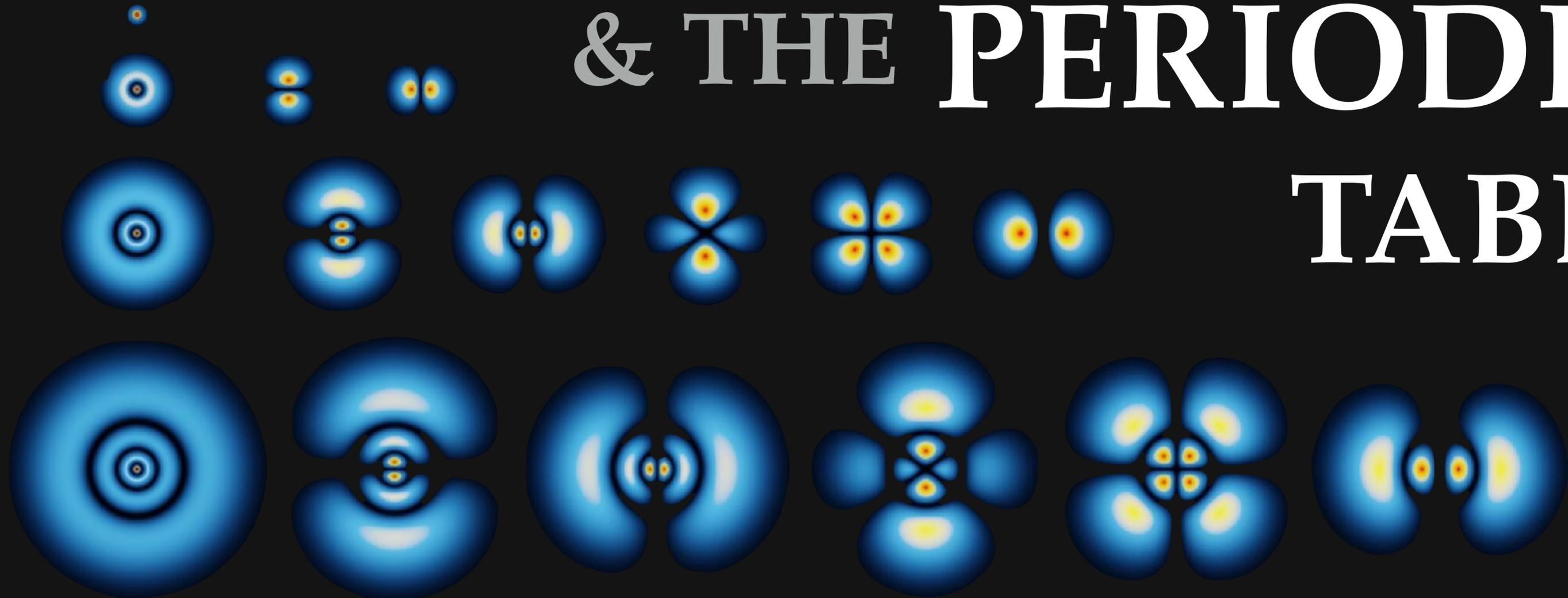


INFINITE DIMENSIONAL



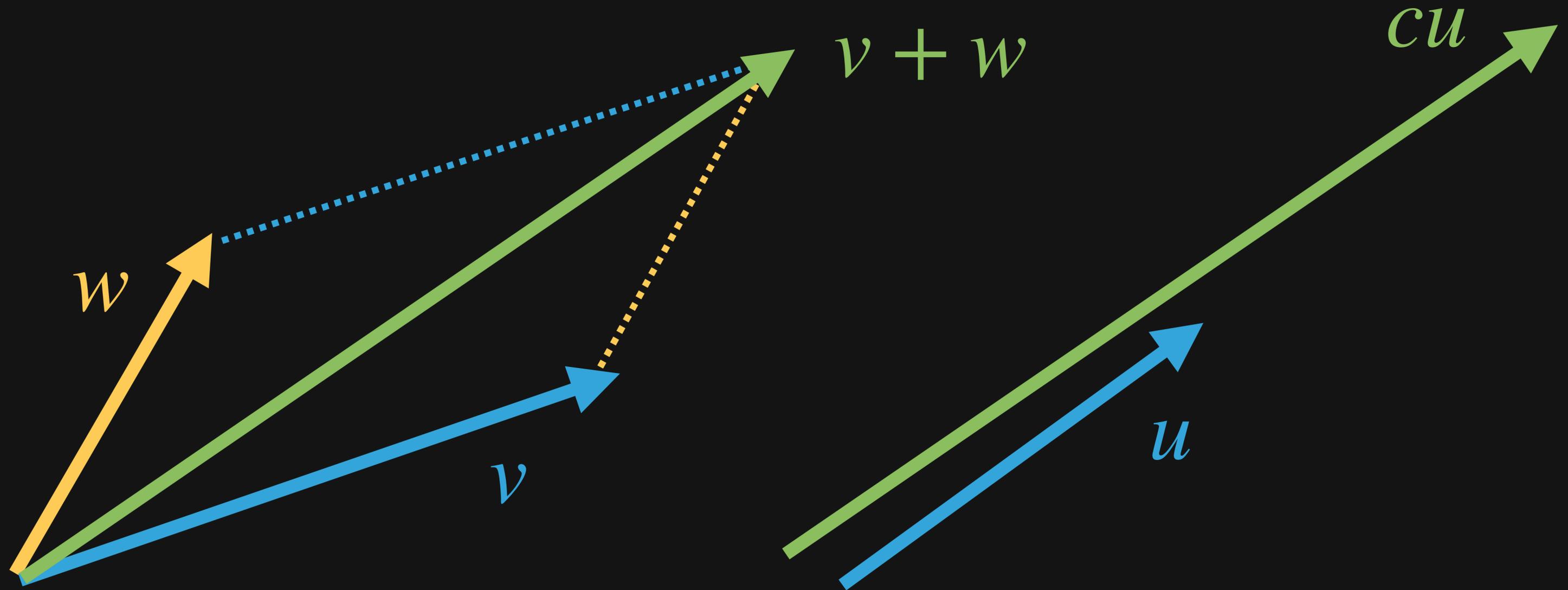
& THE PERIODIC TABLE

TABLE



LINEAR ALGEBRA REVIEW:

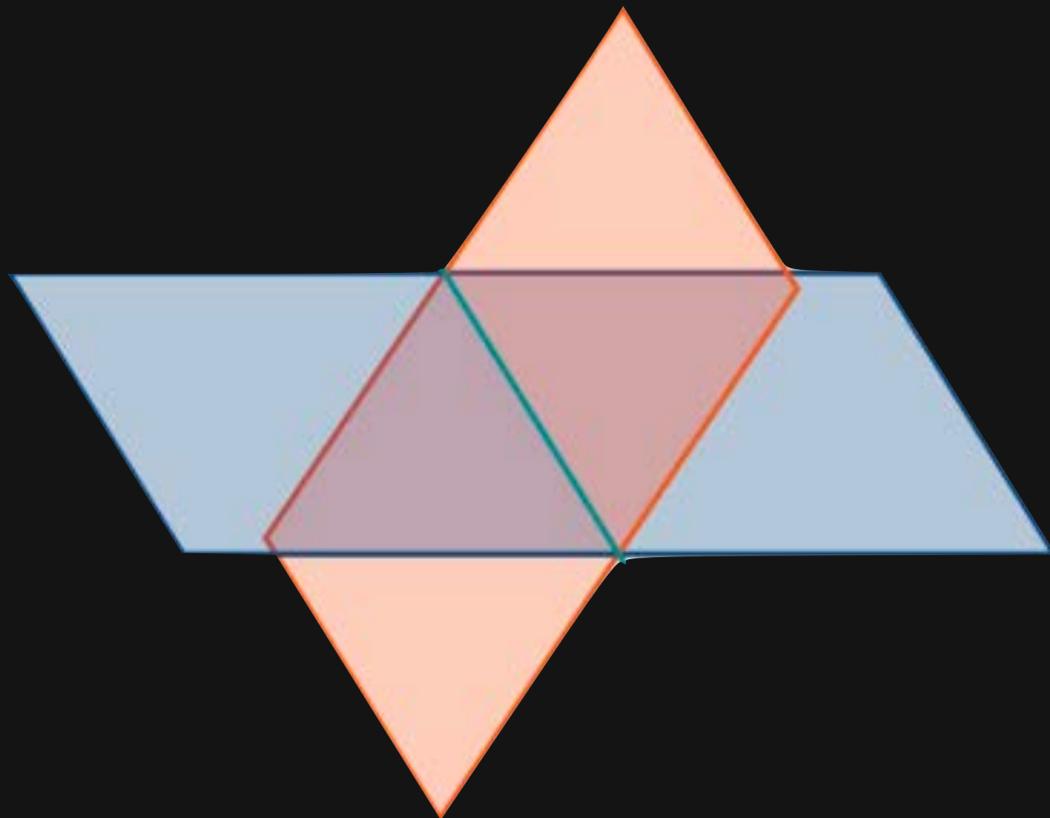
Vectors are things you can add and stretch.



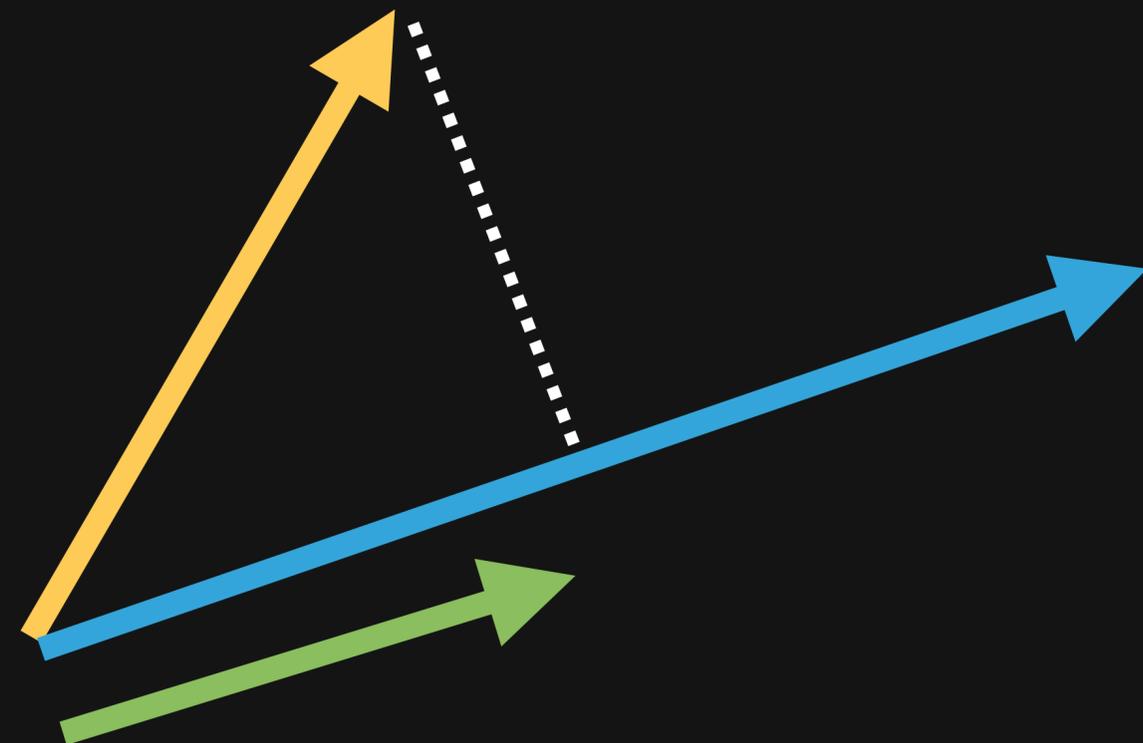
LINEAR ALGEBRA REVIEW:

Once you have some vectors, you can think about them geometrically.

Linear Subspaces

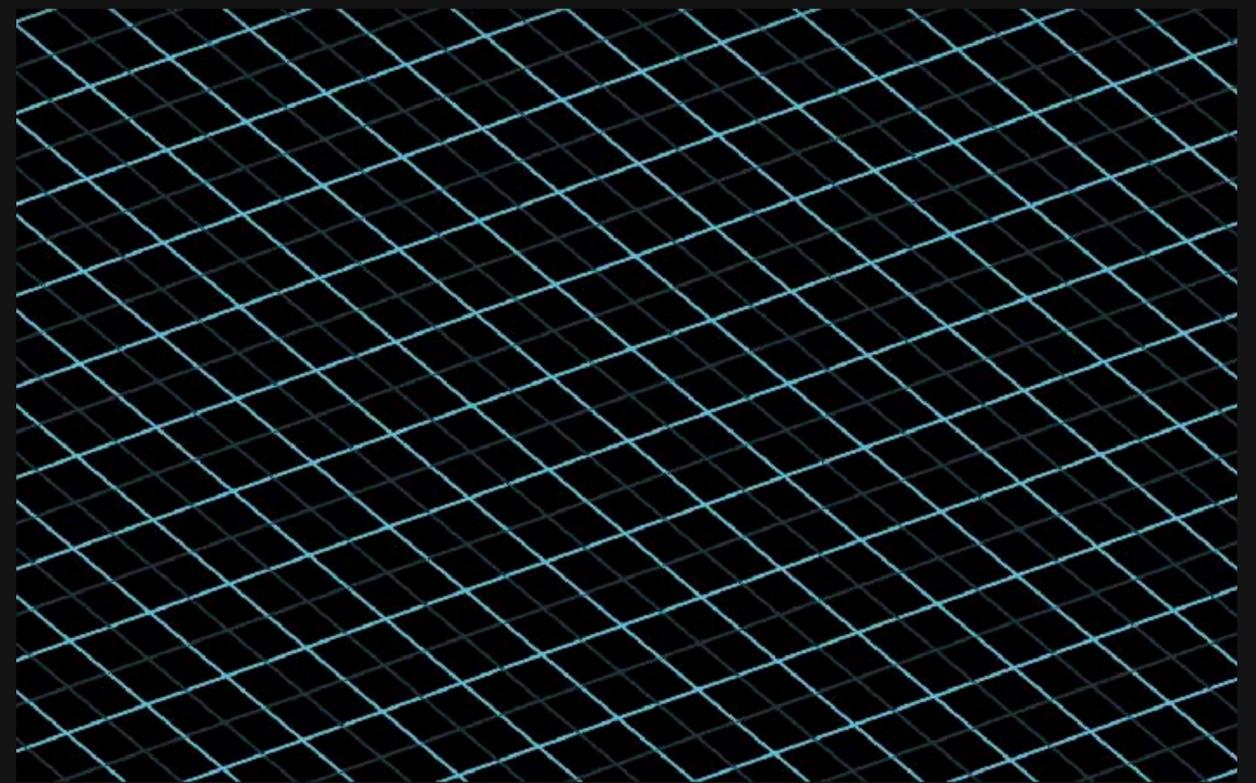
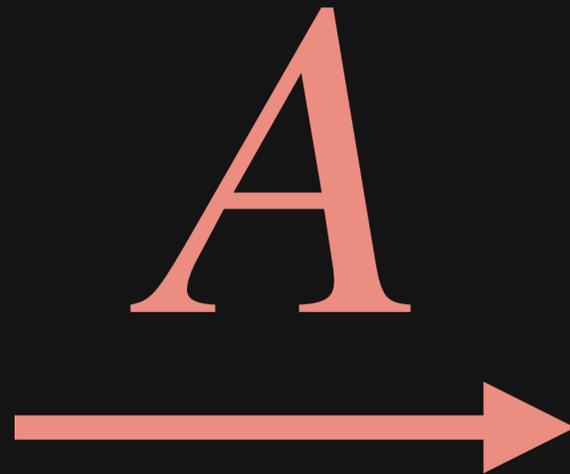
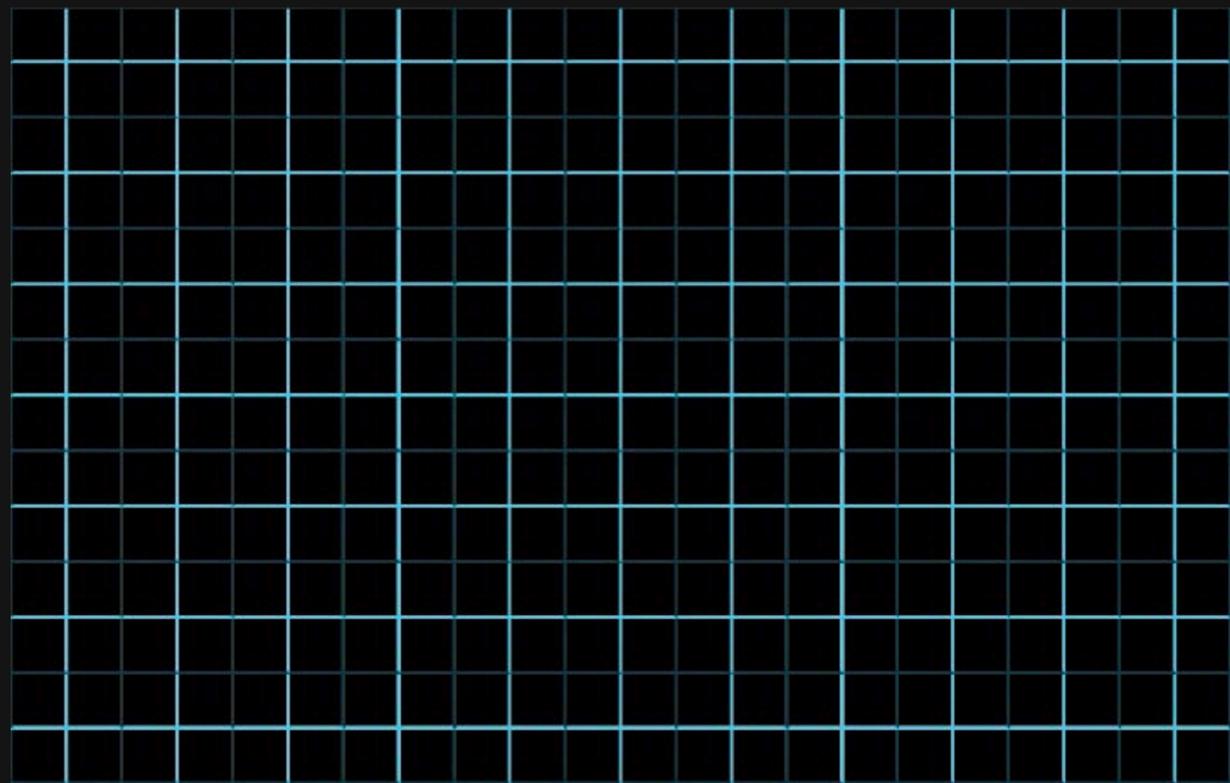


Projections



LINEAR ALGEBRA REVIEW:

Linear transformations take one vector space and send it to another **respecting addition and scalar multiplication.**

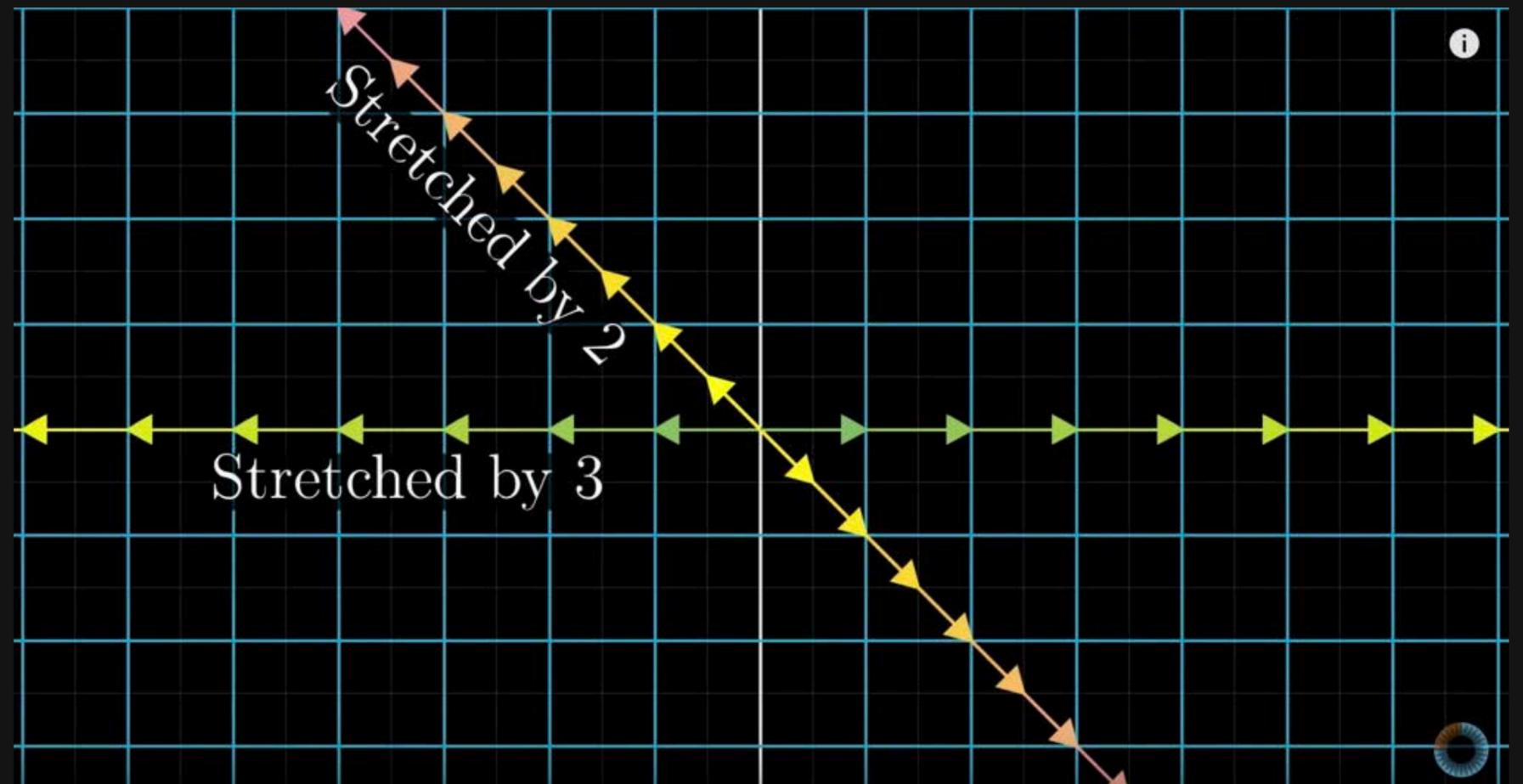


LINEAR ALGEBRA REVIEW:

Eigenvectors give the **directions** in which it is **easy to understand** a particular linear transformation.

$$Av = \lambda v$$

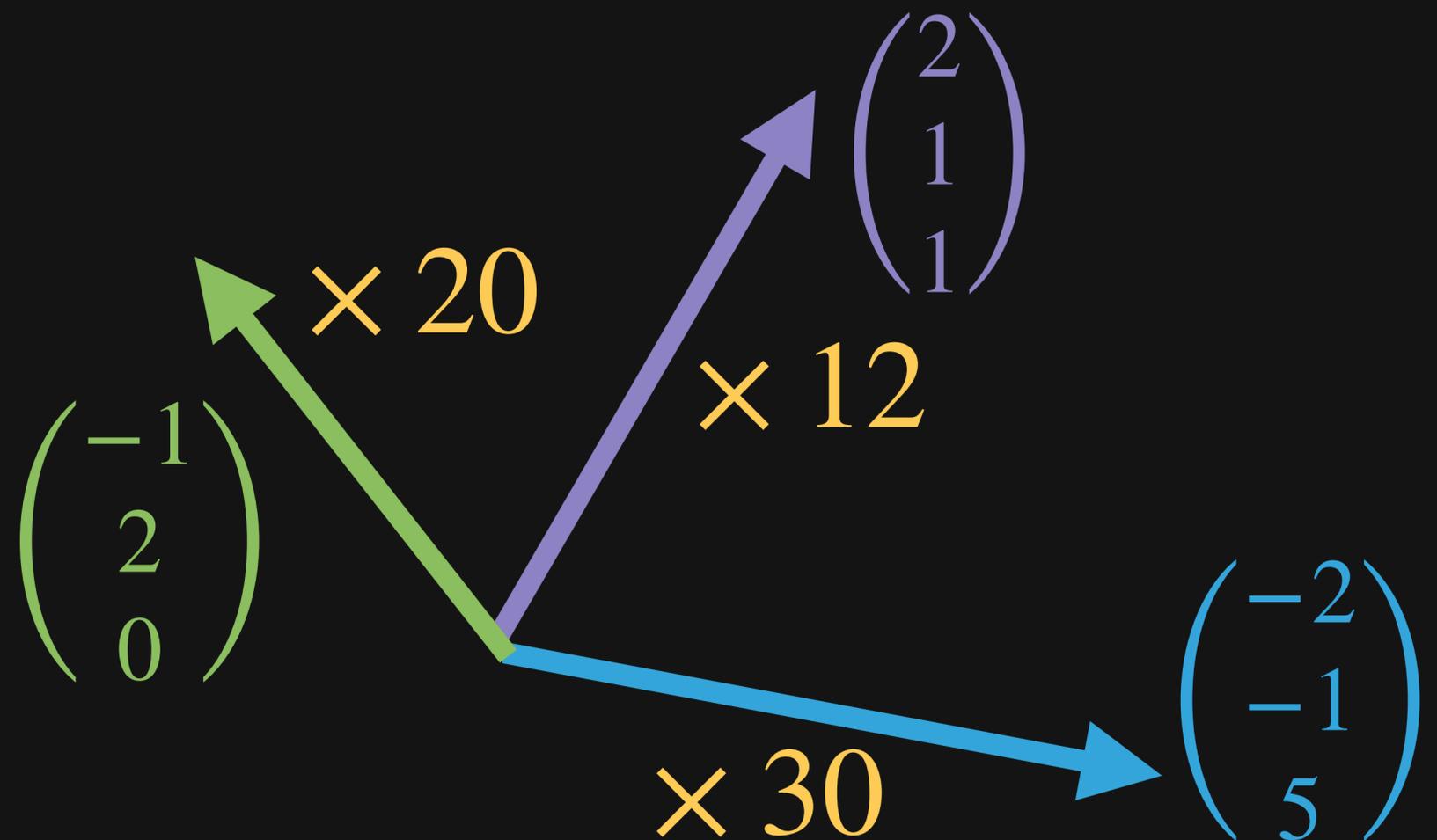
Screenshot from the excellent 3blue1brown series "the Essence of Linear Algebra"



LINEAR ALGEBRA REVIEW:

The **spectral theorem** lets us find an **orthogonal basis of eigenvectors** for certain linear operators.

$$\begin{pmatrix} 16 & -2 & -6 \\ -2 & 19 & -3 \\ -6 & -3 & 27 \end{pmatrix}$$



LINEAR ALGEBRA REVIEW:

An **orthogonal basis of Eigenvectors** lets us compute with linear operators **without matrix multiplication.**

$$\begin{pmatrix} 16 & -2 & -6 \\ -2 & 19 & -3 \\ -6 & -3 & 27 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} = 20 \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$$

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$$\begin{pmatrix} 16 & -2 & -6 \\ -2 & 19 & -3 \\ -6 & -3 & 27 \end{pmatrix} \left(a \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} + b \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + c \begin{pmatrix} -2 \\ -1 \\ 5 \end{pmatrix} \right) =$$

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An **orthogonal basis of Eigenvectors** lets us compute with linear operators **without matrix multiplication.**

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$$20 a \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} + 12 b \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + 30 c \begin{pmatrix} -2 \\ -1 \\ 5 \end{pmatrix}$$

LINEAR ALGEBRA REVIEW:

An **orthogonal basis of Eigenvectors** lets us compute with linear operators **without matrix multiplication.**

$$A \left(a v_1 + b v_2 + c v_3 \right) =$$

$$a \lambda_1 v_1 + b \lambda_2 v_2 + c \lambda_3 v_3$$

LINEAR ALGEBRA REVIEW:

An **orthogonal basis of Eigenvectors** lets us compute with linear operators **without matrix multiplication.**

$$w =$$

$$a v_1 + b v_2 + c v_3$$

LINEAR ALGEBRA REVIEW:

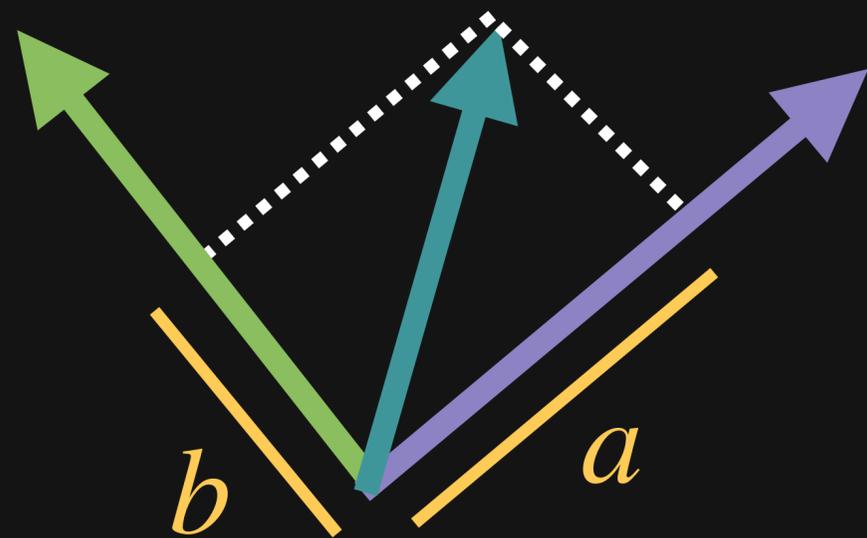
An **orthogonal basis of Eigenvectors** lets us compute with linear operators **without matrix multiplication.**

$$A w =$$
$$A (a v_1 + b v_2 + c v_3) =$$
$$a \lambda_1 v_1 + b \lambda_2 v_2 + c \lambda_3 v_3$$

LINEAR ALGEBRA REVIEW:

An **orthogonal basis of Eigenvectors** lets us compute with linear operators **without matrix multiplication**.

$$W = a v_1 + b v_2 + c v_3$$



$$a v_1 = \text{Proj}_{v_1} W$$

$$b v_2 = \text{Proj}_{v_2} W$$

$$c v_3 = \text{Proj}_{v_3} W$$

**Vector spaces can be
low dimensional**

Lines

Planes

\mathbb{R}^3

Spacetime

Or high dimensional

Data Sets

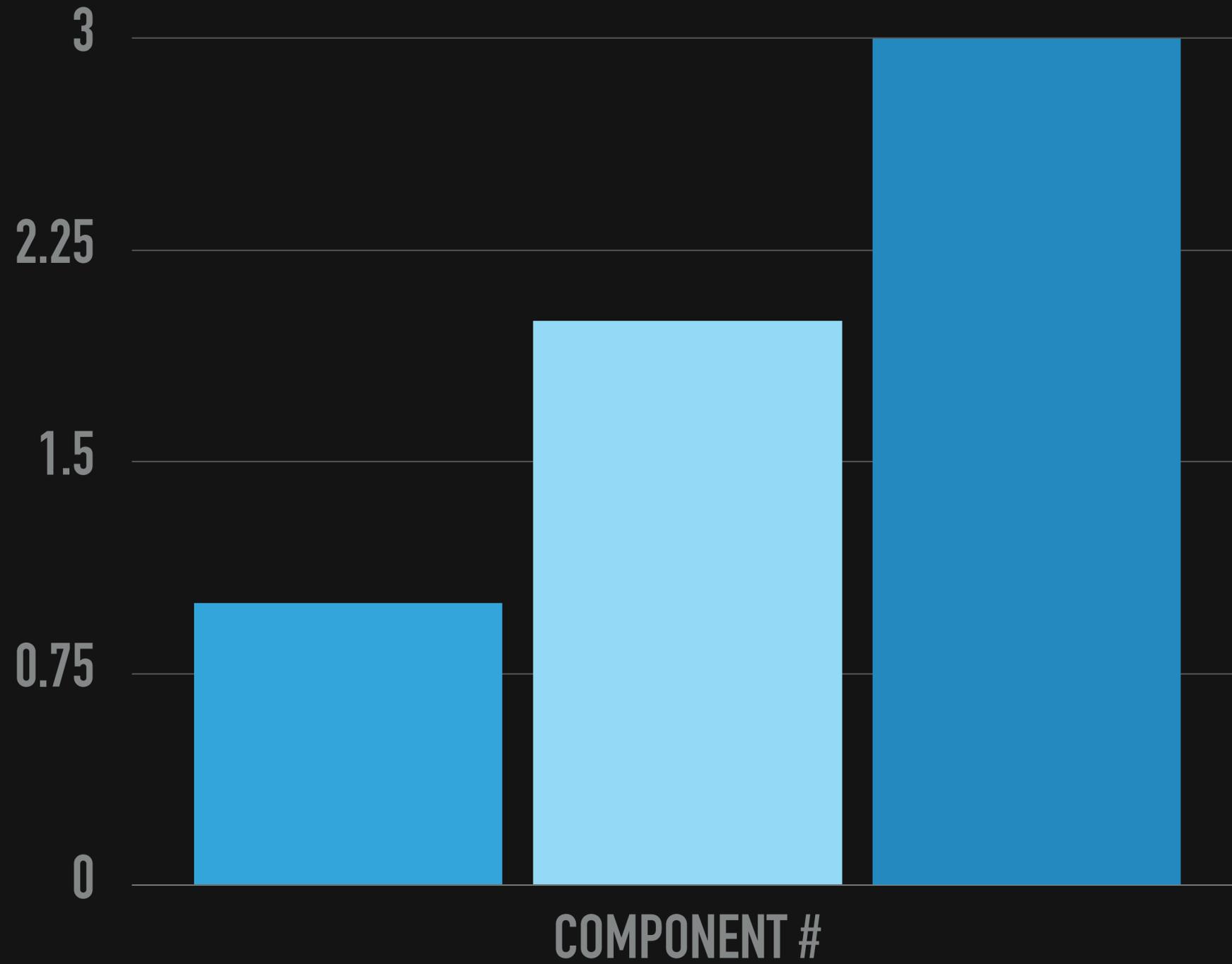
Images

**Preferences /
Rankings**

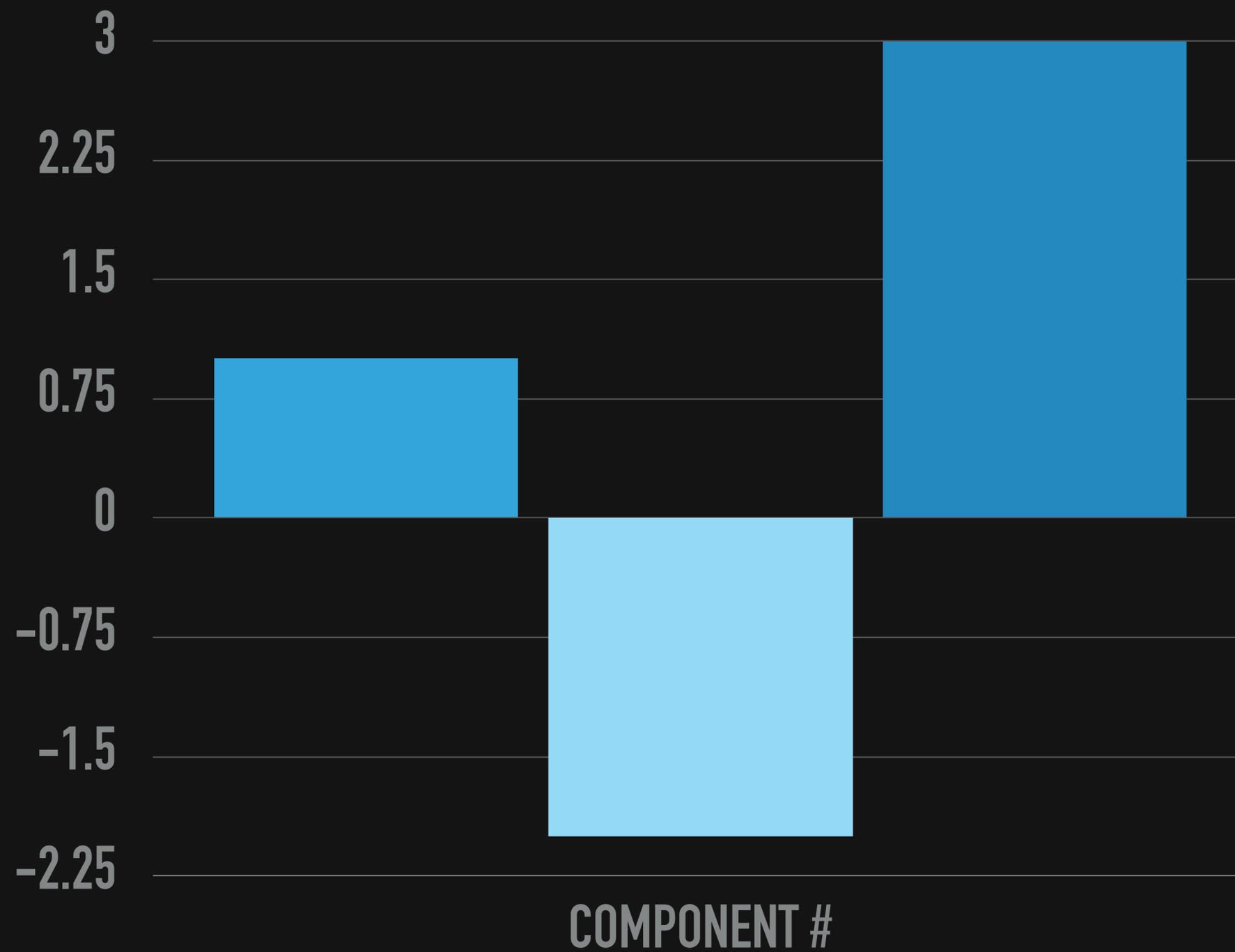
But they can also be

Infinite Dimensional

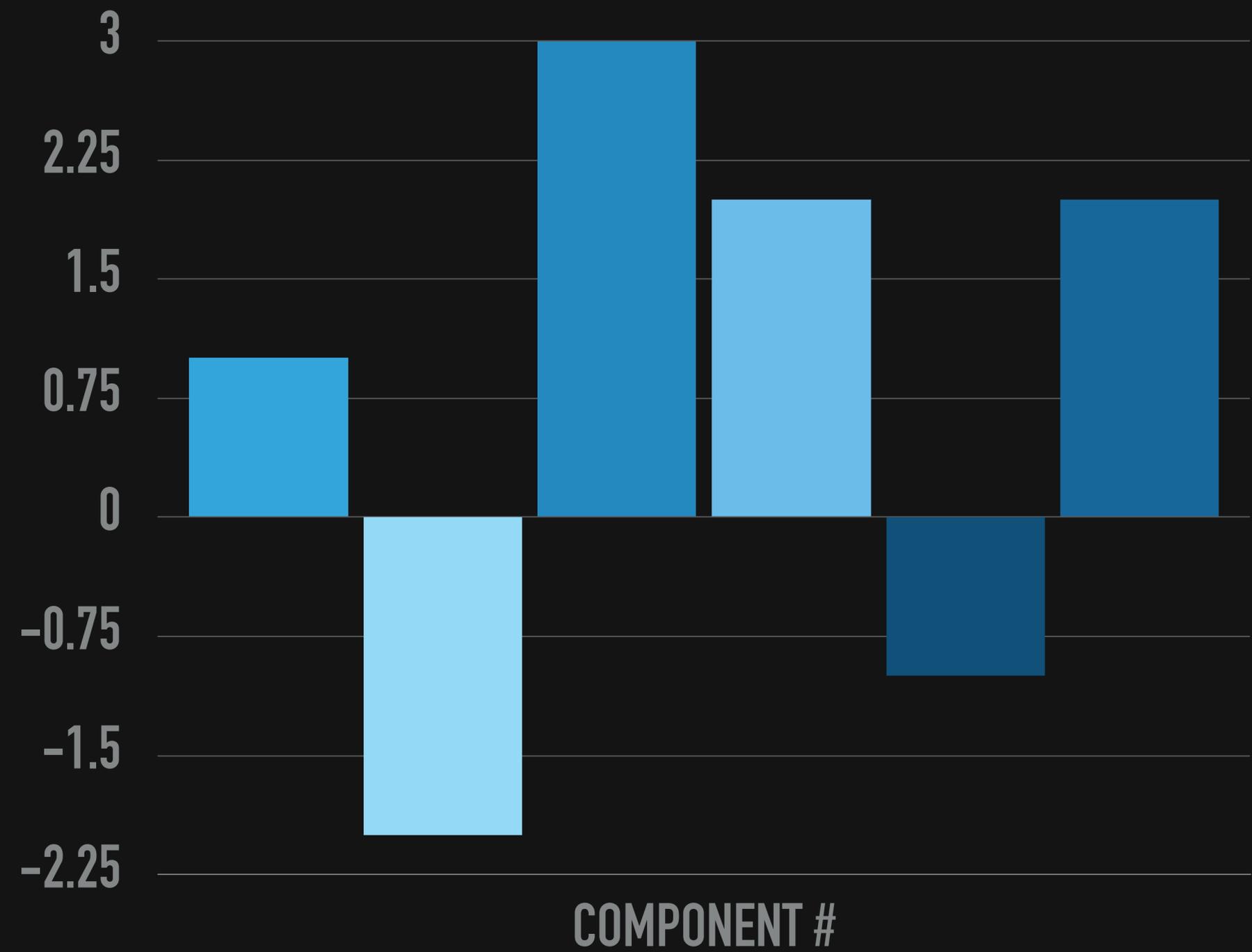
$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$



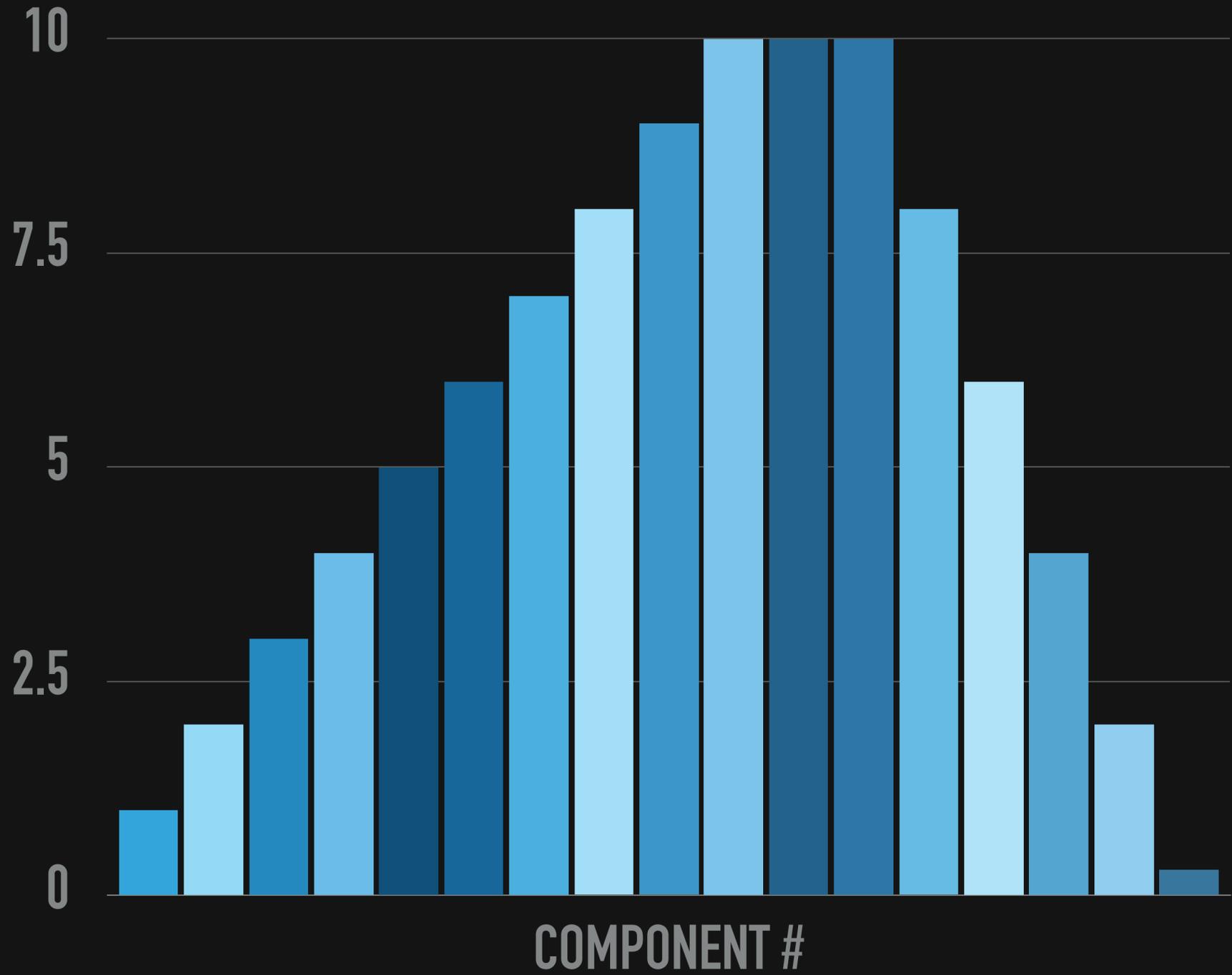
$$\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$$

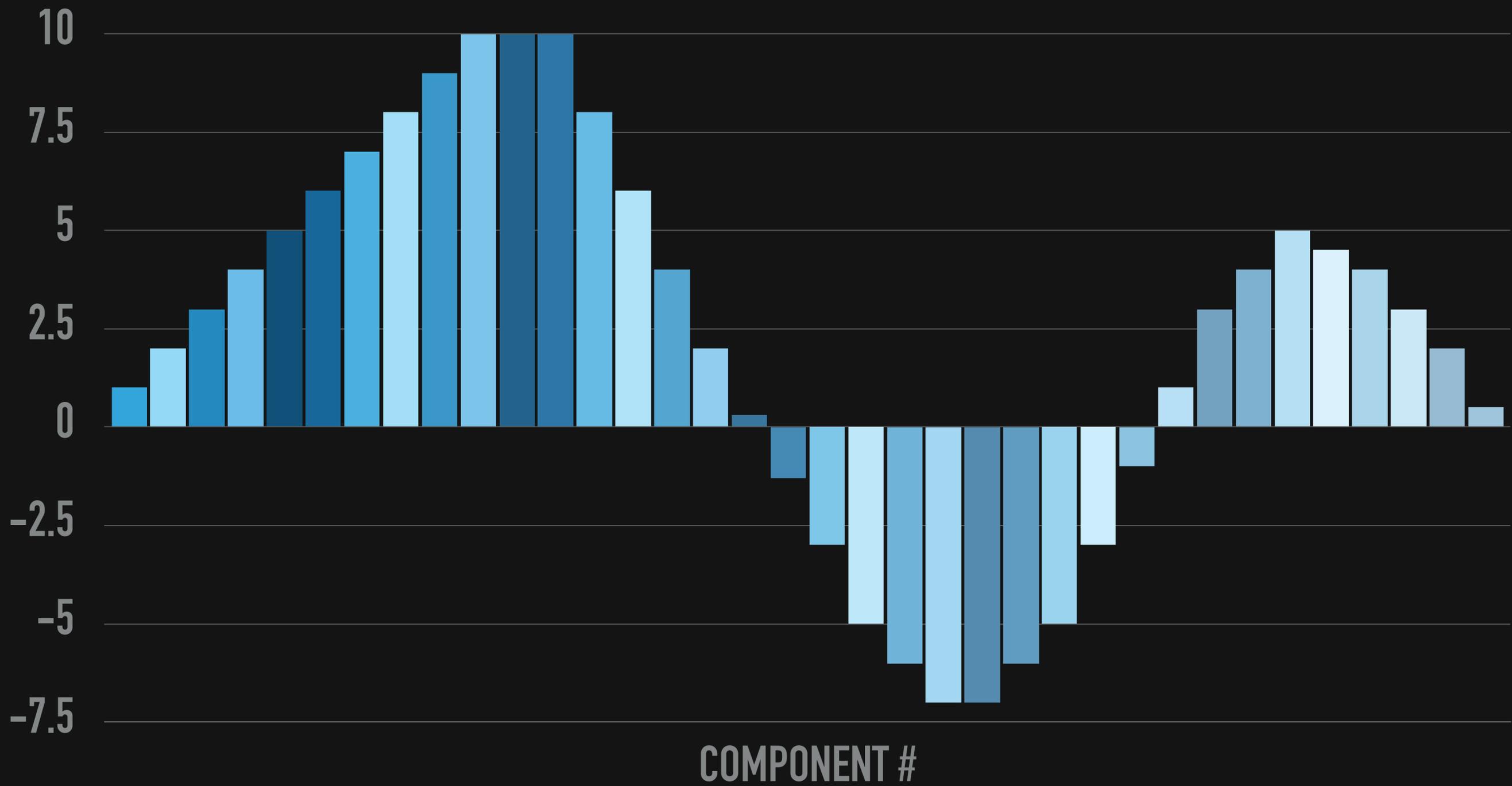


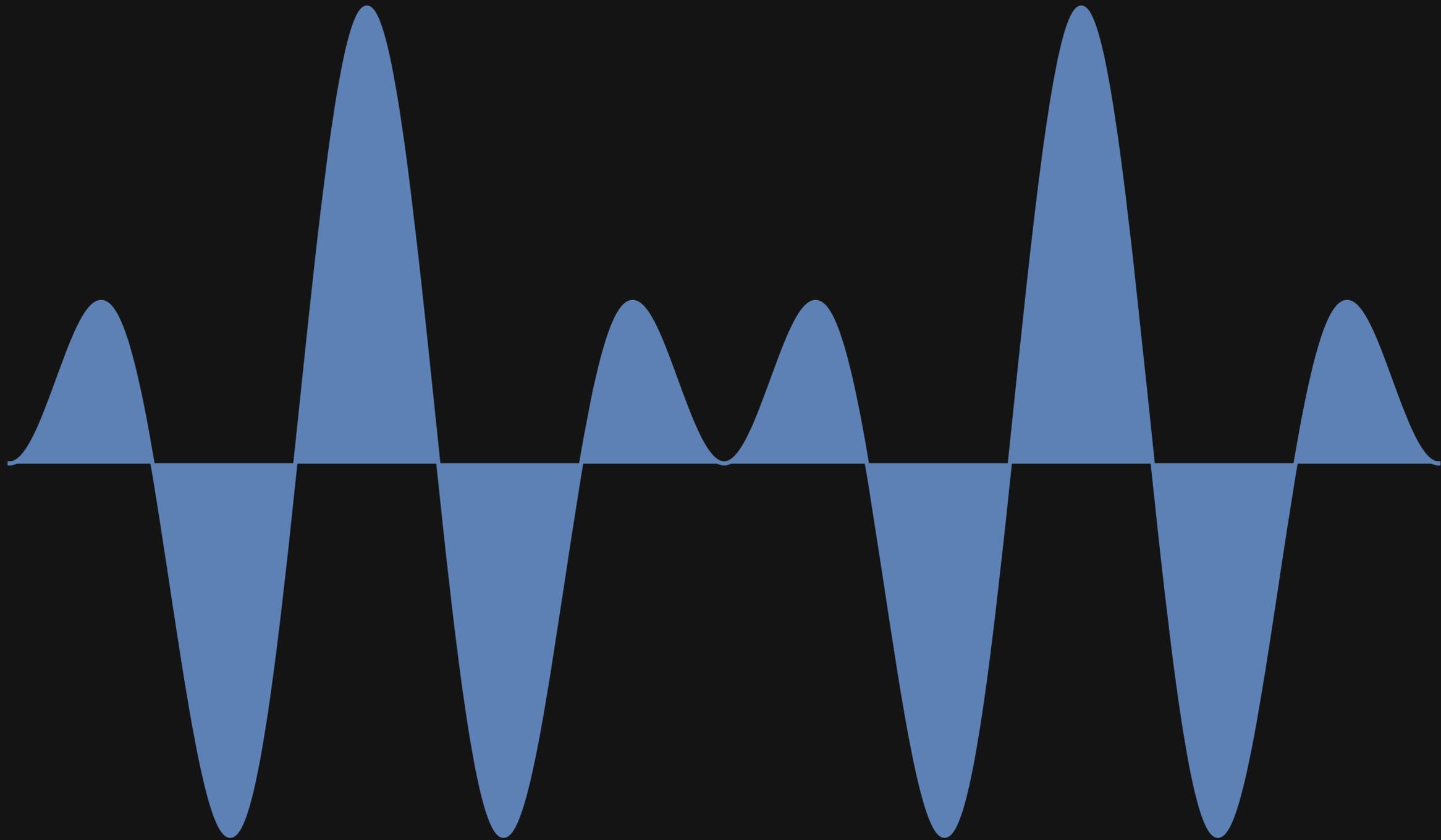
$$\begin{pmatrix} 1 \\ -2 \\ 3 \\ 2 \\ -1 \\ 2 \end{pmatrix}$$



1
2
3
4
5
6
7
8
9
10
10
10
8
6
4
2
0







Are **functions** really...**vectors**?

Vectors are an abstract mathematical concept, defined not so much by what they are, but how they interact with one another.

Go watch the 3Blue1Brown video
"Abstract Vector Spaces" for more!

Vectors are things you can add
and stretch.

$$v, w \mapsto v + w$$

$$u \mapsto cu$$

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“if it walks like a vector, and it talks like a vector, its a vector”

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Functions are things you can add
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$$f(x), g(x) \mapsto f(x) + g(x)$$

$$h(x) \mapsto ch(x)$$

Let's do linear algebra
with functions!

This is called
functional analysis.

$$f + g$$

$$cf$$

$$f \cdot g$$

To make things easy,
for this lecture all of
our functions will be
on the interval $[0, 2\pi]$

Let's do linear algebra
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$$f + g$$



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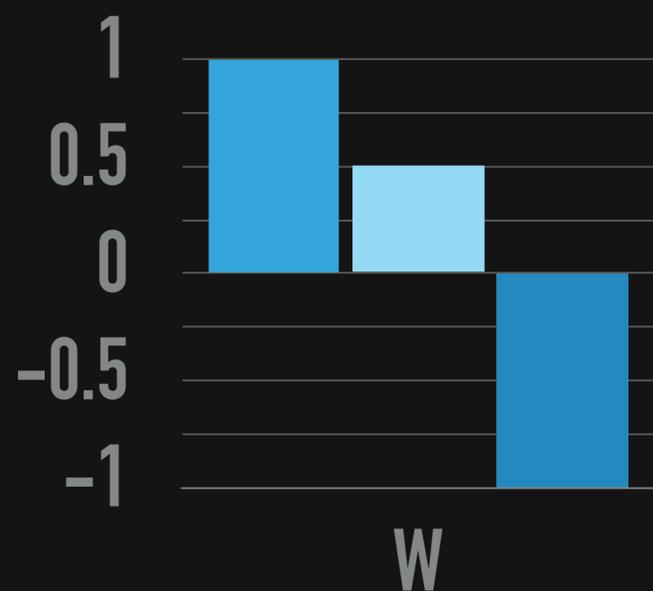
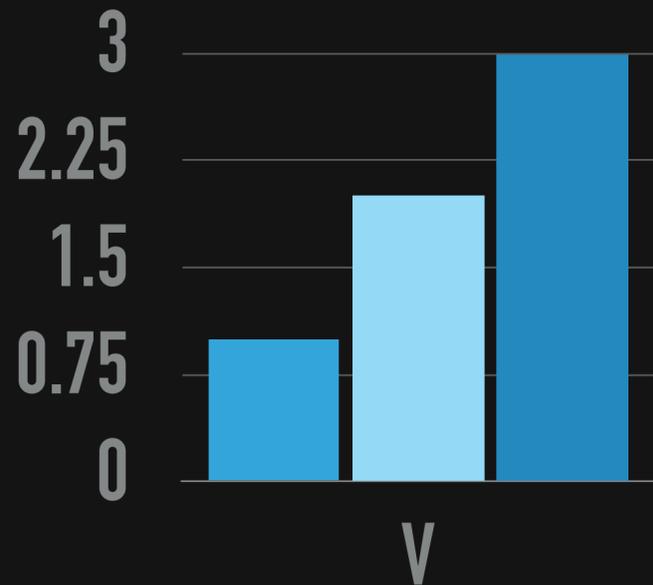


$$f \cdot g$$



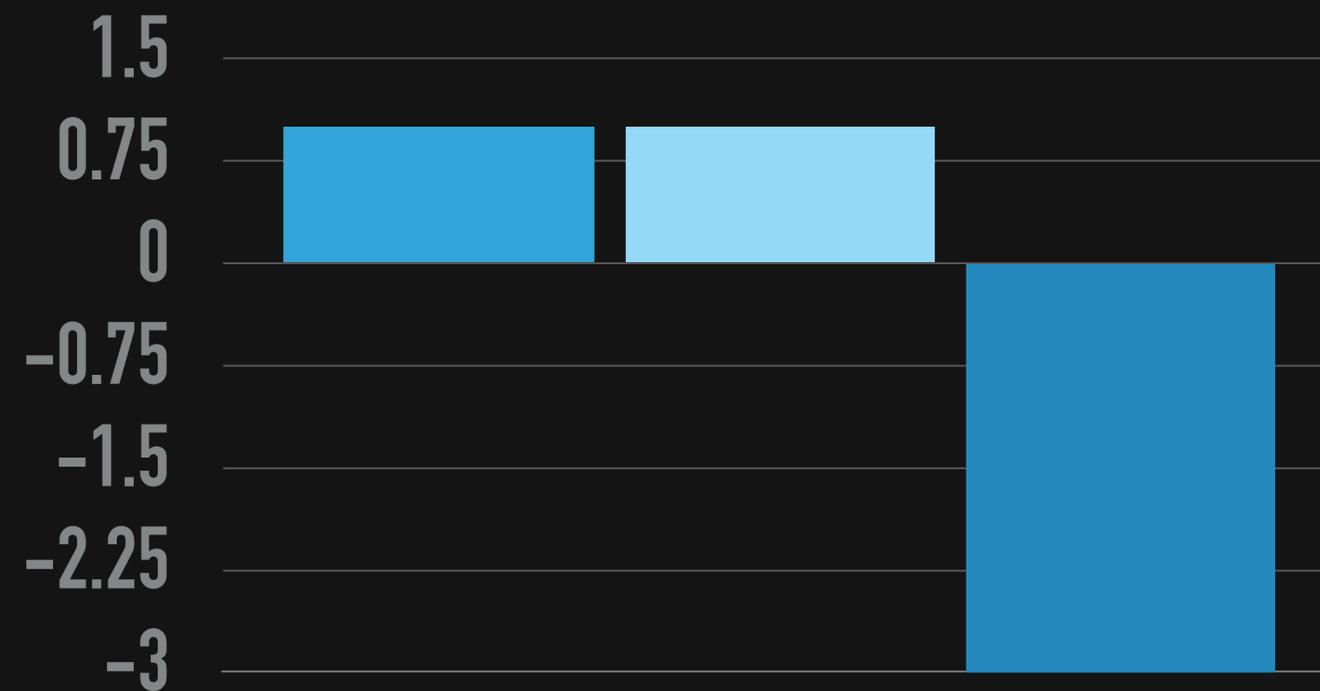
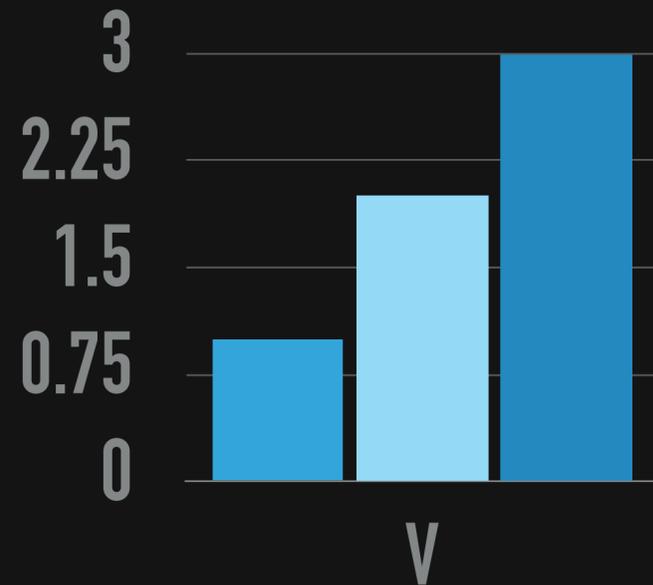
How do we take the dot product of functions?

$$\vec{v} \cdot \vec{w} = \sum_i v_i w_i$$



How do we take the dot product of functions?

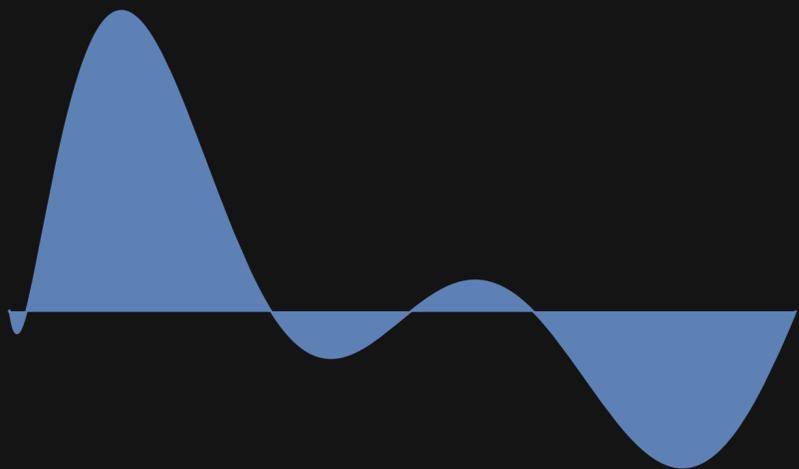
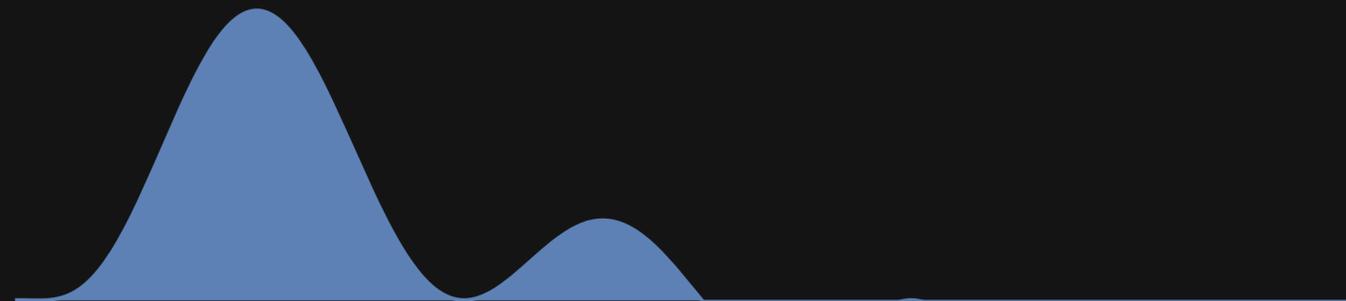
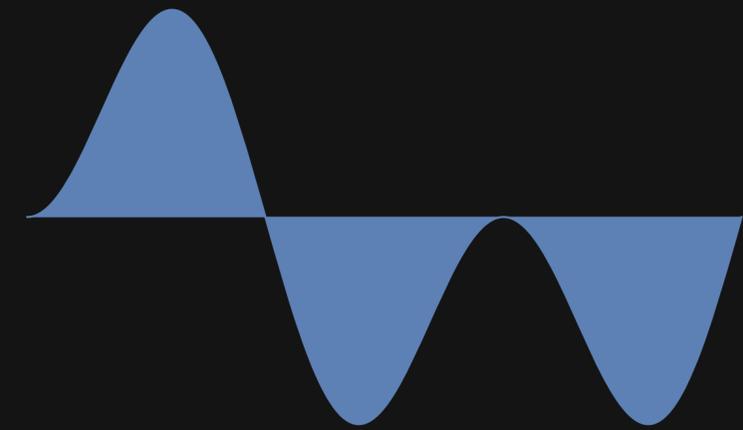
$$\vec{v} \cdot \vec{w} = \sum_i v_i w_i$$



NET AREA = V.W

How do we take the dot product of functions?

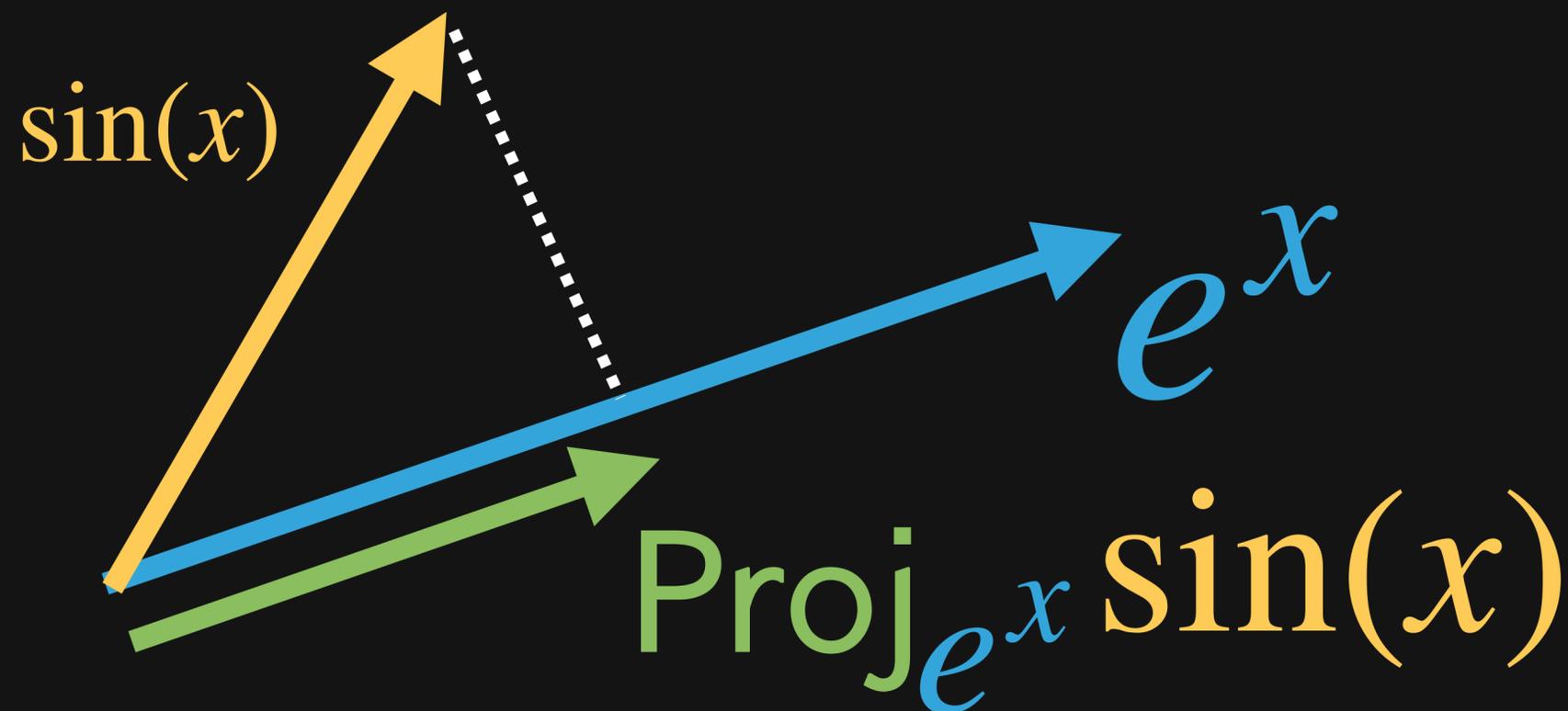
$$f \cdot g = \int_0^{2\pi} f(x)g(x)dx$$



Net Area = $f \cdot g$

One of the greatest things about linear algebra is that we can *think in low dimensions* and *calculate in high dimensions*.

Example: Project $f = \sin(x)$ onto $g = e^x$



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$$\text{Proj}_{e^x} \sin(x) = \text{Proj}_g f = \frac{f \cdot g}{g \cdot g} g$$

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$$f \cdot g = \int_0^{2\pi} e^x \sin(x) dx \quad g \cdot g = \int_0^{2\pi} e^x e^x dx$$

One of the greatest things about linear algebra is that we can *think in low dimensions* and *calculate in high dimensions*.

$$\text{Proj}_{e^x} \sin(x) = \text{Proj}_g f = \frac{f \cdot g}{g \cdot g} g$$

$$f \cdot g = \frac{1 - e^{2\pi}}{2}$$

$$g \cdot g = \frac{e^{4\pi} - 1}{2}$$

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$$\text{Proj}_{e^x} \sin(x) = \frac{1 - e^{2\pi}}{e^{4\pi} - 1} e^x$$

Whats a

Linear Transformation

On the space of functions?

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Linear Transformation

On the space of functions?

$$L(f+g) = L(f) + L(g)$$

$$L(cf) = cL(f)$$

The Derivative

Is a linear map!

$$\frac{d}{dx} (f + g) = \frac{d}{dx} f + \frac{d}{dx} g$$

$$\frac{d}{dx} (cf) = c \frac{d}{dx} f$$

How do we *think about the derivative* as linear algebraists?

Warm up: what's the **null space** of the derivative?

What's an **eigenvector** of the derivative?

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What's an **eigenvector** of the derivative?

$$\frac{d}{dx}f = \lambda f$$

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$$\frac{d}{dx}f = 0$$

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What's an **eigenvector** of the derivative?

$$\frac{d}{dx}f = f$$

$$f(x) = e^x$$

Eigenvectors of the Second Derivative.

Solutions to

$$\frac{d^2}{dx^2} f = \lambda f$$

Eigenvectors of the Second Derivative.

Solutions to

But solutions in *where*?

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Eigenvectors of the Second Derivative.

Solutions to

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But solutions in *where*?

$$f: [0, 2\pi] \rightarrow \mathbb{R}$$

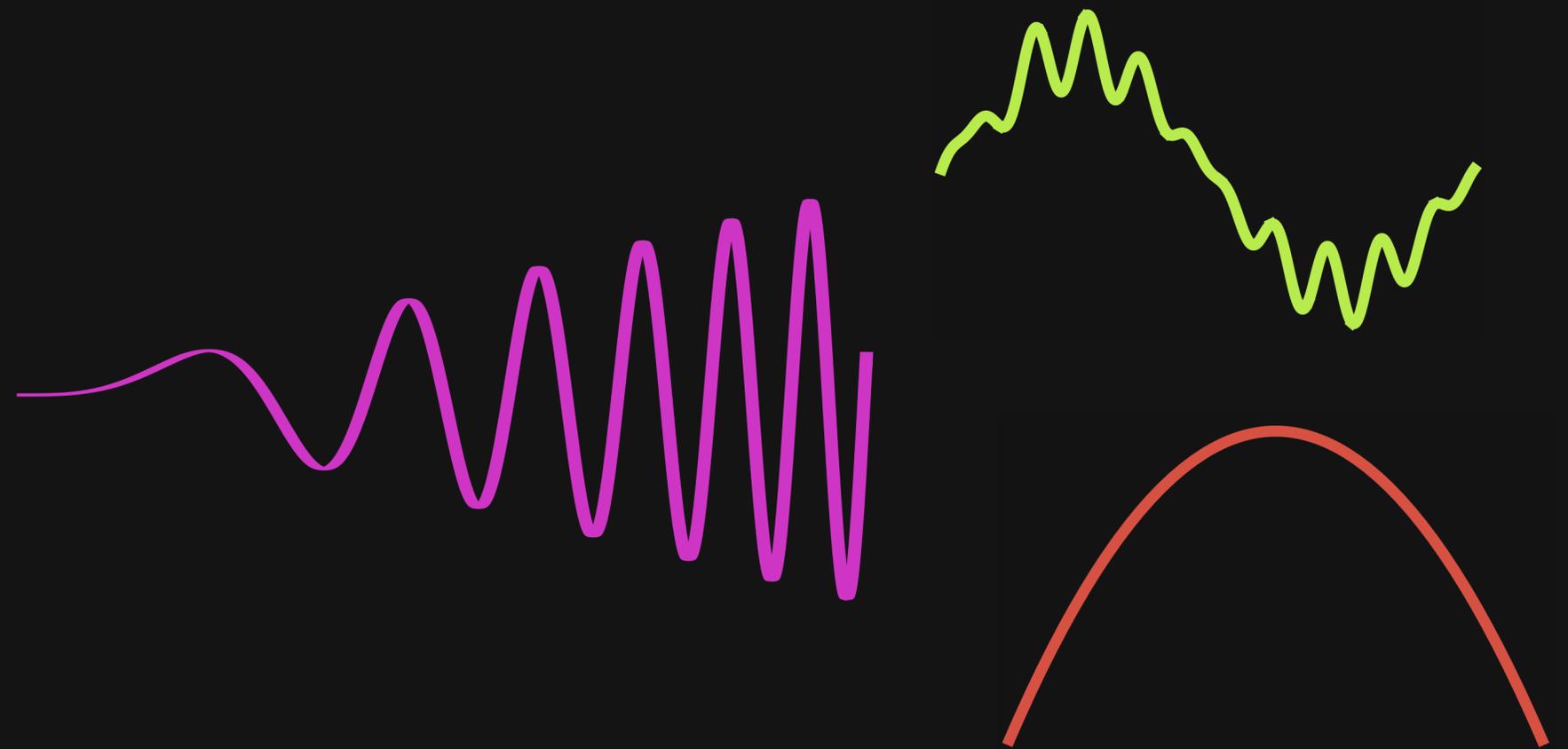
$$f(0) = f(2\pi) = 0$$

Eigenvectors of the Second Derivative.

Solutions to

$$\frac{d^2}{dx^2} f = \lambda f$$

But solutions in *where*?

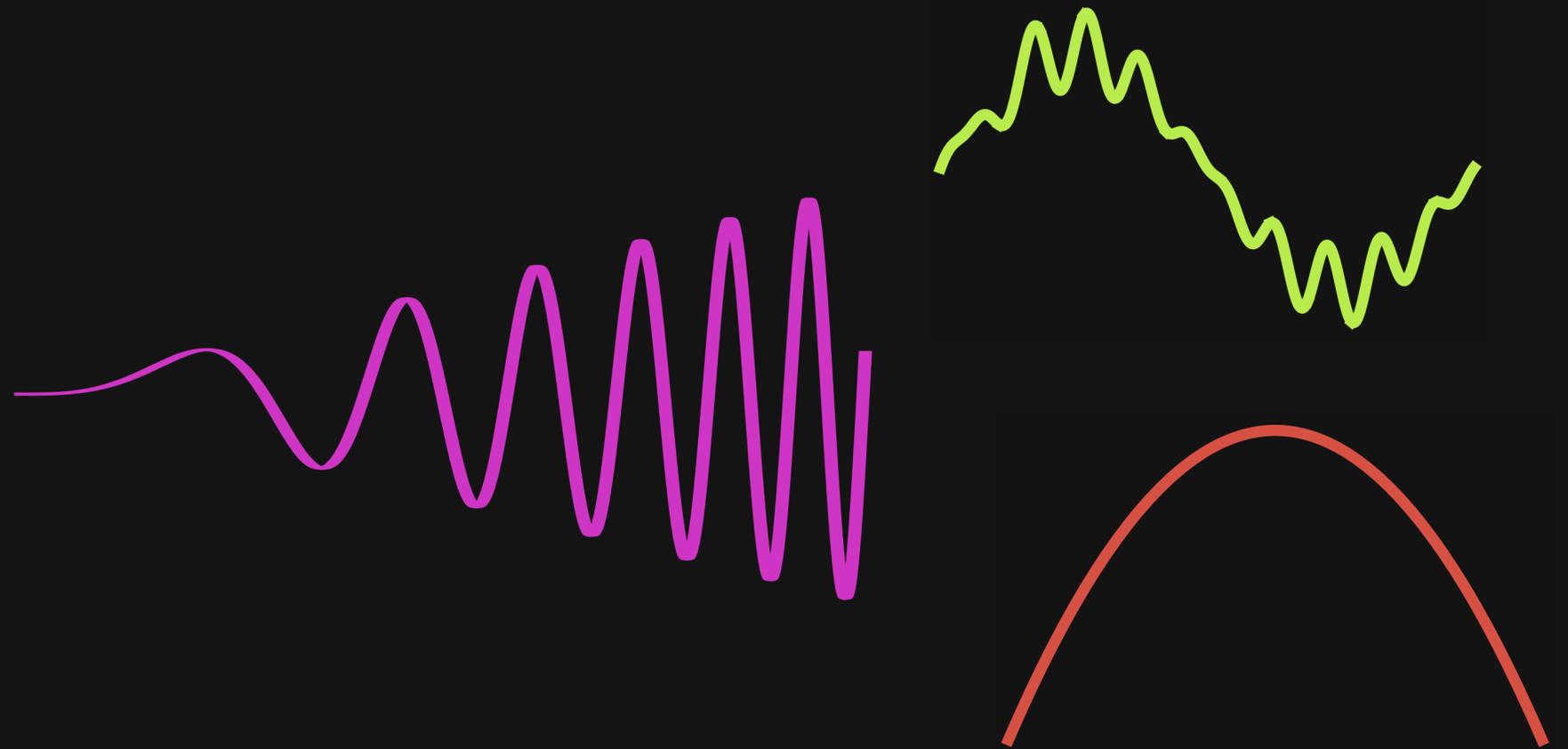


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Solutions to

$$\frac{d^2}{dx^2} f = -1f$$

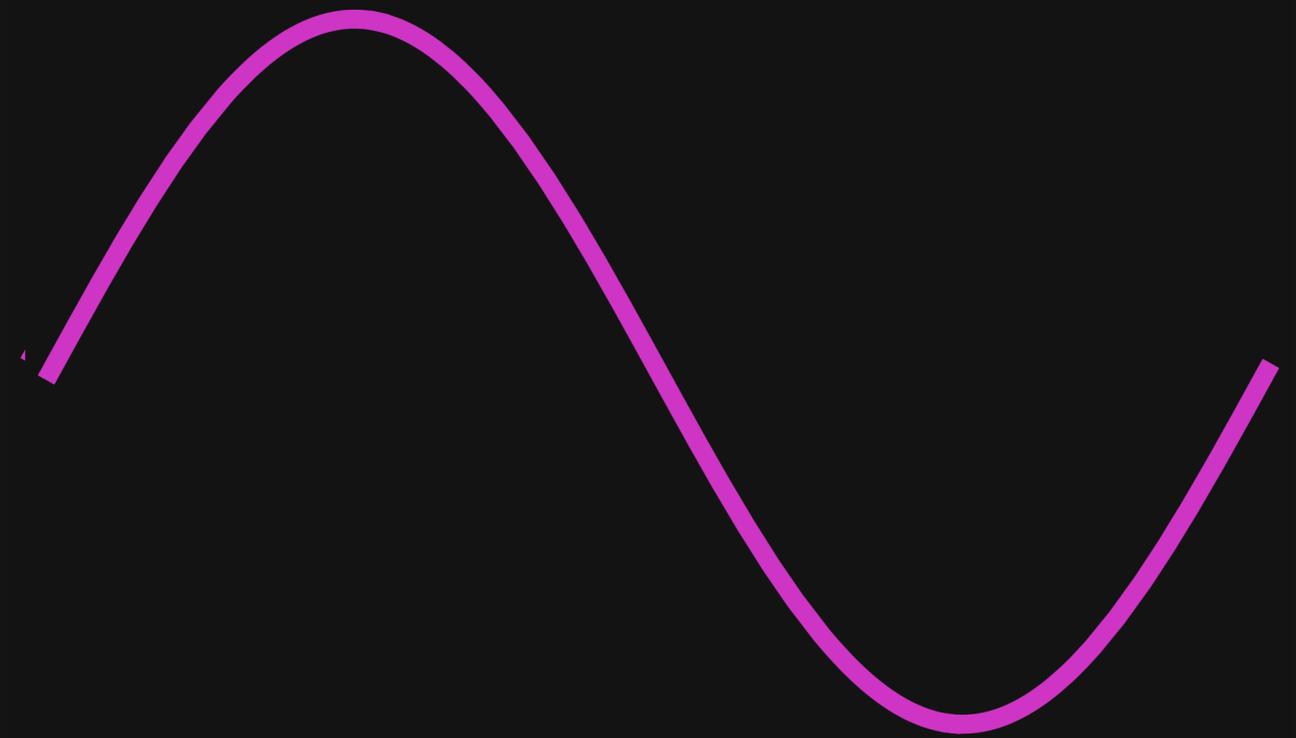
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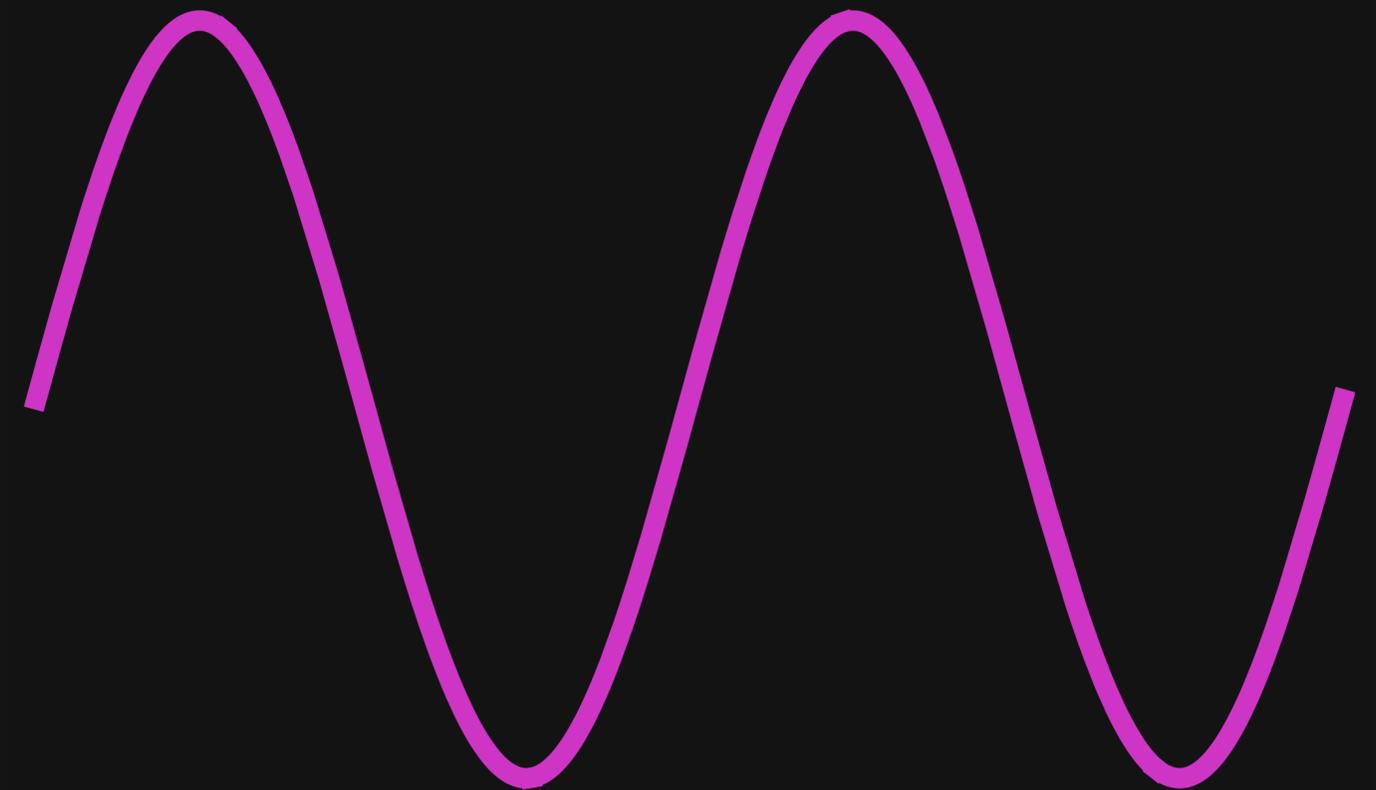


$$f(x) = \sin x$$

Eigenvectors of the Second Derivative.

Solutions to

$$\frac{d^2}{dx^2} f = -4f$$

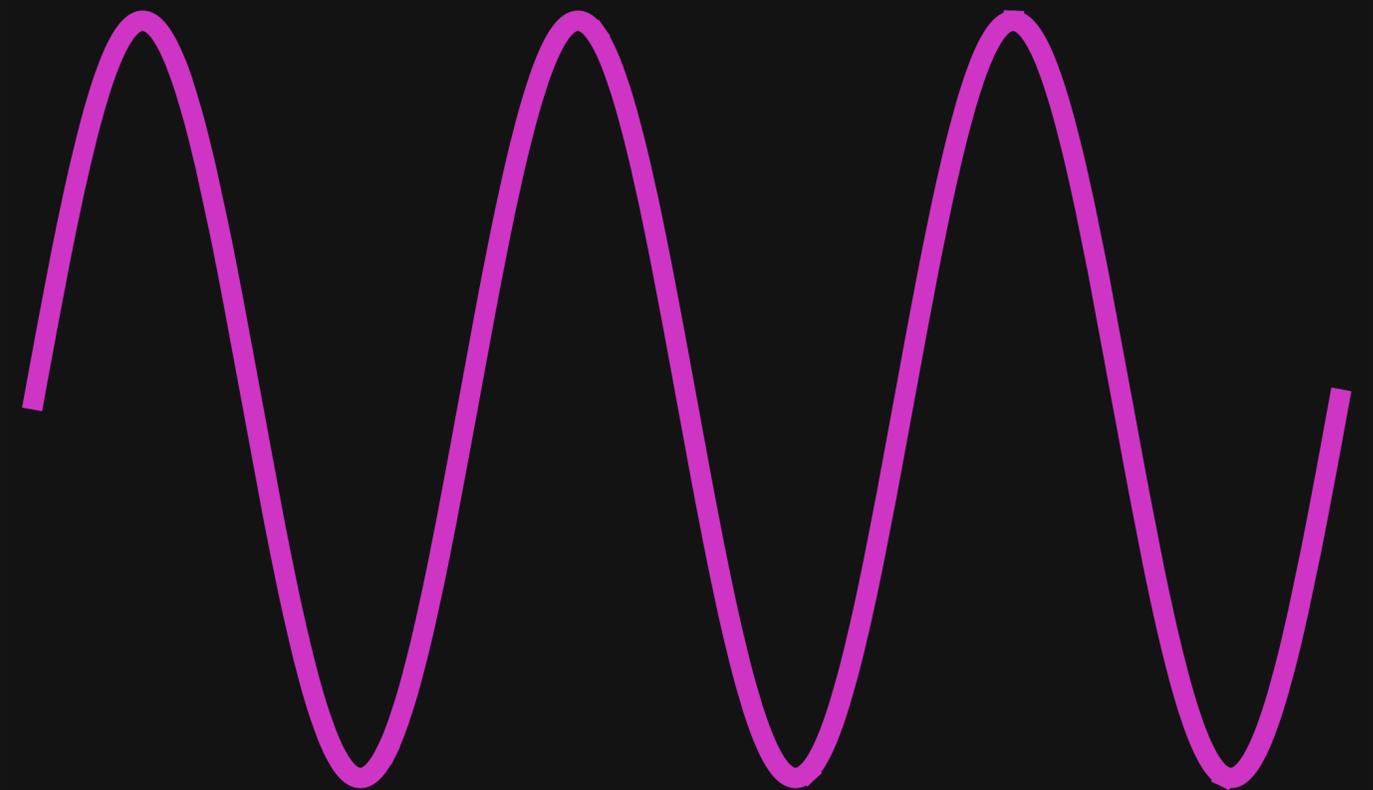


$$f(x) = \sin 2x$$

Eigenvectors of the Second Derivative.

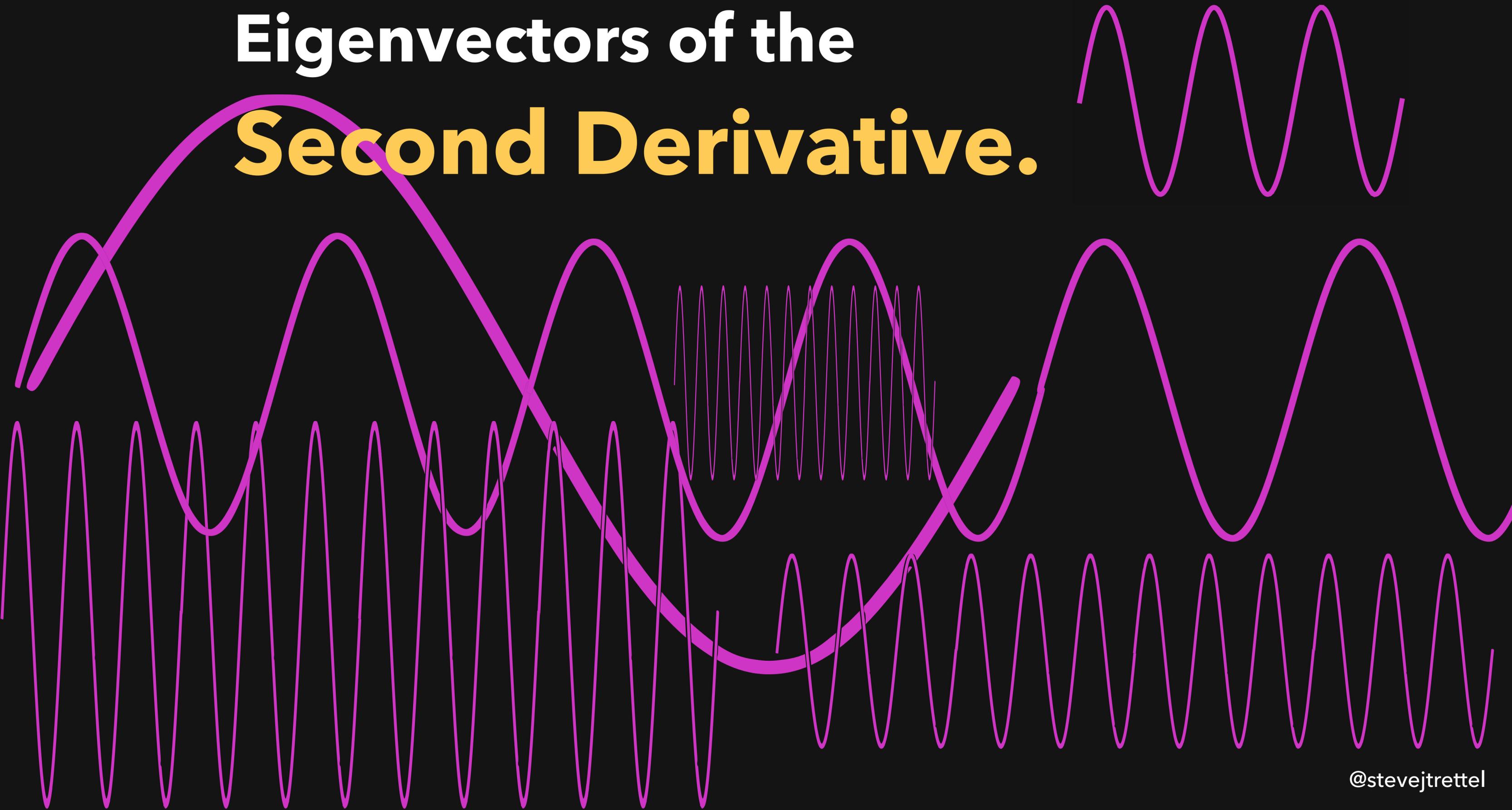
Solutions to

$$\frac{d^2}{dx^2} f = -9f$$



$$f(x) = \sin 3x$$

Eigenvectors of the Second Derivative.



Eigenvectors of the Second Derivative.

There's a version of
the **spectral theorem**
for the second
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Quick check: are these eigenvectors orthogonal?

$$f_n = \sin(nx) \quad f_m = \sin(mx)$$

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$$= 0 \quad \text{When } n \neq m.$$

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Quick check: What's the length of $f_n = \sin(nx)$?

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$$\begin{aligned} \|f_n\|^2 &= f_n \bullet f_n = \int_0^{2\pi} f_n f_n \, dx \\ &= \int_0^{2\pi} \sin(nx) \sin(nx) \, dx \end{aligned}$$

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$$\begin{aligned}\|f_n\|^2 &= f_n \bullet f_n = \int_0^{2\pi} f_n f_n \, dx \\ &= \int_0^{2\pi} \sin(nx) \sin(nx) \, dx \\ &= \pi\end{aligned}$$

**Linear
Transformation**

**Spectral
Theorem?**

**Basis of
eigenvectors**

**Linear
Transformation**

$$\begin{pmatrix} 16 & -2 & -6 \\ -2 & 19 & -3 \\ -6 & -3 & 27 \end{pmatrix}$$

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$$\frac{d^2}{dx^2}$$

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**Spectral
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**Basis of
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$$\begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} -2 \\ -1 \\ 5 \end{pmatrix}$$

$$\left\{ \begin{array}{l} \sin x, \sin 2x, \\ \sin 3x, \dots, \sin nx, \dots \end{array} \right\}$$

Remember how we used the spectral theorem earlier: to **do matrix multiplication without actually multiplying by a matrix.**

$$\begin{pmatrix} 16 & -2 & -6 \\ -2 & 19 & -3 \\ -6 & -3 & 27 \end{pmatrix} \left(\begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \right) = 20 \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} + 12 \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

Remember how we used the spectral theorem earlier: to **do matrix multiplication without actually multiplying by a matrix.**

We can do the same here: **take derivatives without using rules of differentiation.**

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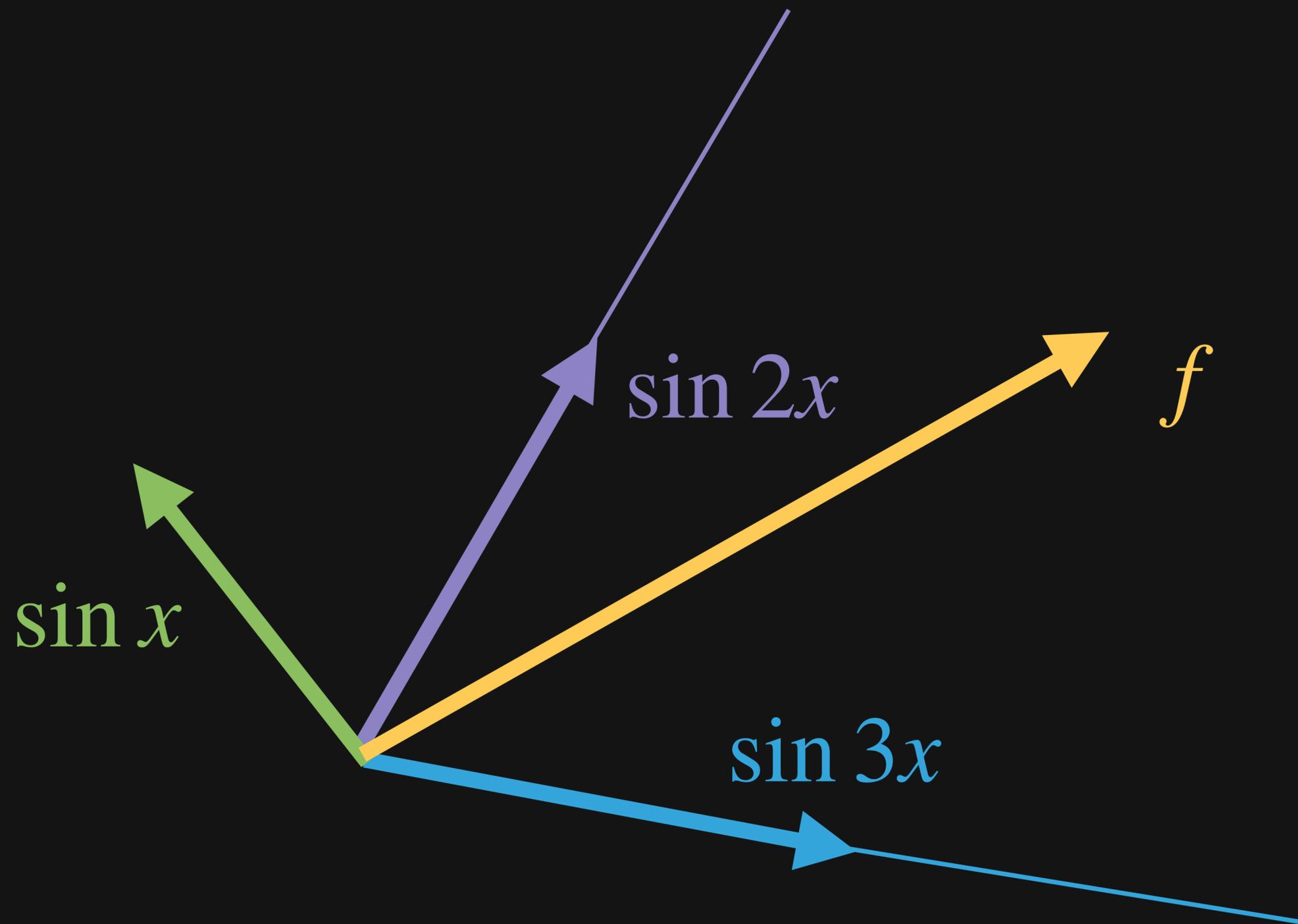
$$\frac{d^2}{dx^2} \left(\sin(2x) + \sin(5x) \right) =$$

Remember how we used the spectral theorem earlier: to **do matrix multiplication without actually multiplying by a matrix.**

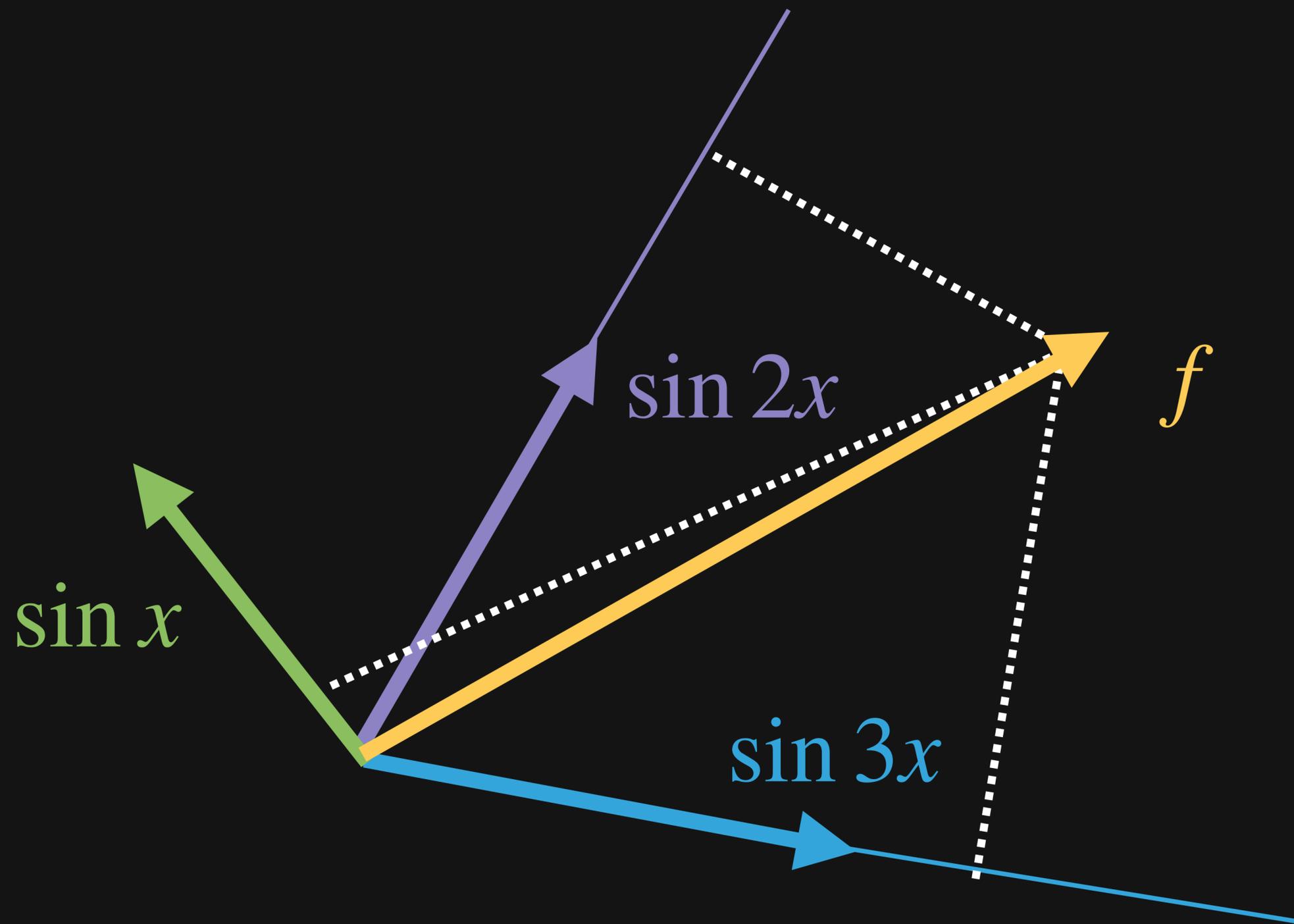
We can do the same here: **take derivatives without using rules of differentiation.**

$$\frac{d^2}{dx^2} \left(\sin(2x) + \sin(5x) \right) = -4 \sin(2x) - 25 \sin(5x)$$

Problem: what happens if you need to deal with other functions?



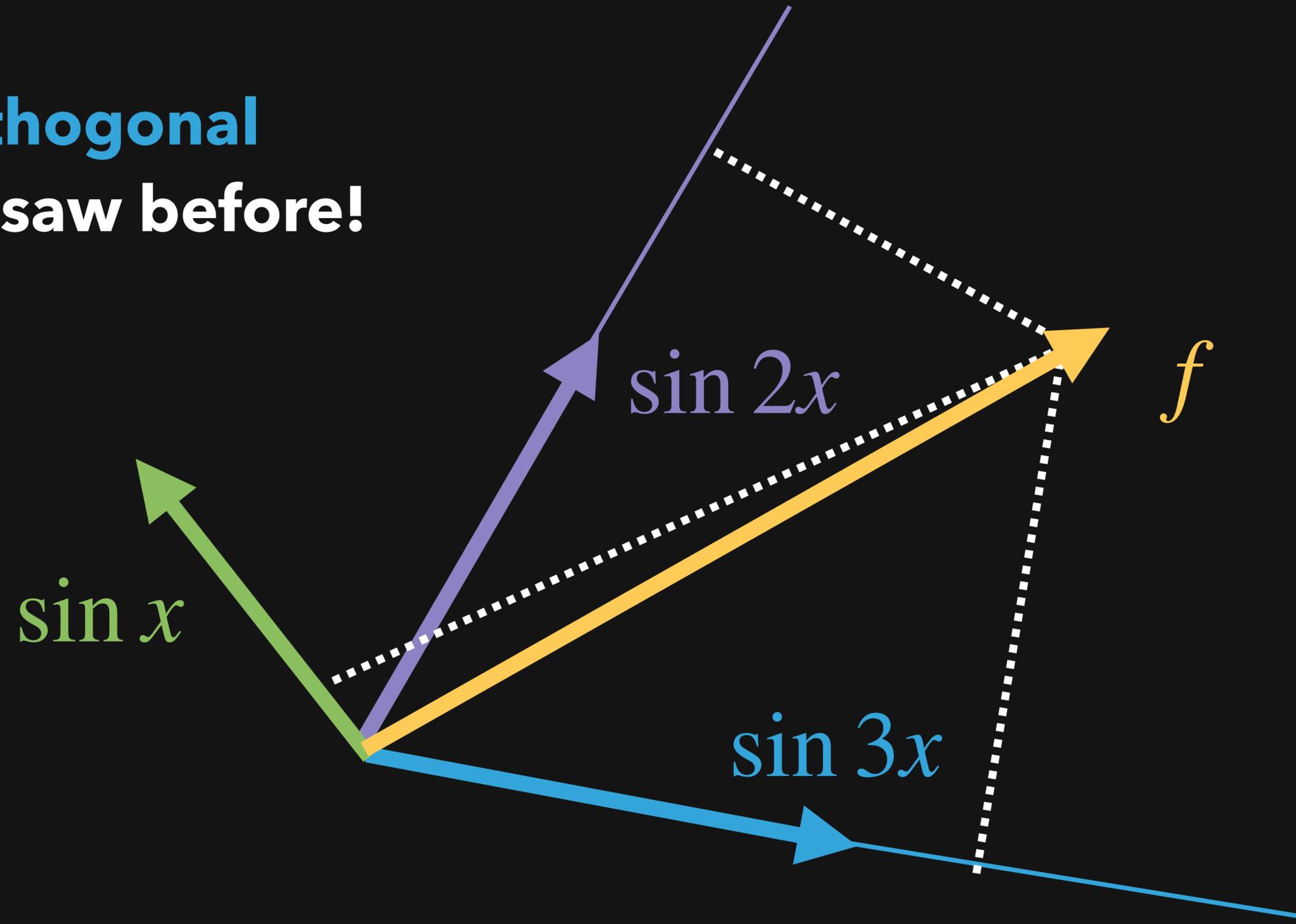
Problem: what happens if you need to deal with other functions?



**Write them in the
basis of eigenvectors!**

Problem: what happens if you need to deal with other functions?

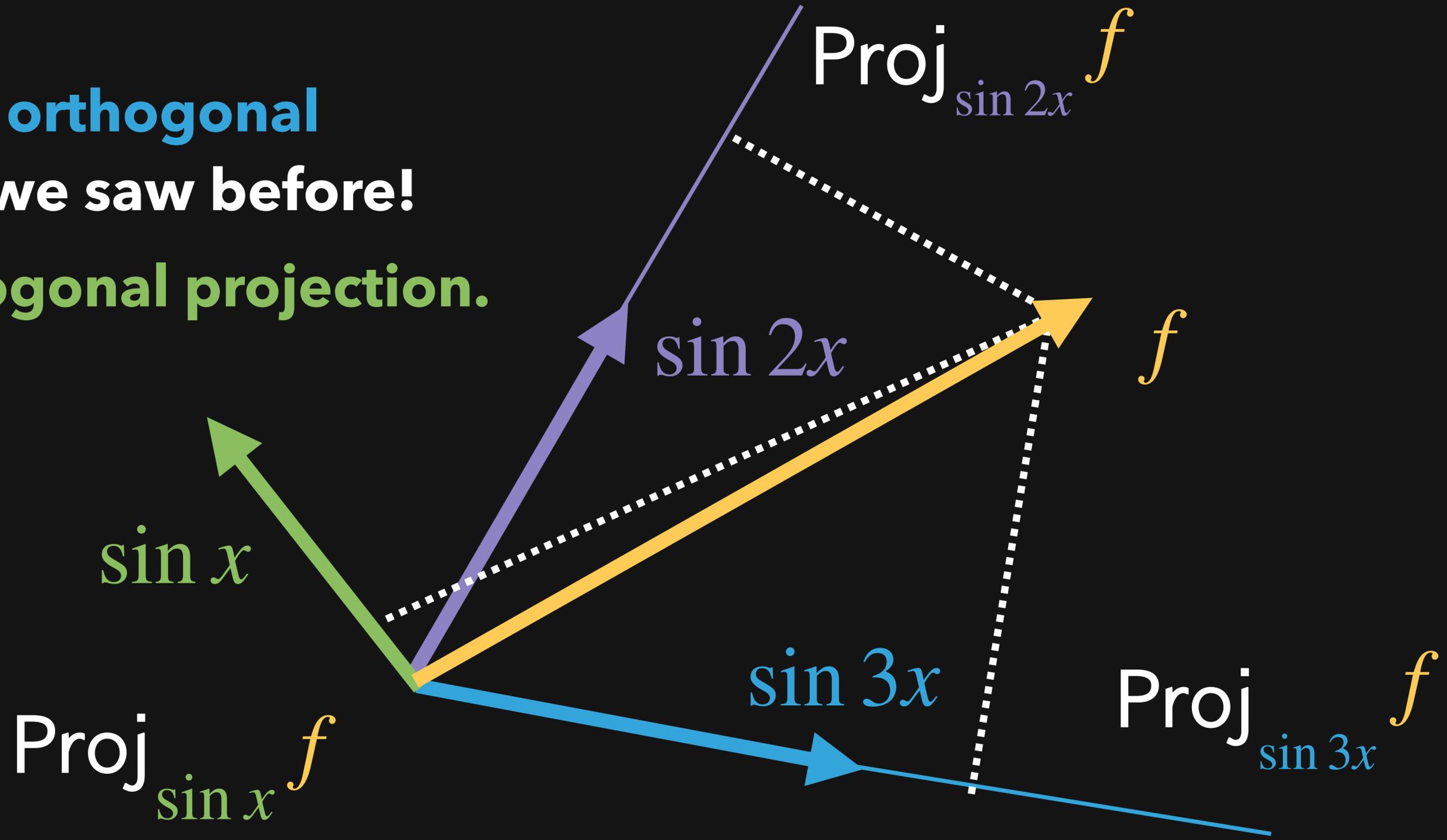
This is an orthogonal basis, as we saw before!



Problem: what happens if you need to deal with other functions?

This is an orthogonal basis, as we saw before!

Use orthogonal projection.



Fun Application #1

Fourier Series.

f

Fun Application #1

Fourier Series.

$$f = \sum_{n=1}^{\infty} c_n \sin(nx)$$

Fun Application #1

Fourier Series.

Writing f in the basis

$$\left\{ \sin x, \sin 2x, \sin 3x, \dots, \sin nx, \dots \right\}$$

$$f = \text{Proj}_{\sin x} f + \text{Proj}_{\sin 2x} f + \dots + \text{Proj}_{\sin nx} f \dots$$

$$= \frac{f \cdot \sin x}{\|\sin x\|^2} \sin x + \frac{f \cdot \sin 2x}{\|\sin 2x\|^2} \sin 2x + \dots + \frac{f \cdot \sin nx}{\|\sin nx\|^2} \sin nx$$

Fun Application #1

Fourier Series.

Writing f in the basis

$$\left\{ \sin x, \sin 2x, \sin 3x, \dots, \sin nx, \dots \right\}$$

$$f = \text{Proj}_{\sin x} f + \text{Proj}_{\sin 2x} f + \dots + \text{Proj}_{\sin nx} f \dots$$

$$= \frac{f \cdot \sin x}{\pi} \sin x + \frac{f \cdot \sin 2x}{\pi} \sin 2x + \dots + \frac{f \cdot \sin nx}{\pi} \sin nx$$

Fun Application #1

Fourier Series.

$$f = \frac{f \cdot \sin x}{\pi} \sin x + \frac{f \cdot \sin 2x}{\pi} \sin 2x + \dots + \frac{f \cdot \sin nx}{\pi} \sin nx$$

Fun Application #1

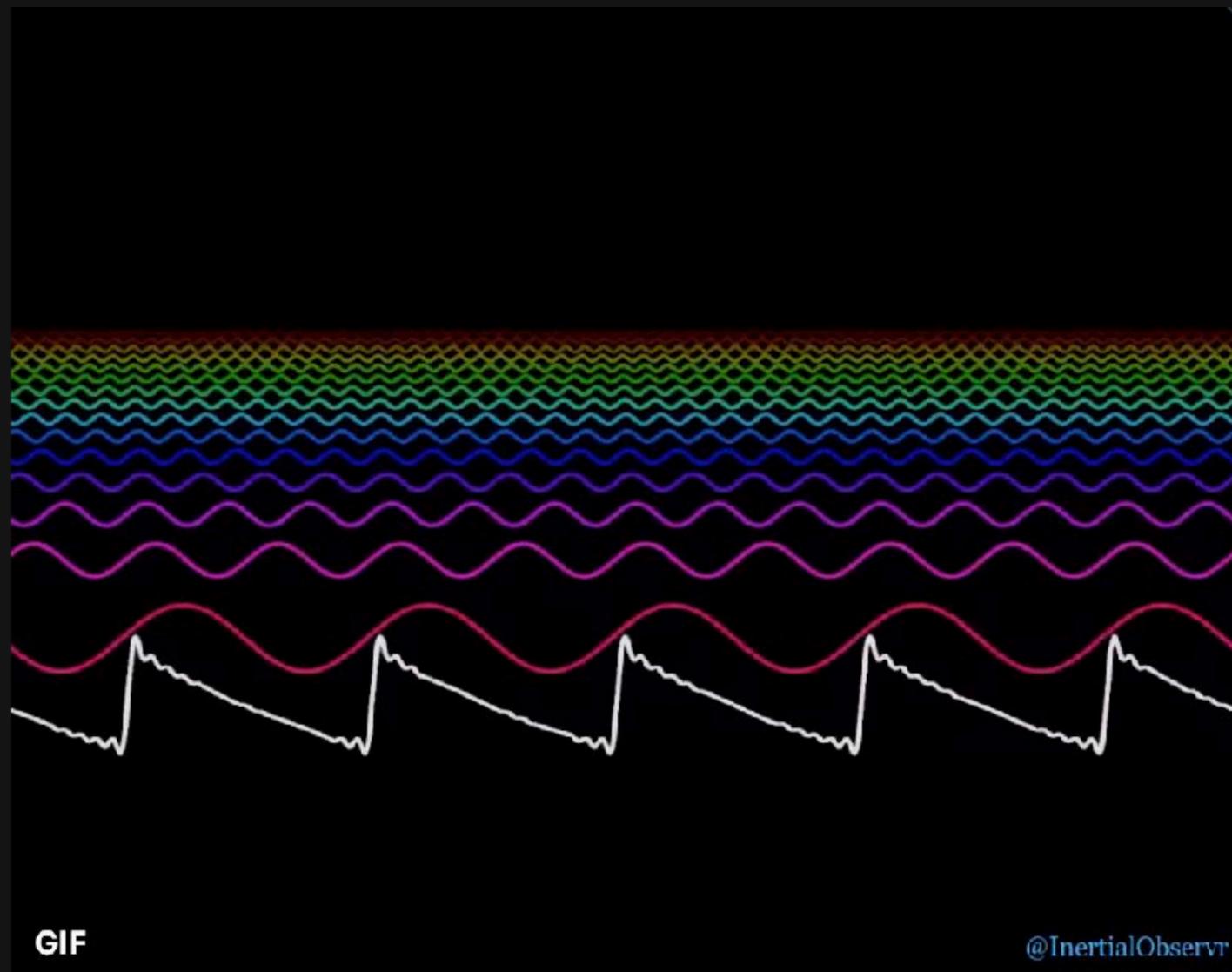
Fourier Series.

$$f = \frac{f \cdot \sin x}{\pi} \sin x + \frac{f \cdot \sin 2x}{\pi} \sin 2x + \dots + \frac{f \cdot \sin nx}{\pi} \sin nx$$

$$f \cdot \sin nx = \int_0^{2\pi} f(x) \sin nx \, dx$$

Fun Application #1

Fourier Series.



Here's a decomposition of the 'sawtooth wave' as a **linear combination of sine curves.**

Check out @InertialObservr on Twitter for more great mathematics and physics content.

Quick Summary:

When studying something involving $\frac{d^2}{dx^2}$, we can first look at the **simple cases**, given by the **eigenvectors** of $\frac{d^2}{dx^2}$.

And if we **understand these**, we **understand everything**, because **every function can be written as an infinite linear combination of these**.

Eigenvectors of derivatives in the real world



Vibrations of Strings and Surfaces

Equation of
motion for a
vibrating string.

$$\frac{\partial^2 f}{\partial t^2} = \frac{\partial^2 f}{\partial x^2}$$

Vibrations of Strings and Surfaces

Look for simple
solutions:

Separate the
dependence of
space and time

$$f(x, t) = g(t) h(x)$$

$$\frac{\partial^2}{\partial t^2} f = \frac{\partial^2}{\partial x^2} f$$

Vibrations of Strings and Surfaces

Look for simple
solutions:

Separate the
dependence of
space and time

$$f(x, t) = g(t) h(x)$$

$$\frac{\partial^2}{\partial t^2} g h = \frac{\partial^2}{\partial x^2} g h$$

Vibrations of Strings and Surfaces

Look for simple
solutions:

Separate the
dependence of
space and time

$$f(x, t) = g(t) h(x)$$

$$h \frac{\partial^2}{\partial t^2} g = g \frac{\partial^2}{\partial x^2} h$$

Vibrations of Strings and Surfaces

Look for simple
solutions:

Separate the
dependence of
space and time

$$f(x, t) = g(t) h(x)$$

$$\frac{1}{g} \frac{\partial^2 g}{\partial t^2} = \frac{1}{h} \frac{\partial^2 h}{\partial x^2}$$

Vibrations of Strings and Surfaces

Look for simple
solutions:

Separate the
dependence of
space and time

$$\frac{1}{g} \frac{\partial^2 g}{\partial t^2} = \frac{1}{h} \frac{\partial^2 h}{\partial x^2}$$

Only depends
on t

Only depends
on x

Vibrations of Strings and Surfaces

Look for simple
solutions:

Separate the
dependence of
space and time

$$\frac{1}{g} \frac{\partial^2 g}{\partial t^2} = \frac{1}{h} \frac{\partial^2 h}{\partial x^2}$$

Only depends
on t

Only depends
on x

Must be constant!

Vibrations of Strings and Surfaces

$$\frac{1}{g} \frac{\partial^2 g}{\partial t^2} = \lambda = \frac{1}{h} \frac{\partial^2 h}{\partial x^2}$$

Vibrations of Strings and Surfaces

$$\frac{1}{g} \frac{\partial^2 g}{\partial t^2} = \lambda$$

$$\frac{1}{h} \frac{\partial^2 h}{\partial x^2} = \lambda$$

Vibrations of Strings and Surfaces

$$\frac{1}{g} \frac{d^2 g}{dt^2} = \lambda$$

$$\frac{1}{h} \frac{d^2 h}{dx^2} = \lambda$$

Vibrations of Strings and Surfaces

**Eigenvector
equations for the
second derivatives.**

$$\frac{d^2 g}{dt^2} = \lambda g$$

$$\frac{d^2 h}{dx^2} = \lambda h$$

Vibrations of Strings and Surfaces

Eigenvector
equations for the
second derivatives.

The **simplest motions**
of a vibrating string
are **eigenvectors** in
space and in **time**.

$$\frac{d^2 g}{dt^2} = \lambda g$$

$$\frac{d^2 h}{dx^2} = \lambda h$$

Vibrations of Strings and Surfaces

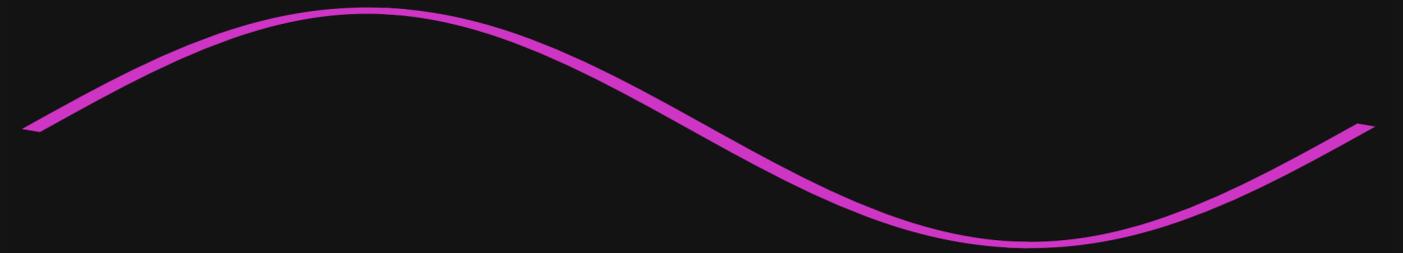
$$\lambda = -1 \quad \sin(x)$$

$$\lambda = -4 \quad \sin(2x)$$

$$\lambda = -9 \quad \sin(3x)$$

Vibrations of Strings and Surfaces

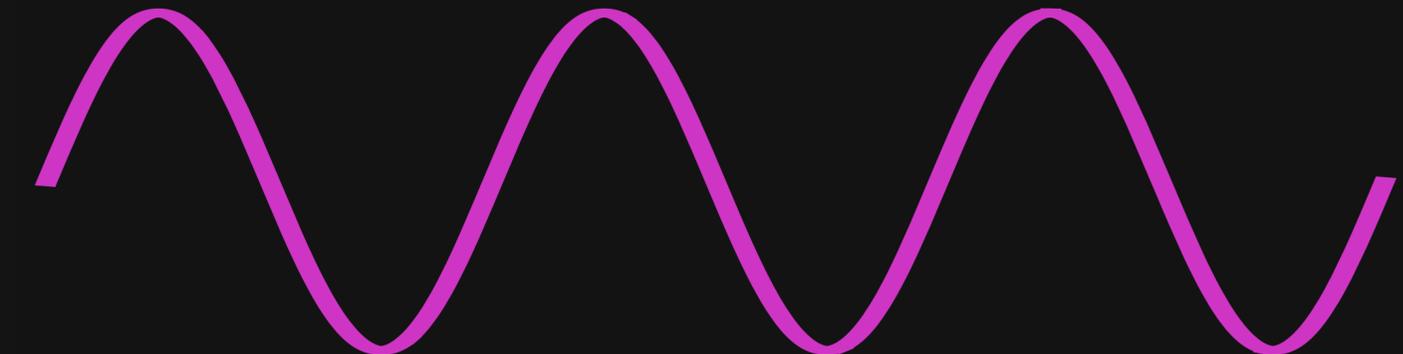
$$\lambda = -1$$



$$\lambda = -4$$



$$\lambda = -9$$

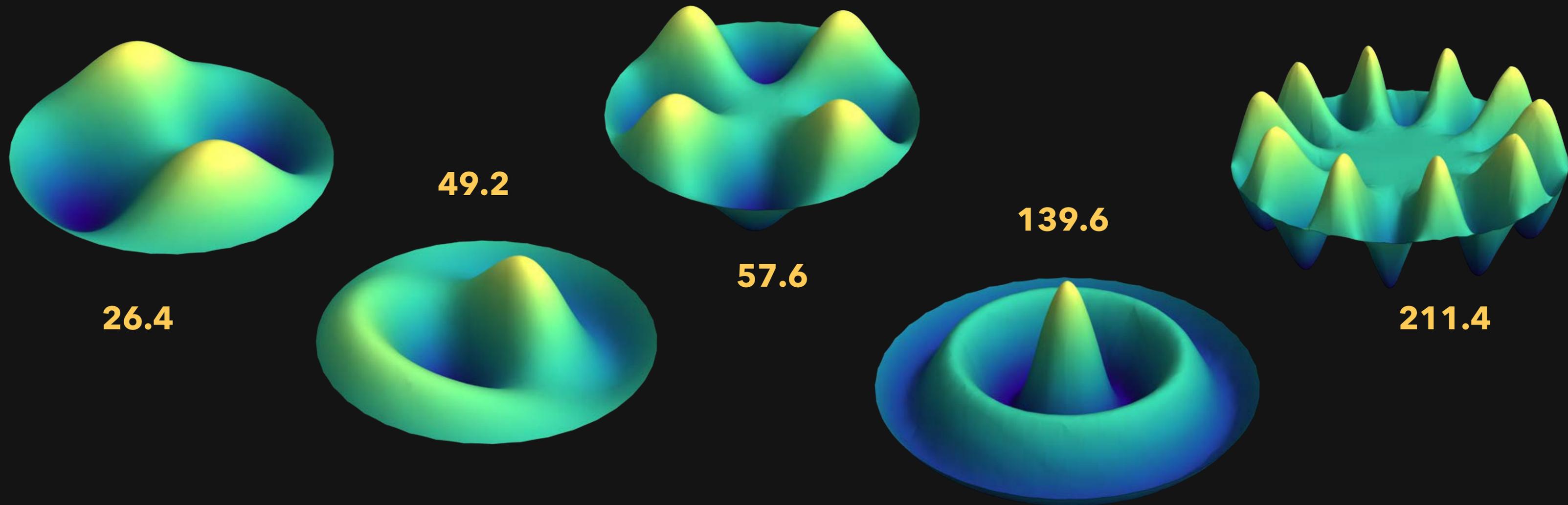


Interpretation of this:

Eigenvectors of $\frac{d^2}{dx^2}$ are the basic vibrational modes of a string.
The **eigenvalues** are the **frequencies**.

General solutions are **linear combinations of these simple solutions**, and we can find all solutions this way because the **simple solutions form a basis**.

The same thing works in higher dimensions:
eigenvectors of the 2nd derivative give simple
vibrational modes of a drum. Eigenvalues give the
frequencies (tones).



Really cool example:

Atomic Structure.

The equation for the wave function of an electron orbiting a nucleus.

$$i \frac{\partial \Psi}{\partial t} = \left(\frac{\nabla^2}{2m} + \frac{1}{4\pi\epsilon_0 r} \right) \Psi$$

Really cool example:

Atomic Structure.

The equation for the wave function of an electron orbiting a nucleus.

$$i \frac{\partial}{\partial t} \Psi = \left(\frac{\nabla^2}{2m} + \frac{1}{4\pi\epsilon_0 r} \right) \Psi$$

Really cool example:

Atomic Structure.

The equation for the wave function of an electron orbiting a nucleus.

$$i \frac{\partial}{\partial t} \Psi = H \Psi$$

Really cool example:

Atomic Structure.

The equation for the wave function of an electron orbiting a nucleus.

Look for simple solutions!

$$\Psi(\vec{p}, t) = v(t) \psi(\vec{p})$$

Really cool example:

Atomic Structure.

The equation for the wave function of an electron orbiting a nucleus.

$$i \frac{\partial}{\partial t} \Psi = H \Psi$$

Really cool example:

Atomic Structure.

The equation for the wave function of an electron orbiting a nucleus.

$$i \frac{\partial}{\partial t} \psi = H \psi$$

Really cool example:

Atomic Structure.

The equation for the wave function of an electron orbiting a nucleus.

$$\psi \ i \frac{\partial}{\partial t} \nu = \nu \ H \ \psi$$

Really cool example:

Atomic Structure.

Associated eigenvector equation

$$\frac{\partial}{\partial t} v = -i\lambda v$$

$$H\psi = \lambda\psi$$

Really cool example:

Atomic Structure.

Associated eigenvector equation

$$v = e^{-i\lambda t}$$

$$H\psi = \lambda\psi$$

Really cool example:

Atomic Structure.

Associated eigenvector equation

$$H\psi = \lambda \psi$$

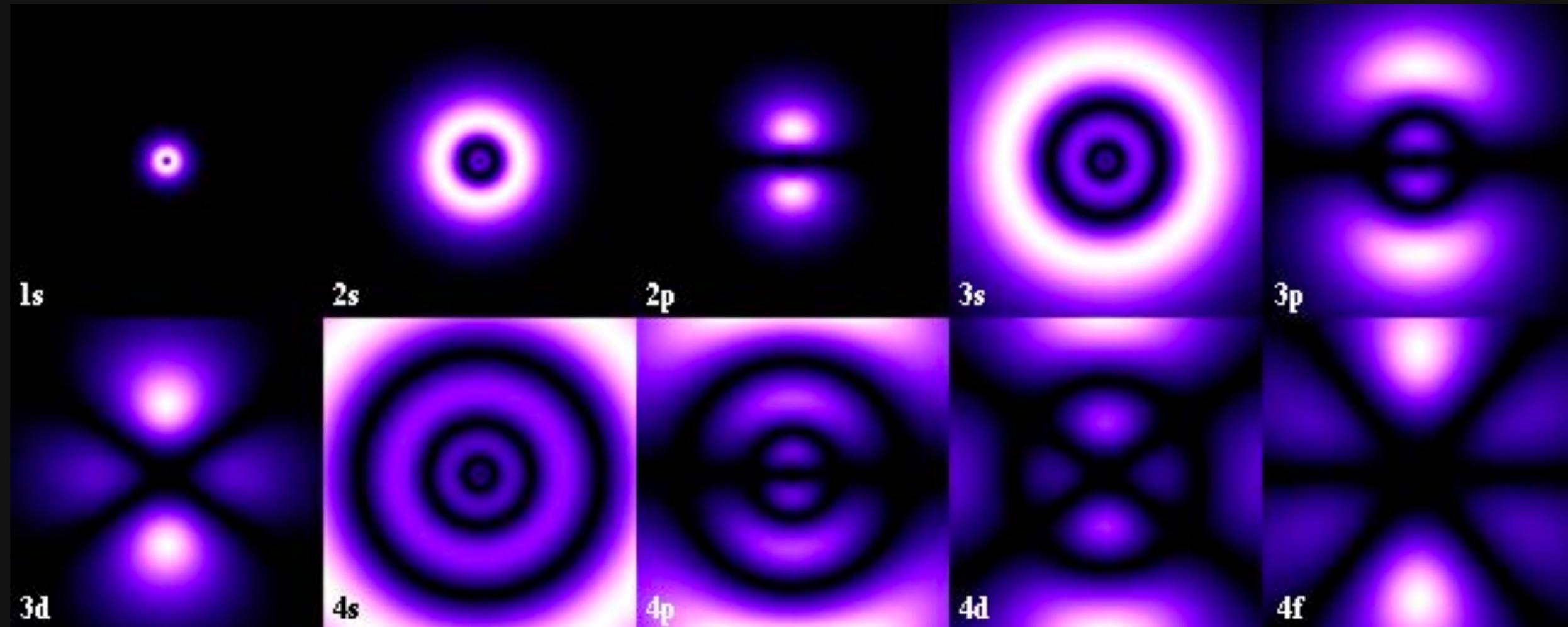
Solutions give *basic states* for the electron.

Eigenvalue gives the energy of that state.

Really cool example:

Atomic Structure.

A few of the eigenvectors.



Really cool example:

Atomic Structure.

To understand the simple states of an electron in an atom, we need to find:

Eigenvalues of H

Eigenvectors of H

Really cool example:

Atomic Structure.

First lets look at the eigenvalues:

Eigenvalues of H $\lambda_n = -\frac{13.6}{n^2} \quad n = 1, 2, 3, \dots$

This tells us the **amount of energy that electrons are allowed to have** when orbiting a nucleus.

Really cool example:

Atomic Structure.

How does an electron change from **one basic state to another**? It needs a **very precise amount of energy** to do so.

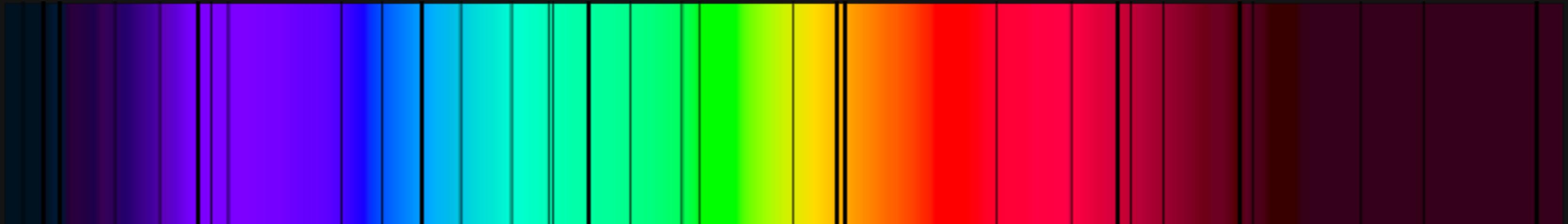
$$\lambda_n - \lambda_m = \frac{13.6}{n^2 - m^2}$$

Really cool example:

Atomic Structure.

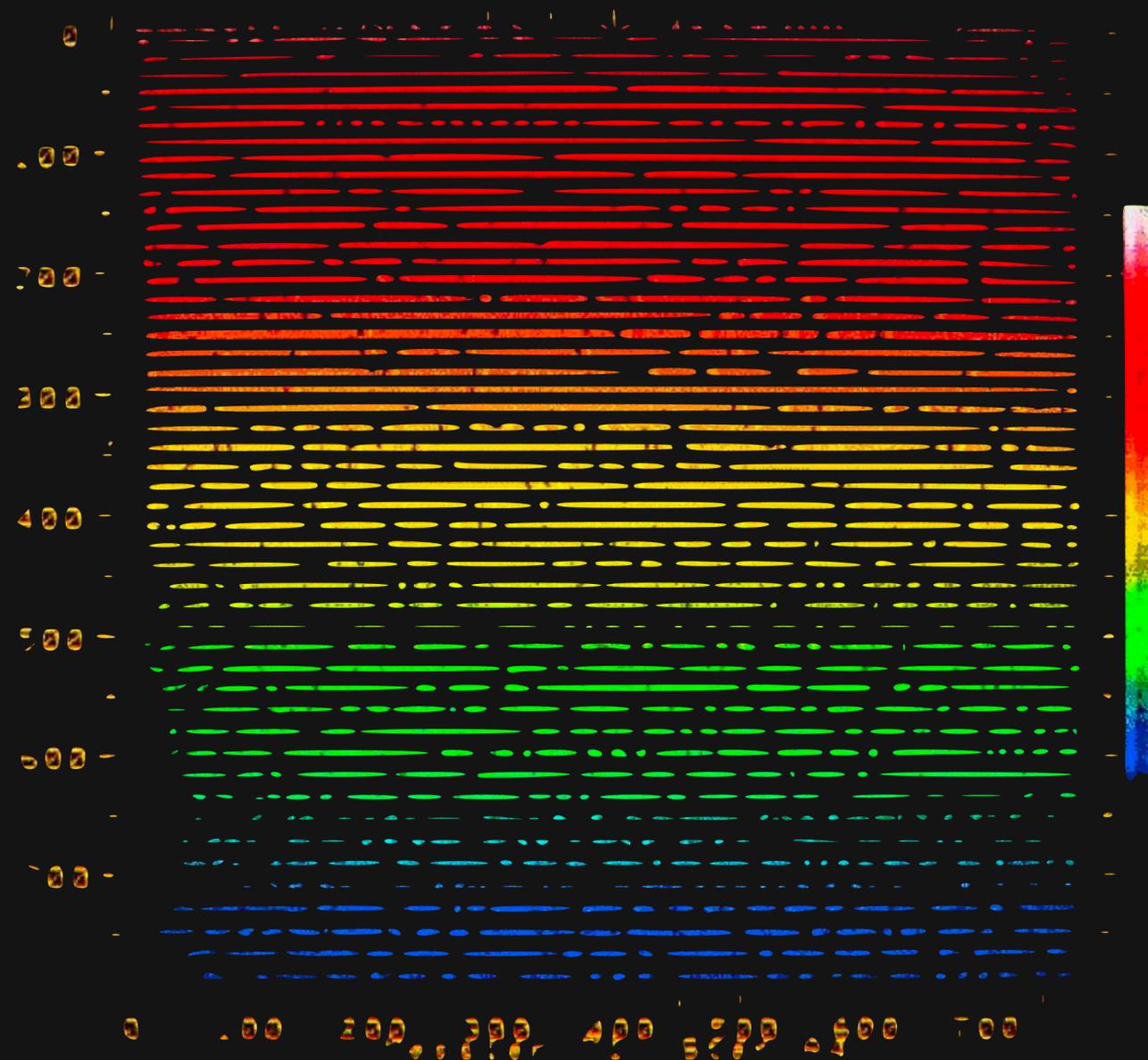
This list of energies $\lambda_n - \lambda_m$ shows up in
sunlight!

It's how we **discovered helium.**



Really cool example:

Atomic Structure.



Carefully measuring the spectrum of a star as an exoplanet passes in front of it tells us which molecules are in its atmosphere.

Really cool example:

Atomic Structure.

There's still more to learn from the eigenvectors of the Schrodinger equation for the atom.

What is the dimension of each eigenspace?

Really cool example:

Atomic Structure.

 λ_1

$$\dim E_{\lambda_1} = 1$$

 λ_2

$$\dim E_{\lambda_2} = 4$$

 \vdots \vdots λ_n

$$\dim E_{\lambda_n} = n^2$$

Really cool example:

Atomic Structure.

$$\Psi_1^1$$

$$\Psi_1^2, \Psi_2^2, \Psi_3^2, \Psi_4^2$$

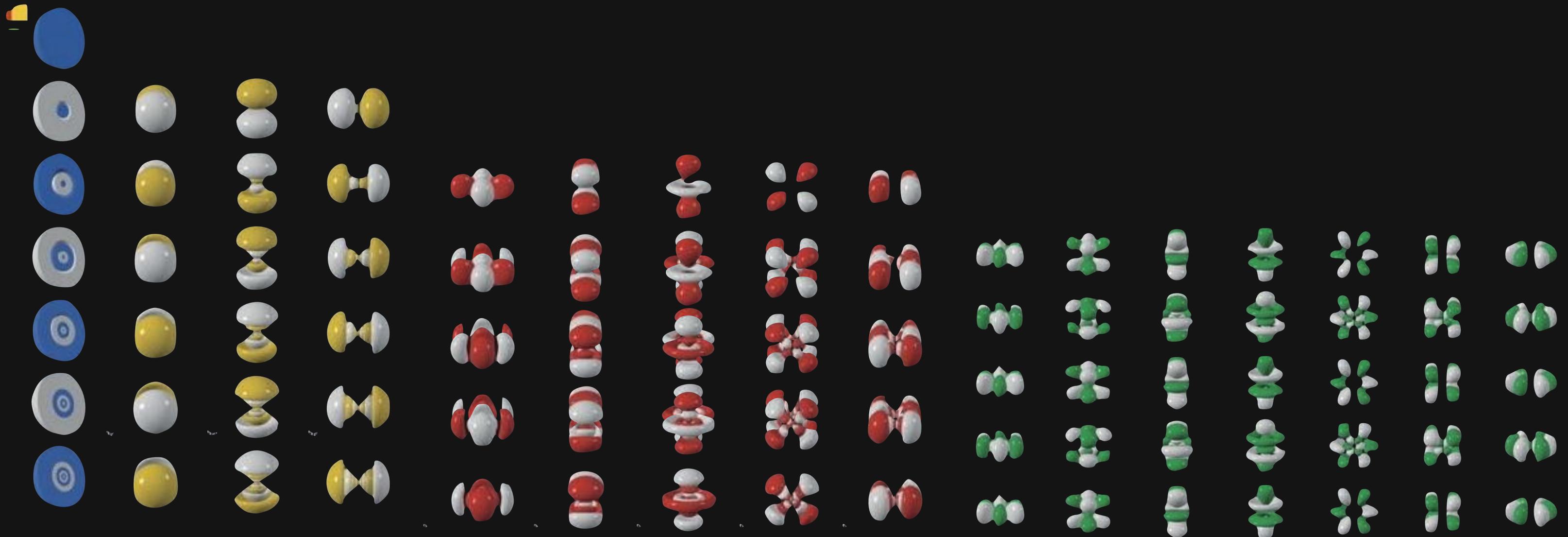
$$\Psi_1^3, \Psi_2^3, \Psi_3^3, \Psi_4^3, \Psi_5^3, \Psi_6^3, \Psi_7^3, \Psi_8^3, \Psi_9^3$$

$$\Psi_1^n, \Psi_2^n, \Psi_3^n, \Psi_4^n, \Psi_5^n \dots \dots \dots \Psi_{n^2-1}^n, \Psi_{n^2}^n$$

Really cool example:

Atomic Structure.

When we list out the eigenspaces...

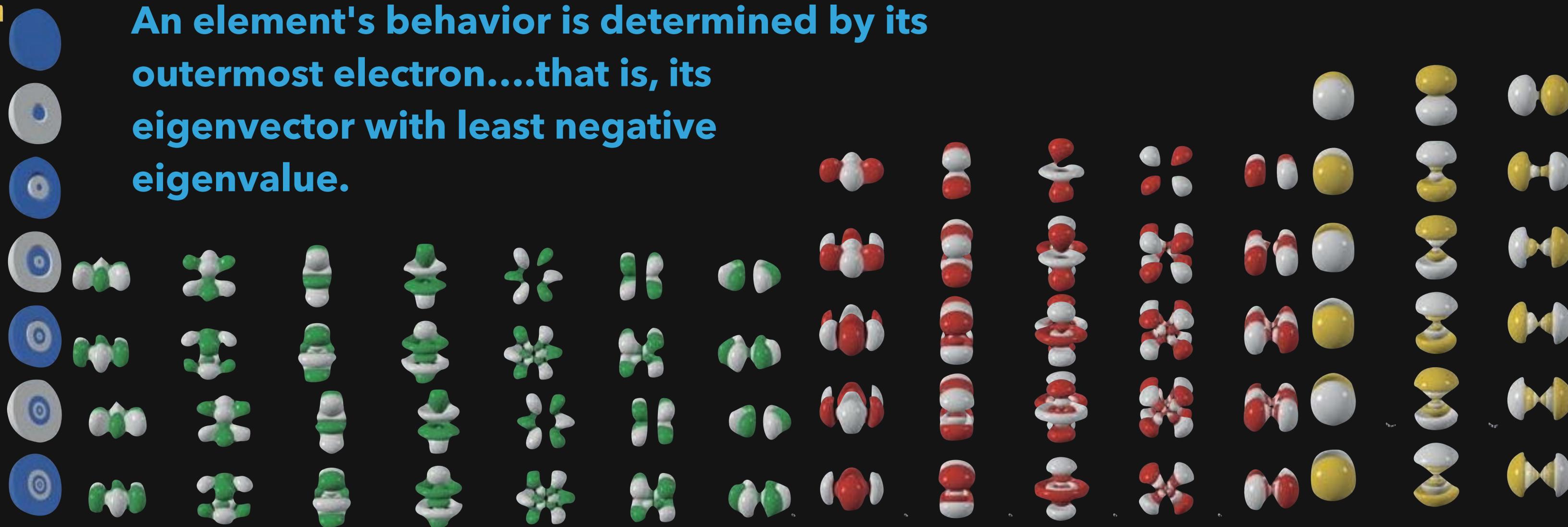


Really cool example:

Atomic Structure.

When we list out the eigenspaces...

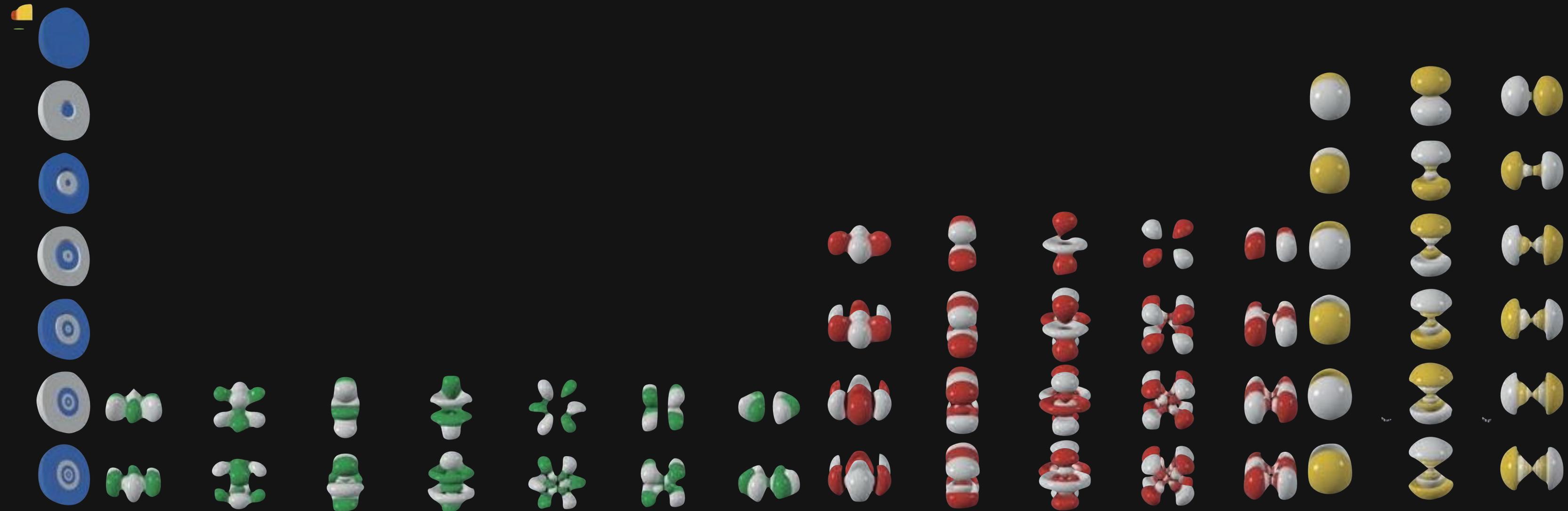
An element's behavior is determined by its outermost electron...that is, its eigenvector with least negative eigenvalue.



Really cool example:

Atomic Structure.

When we list out the eigenspaces...

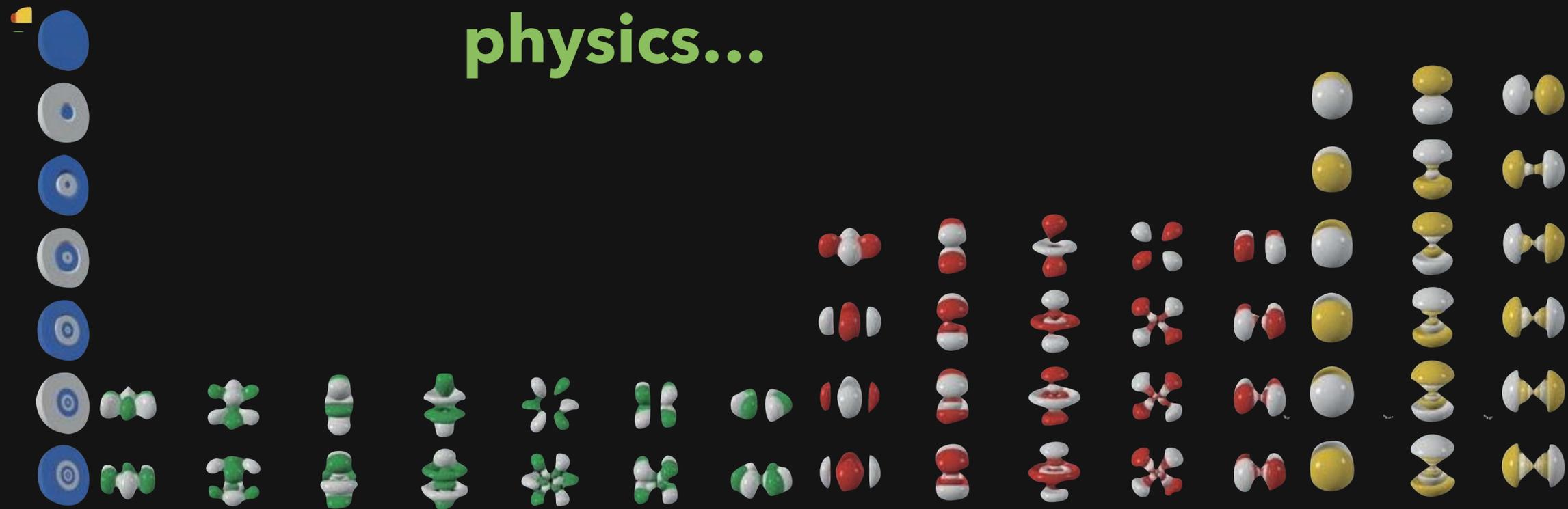


Really cool example:

Atomic Structure.

When we list out the eigenspaces...

And then pay a little closer attention to the physics...



Really cool example:

Atomic Structure.

Group →	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
Period ↓	1 1 H																	2 He
2	3 Li	4 Be											5 B	6 C	7 N	8 O	9 F	10 Ne
3	11 Na	12 Mg											13 Al	14 Si	15 P	16 S	17 Cl	18 Ar
4	19 K	20 Ca	21 Sc	22 Ti	23 V	24 Cr	25 Mn	26 Fe	27 Co	28 Ni	29 Cu	30 Zn	31 Ga	32 Ge	33 As	34 Se	35 Br	36 Kr
5	37 Rb	38 Sr	39 Y	40 Zr	41 Nb	42 Mo	43 Tc	44 Ru	45 Rh	46 Pd	47 Ag	48 Cd	49 In	50 Sn	51 Sb	52 Te	53 I	54 Xe
6	55 Cs	56 Ba	57 La	72 Hf	73 Ta	74 W	75 Re	76 Os	77 Ir	78 Pt	79 Au	80 Hg	81 Tl	82 Pb	83 Bi	84 Po	85 At	86 Rn
7	87 Fr	88 Ra	89 Ac	104 Rf	105 Db	106 Sg	107 Bh	108 Hs	109 Mt	110 Ds	111 Rg	112 Cn	113 Nh	114 Fl	115 Mc	116 Lv	117 Ts	118 Og
				58 Ce	59 Pr	60 Nd	61 Pm	62 Sm	63 Eu	64 Gd	65 Tb	66 Dy	67 Ho	68 Er	69 Tm	70 Yb	71 Lu	
				90 Th	91 Pa	92 U	93 Np	94 Pu	95 Am	96 Cm	97 Bk	98 Cf	99 Es	100 Fm	101 Md	102 No	103 Lr	

This is just the beginning....

**There's always
mysteries pointing
towards exciting
new mathematics...**



**One of my favorites:
gold is yellow.**