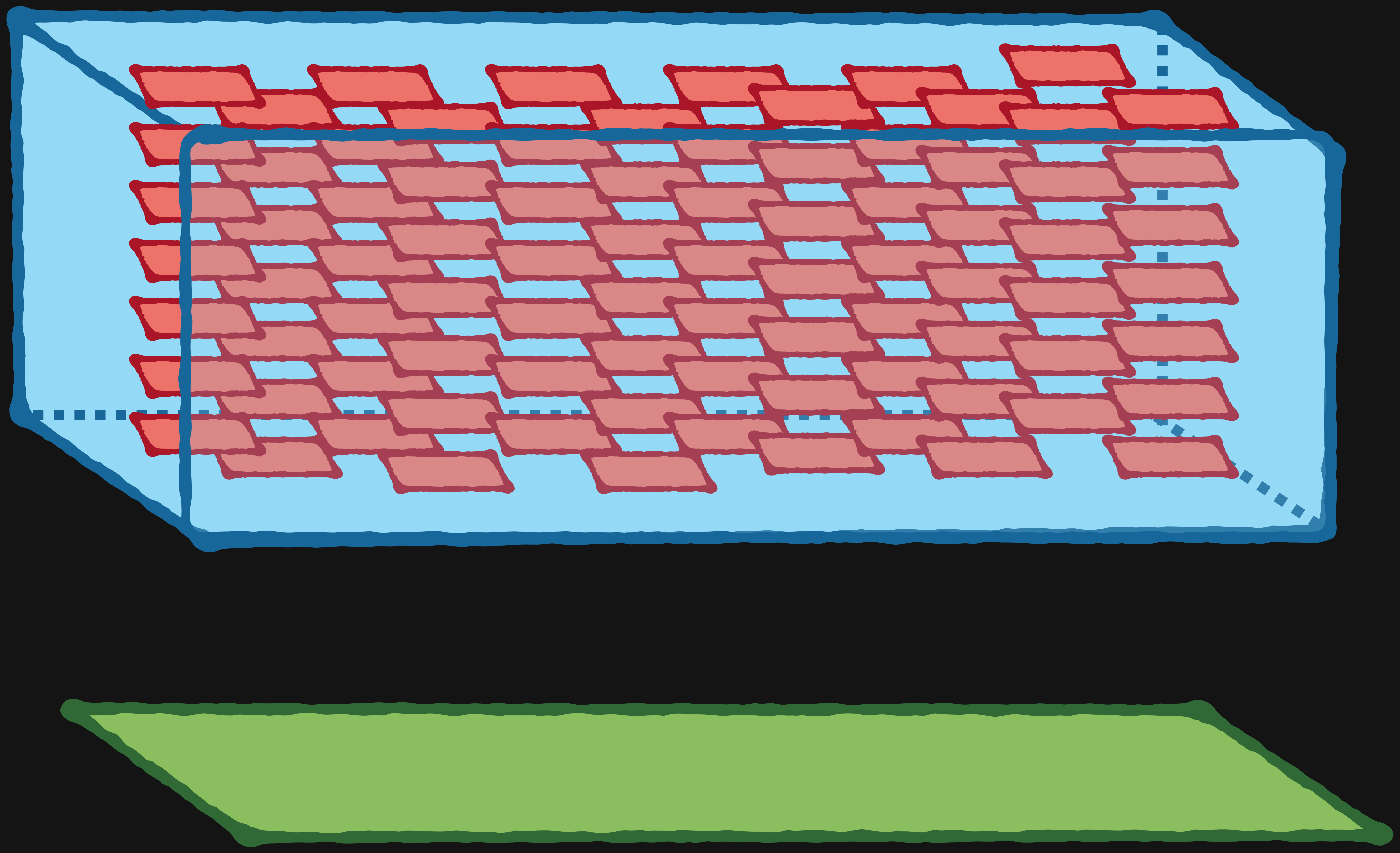
Euclidean Snakes:

a proof you’re safe

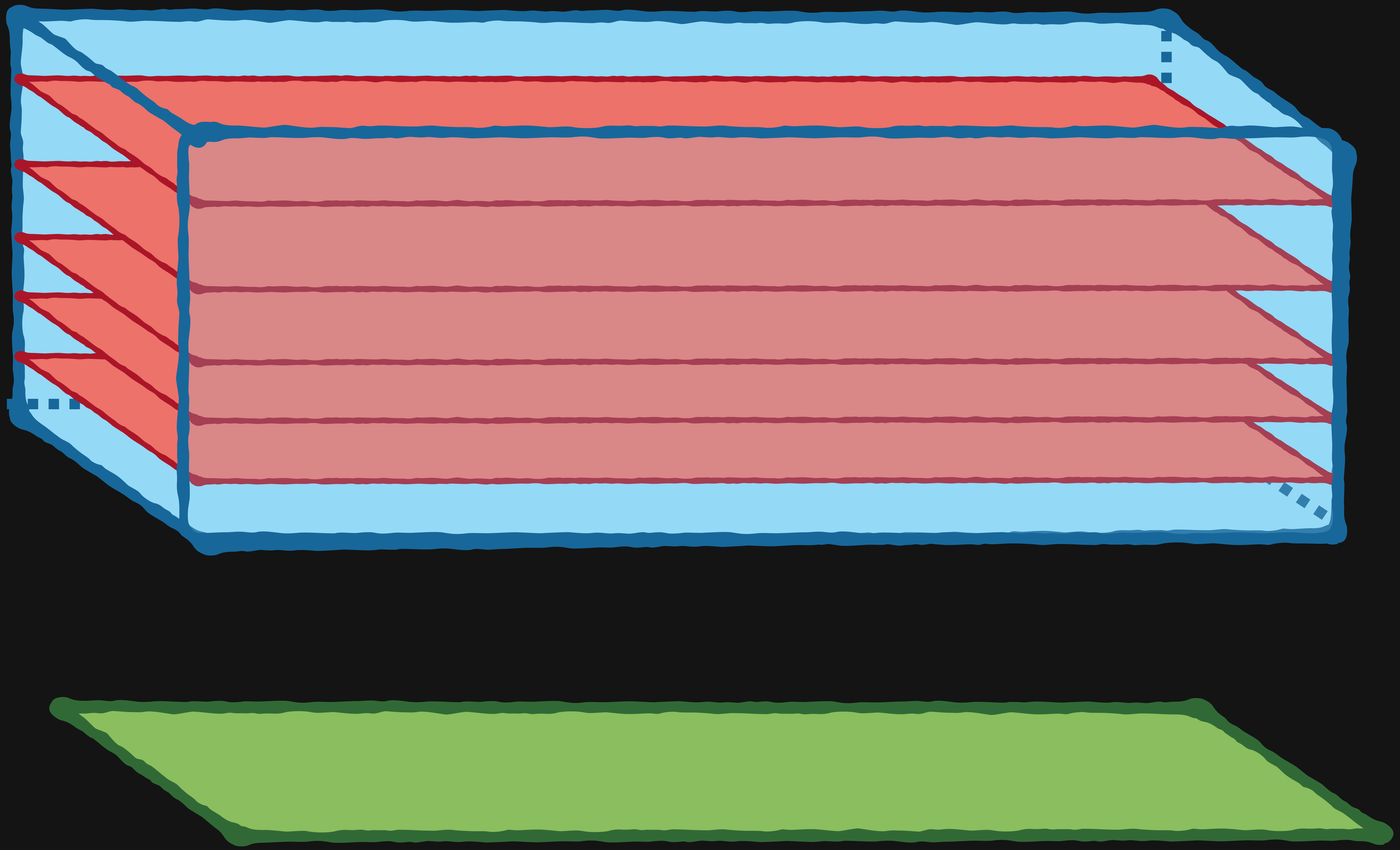
Group of translations

$\cong \mathbb{R}^3$



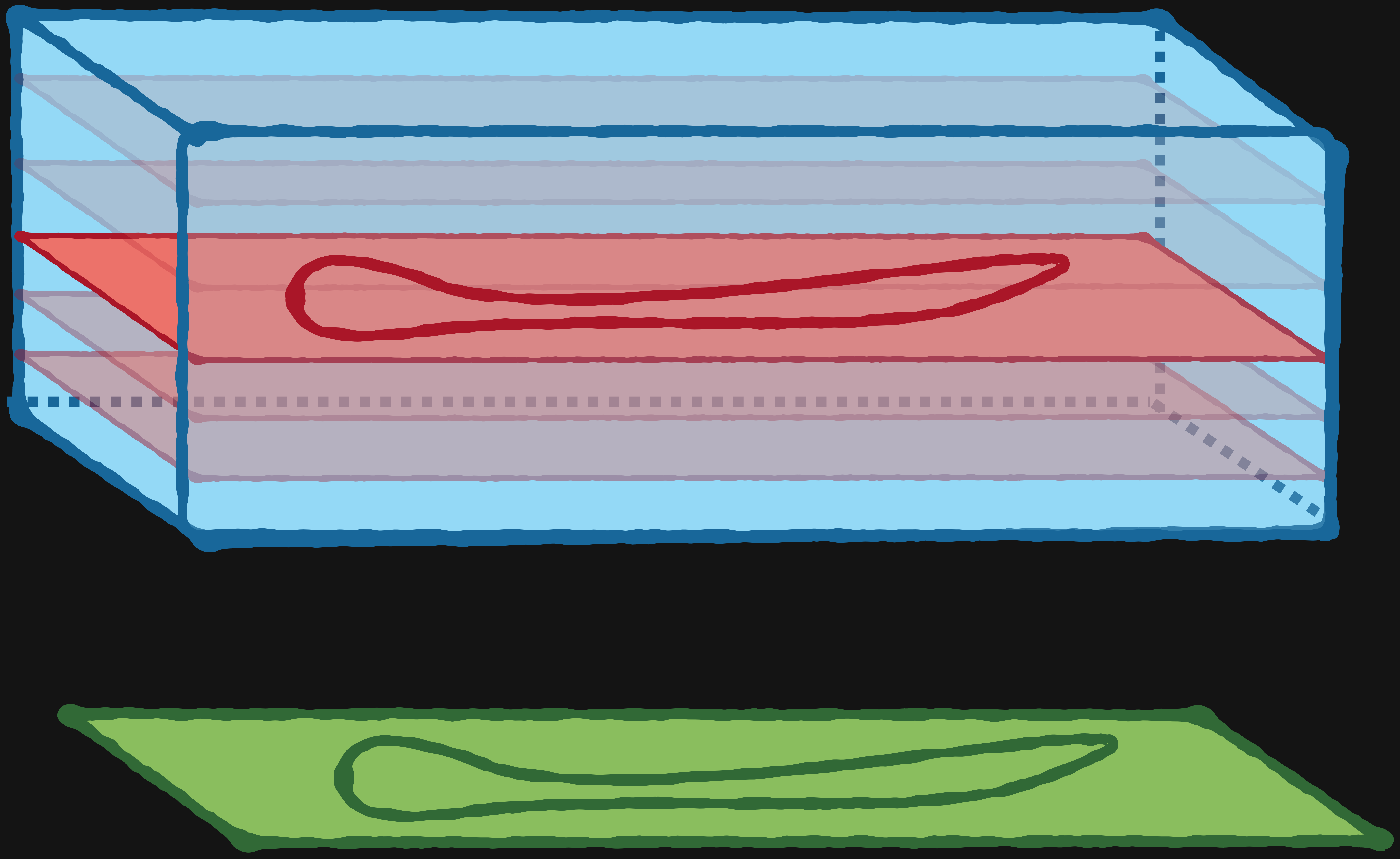


The plane distribution is integral: they are tangent planes to a foliation by surfaces!



The connection is flat.

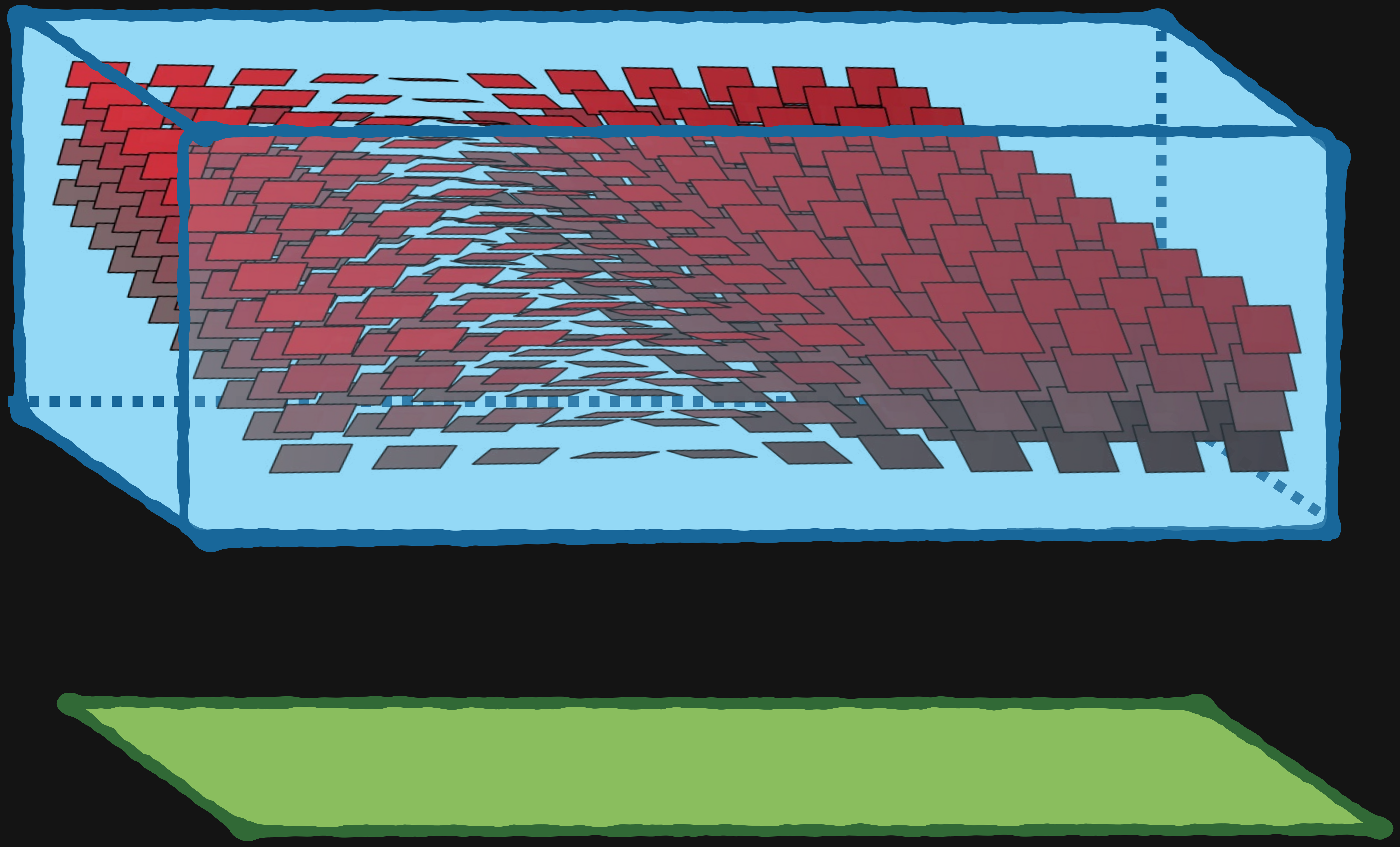
Thus, any periodic internal motion returns you to your starting position

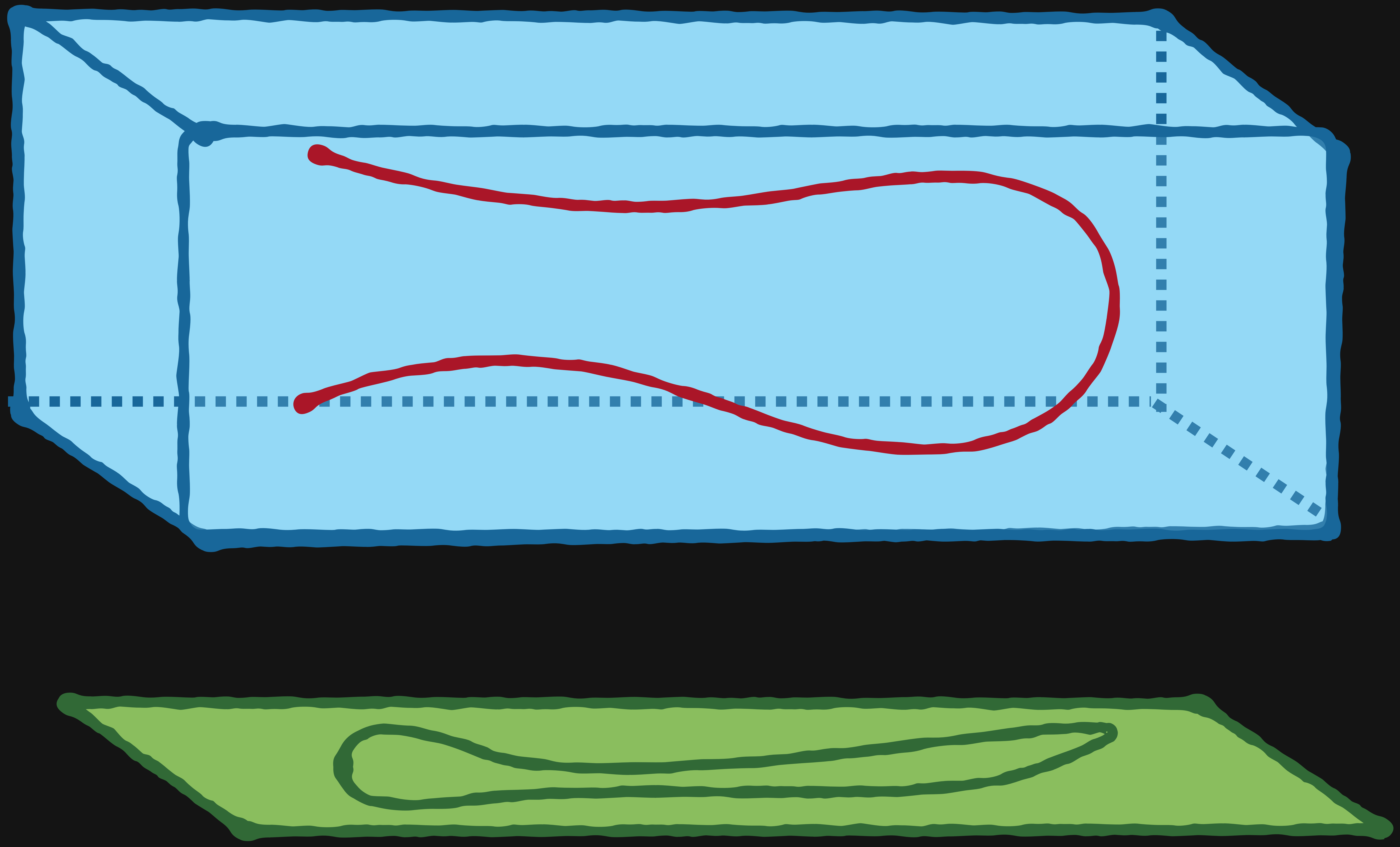


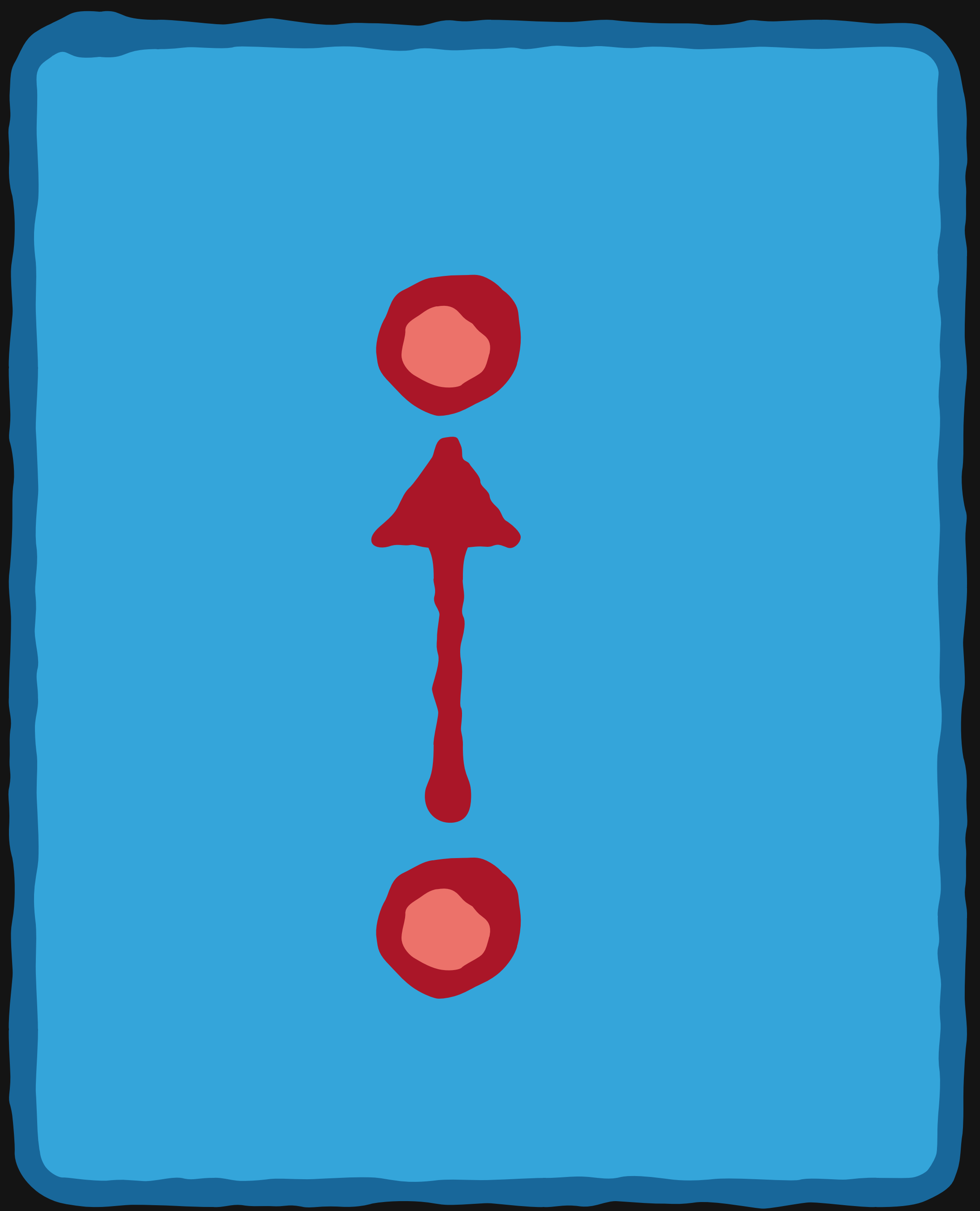
Hyperbolic Snakes:

be scared, very scared

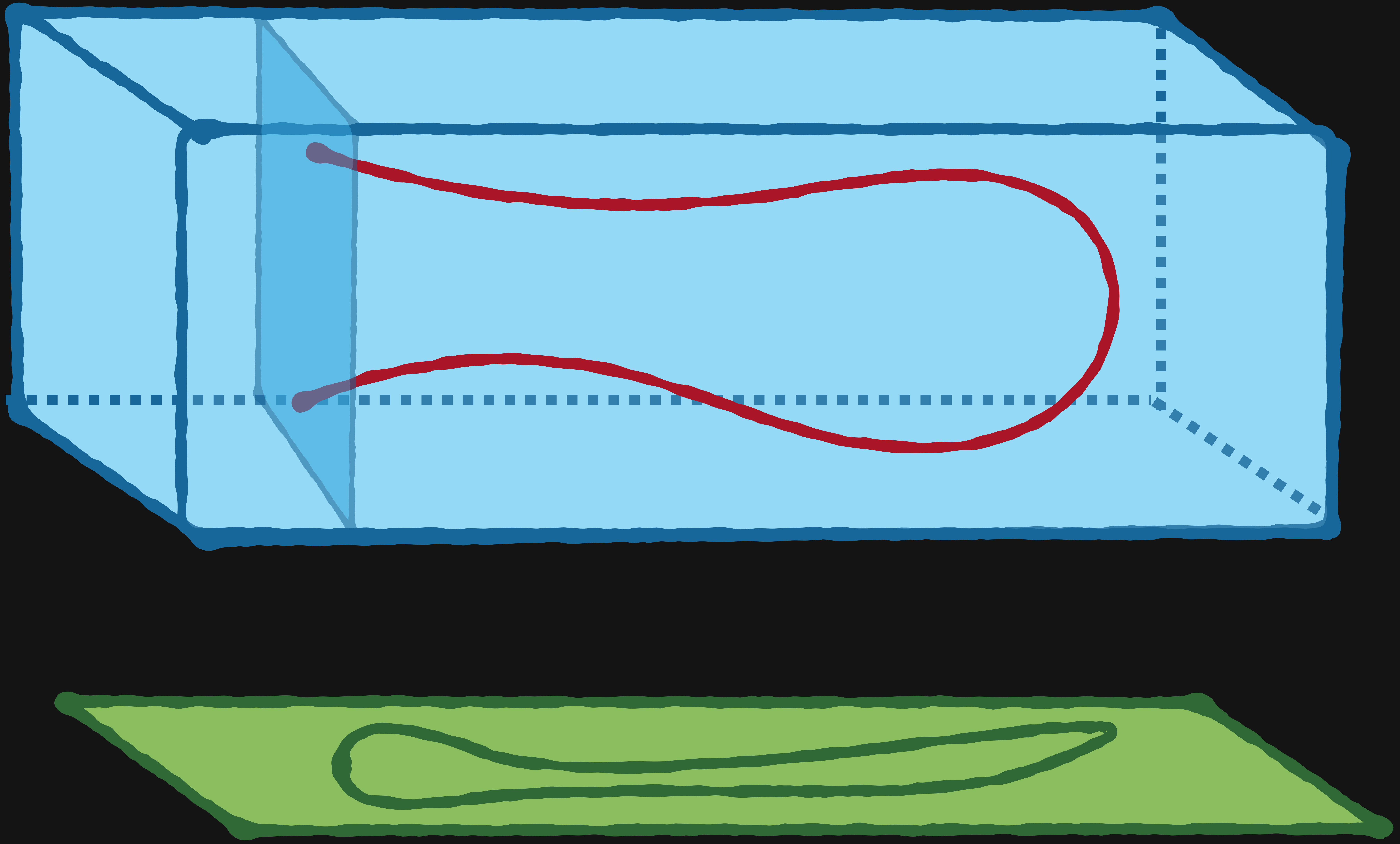
The distribution on the SO(3,1) bundle is totally non-integrable



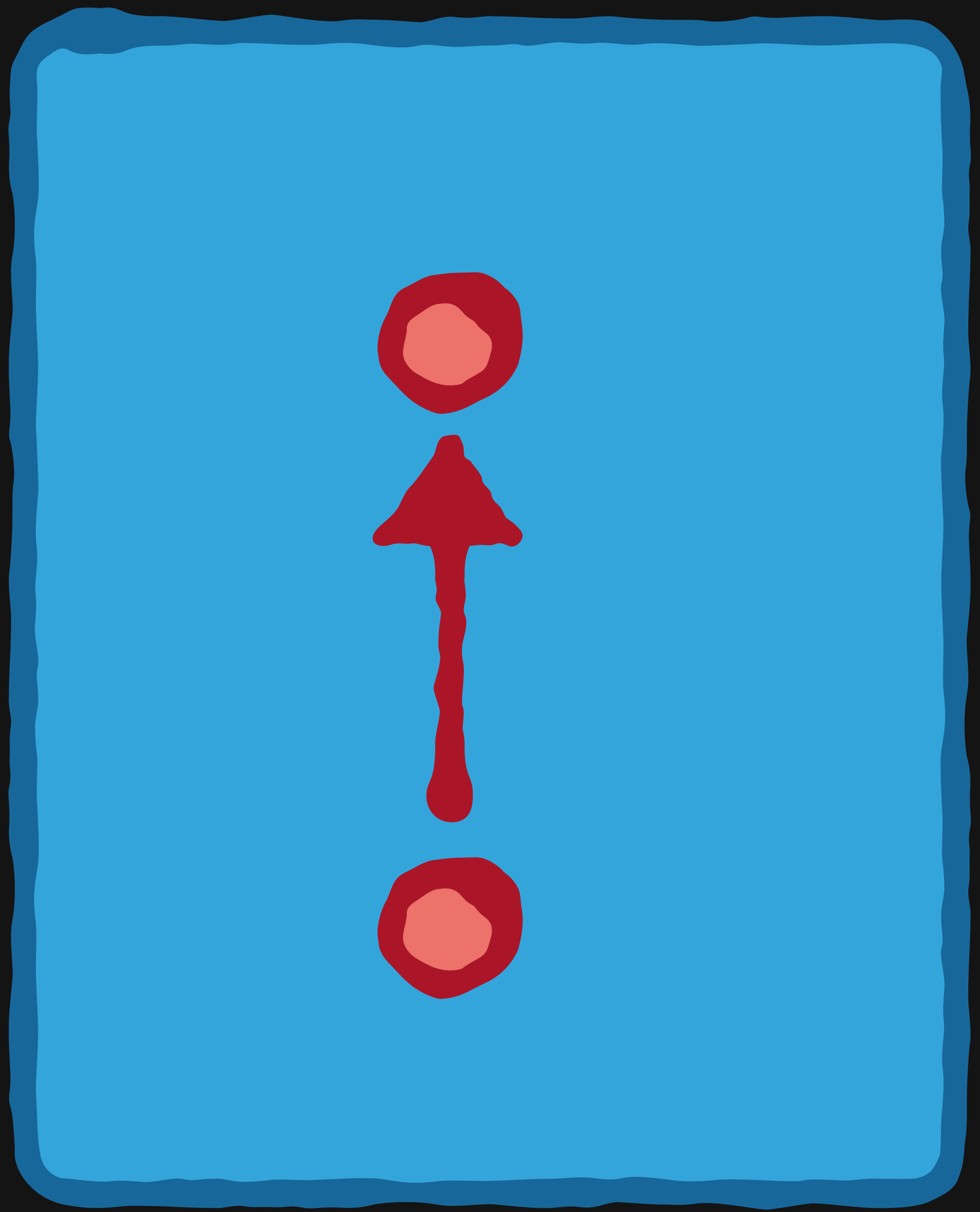




Fiber $\cong SO(3,1)$

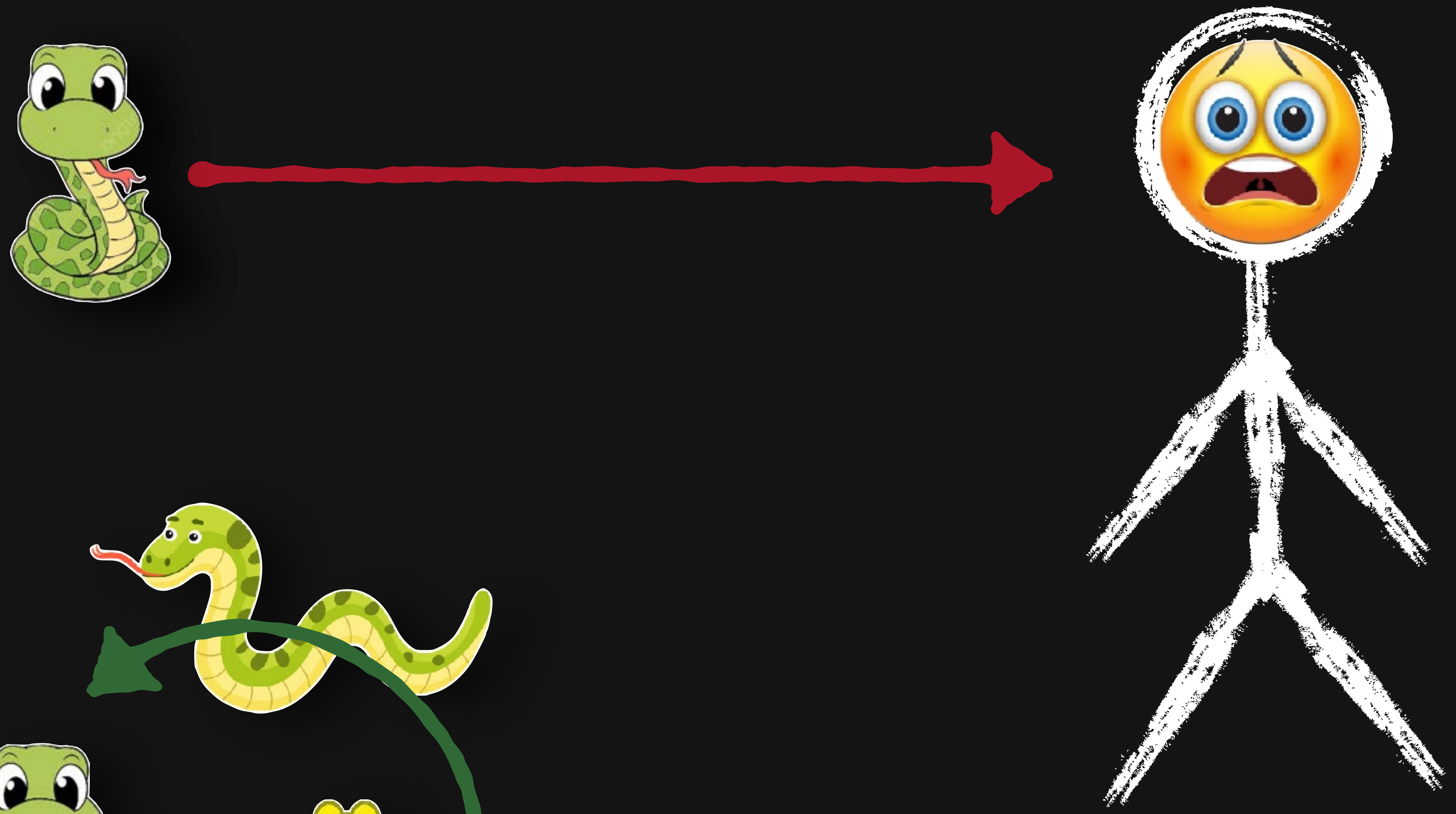


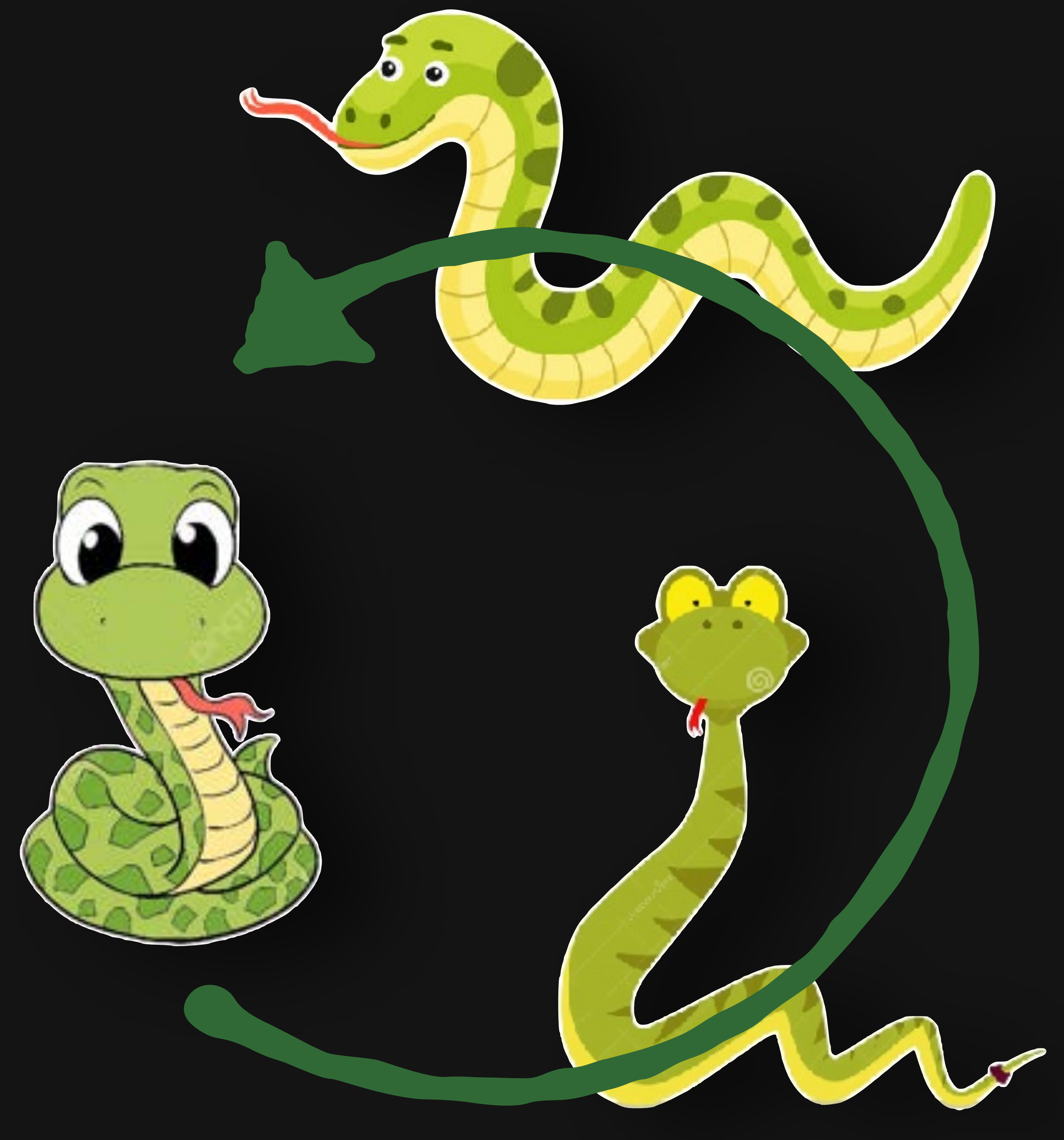
The difference between endpoints is a the **holonomy** of the **connection** for this loop!



Fiber $\cong SO(3,1)$

The difference between endpoints is a the **holonomy** of the **connection** for this loop!





Every element is the holnomy of some loop.

Internal deformations can lead to net translation without external force

How to
Swim
in the vacuum