The Mathematics of Beautiful Graphics

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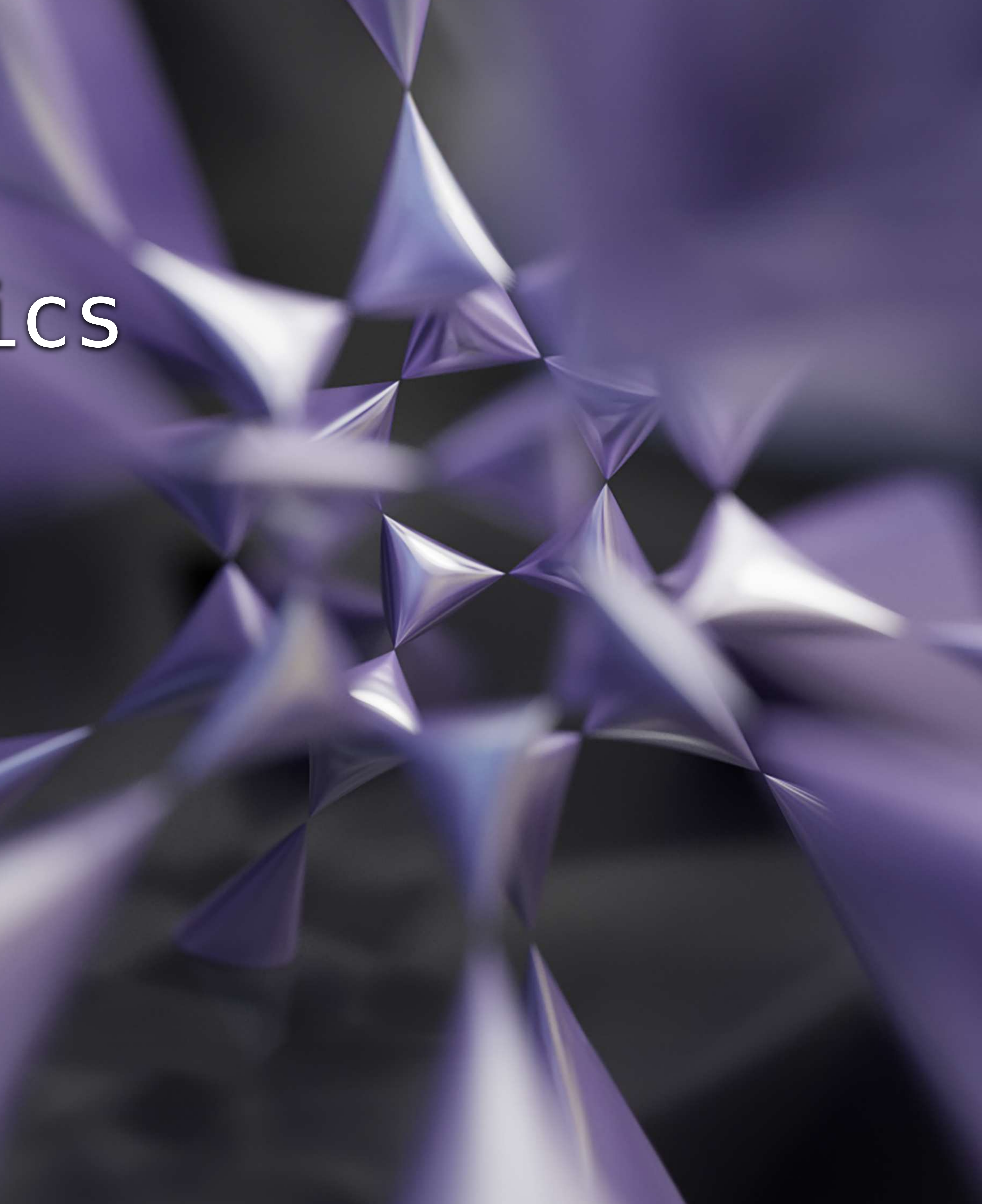


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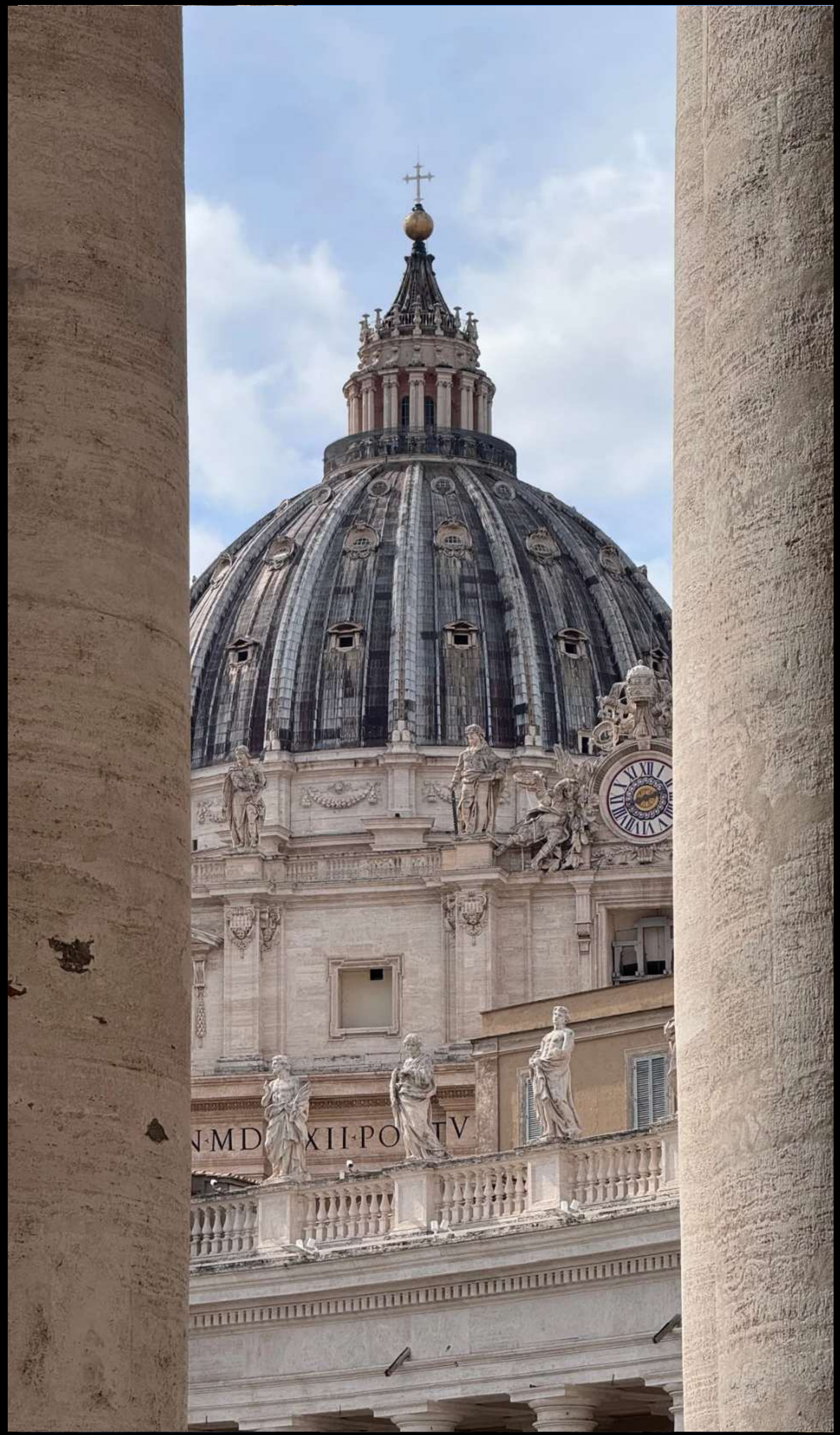
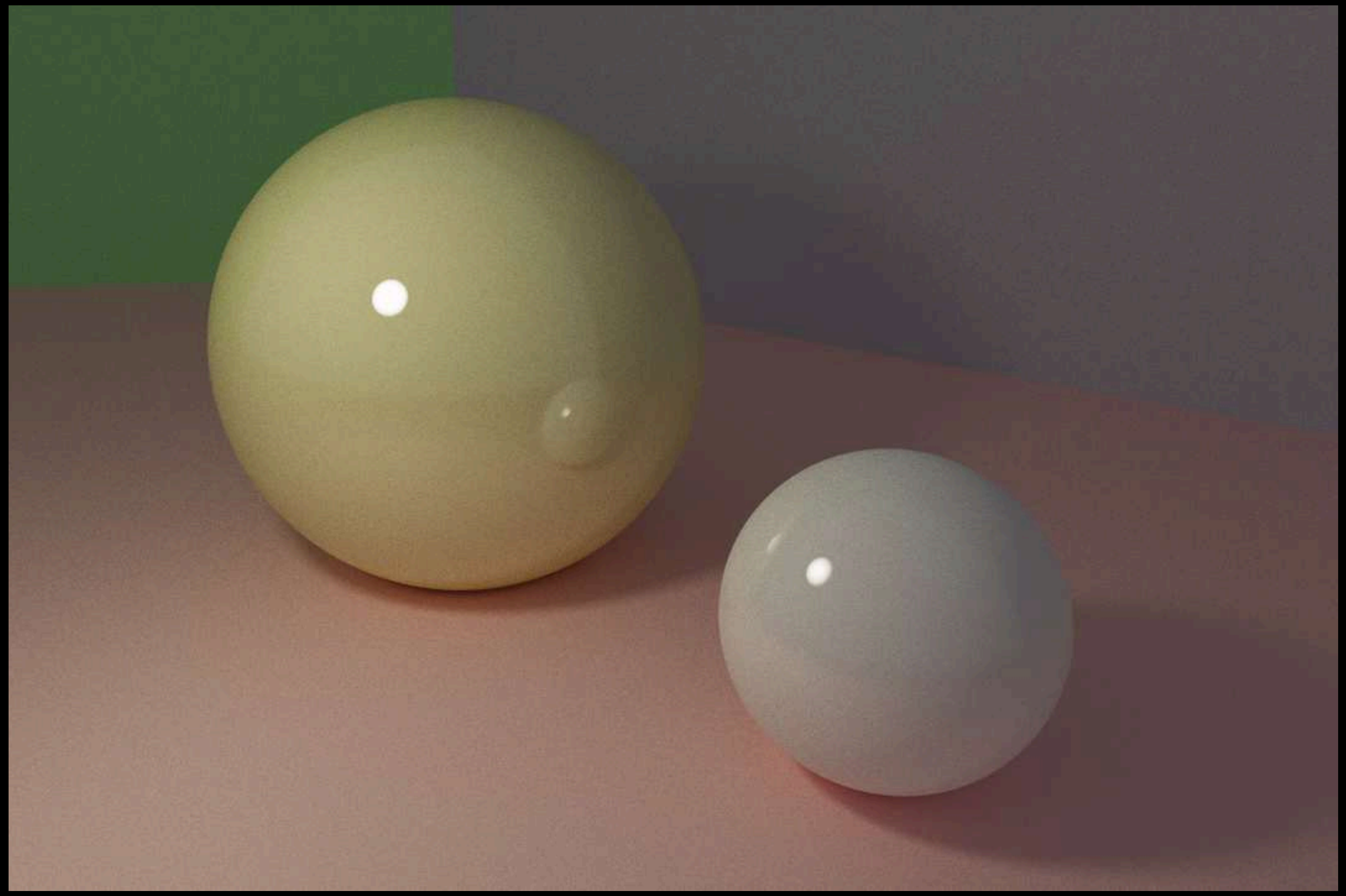
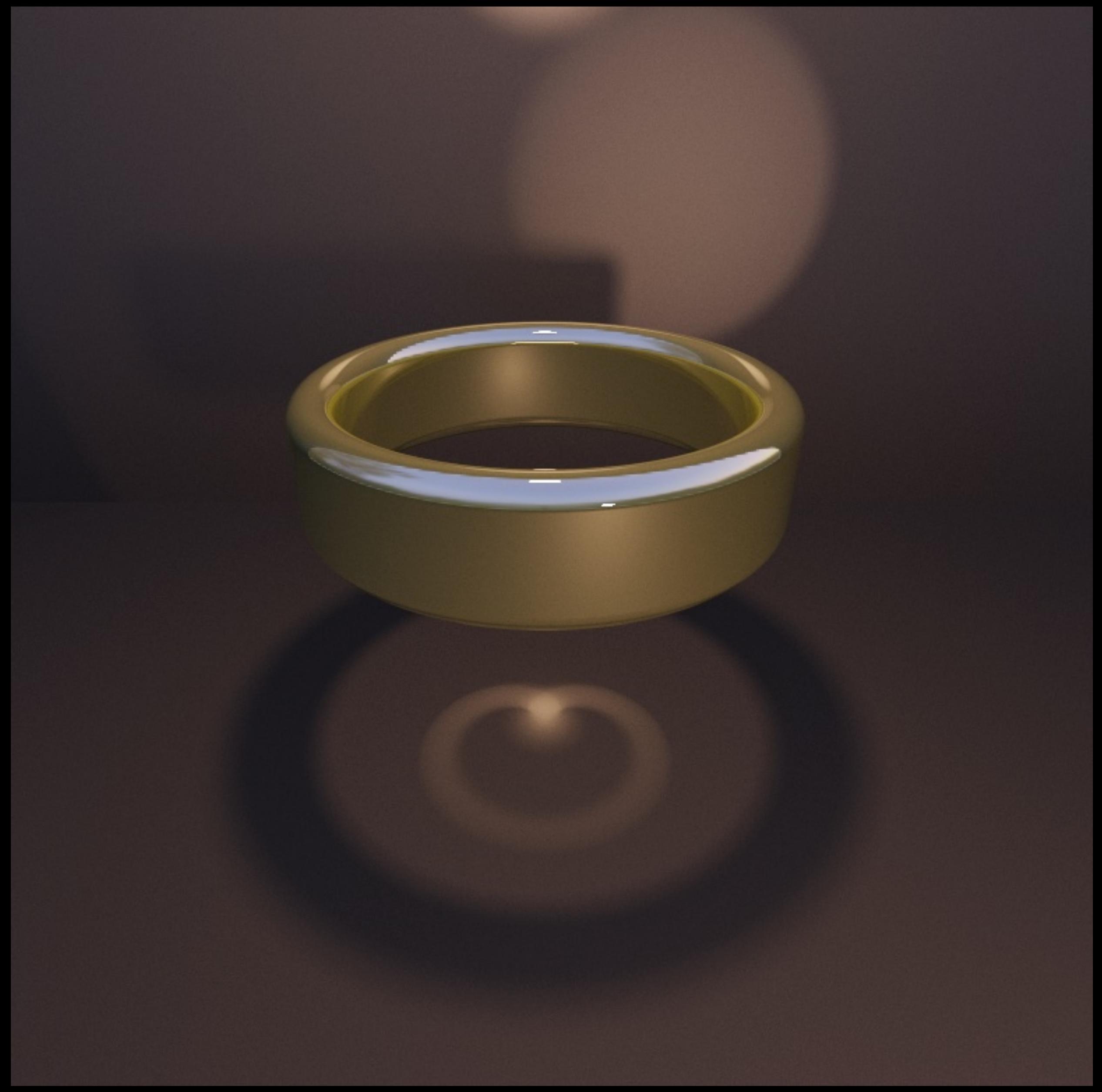
Photography’s a hobby of mine

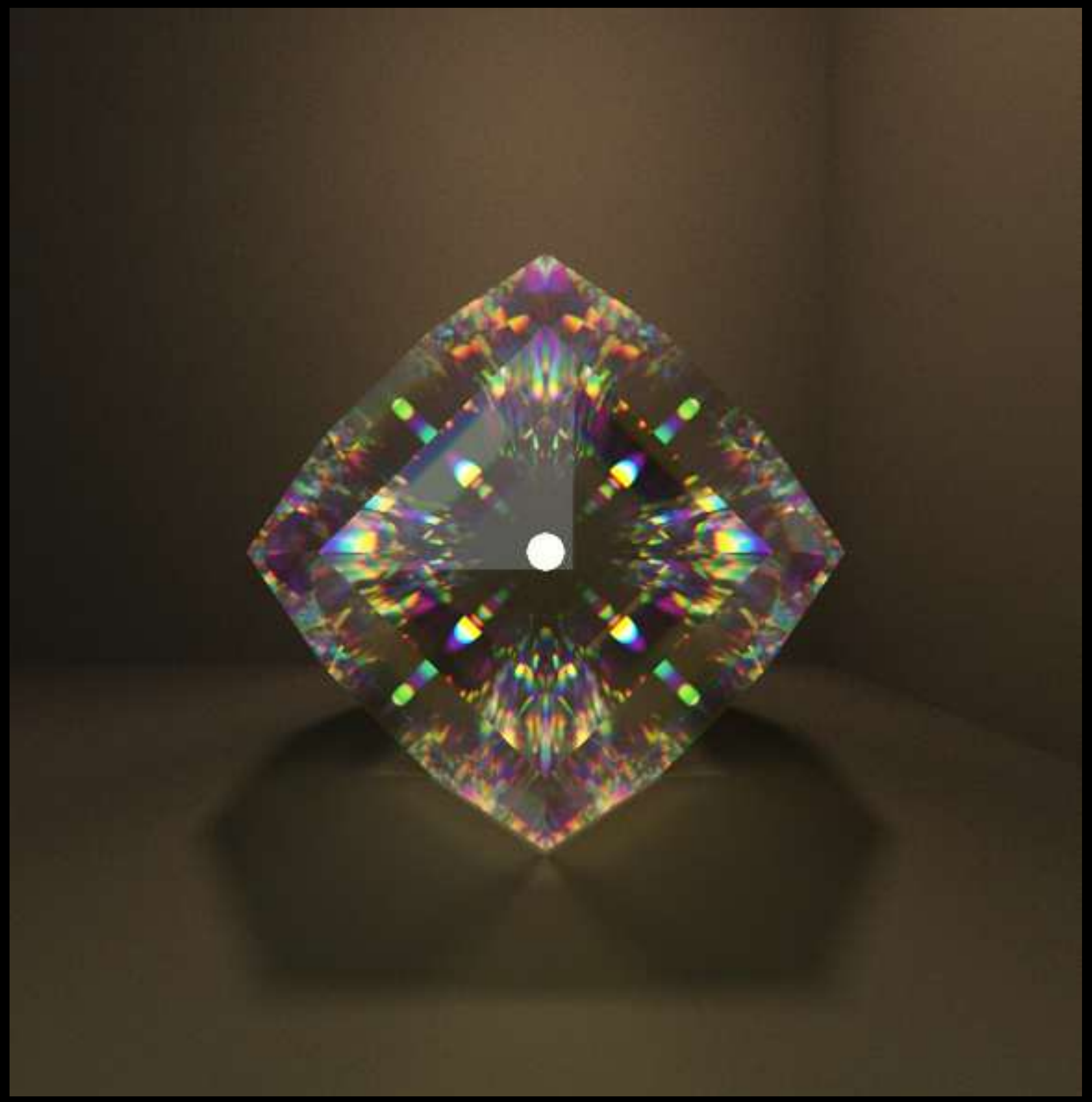


then
came
the
pandemic…

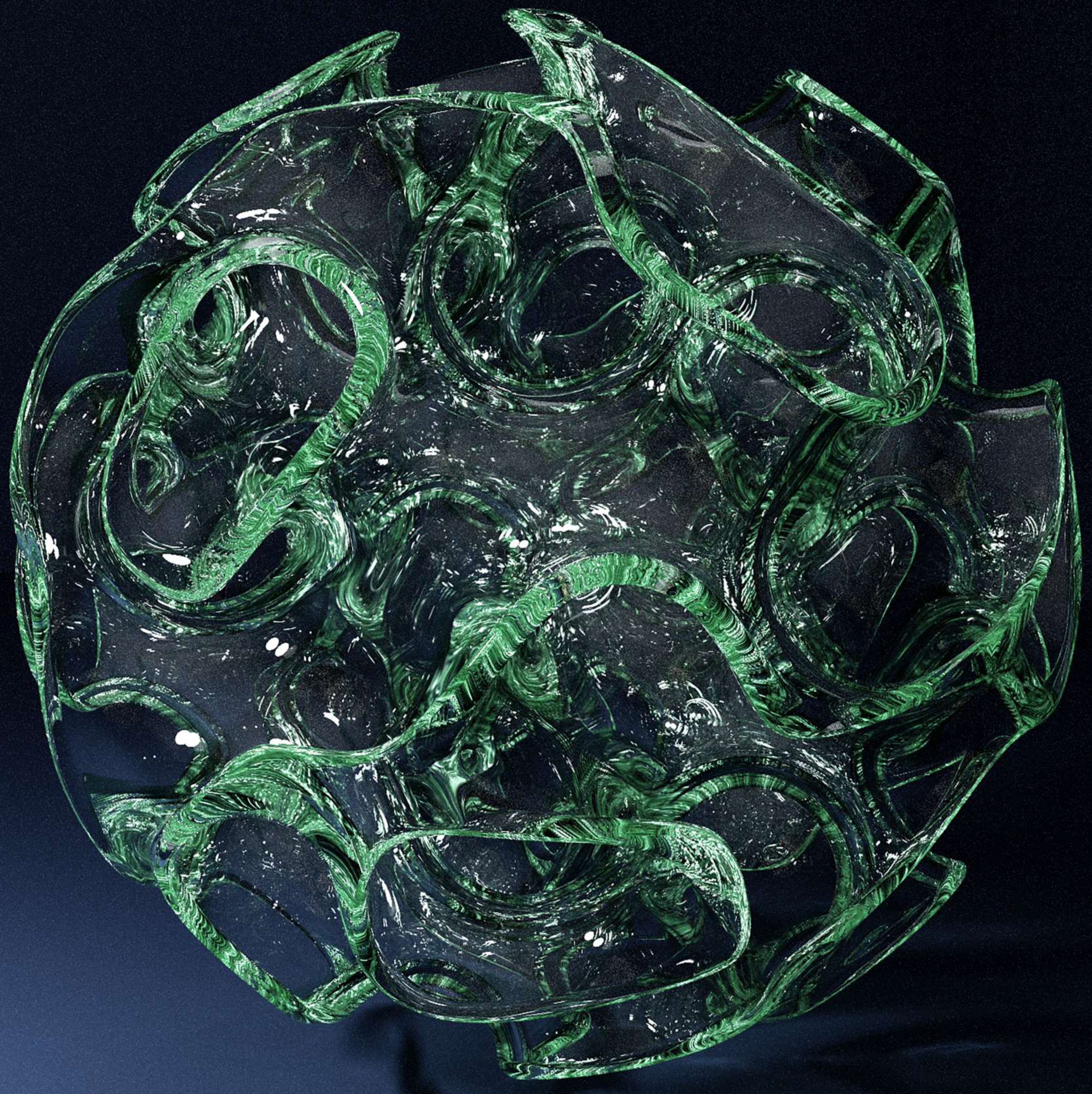
I was excited from the beginning



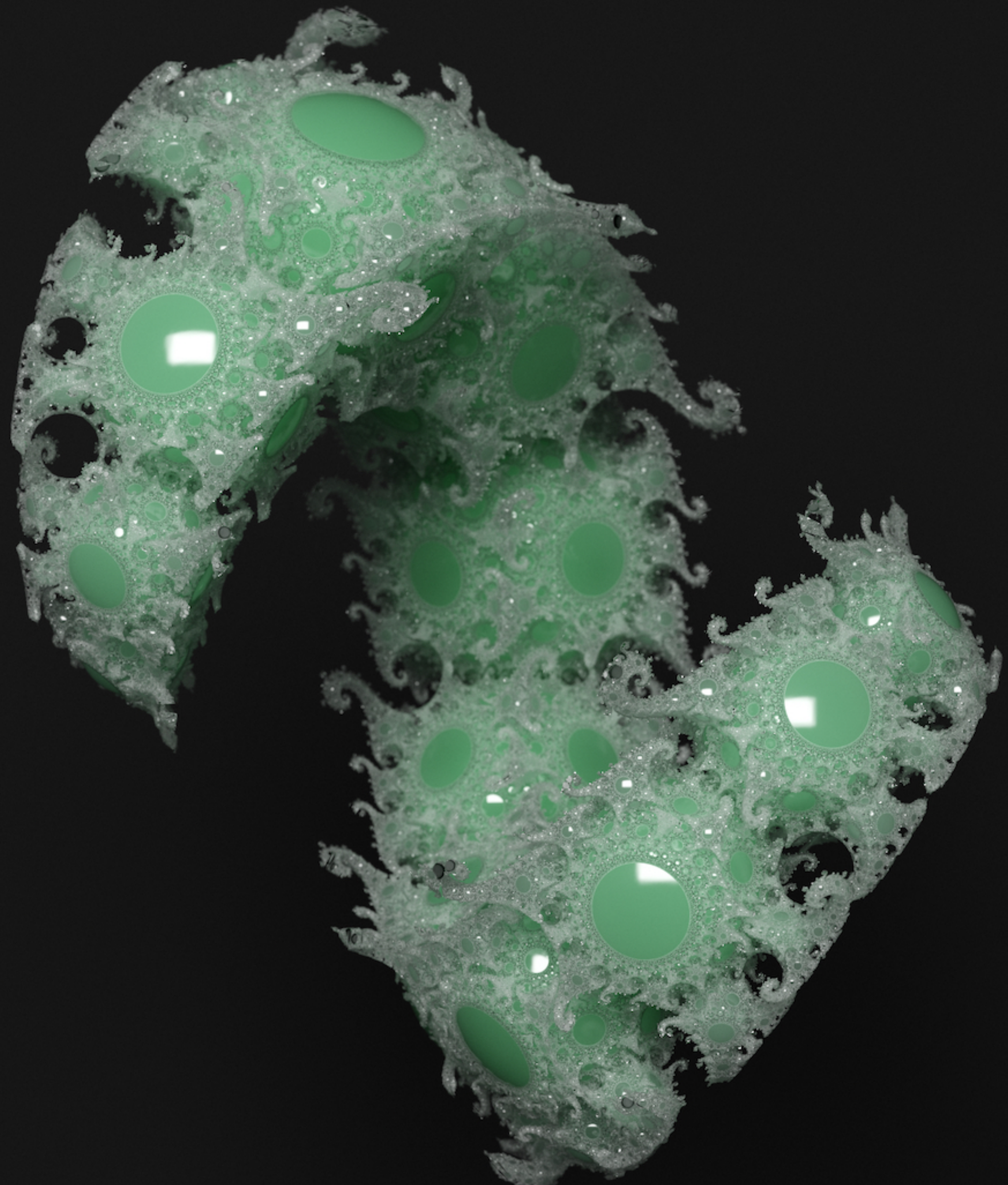


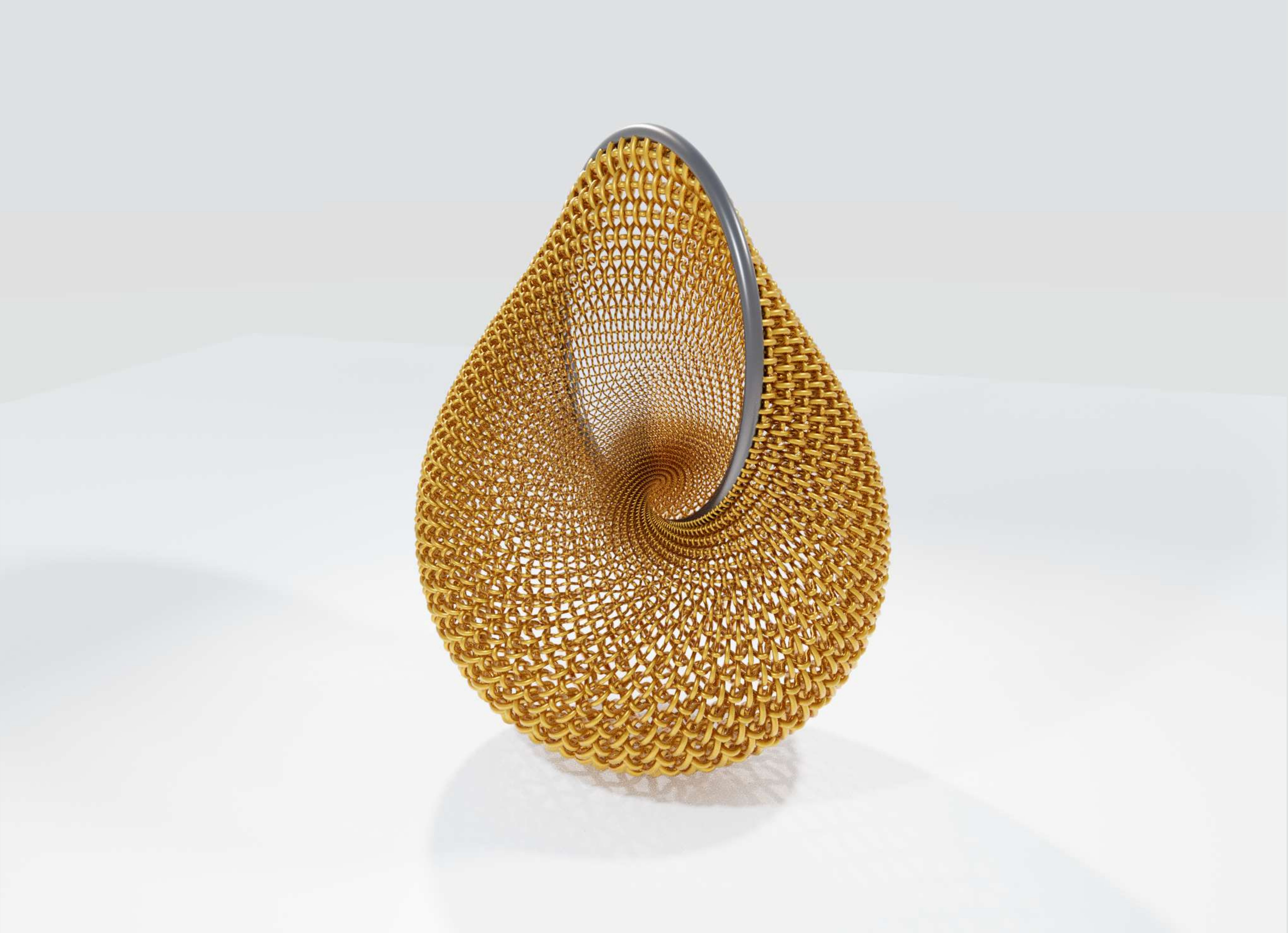


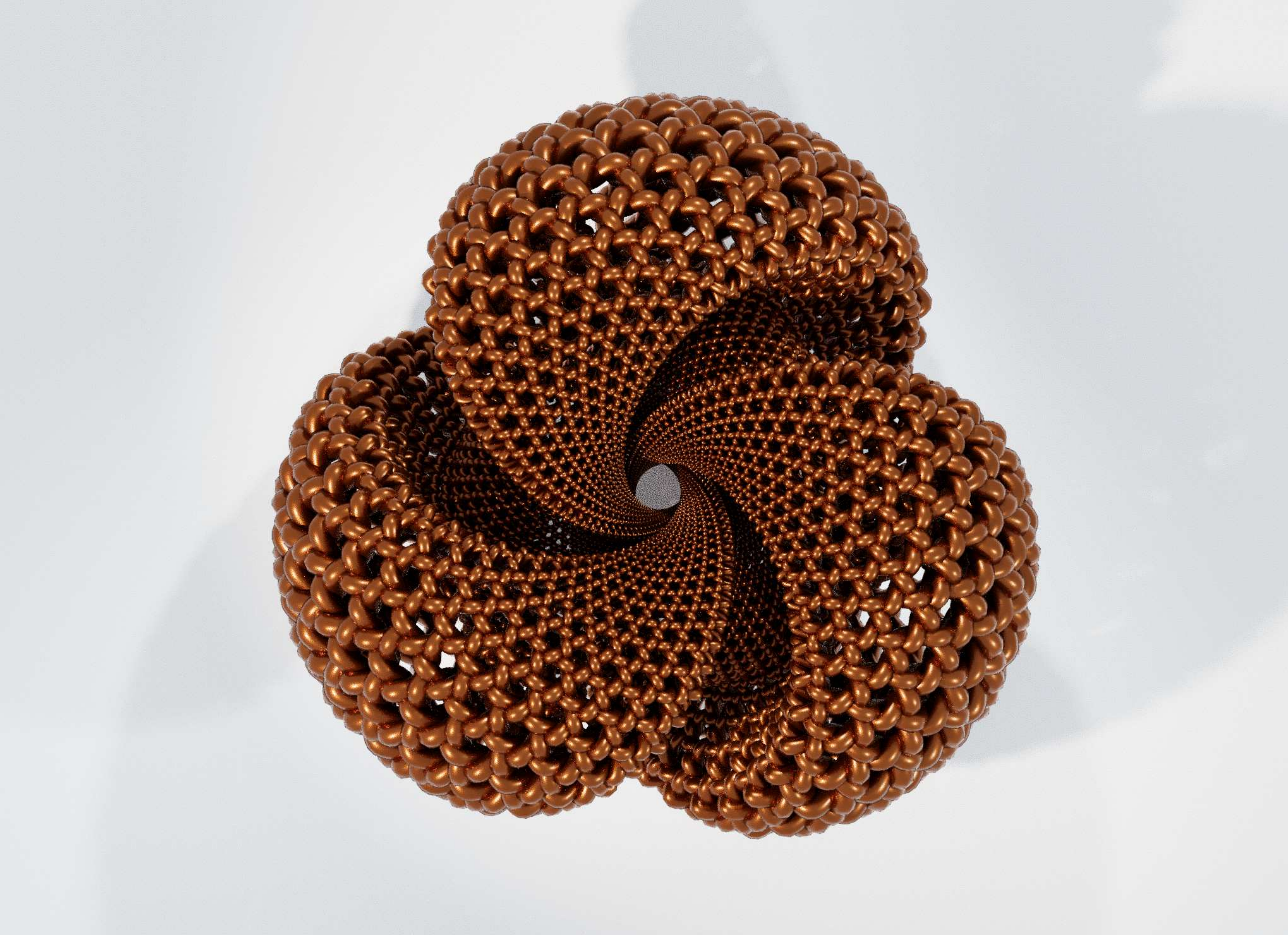
And surprised how much math was involved



learning more
math really did
lead to better
graphics



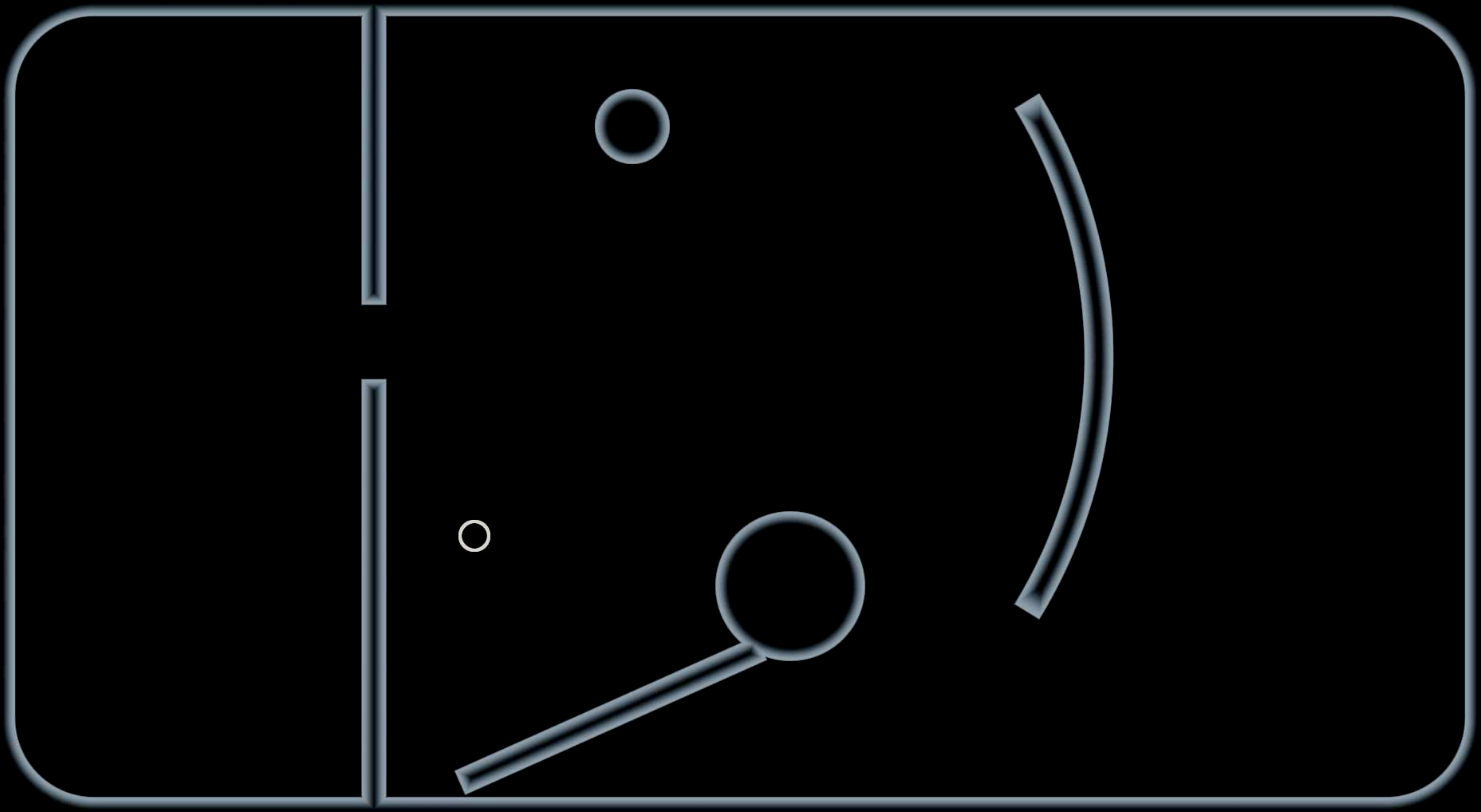




Part I

What is a Photograph?

What is Light?



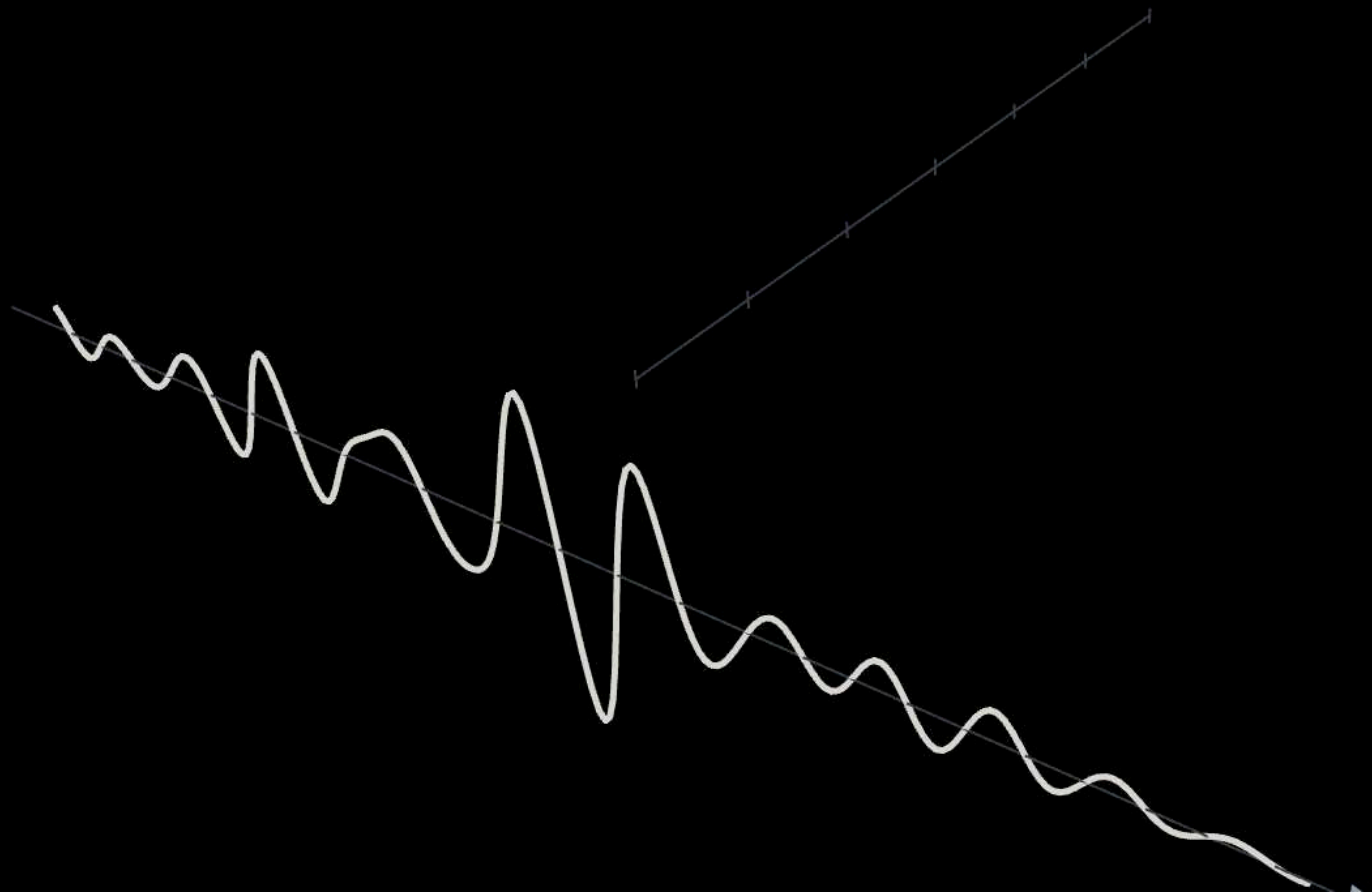
Light is a wave solution to Maxwell's equations

$$\nabla \cdot \mathbf{E} = 0$$
$$\nabla \cdot \mathbf{B} = 0,$$
$$\nabla \times \mathbf{E} = -\partial_t \mathbf{B},$$
$$\nabla \times \mathbf{B} = \partial_t \mathbf{E}$$

What is Light?

Small wavelength
converges towards
ray optics

What is Light?



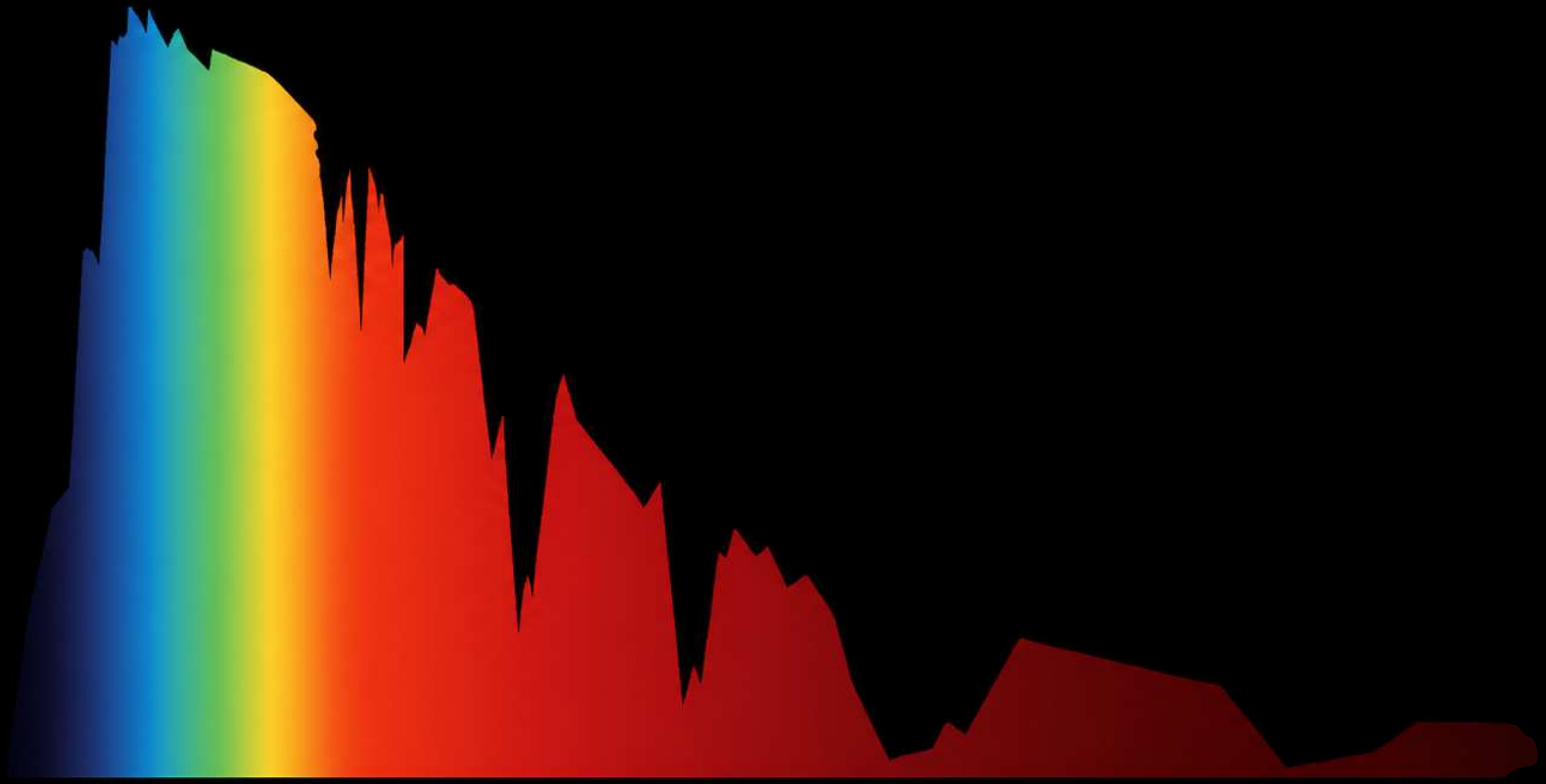
Light along a ray decomposes into a **spectrum** along **the space Λ of wavelengths**

What is Light?

The spectrum of an
incandescent lamp

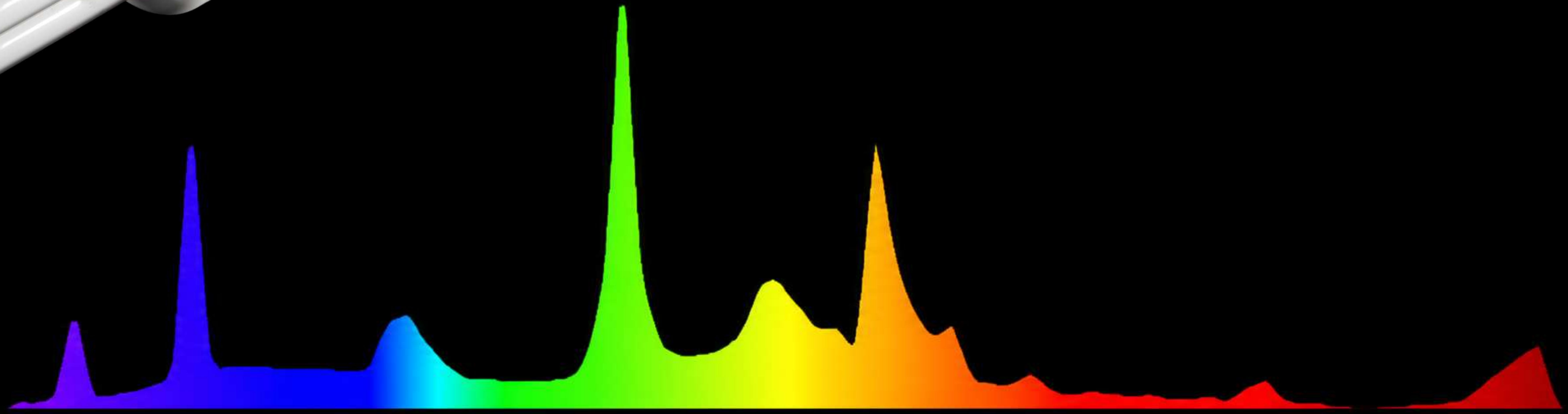
What is Light?

The solar spectrum



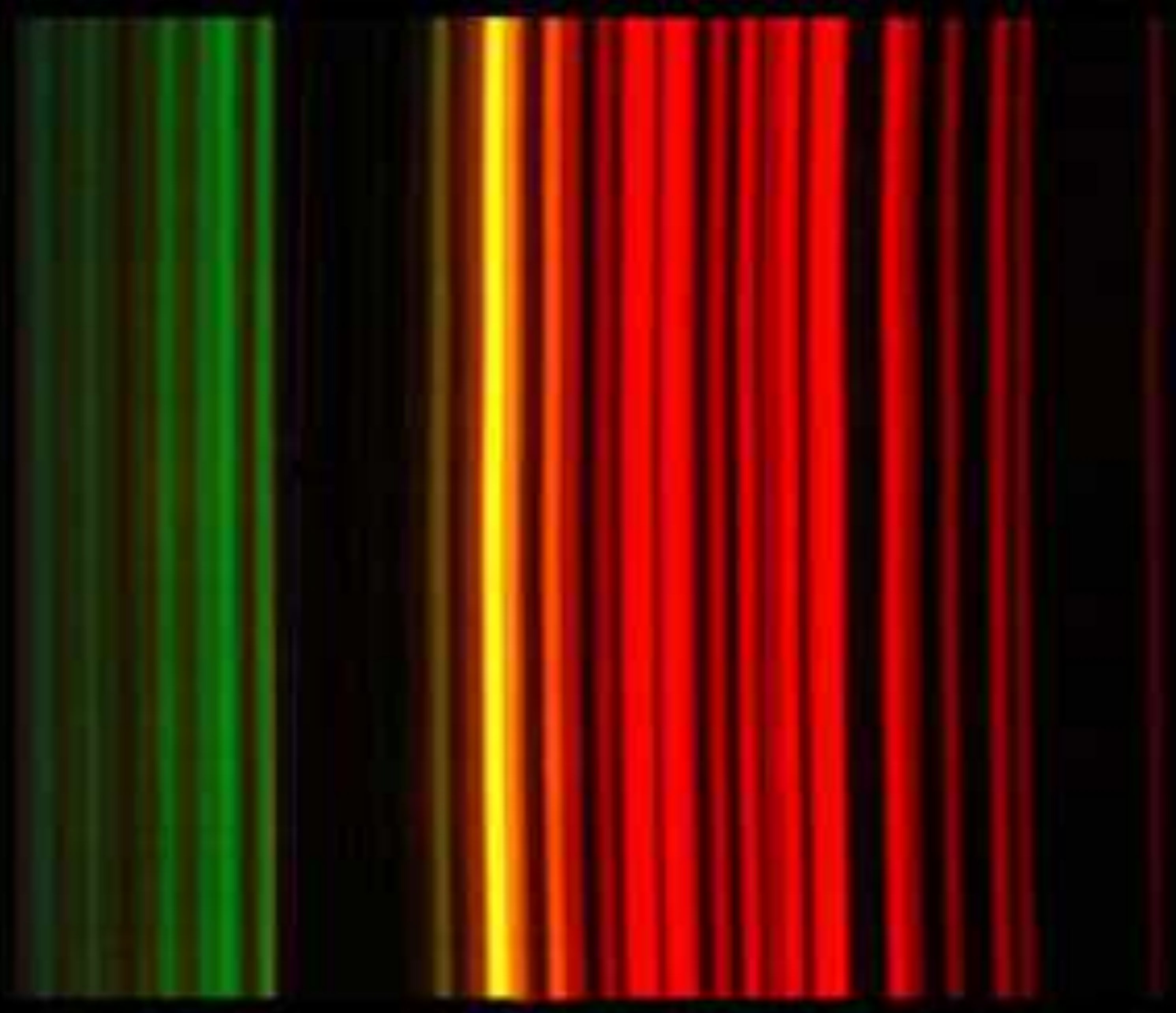
What is Light?

The spectrum of a flouescent light



What is Light?

The spectrum of neon is a discrete et of lines, due to quantum mechanical energy transitions.



What is Light?

A spectrum of light can be *continuous* like the blackbody, or *discrete* like a neon lamp.

The natural mathematical objects that describe such things are *measures*.

Definition:

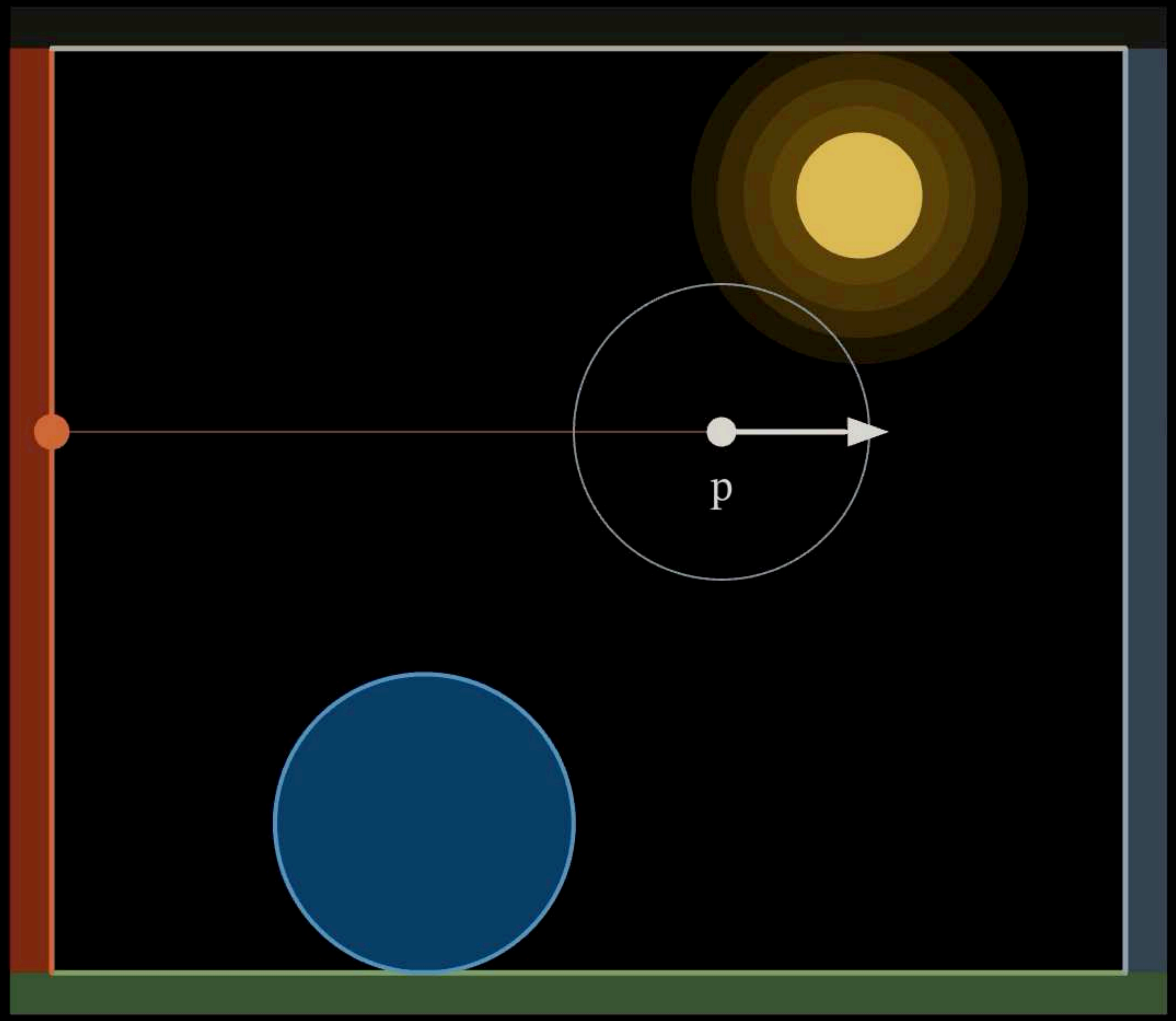
Let $\Lambda \subset \mathbb{R}_+$ be a set of relevant wavelengths.

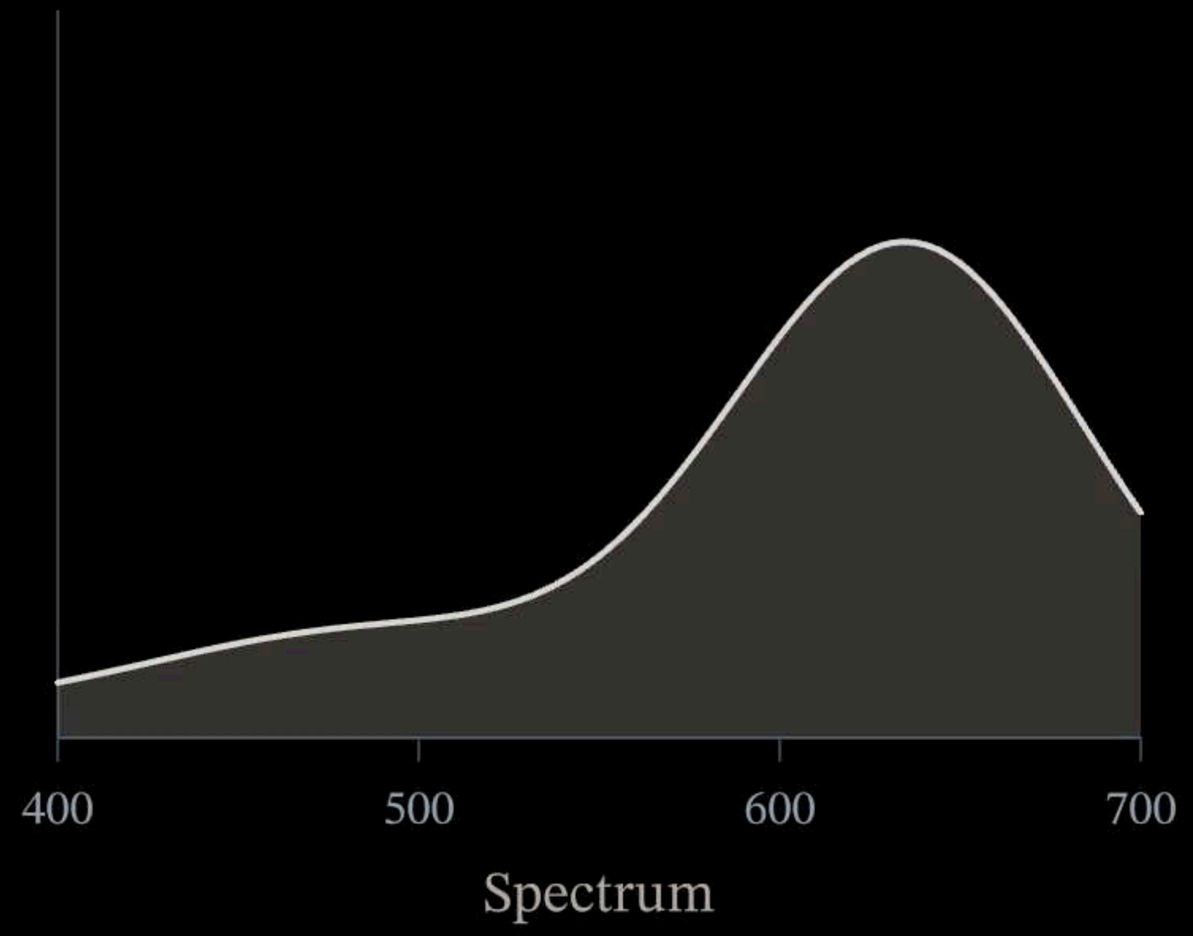
A spectrum is a positive finite measure σ on Λ

$\mathcal{S}$ = set of finite measures

$\mathcal{S}_+$ = set of spectra

What is Light?



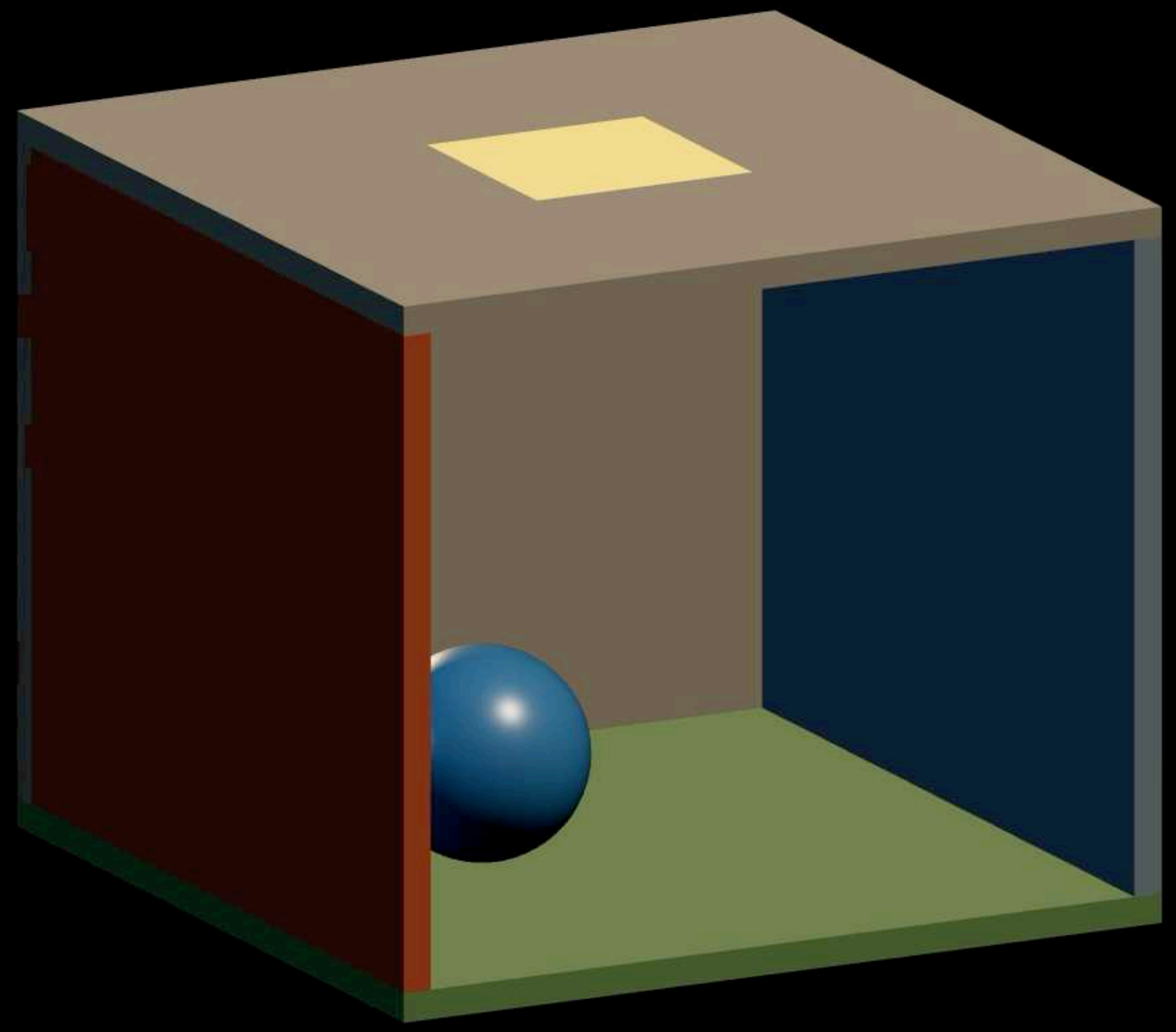


Definition:

A Light Field on M is a function

$$T^1M \to \mathcal{S}_+$$

Lighting a Room

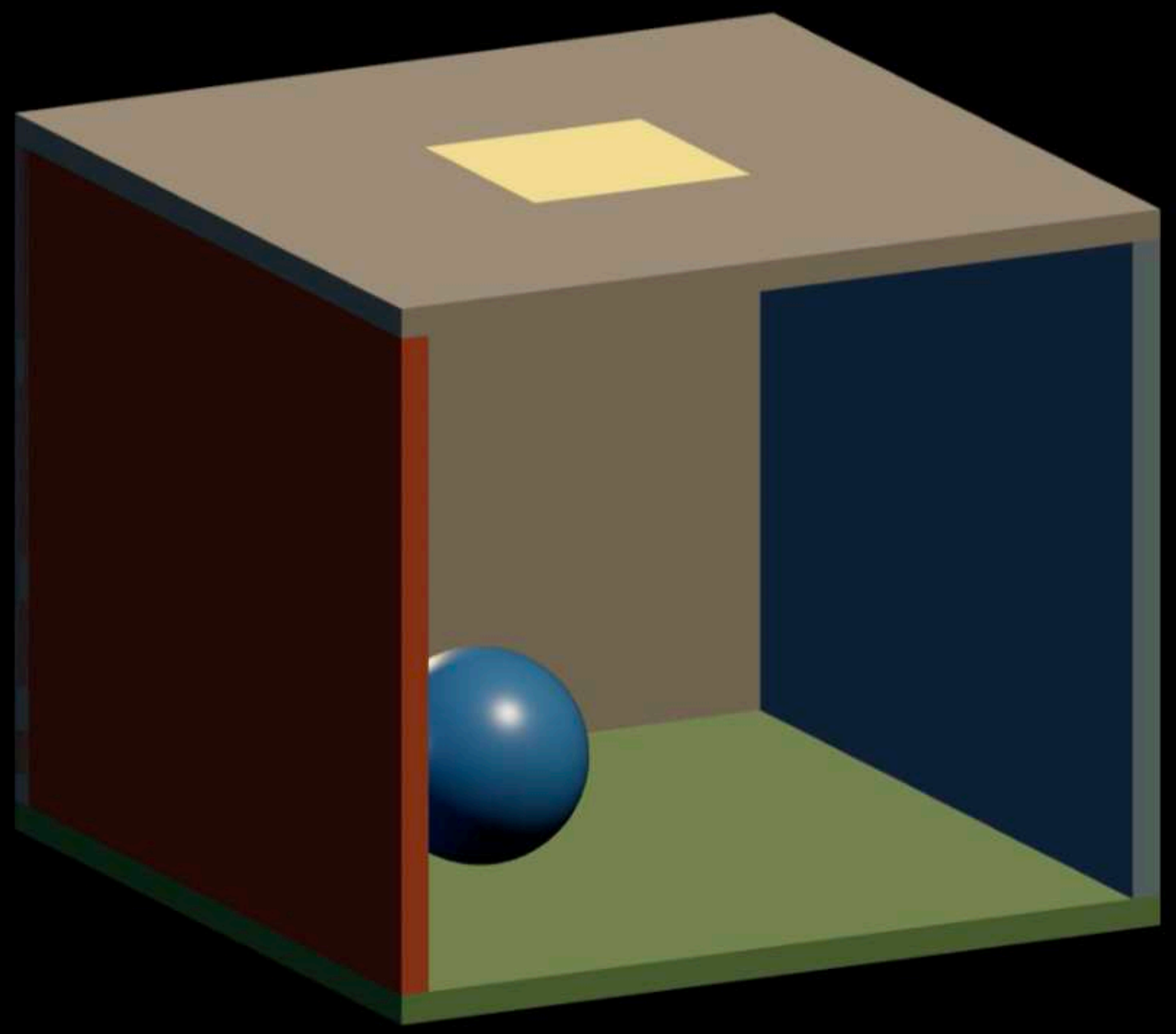


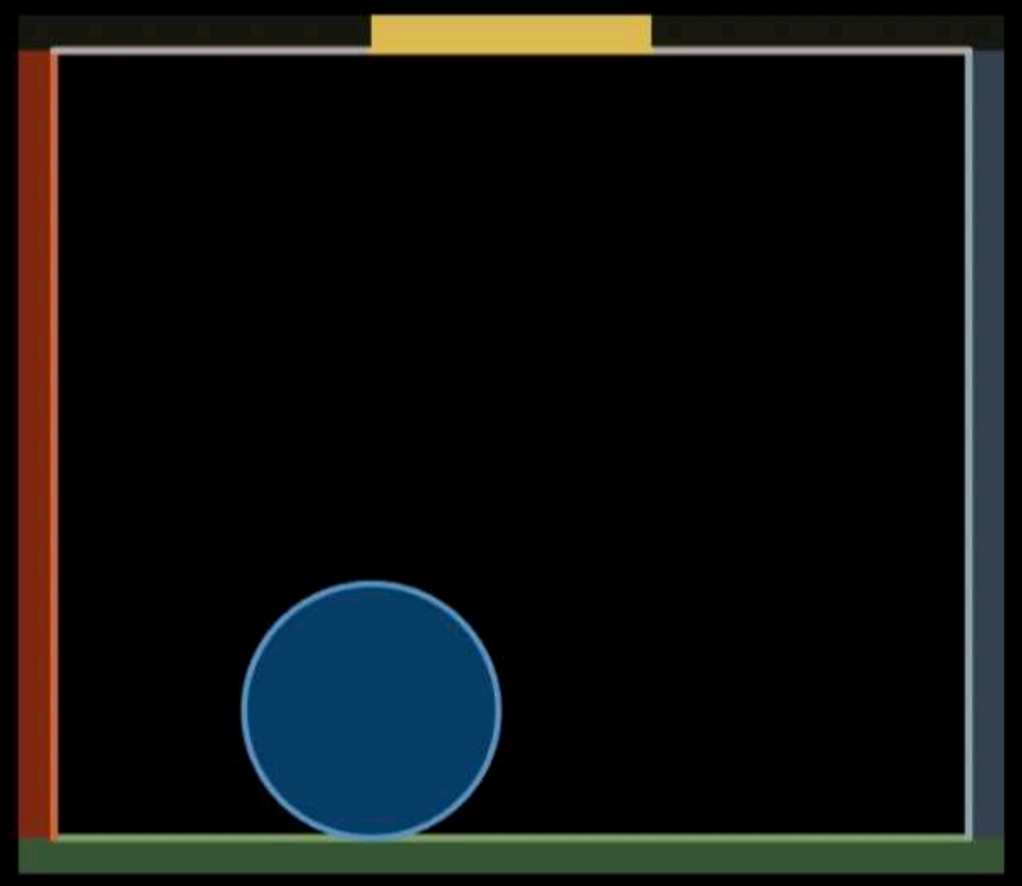
An *object* is a subset of M.

A scene is the complement of a collection of objects.

We assume (for technical reasons) that the scene & its boundary are compact.

Lighting a Room





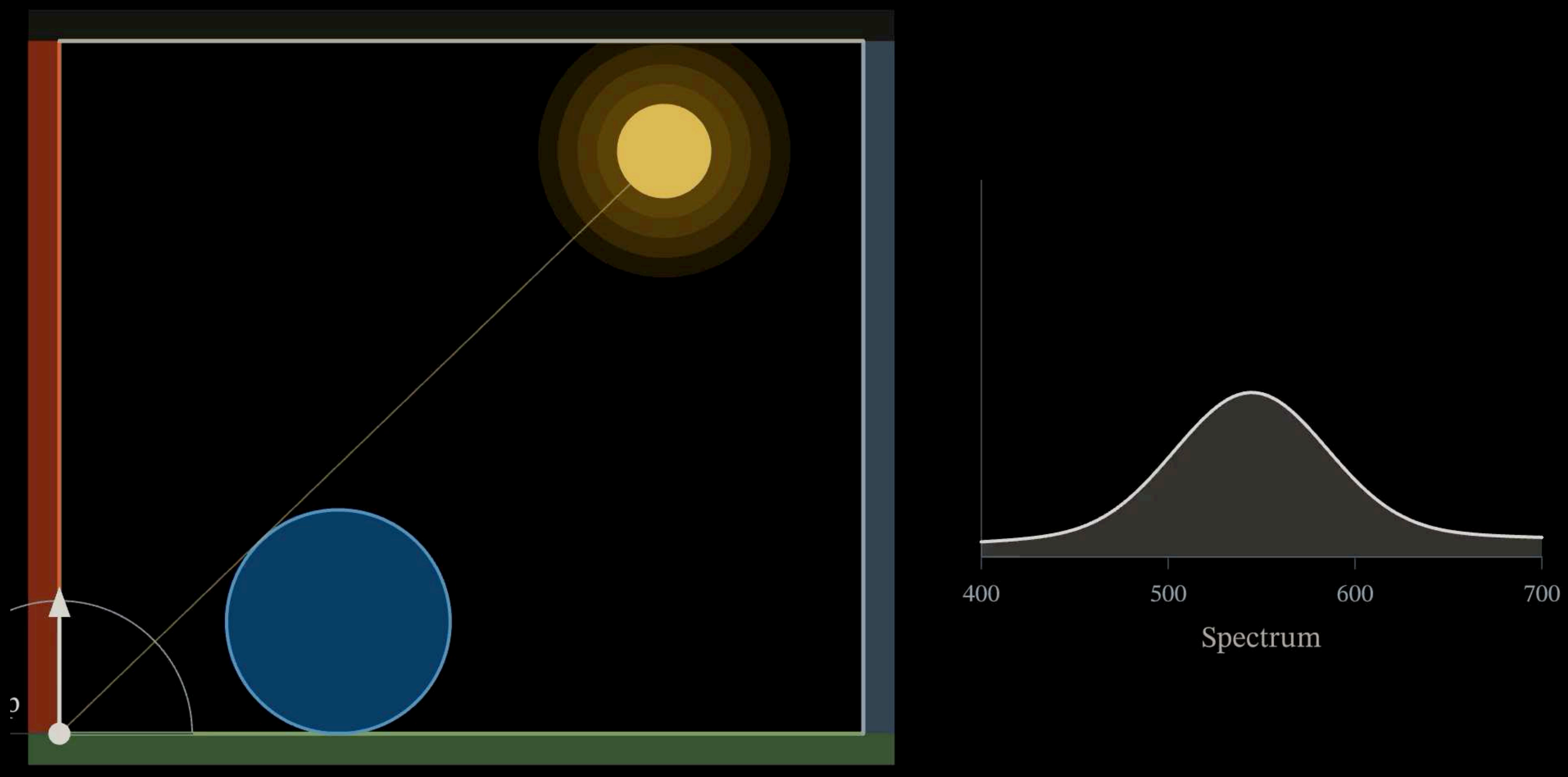
Notation:

$$\mathcal{O} = \text{set of objects}$$

$$\Sigma = \partial\mathcal{O}$$

$$\mathcal{X} = T^1M\Big|_{\Sigma}$$

Lighting a Room



Nothing happens to light in-between objects

A light field L is fully determined by its restriction to $\mathscr{X} = T^1M\Big|_{\partial\mathscr{O}}$

Lighting a Room

Definition:

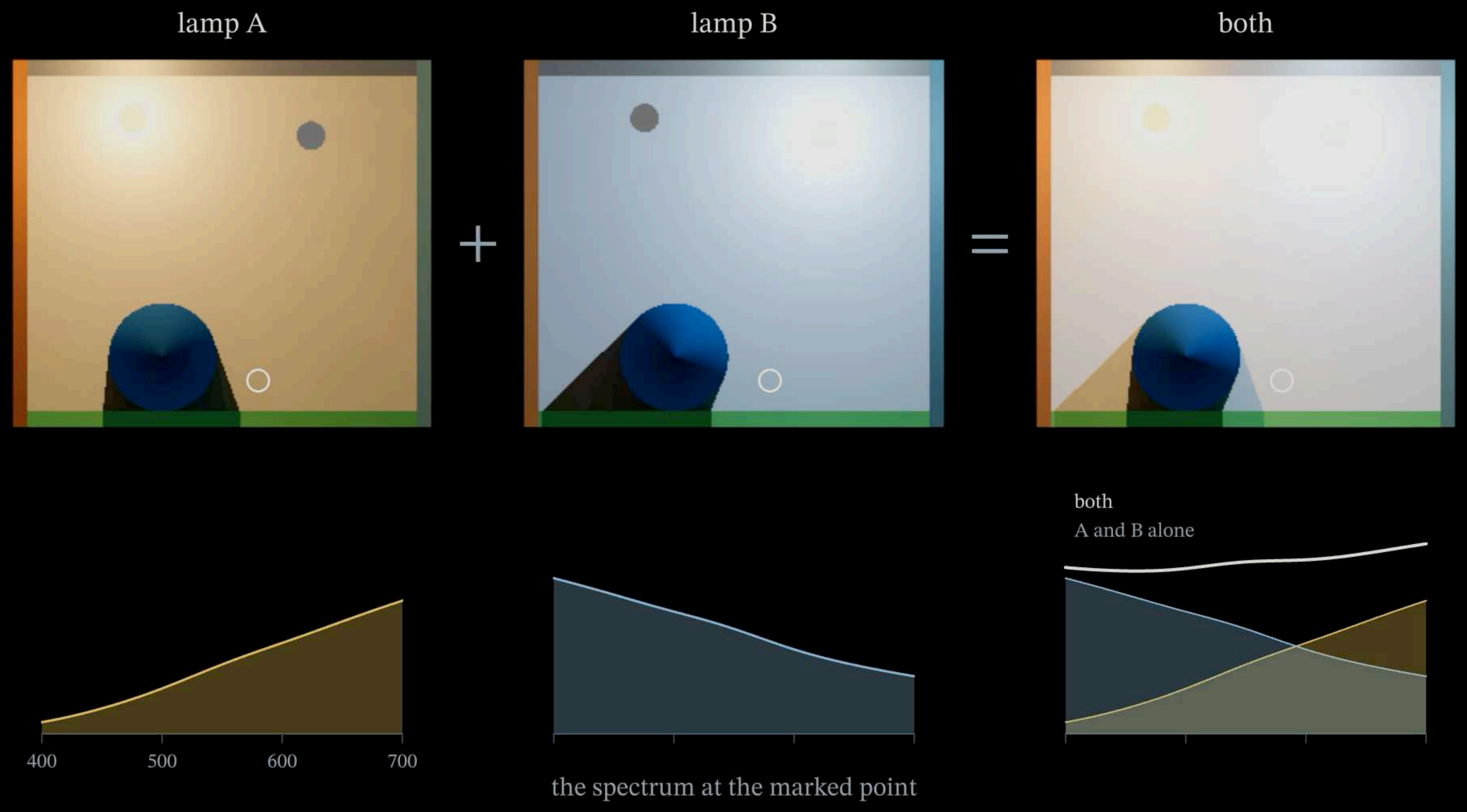
A *light field* L *on a scene is a map* $\mathscr{X} \to \mathscr{S}_+$

That is, an assignment of a spectrum to each ambient tangent vector based at a point of the scene boundary.

The space of light fields on the scene is denoted $\mathscr{L}$

This is a Banach space, with norm $\|F\| = \sup_{x \in \mathscr{X}} \|L(x)\|$, for $\|L(x)\|$ the total variation of this measure.

Lighting a Room



What happens when light hits an object?

Theorem:

*Incoherent light: brightness responds *linearly* to interaction*

Lighting a Room

Corollary:

A *scene acts as a linear map on the space of light fields on it*

$$T\colon \mathscr{L} \to \mathscr{L}$$

Assumption:

In the real world, all interactions dissipate some energy. So, we require $\|T\| < 1$

Lighting a Room

Objects can also *emit* light:
this is a light field E on $\mathscr{X}$

Emission and interaction are independent processes

Corollary:

The scene is an *affine map*

$$L \mapsto E + T(L)$$

What are Materials?

The scene acts as a linear, positive map $T\colon \mathscr{L} \to \mathscr{L}$ on light fields.

Apply the Riesz representation theorem: T is represented by integration against a measure!

For $x \in \mathscr{X}$ there is a measure $K(x)$ valued in operators on $\mathscr{S}$, such that

$$(TL)(x) = \int_{\mathscr{X}} dK(x)\, L$$

What are Materials?

Thus, K stores *what the scene does at each point*.

But we can decompose even better: since light remains constant away from the scene, T factors

$$T = RP$$

For P the geodesic flow.

Applying Riesz to R, we get the kernel describing *interaction with the light field at a point*

$$(RL)(x) = \int_{\mathscr{X}} d\mu(x)\, L$$

What are Materials?

Simplifying assumptions about materials (for this talk):

Interactions are Local:

Evaluation reduces to the unit sphere of tangents at a point:

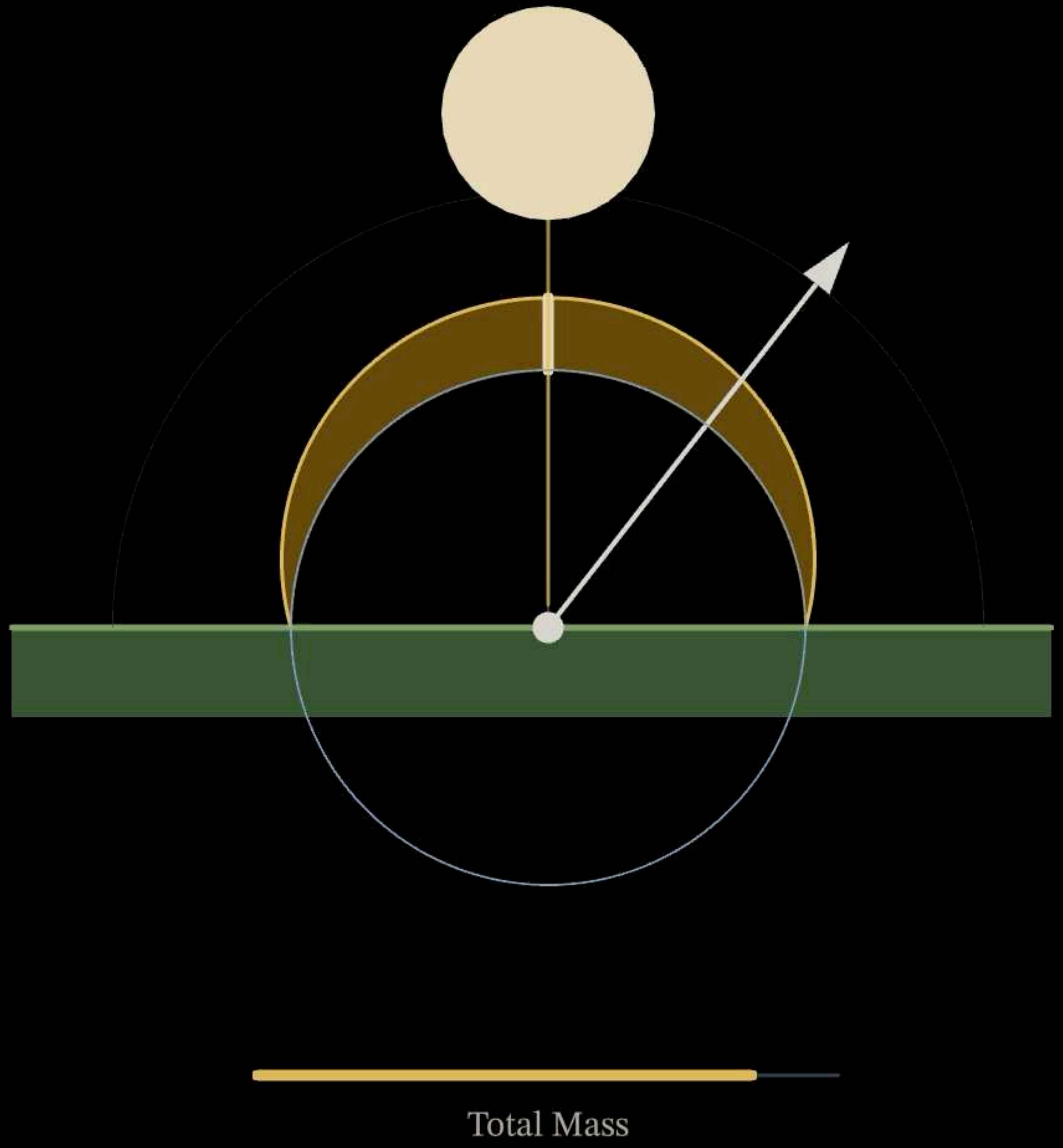
$$RL(p,v) = \int_{\mathbb{S}^2_p} d\mu(p,v)\, L$$

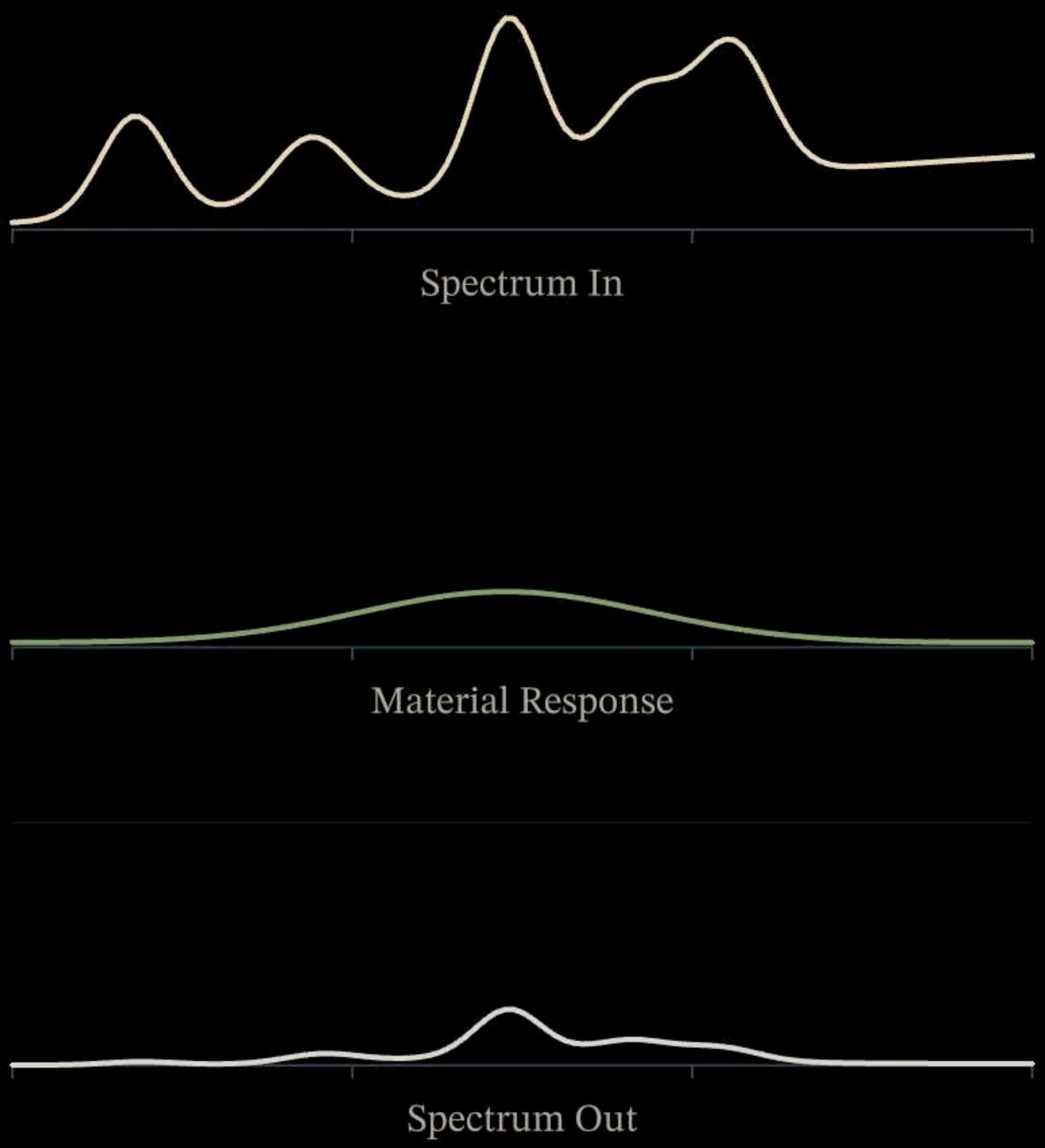
Wavelength is Preserved:

The kernels $K(x), \mu(x)$ are multiplication of the spectrum by a scalar function.

$$R(x)L = a(\,\cdot\,)L(\,\cdot\,)$$

What are Materials?



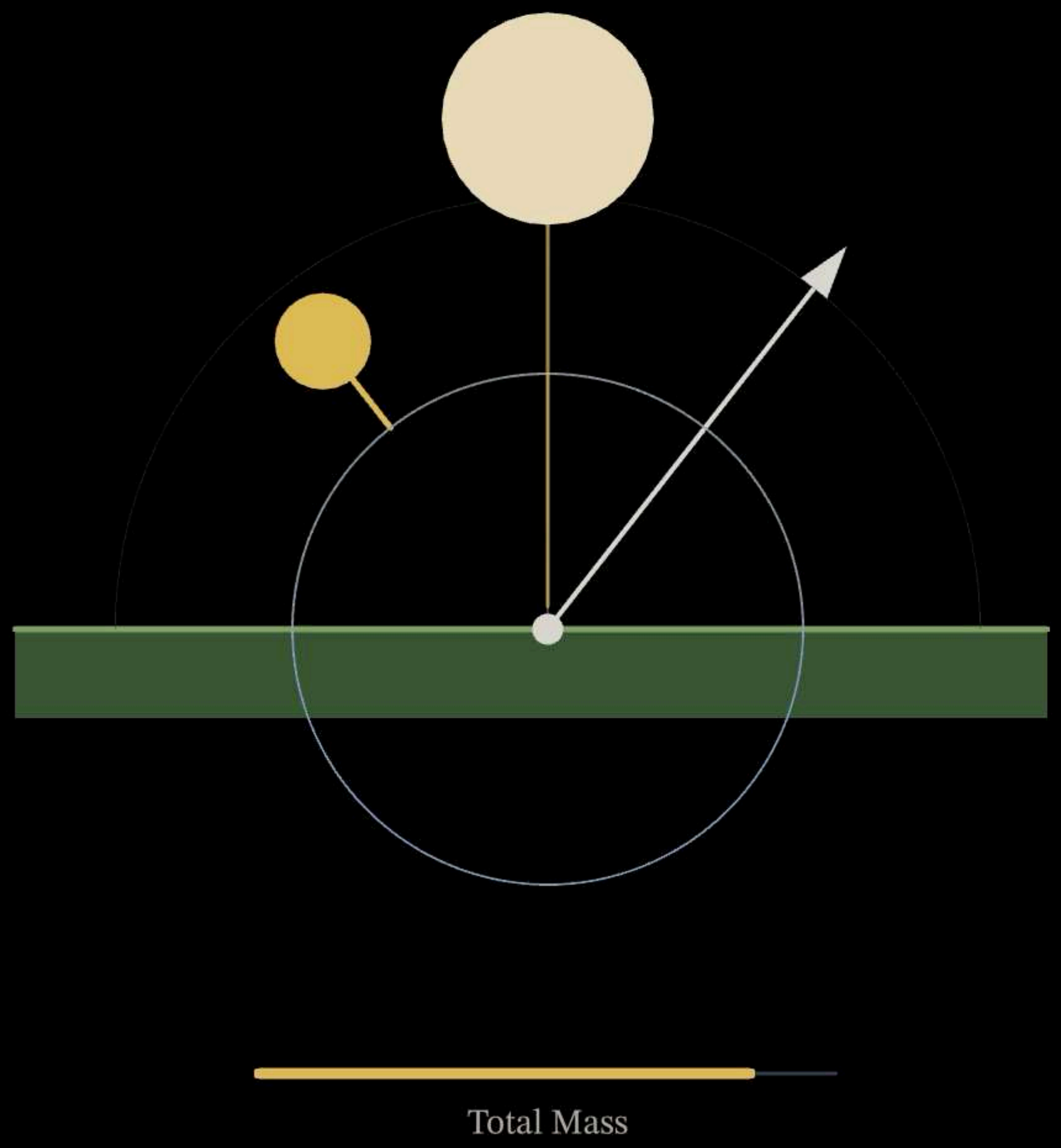


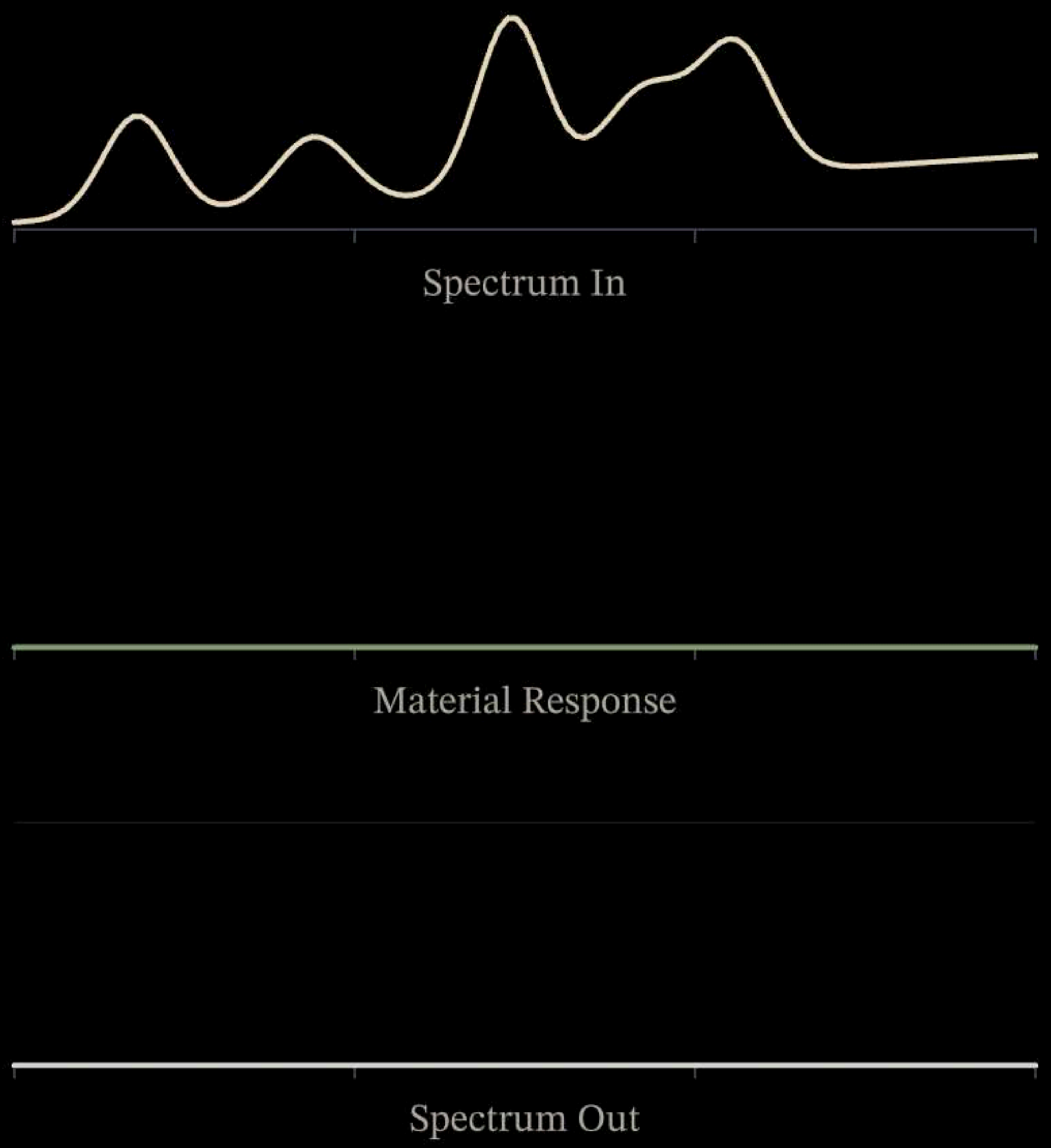
Diffuse Material

For incoming direction $\omega \in \mathbb{S}^2$

$$\mu(x) \propto |n \cdot \omega| \, d\omega$$

What are Materials?



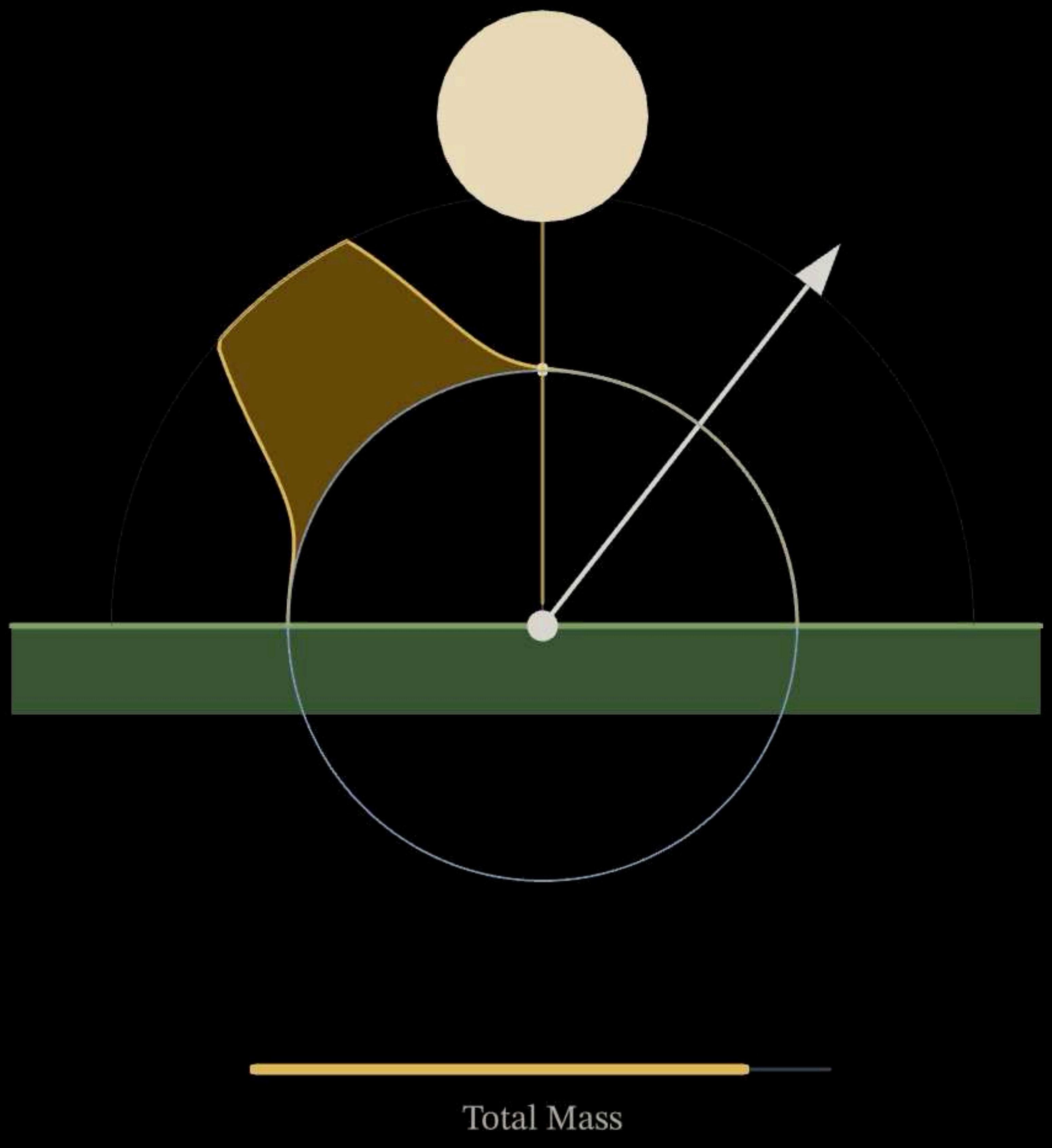


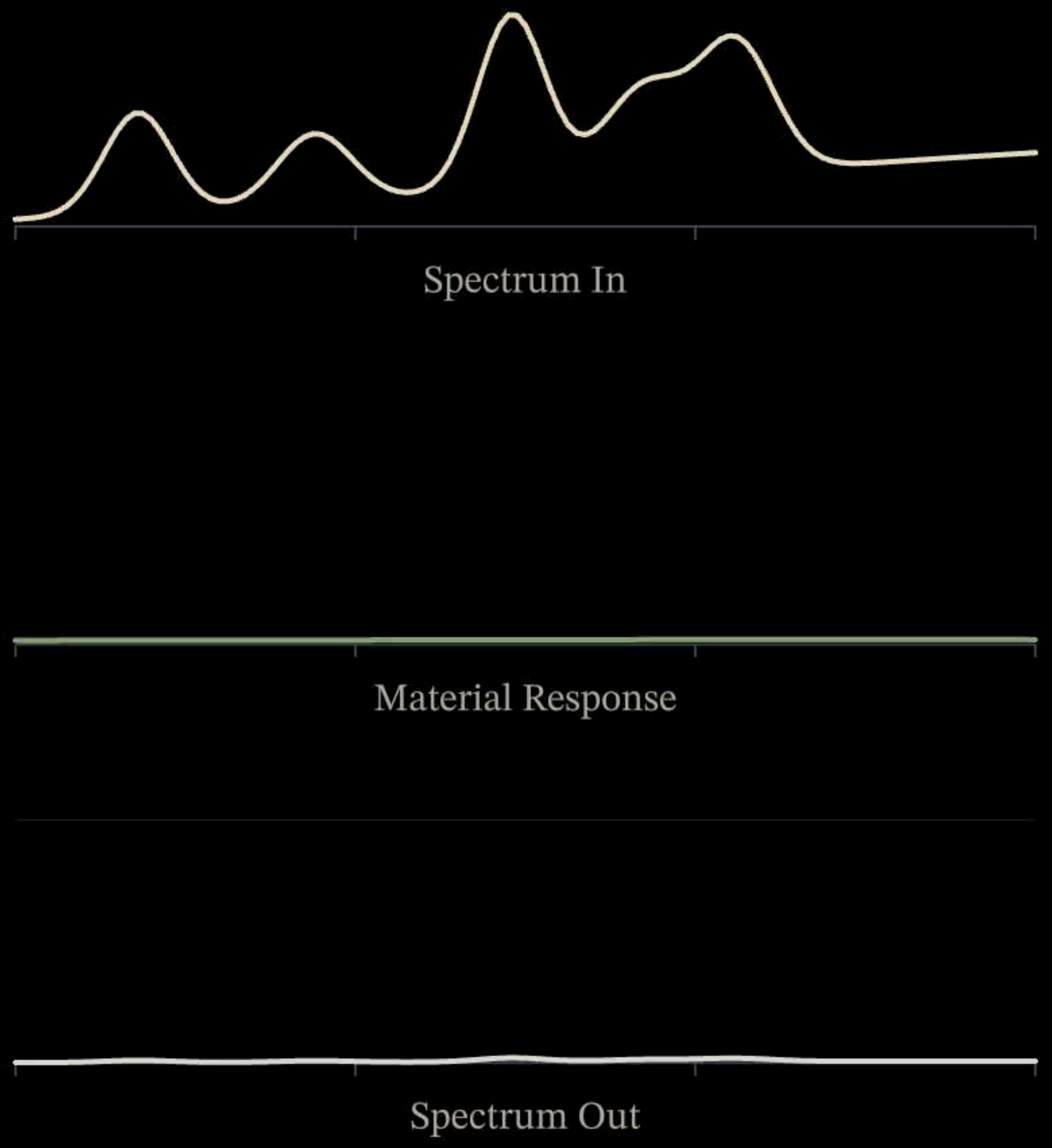
Mirror-like Material

For incoming direction $\omega \in \mathbb{S}^2$

$$\mu(x) \propto \delta_{2(n\cdot\omega_o)\,n-\omega_o}$$

What are Materials?





Glossy Material

For incoming direction $\omega \in \mathbb{S}^2$

$$\mu(x) = \alpha \,|\, n \cdot \omega \,|\, d\omega + \delta_{2(n\cdot\omega_o)\,n-\omega_o}$$

Realistic Lighting

Light in a room is in a stable state, not changing in time. How do we model this?

Recall, the scene is an affine map

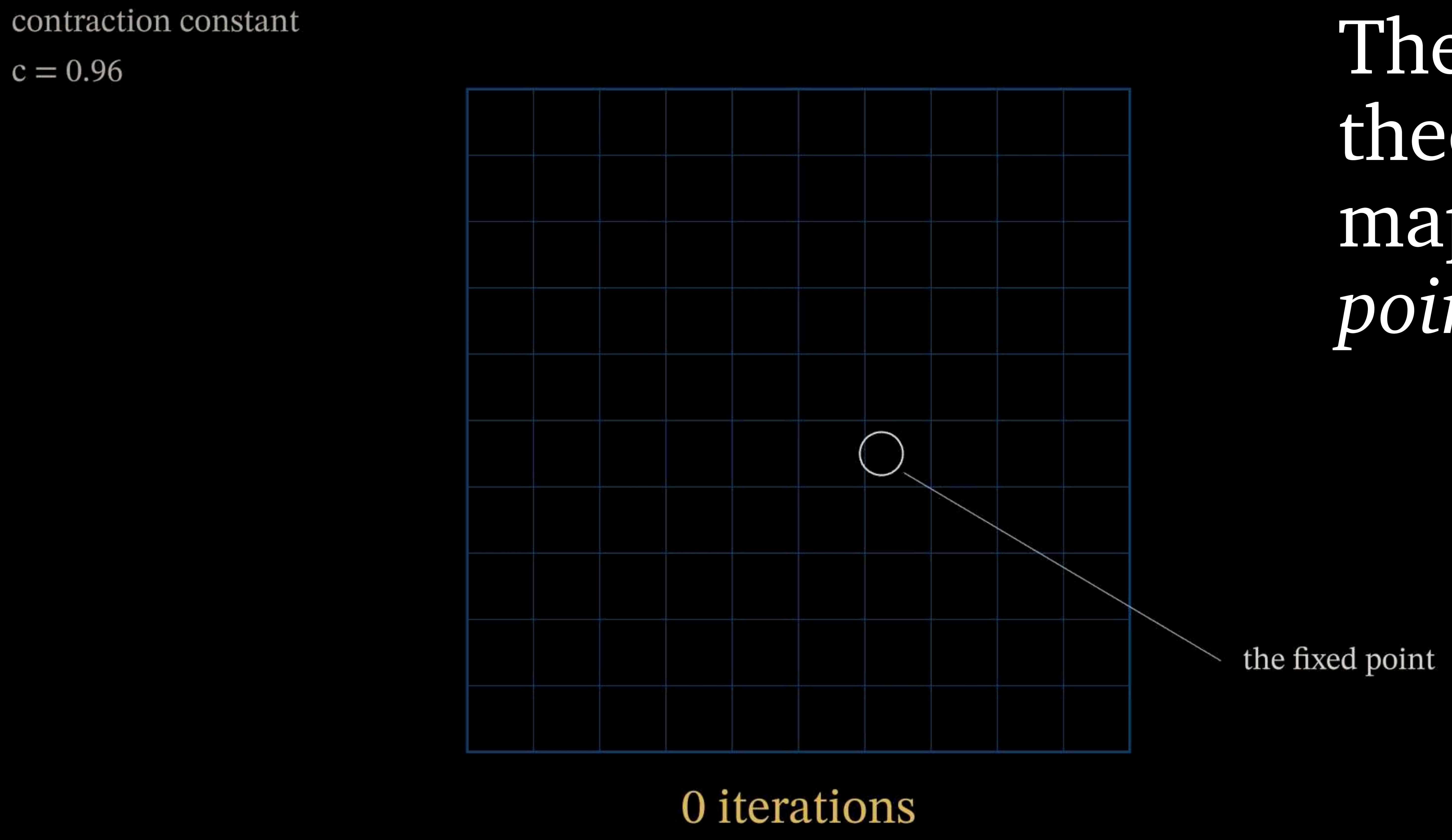
$$L \mapsto E + T(L)$$

And also, that $\|T\| < 1$

Observation:

The scene is a *contraction map!*

Realistic Lighting



The contraction mapping theorem implies such maps have *unique fixed points!*

This applies quite generally, including to the Banach space of Light fields

Realistic Lighting

Theorem:

Starting from any light field, repeatedly applying the scene $L \mapsto E + T(L)$ converges to a *unique fixed point* $L_\star$

$$L_\star = E + T(L_\star)$$

This is the *equilibrium light field* for the scene.

Let’s Simulate This!

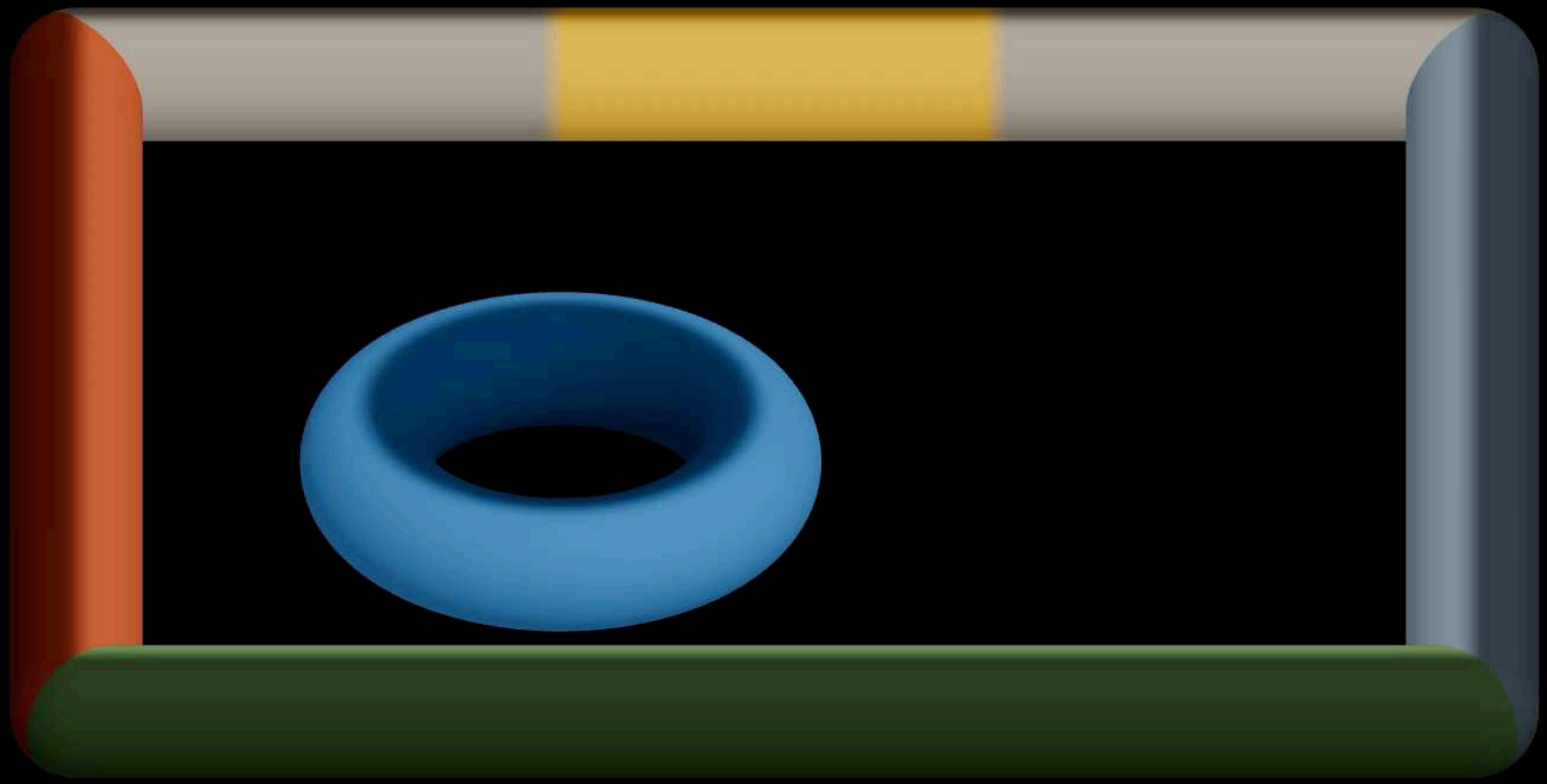
Can we build a test case, where we can really watch light in a closed room converge on a stable, natural looking state?

Let’s Simulate This!

Our 2D room is two closed loops: a rectangular box and a circular disk

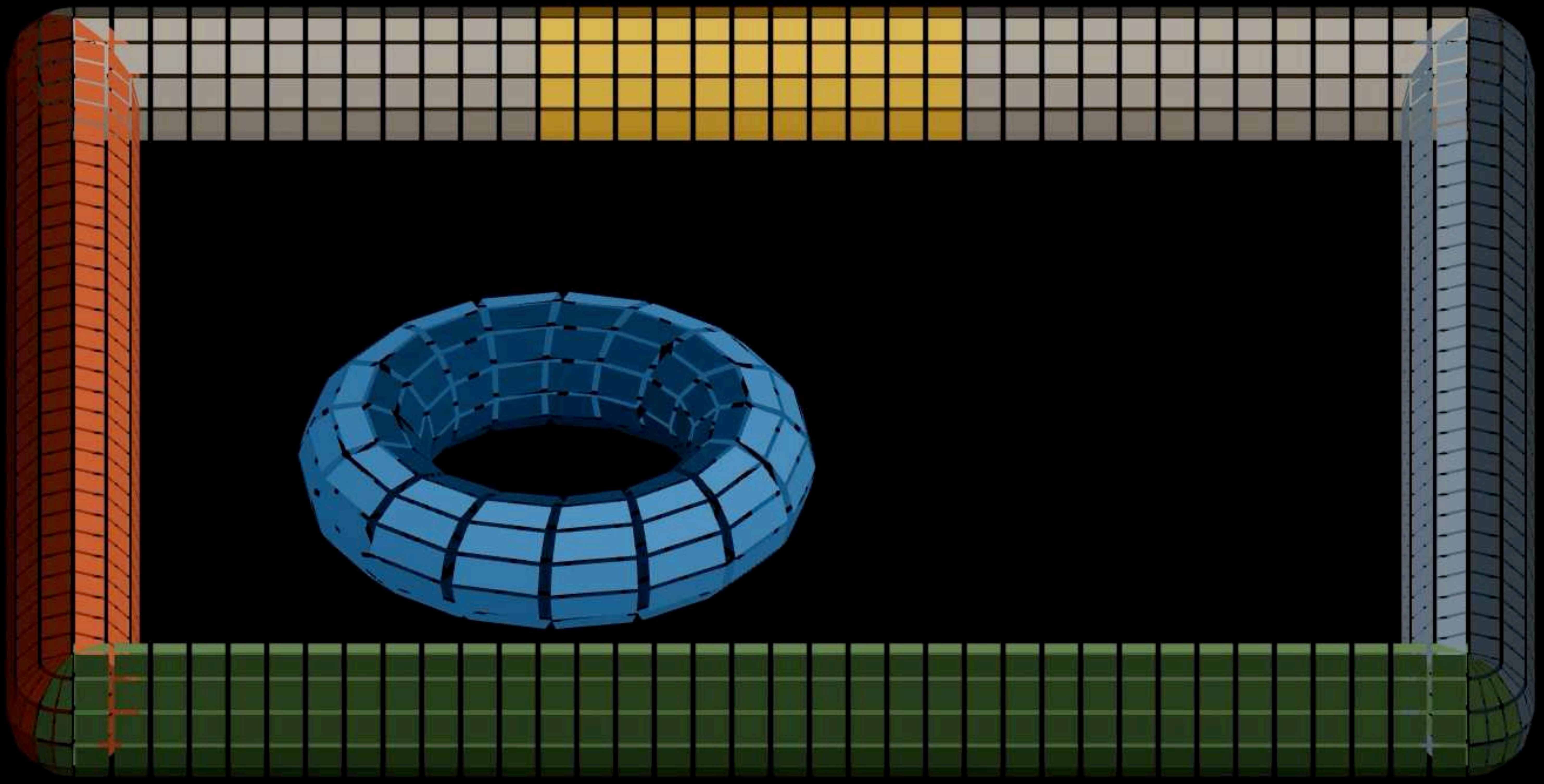
The unit tangent bundle $\mathscr{X}$ is two tori.

Let’s Simulate This!



Discretize these tori in preparation for computation.

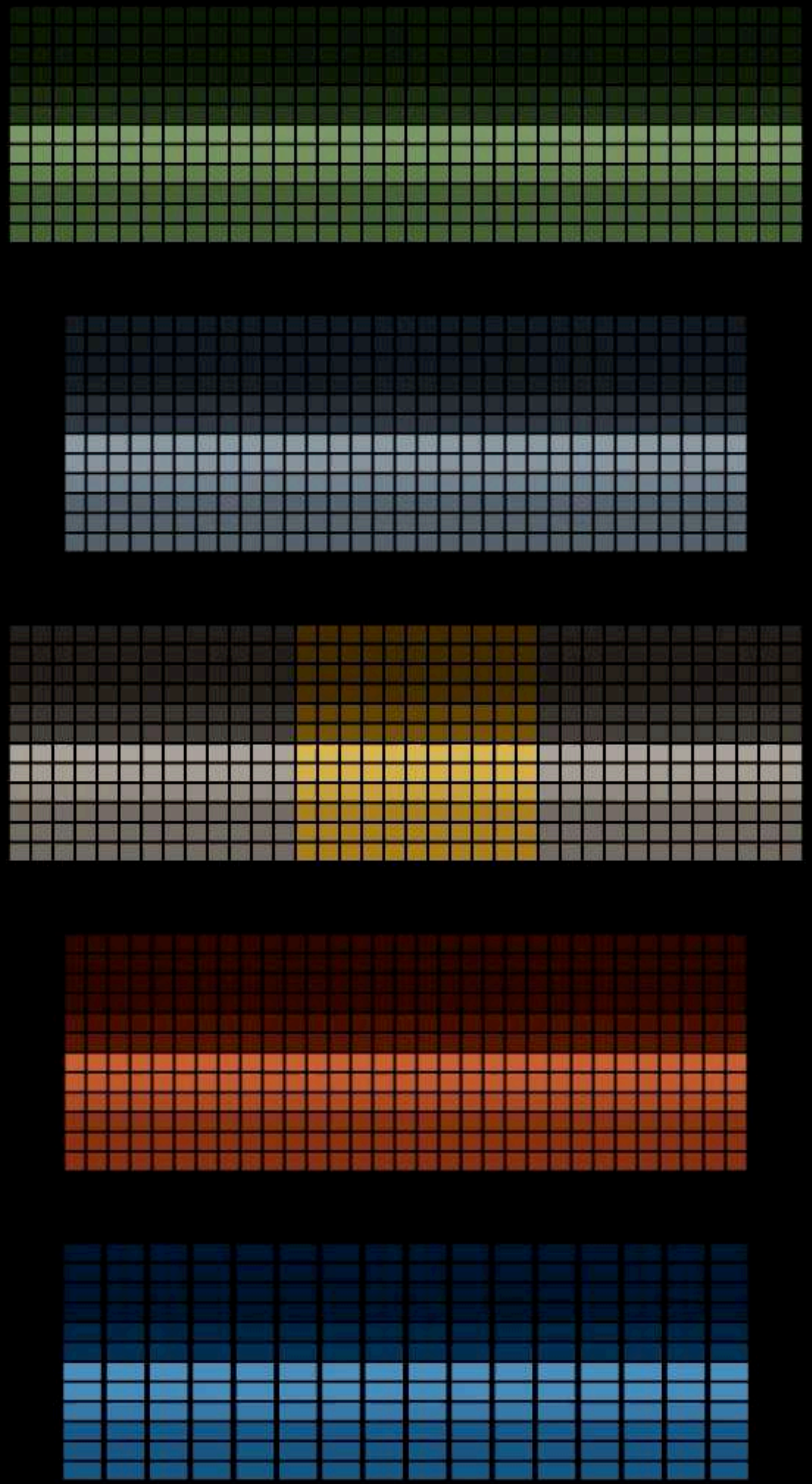
Let’s Simulate This!



Unroll into a grid.

This is 1800 sites, so 1800 spectra.

Let’s Simulate This!



If a spectrum is discretized with 10 wavelengths, the space of light fields is 18,000 dimensional.

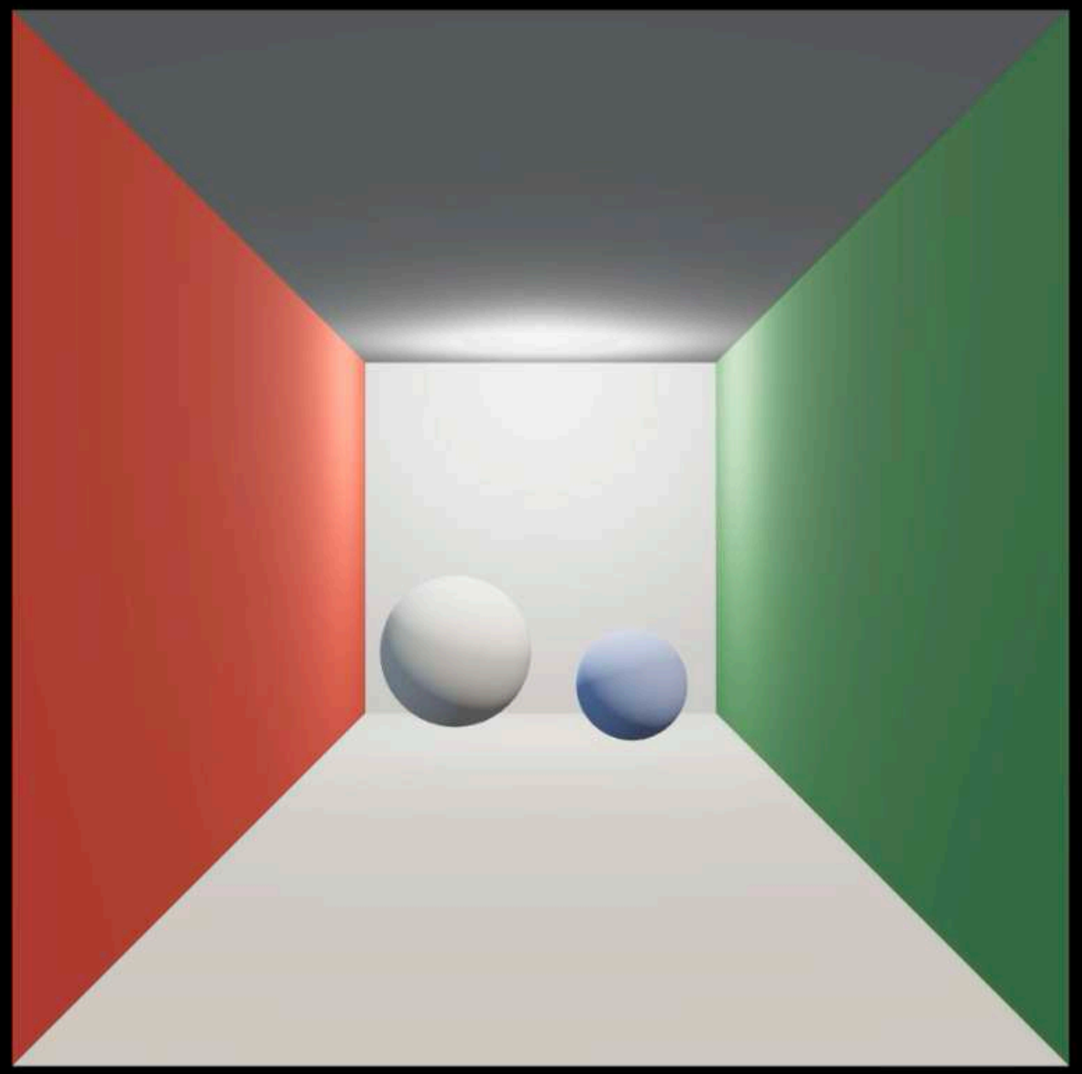
So our scene is an 18,000 x 18,000 matrix

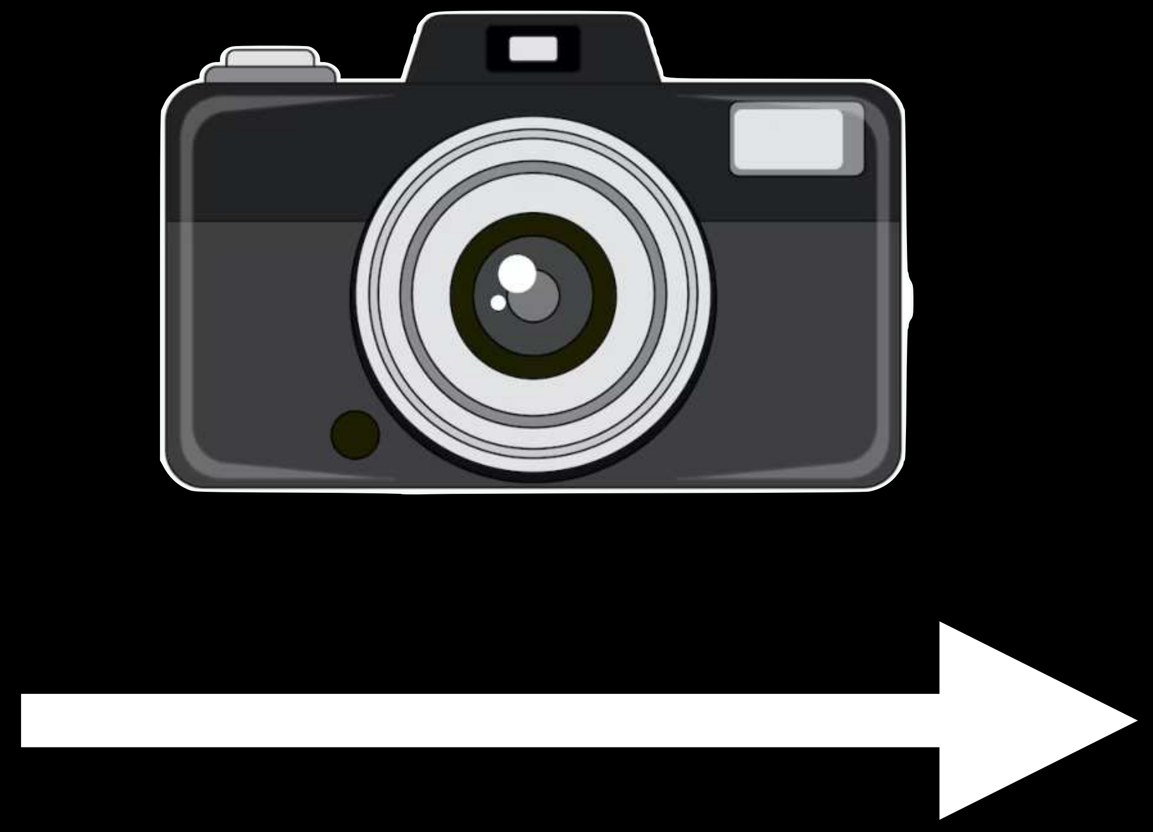
Let’s Simulate This!

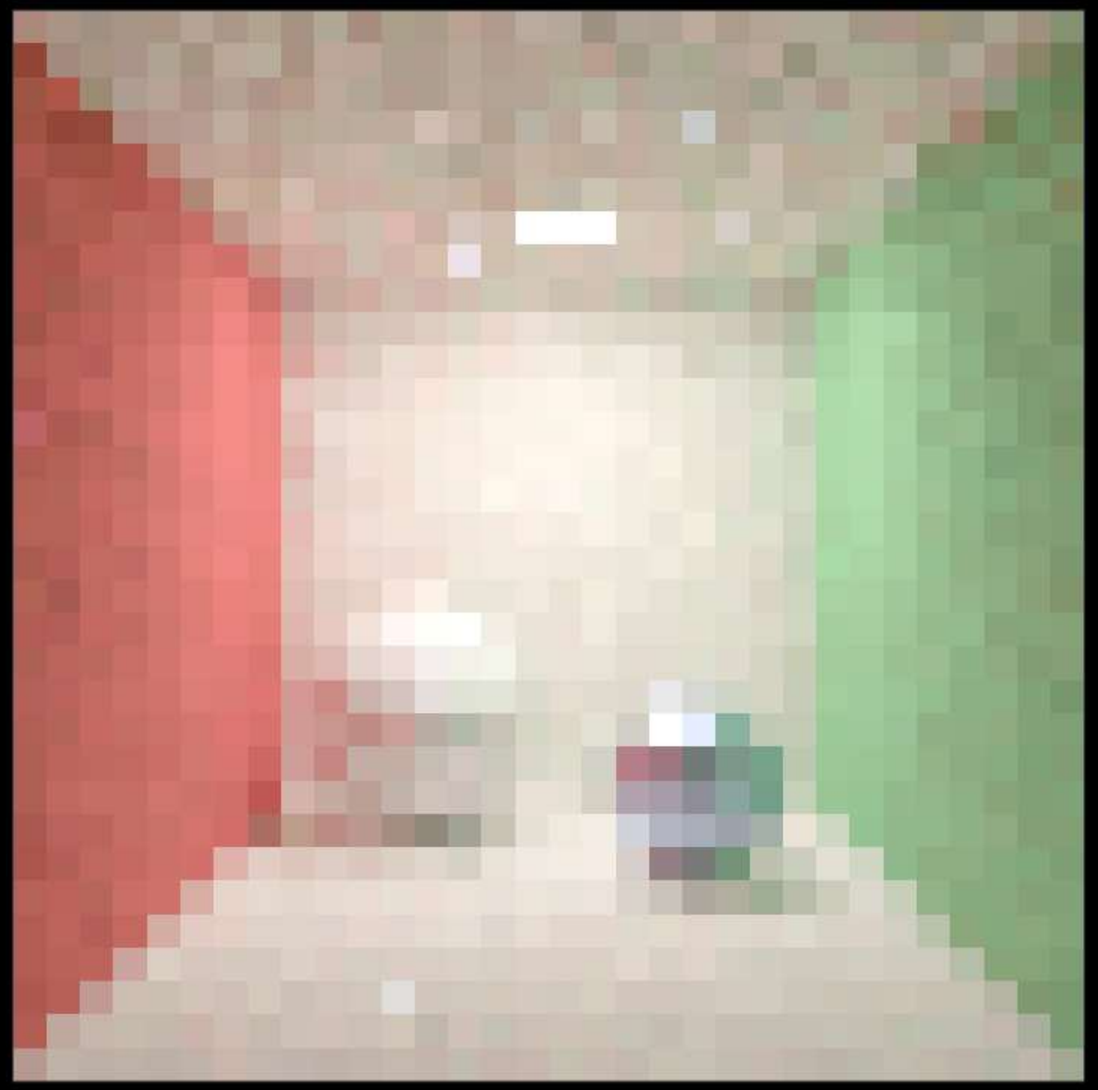
Iterate the scene, and an equilibrium emerges!

0

What is a Camera?







Breaks into two components:
the sensor (made of pixels) and the lens

What is a Camera?

$$\mathscr{L} \xrightarrow{C} V^J$$

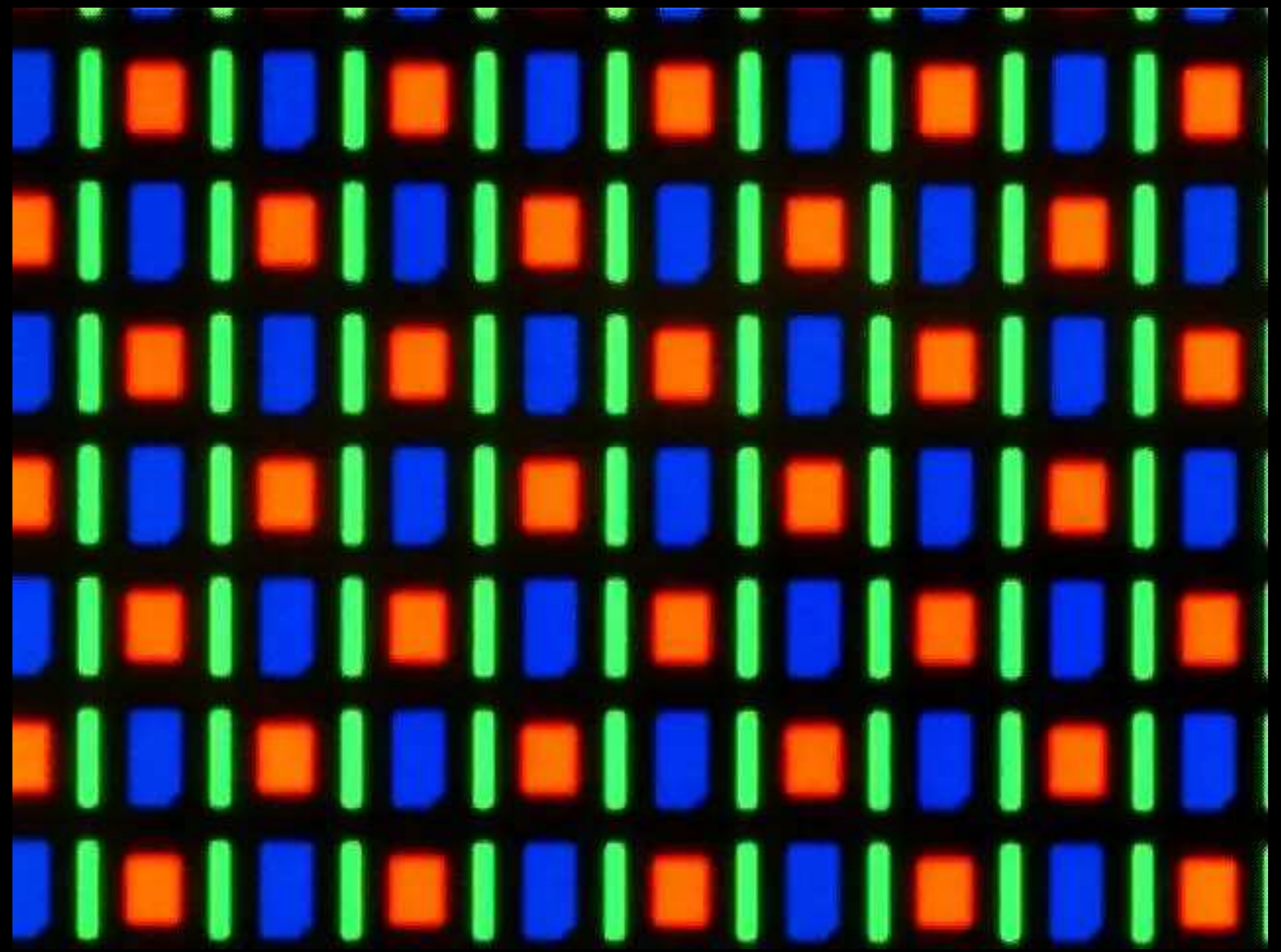
Space of light fields

V = values a pixel records

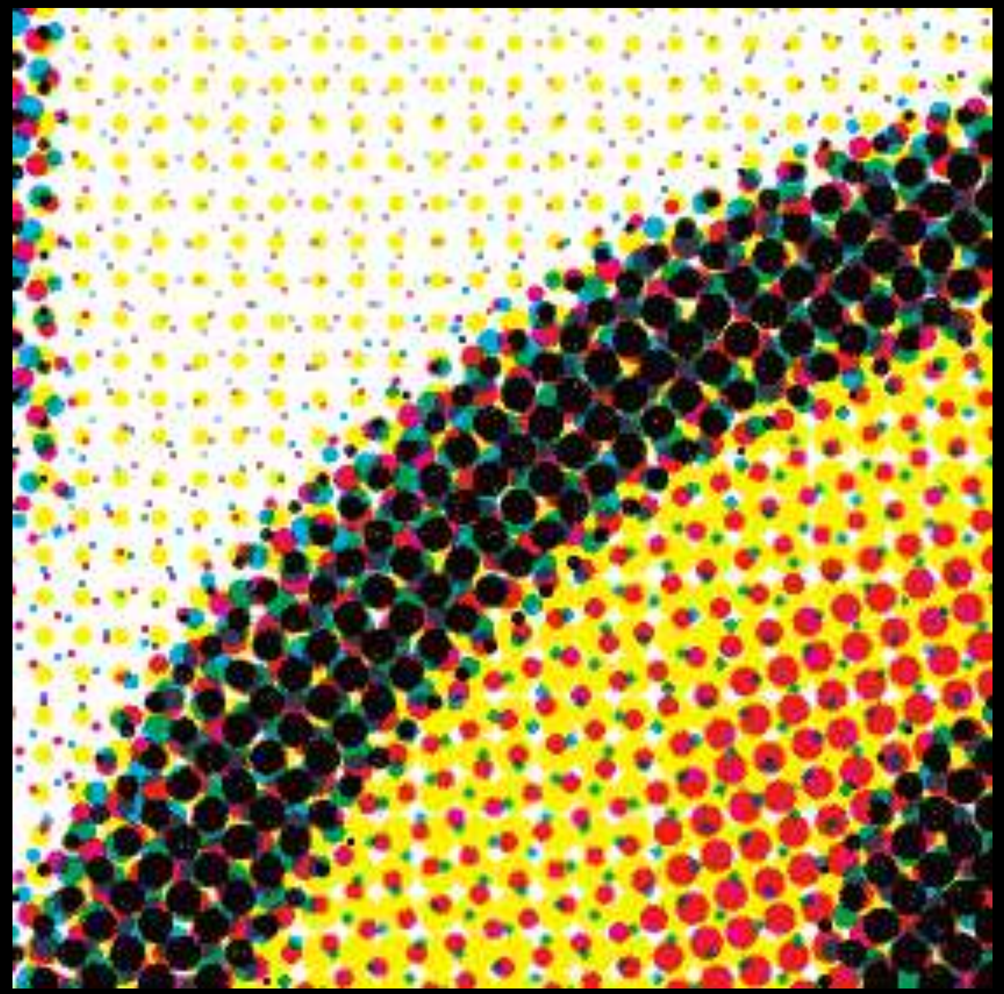
J = pixel index set

What is a Camera?

A pixel has a value space V, where it stores a value.



V=RGB

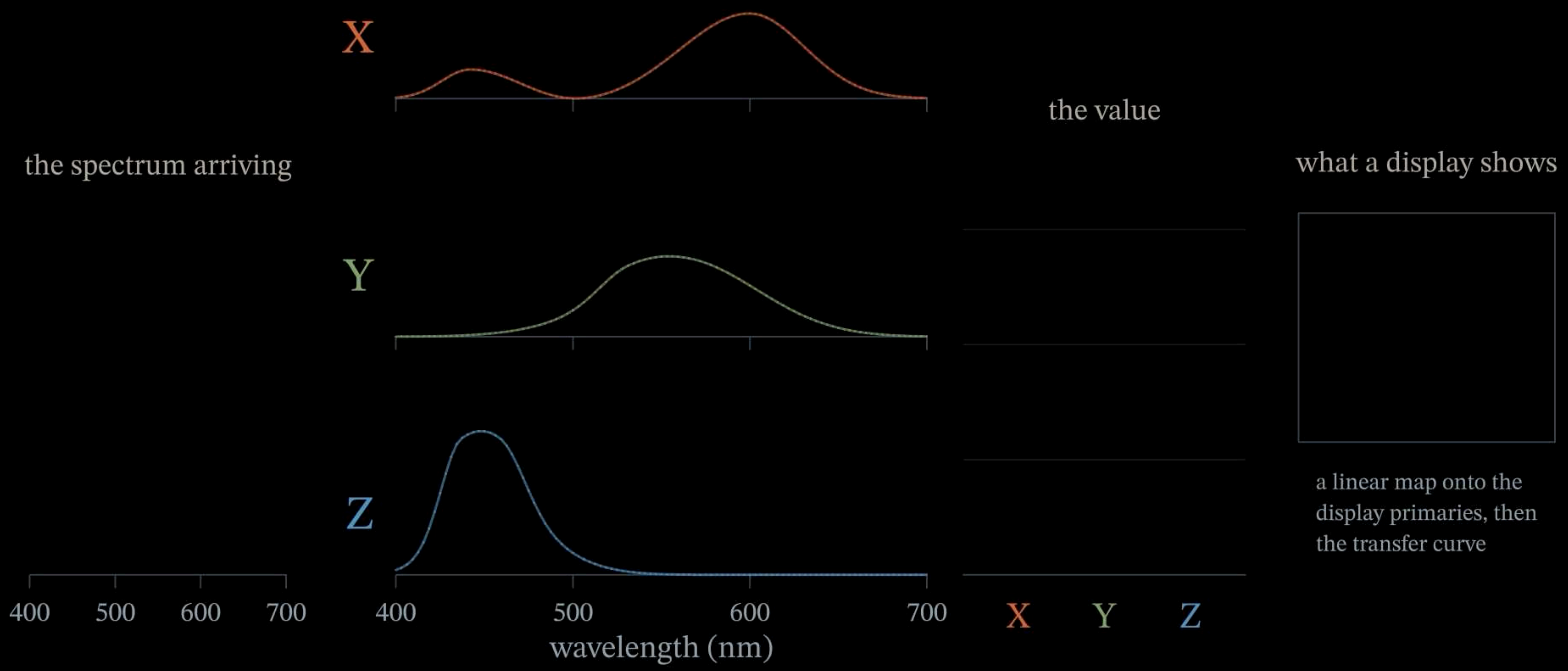


V= CMYK

Thus a single pixel is a map $\mathcal{S} \to V$

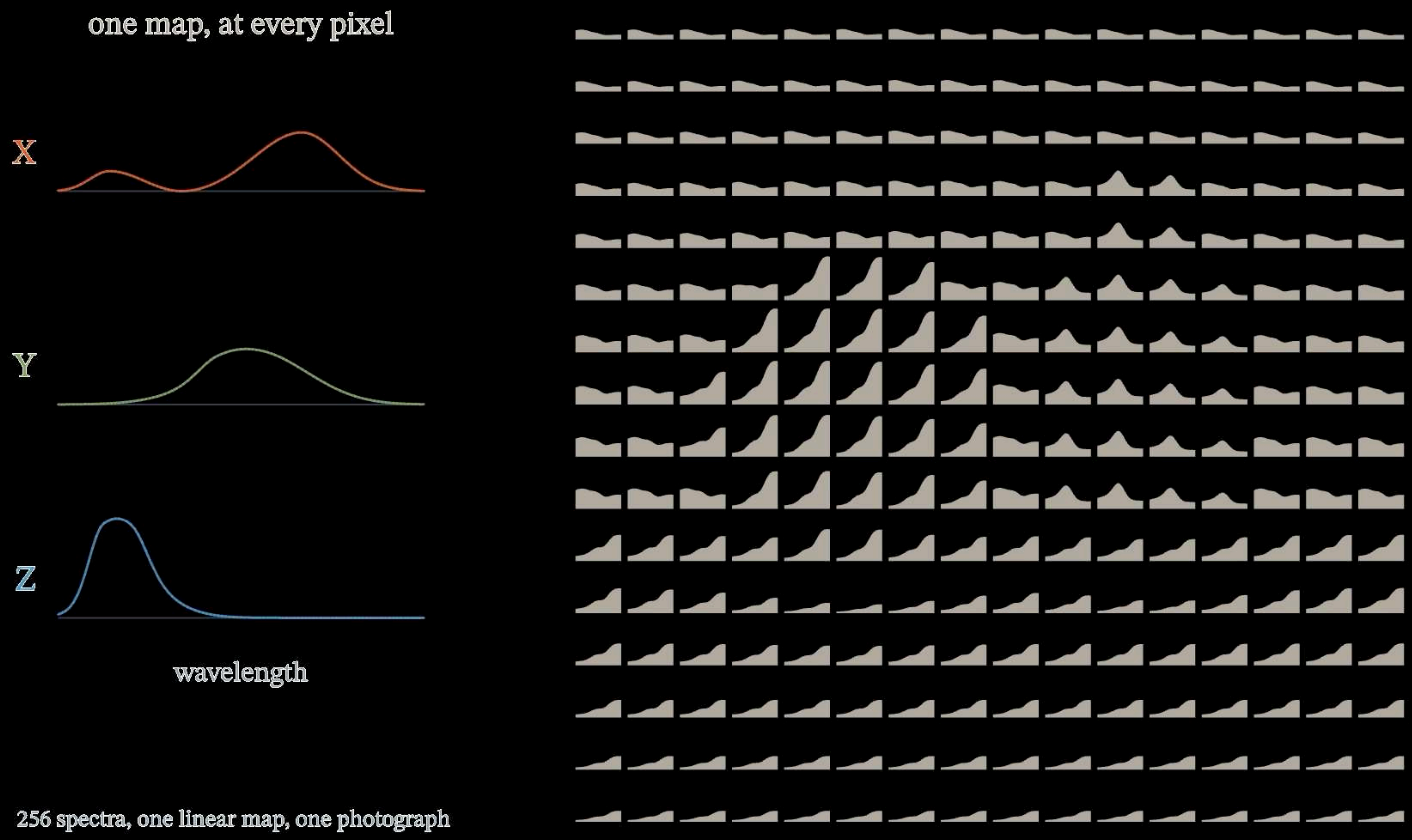
This can be represented as integration against a measure

What is a Camera?



CIE 1931 standard observer - the value is three integrals

What is a Camera?



String a bunch of these together and get an image!

What is a Camera?

The Lens of a camera is a map

$$\mathscr{L} \to \mathscr{S}$$

This admits an integral representation, like the thin lens camera here.

Defining a Photograph

(finally!)

Definition:

A photograph of a scene is the image $I = C(L_\star)$ under a camera, of the equilibrium light field $L_\star = E + T(L_\star)$ for that scene

Part II

Photography is Integration

Defining Photography

Definition:

Photography is the act of taking a photograph.

An algorithm which from T, E and C computes $C(L_\star)$ for $L_\star = E + T(L_\star)$

Solving for the Light Field

We can try to solve for $L_\star$:

$$L_\star = E + T(L_\star)$$

$$L_\star - T(L_\star) = E$$

$$(\mathrm{Id} - T)L_\star = E$$

$$L_\star = (\mathrm{Id} - T)^{-1}E$$

Observation:

$\|T\| < 1$ so $(\mathrm{Id} - T)$ is invertible!

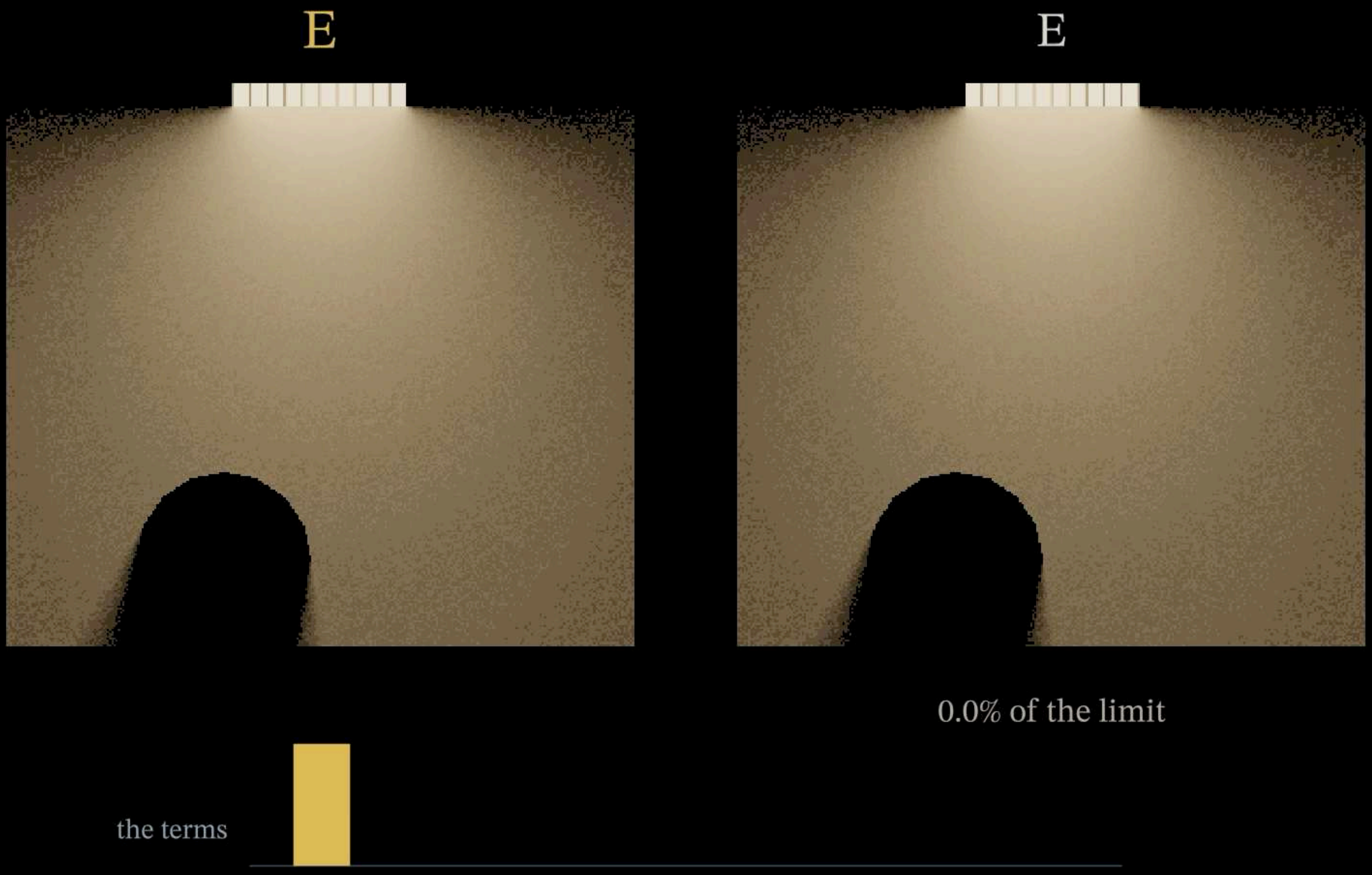
Solving for the Light Field

Inverting $\mathrm{Id} - T$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$$

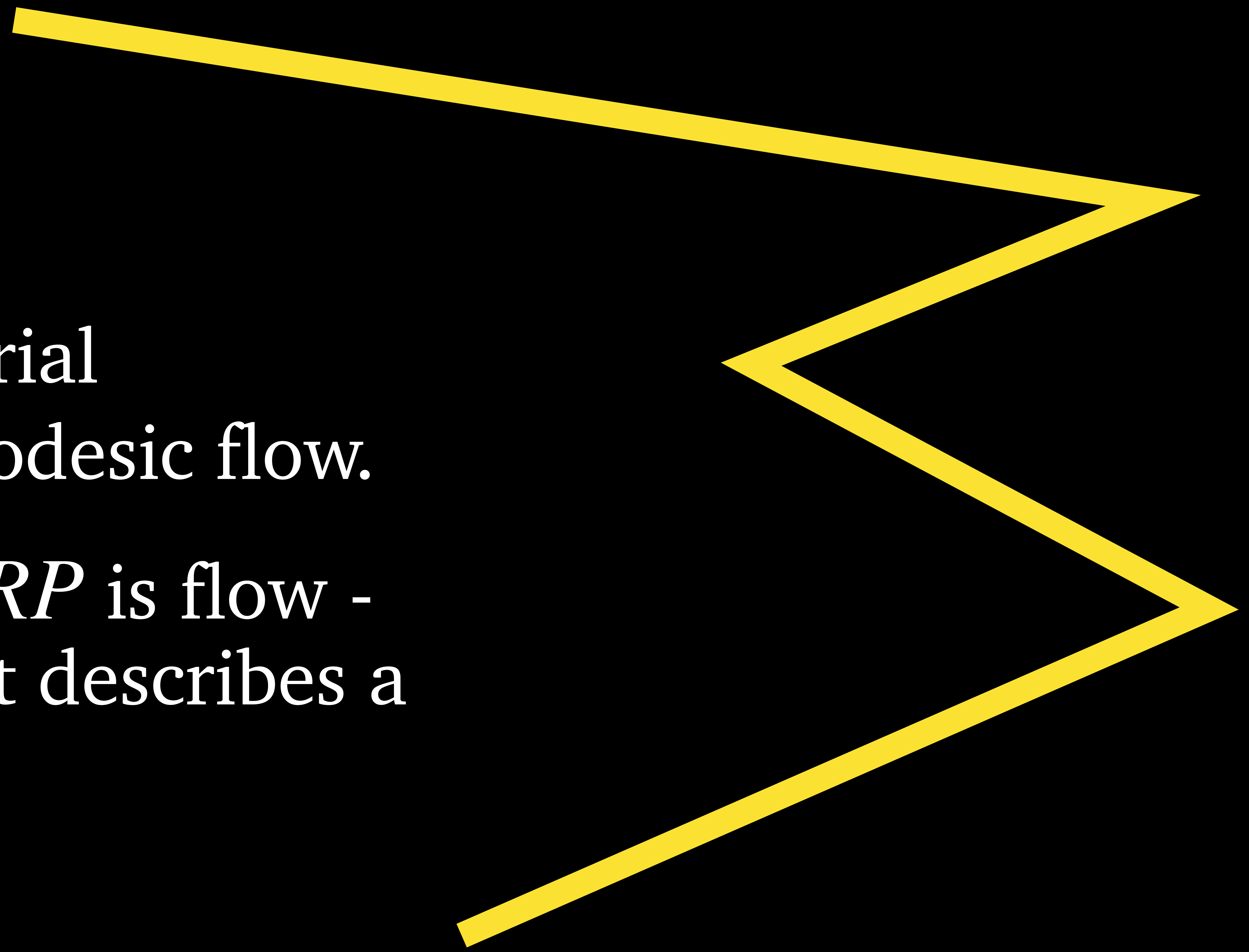
$$(\mathrm{Id} - T)^{-1} = 1 + T + T^2 + T^3 + \cdots$$

Solving for the Light Field



In our discrete toy model, we can try this! And we get the same equalibrium as iteration.

Solving for the Light Field



Interpreting $T^n E$

$T = RP$ where R is a material interaction and P is the geodesic flow.

Thus $T^n = RP \circ RP \circ \cdots \circ RP$ is flow - interact -flow - interact… it describes a path!

Solving for the Light Field

We see this explicitly by expanding out T as an integral

$$(TE)(x_0) = \int_{\mathcal{X}} K(x_0, dx_1)\, E(x_1)\,.$$

Do it again: the two integrals nest

$$(T^2E)(x_0) = \int_{\mathcal{X}} K(x_0, dx_1) \int_{\mathcal{X}} K(x_1, dx_2)\, E(x_2)\,.$$

Solving for the Light Field

After n generations

$$(T^n E)(x_0) = \int_{\mathcal{X}} K(x_0, dx_1) \int_{\mathcal{X}} K(x_1, dx_2) \cdots \int_{\mathcal{X}} K(x_{n-1}, dx_n)\, E(x_n)\,.$$

If we're careful everything checks out, and these turn into a single integral over $\mathcal{X}^n$

$$(T^n E)(x_0) = \int_{\mathcal{X}^n} K(x_0, dx_1) K(x_1, dx_2) \cdots K(x_{n-1}, dx_n)\, E(x_n)\,.$$

Solving for the Light Field

$$(T^n E)(x_0) = \int_{\mathcal{X}^n} K(x_0, dx_1)K(x_1, dx_2)\cdots K(x_{n-1}, dx_n)\; E(x_n)\,.$$

A point of $\mathcal{X}^n$ is a sequence of points $(x_0, x_1, x_2, \ldots x_{n-1})$.
This is a path in our scene!

The integrand is a *measure* ν_n on this space of paths

$$(T^n E)(x_0) = \int_{\mathcal{X}^n} \nu_n\,.$$

Solving for the Light Field

Putting it all together:

$$L_\star = (\mathrm{Id} - T)^{-1} E$$

$$= \sum_{n \geq 0} T^n E$$

$$= \sum_{n \geq 0} \int_{\mathcal{X}^n} \nu_n$$

$$= \int_{\bigcup_{n \geq 0} \mathcal{X}^n} \sum_{n \geq 0} \nu_n$$

$$= \int_{\text{Finite Paths}} \nu$$

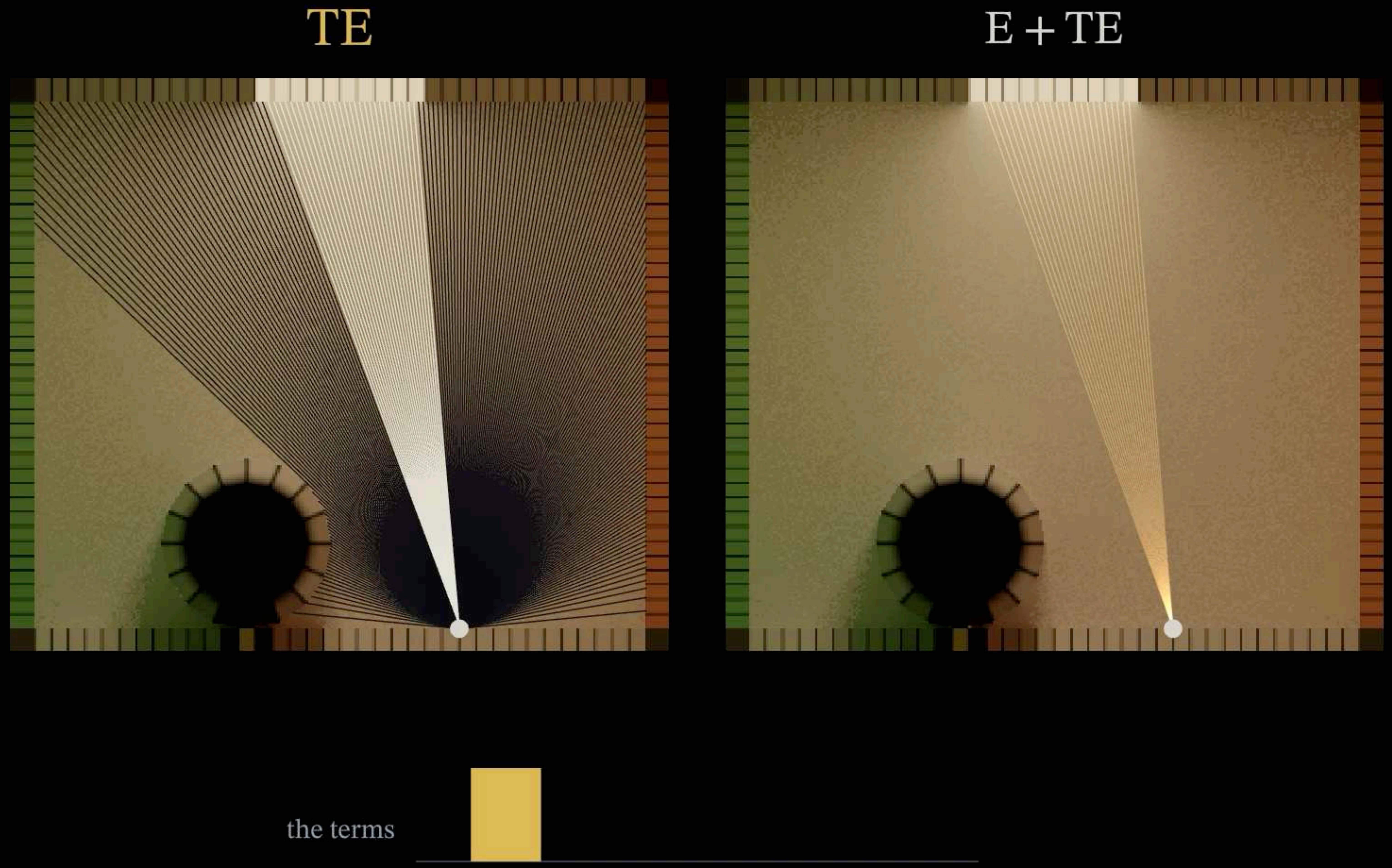
Solving for the Light Field

Putting it all together:

> *The equilibrium light field at any point, and in any direction is computable as the total measure of all finite paths with that starting data, with respect to a measure defined by the scene.*

> Equilibrium distributions are the solution to a path integral!

Solving for the Light Field



The geometric sum interpreted as integration over the stratified space of paths.

Calculating Big Integrals

Both halves of the photography problem are integrals!

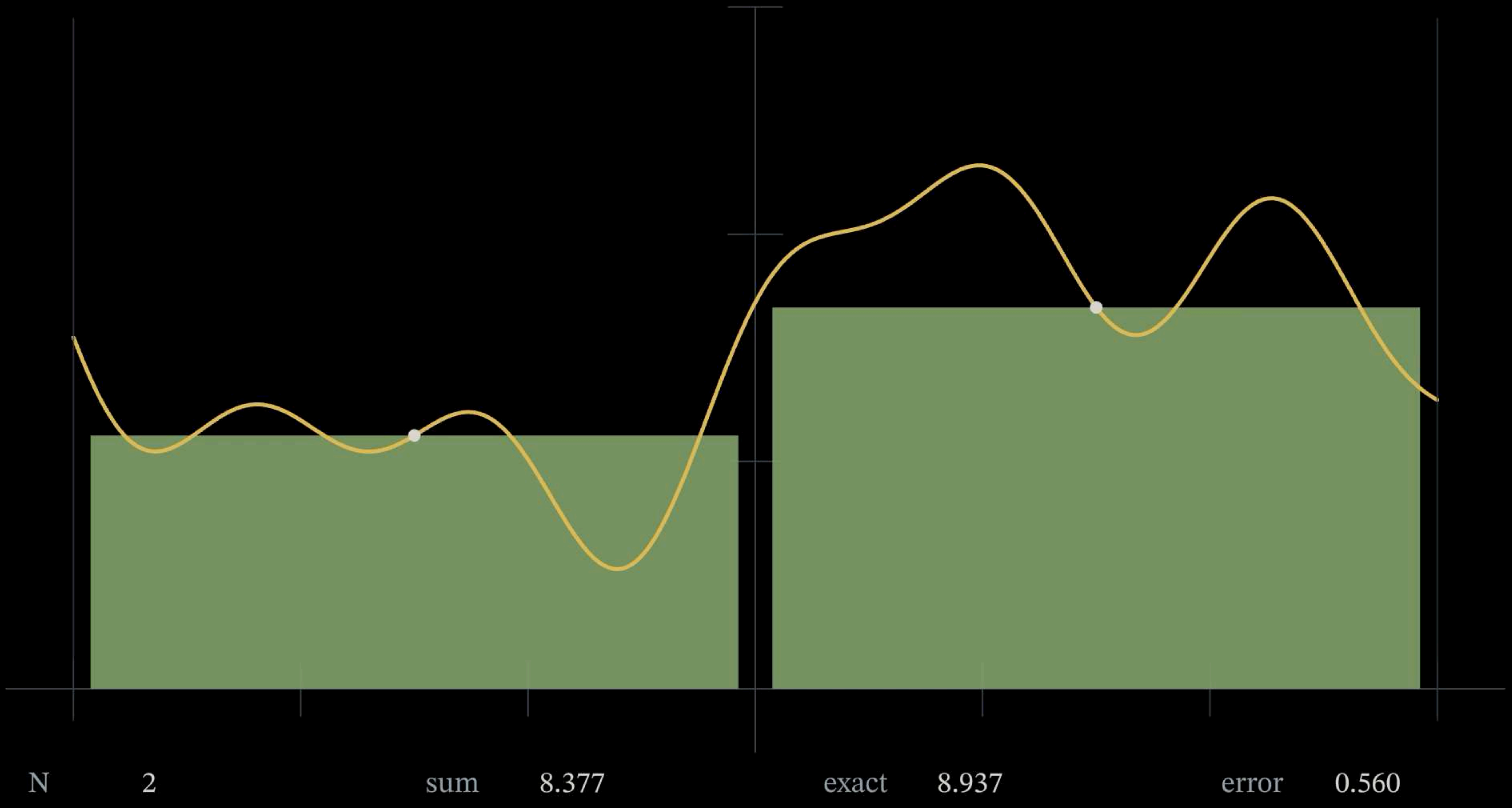
The **Camera** is a low dimensional integral

$$I_c(j) = \int_{P_j \times A} \int_{\Lambda} r_c(\lambda)\, L_\star\Big(\kappa(u,a)\Big)(d\lambda)\, w(u,a)\, dA(u)\, dA(a)\,.$$

The equilibrium light field is a path integral

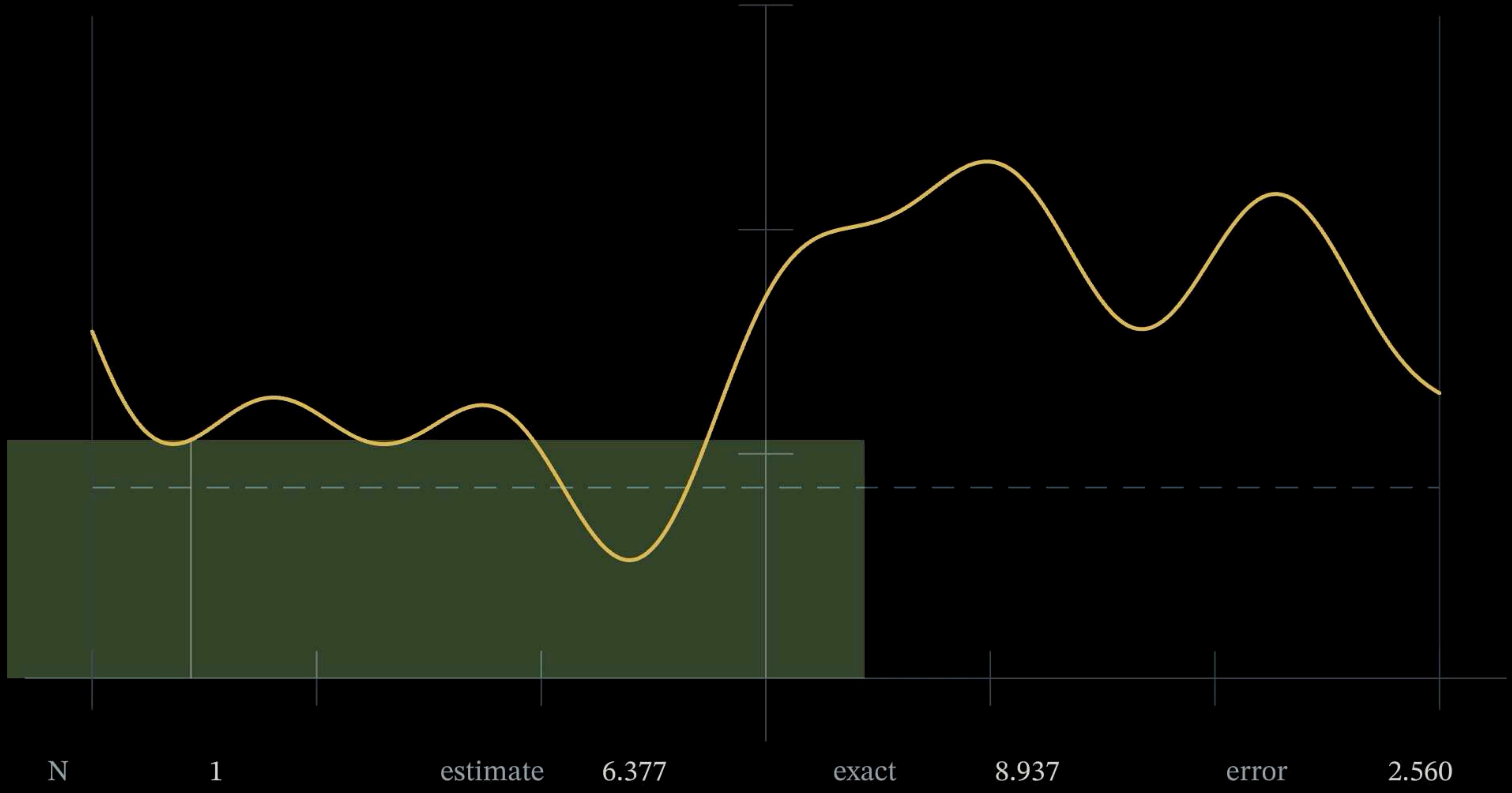
$$L_\star(x_0) = \int_{\text{Finite Paths})} d\boldsymbol{\nu}_{x_0} = \sum_{n \geq 0} \int_{\mathrm{X}^n} \boldsymbol{\nu}_{x_0,n}$$

Calculating Big Integrals



The one way we all know how to numerically integrate!

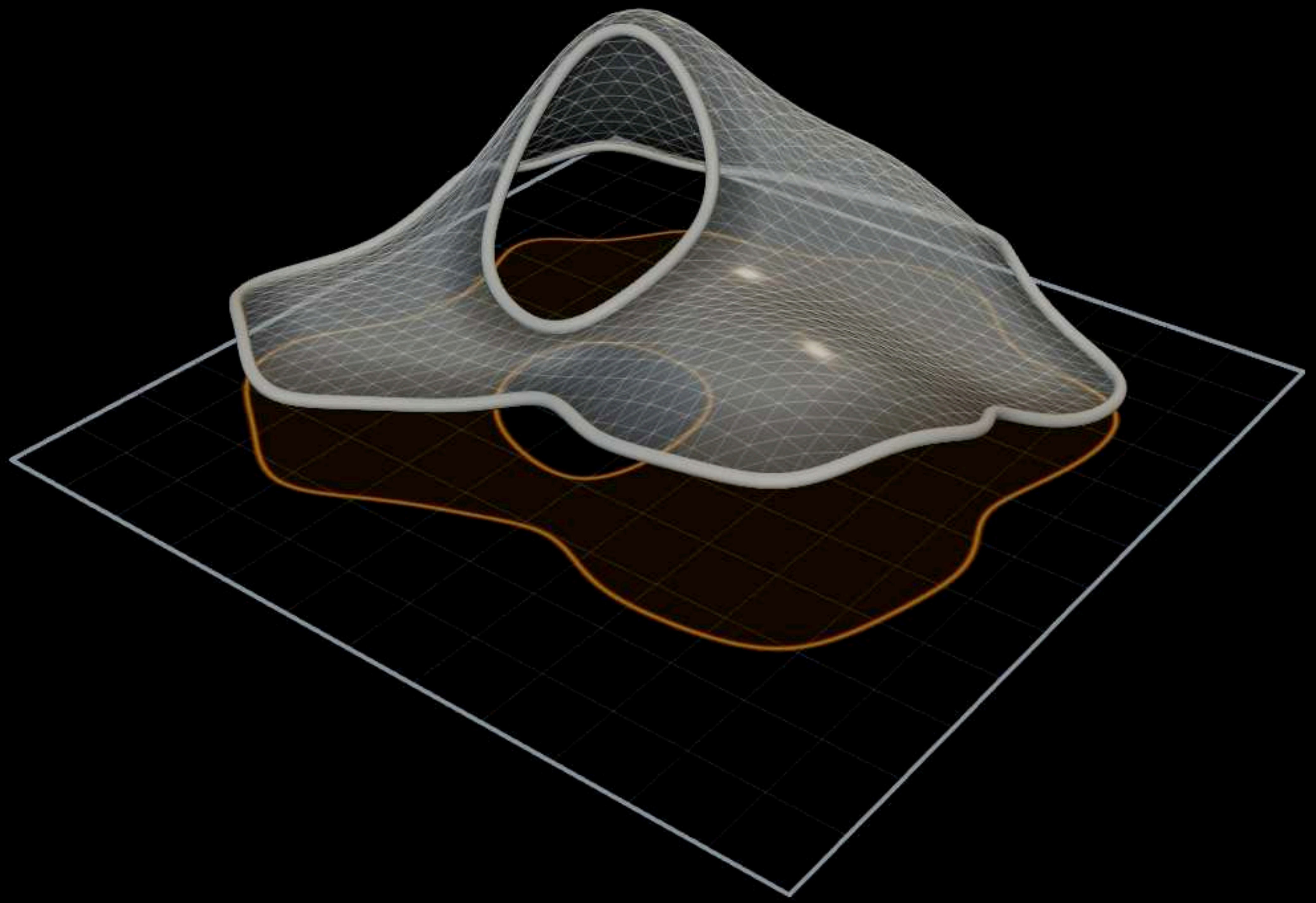
Calculating Big Integrals



A more flexible approach: draw samples randomly instead of from a lattice

Calculating Big Integrals

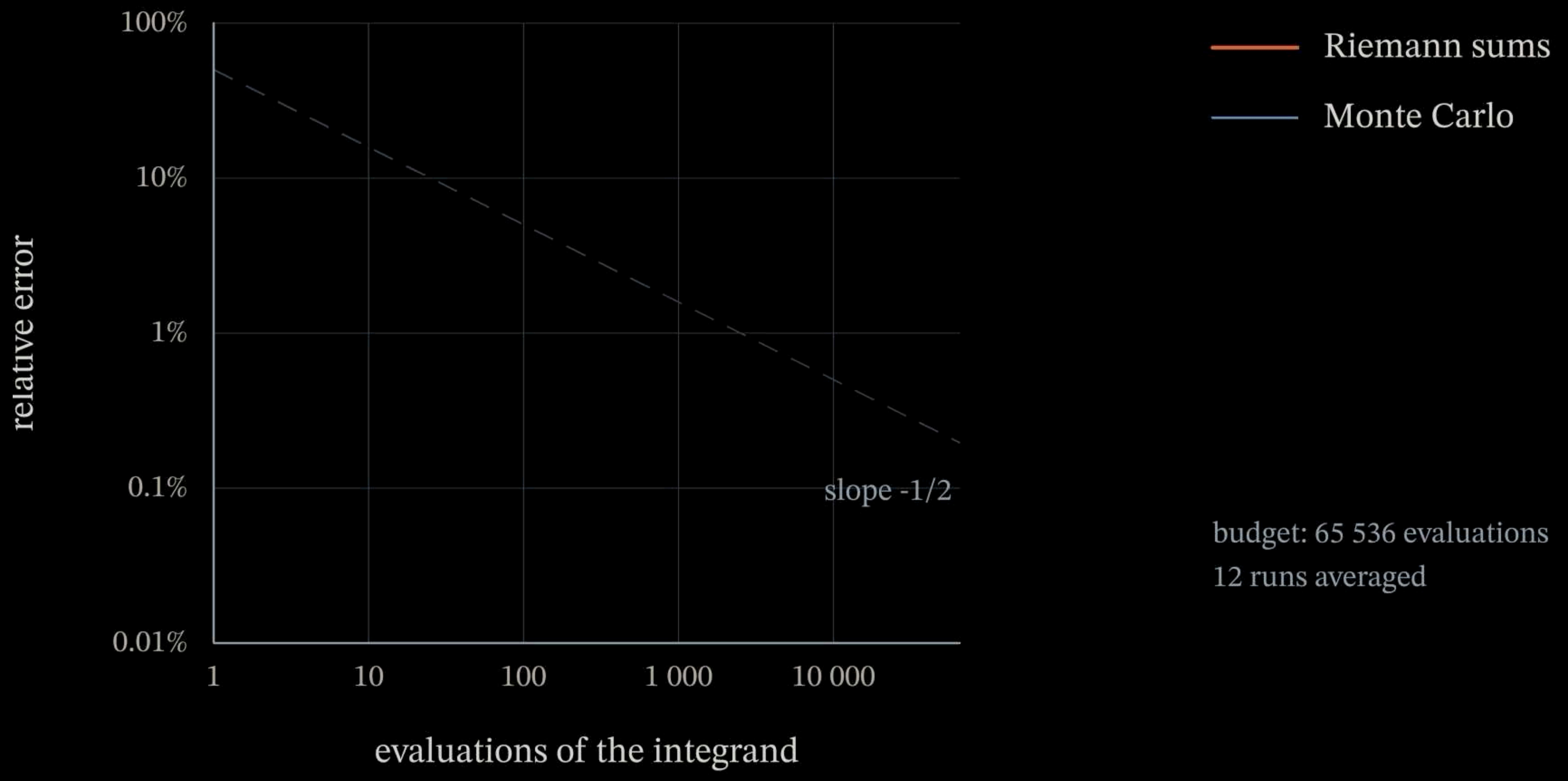
samples	0
region area	16.342
estimate	-
exact	13.649



This can handle higher dimensional functions with irregular domains

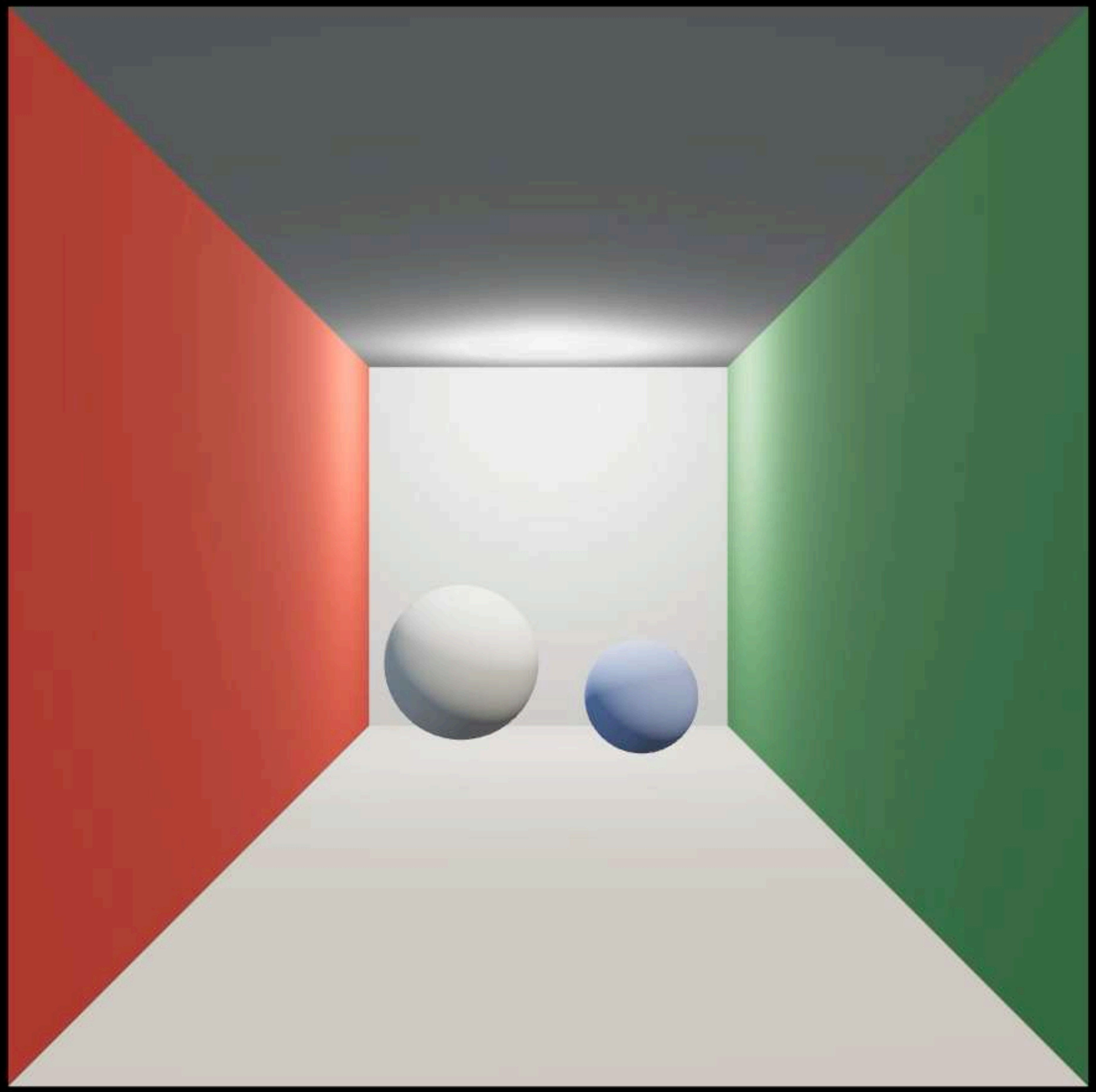
Monte Carlo

Calculating Big Integrals



Amazing:
the rate of convergence goes with $\sqrt{N}$, *independent of the domain's dimension.*

Calculating Big Integrals

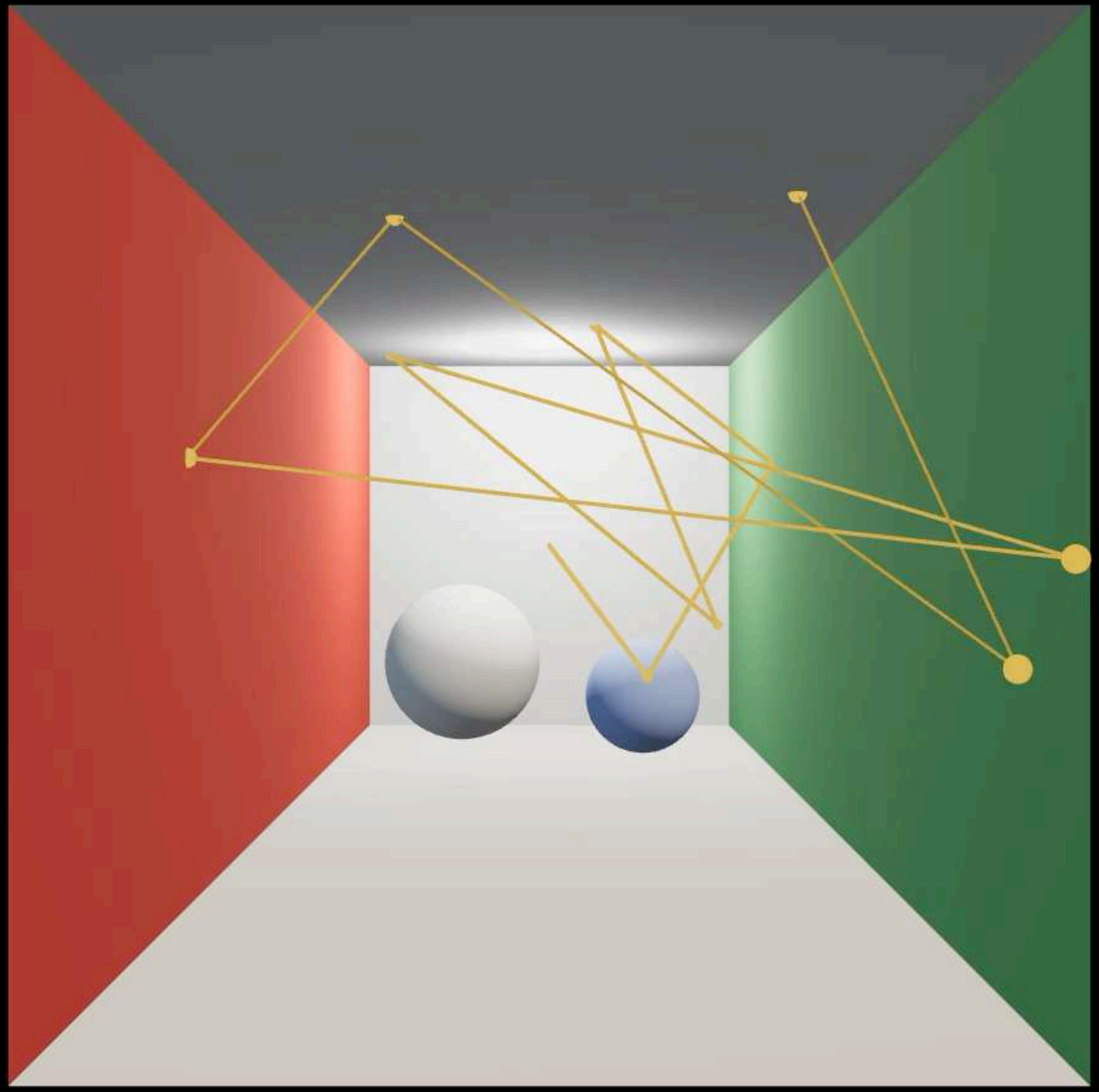


Monte Carlo Path Integrals

Sample a path length n randomly.

Then sample a random direction at each vertex

Calculating Big Integrals



Monte Carlo Path Integrals

Sample a path length n randomly.

Then sample a random direction at each vertex

Our First Photograph

We need a procedure: given C, E, T compute

$$I_\star = C(L_\star) \quad \text{for}$$

$$L_\star = E + T(L_\star)$$

Monte Carlo teaches us, if we can write this *entire thing as one big integral* we can compute it!

Our First Photograph

Both the camera and the
light field are integrals.

The image is a *composition*

$$I = \int_{\Lambda} \int_{\mathscr{C}} \int_{\mathscr{P}} (\text{ScaryStuff})$$

Wavelength

Camera Parameters

Finite Paths

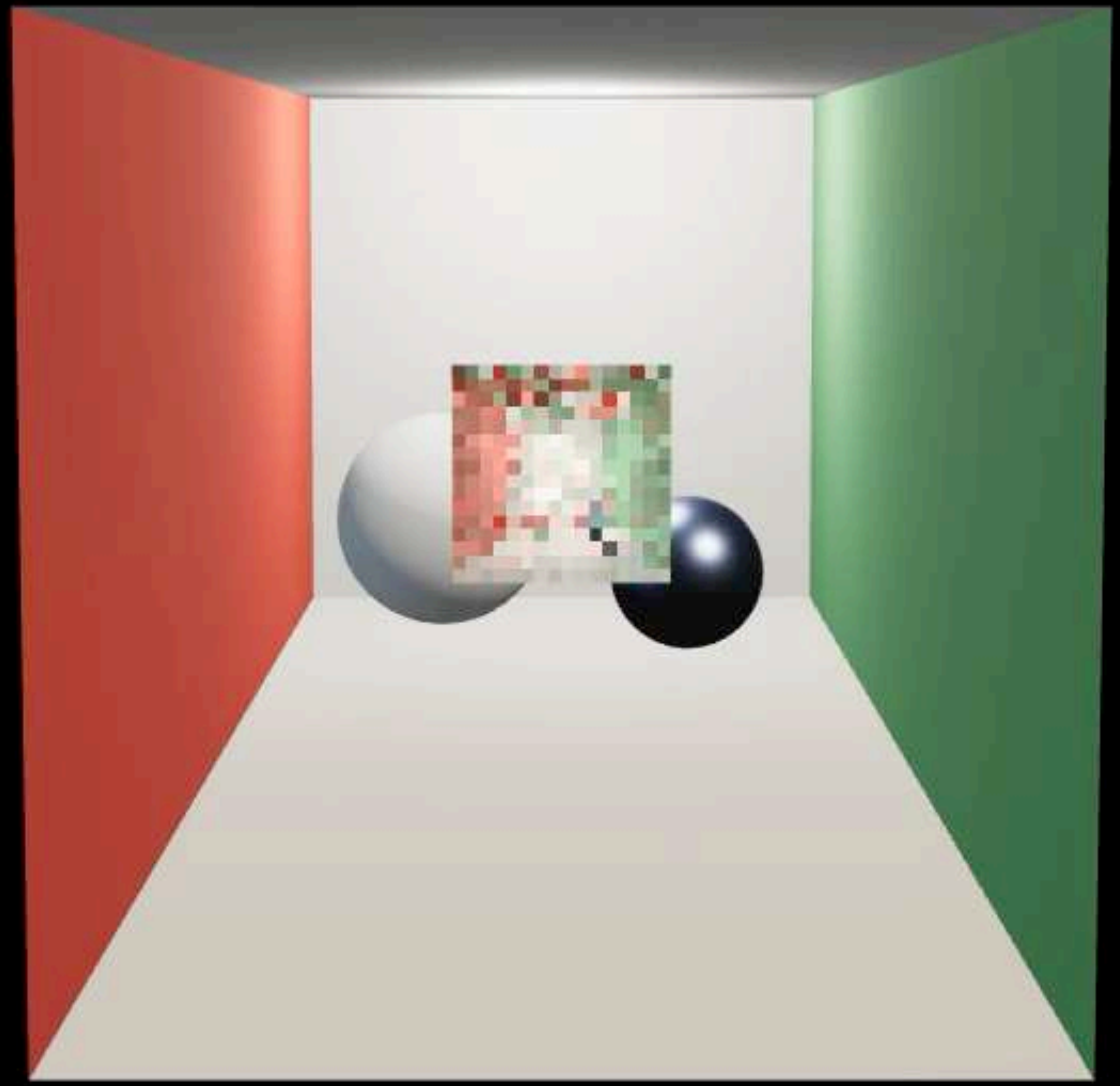
Our First Photograph

Both the camera and the
light field are integrals.

The image is a *composition*

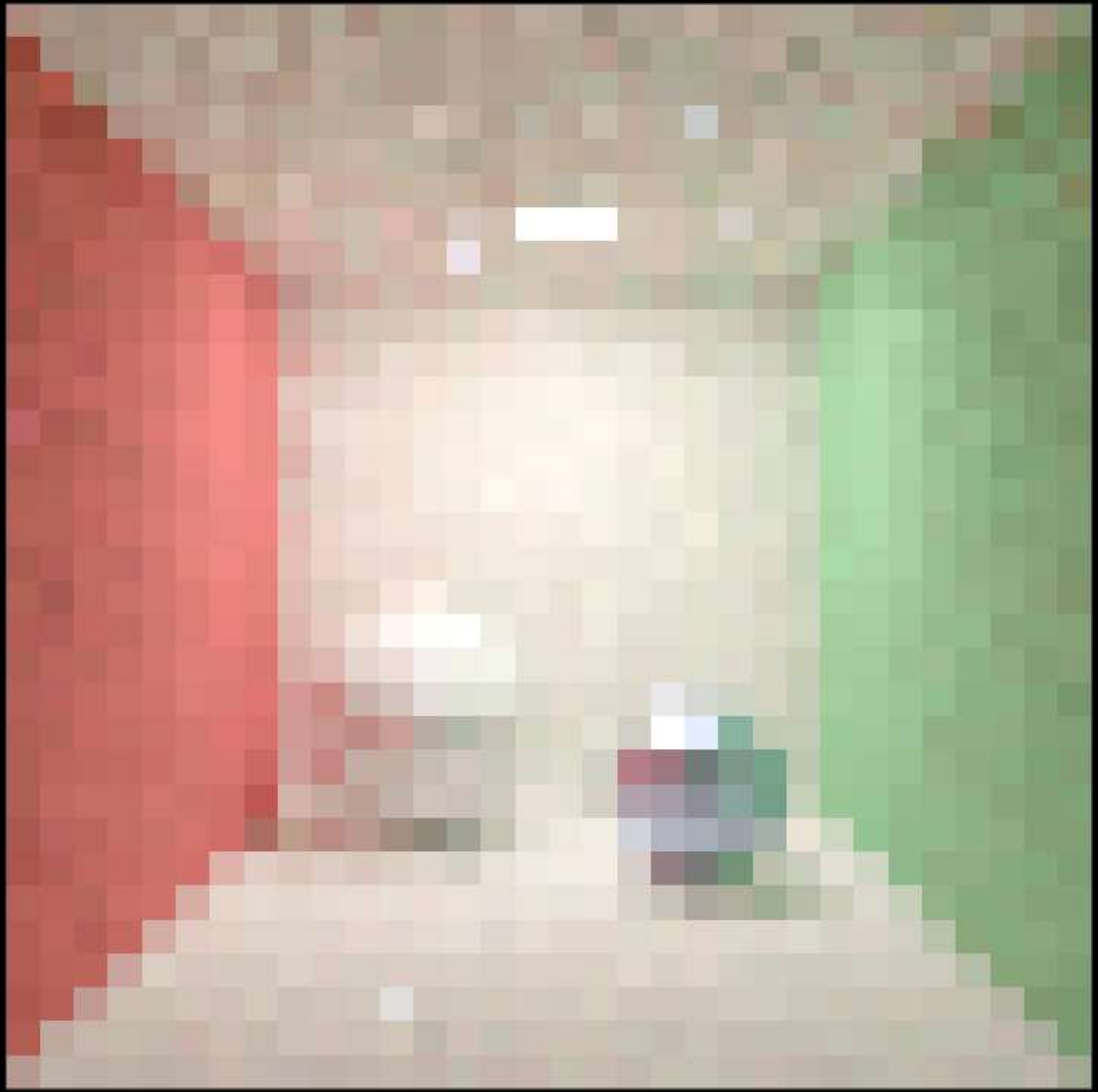
$$I_c(j) \;=\; \sum_{n\geq 1} \int_{P_j}\int_A\int_\Lambda\int_{\Omega_n(x_0)} r_c(\lambda)\; w(u,a)\; K_\lambda(x_0, dx_1)\, K_\lambda(x_1, dx_2)\cdots K_\lambda(x_{n-2}, dx_{n-1})\; E(x_{n-1}, d\lambda)\; dA(u)\, dA(a),$$

Our First Photograph

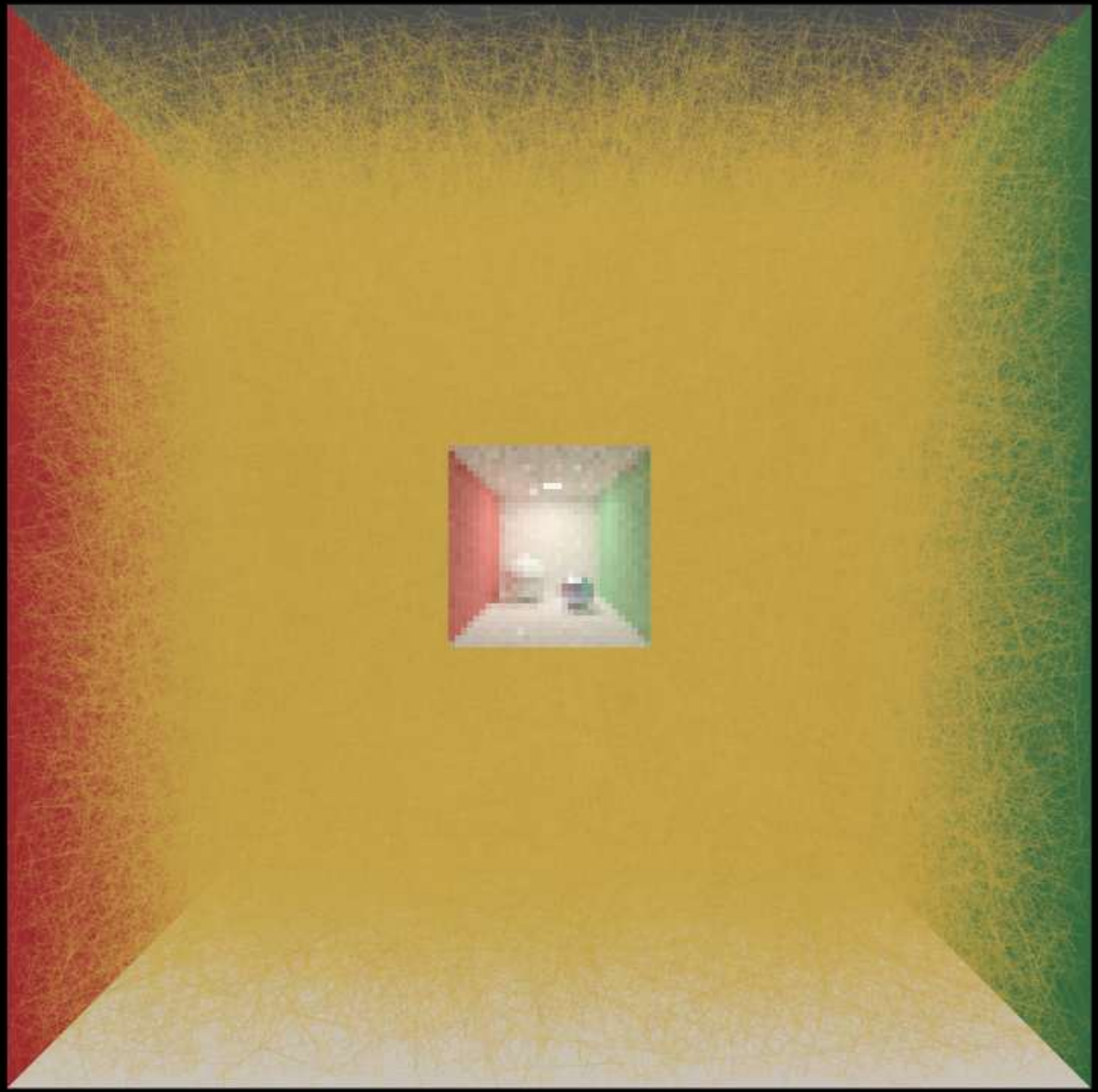


Hundreds of rays per pixel per frame

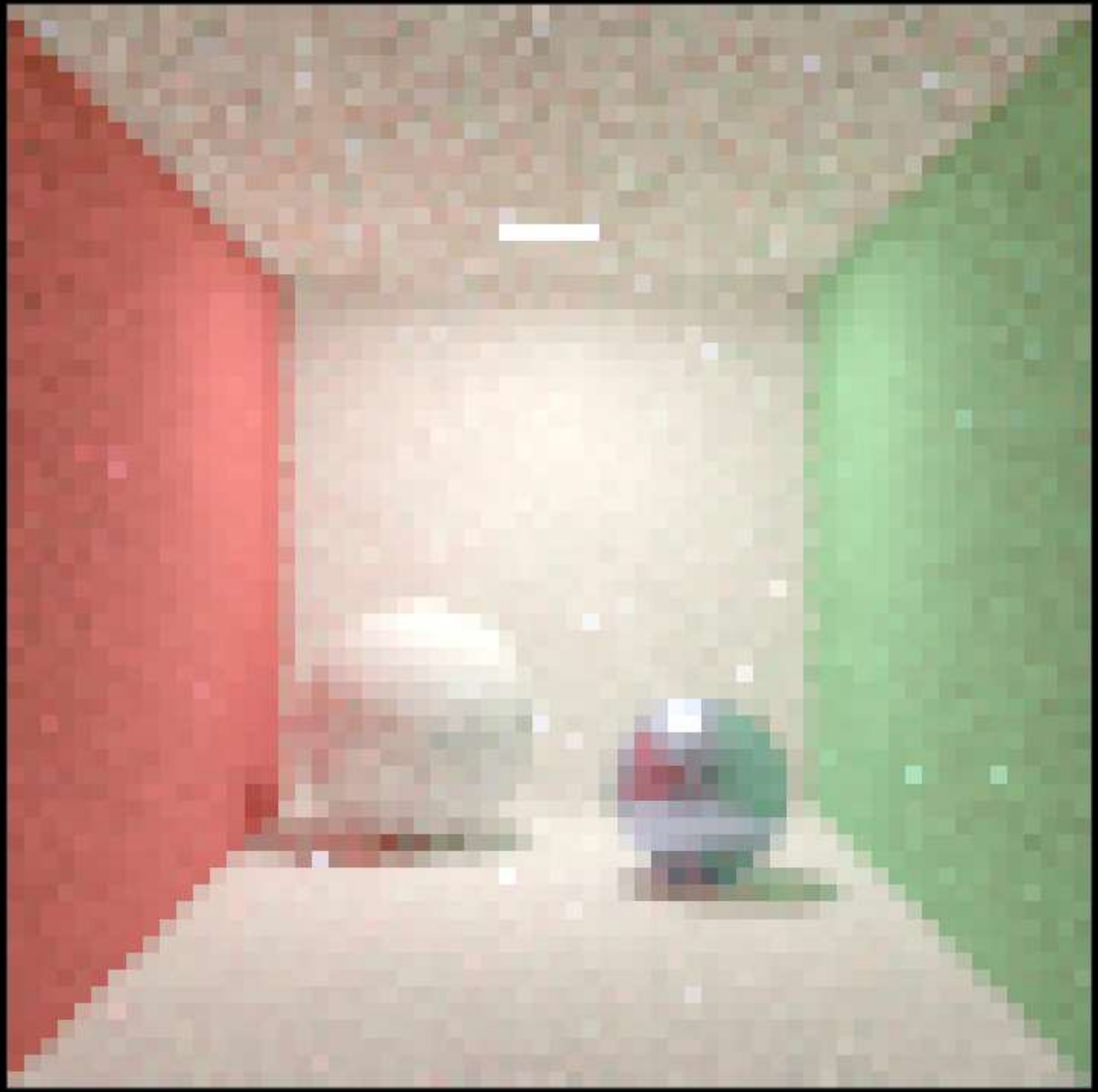
Our First Photograph



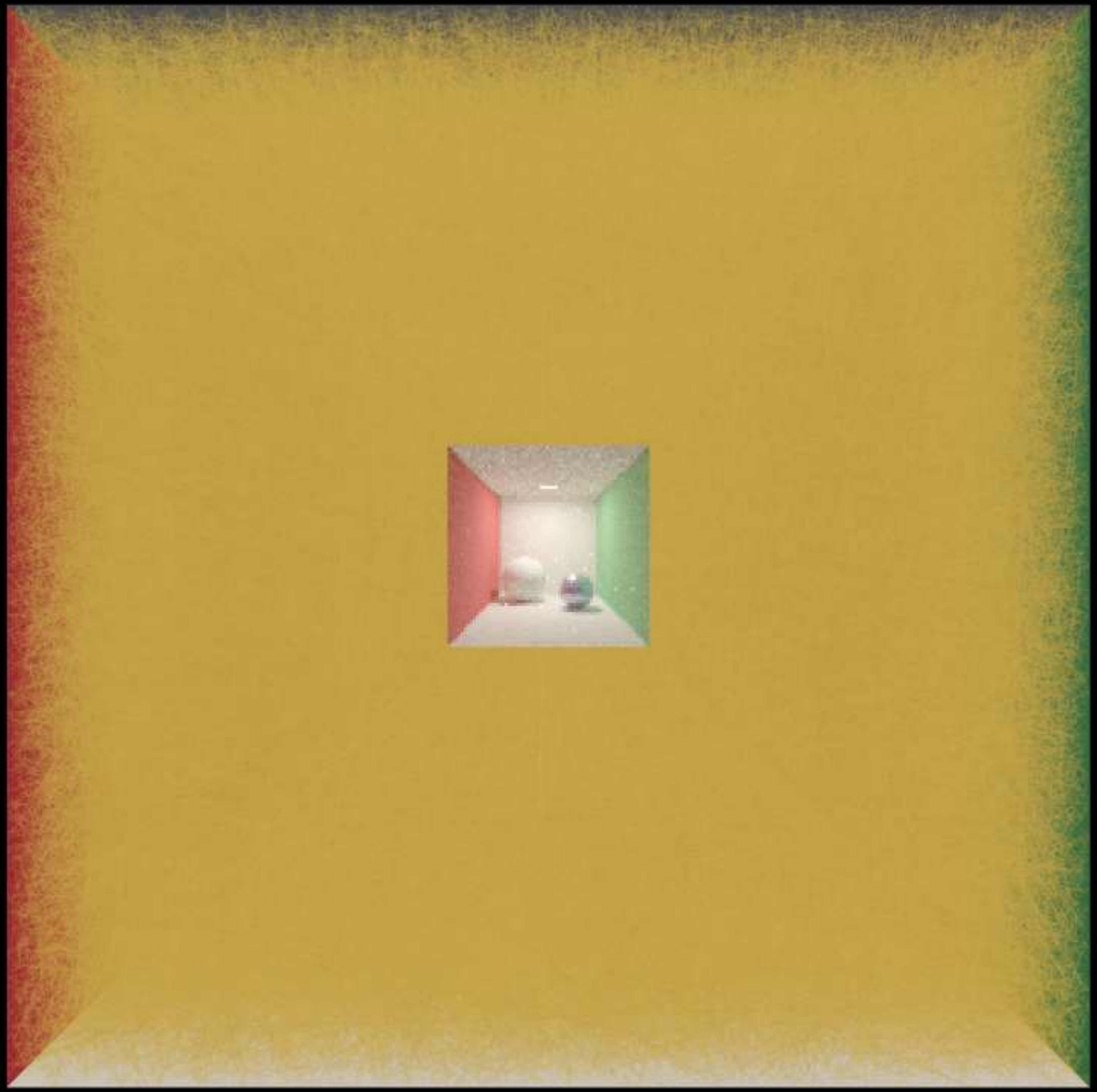
32
x
32



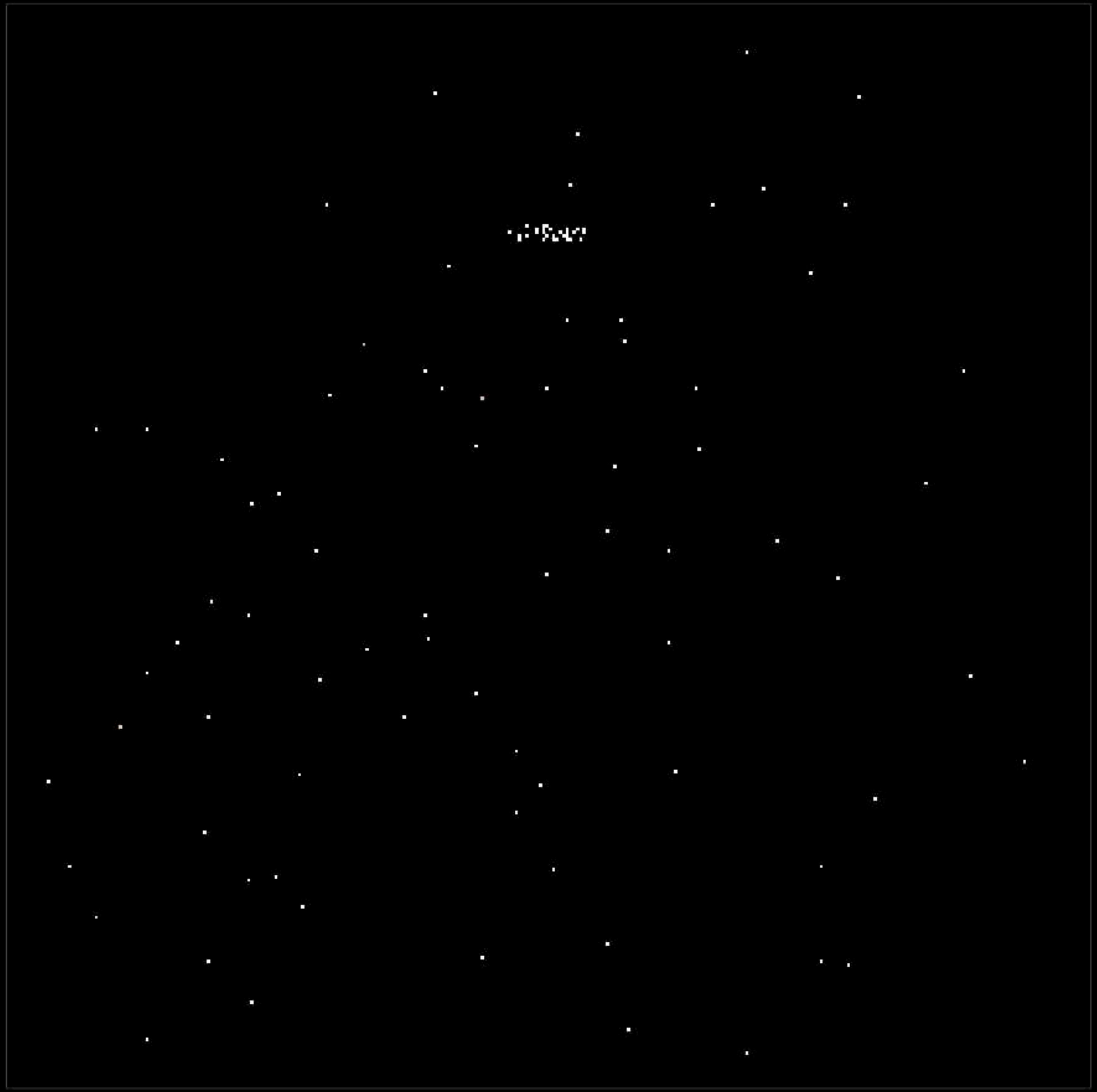
Our First Photograph



64
x
64



Our First Photograph



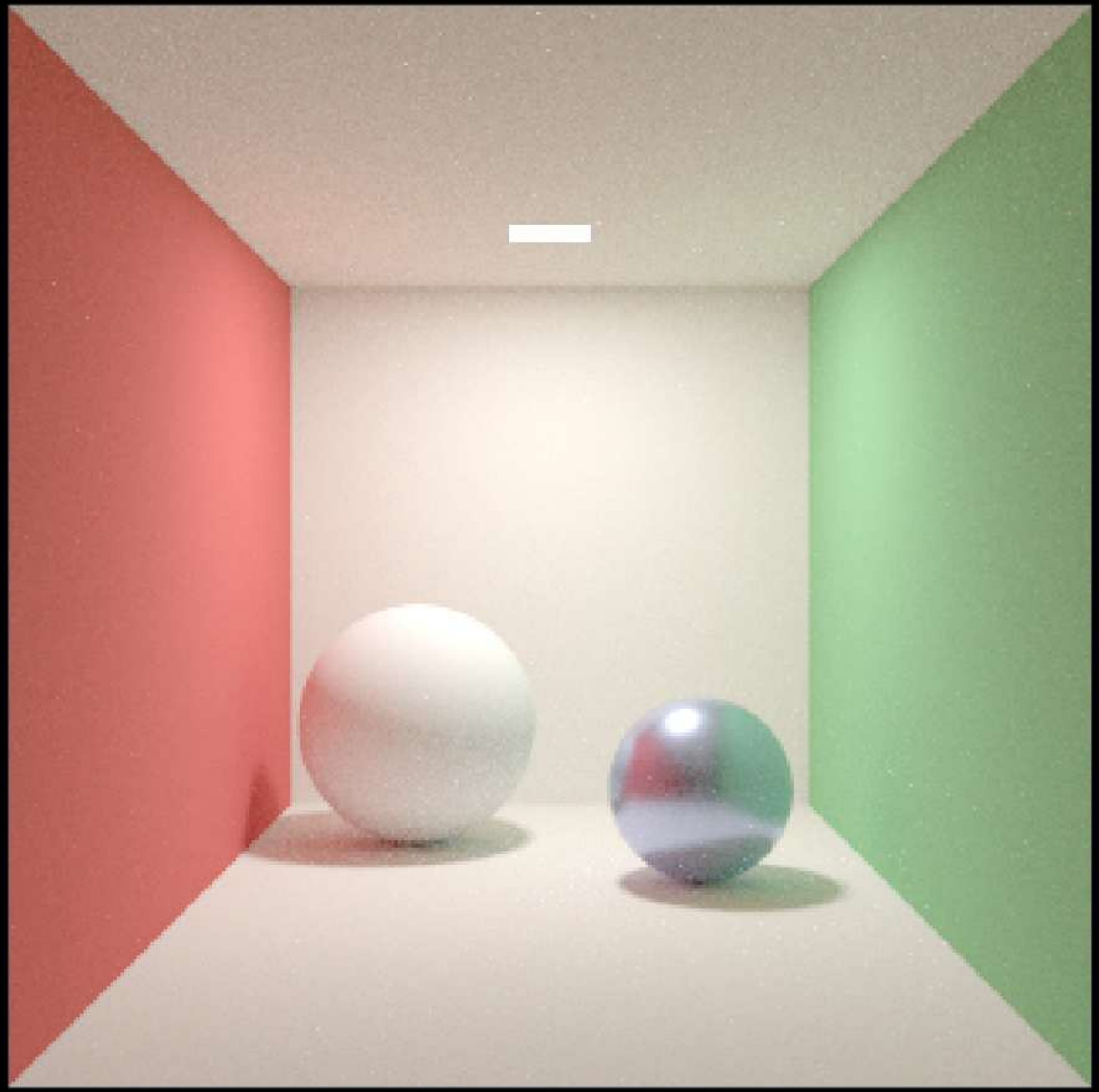
pt-naive · 1 sample per pixel

One Eternity Later

Our First Photograph

320x320

10,000
Paths per Pixel



1,024,000,000
Rays Computed

Thanks!