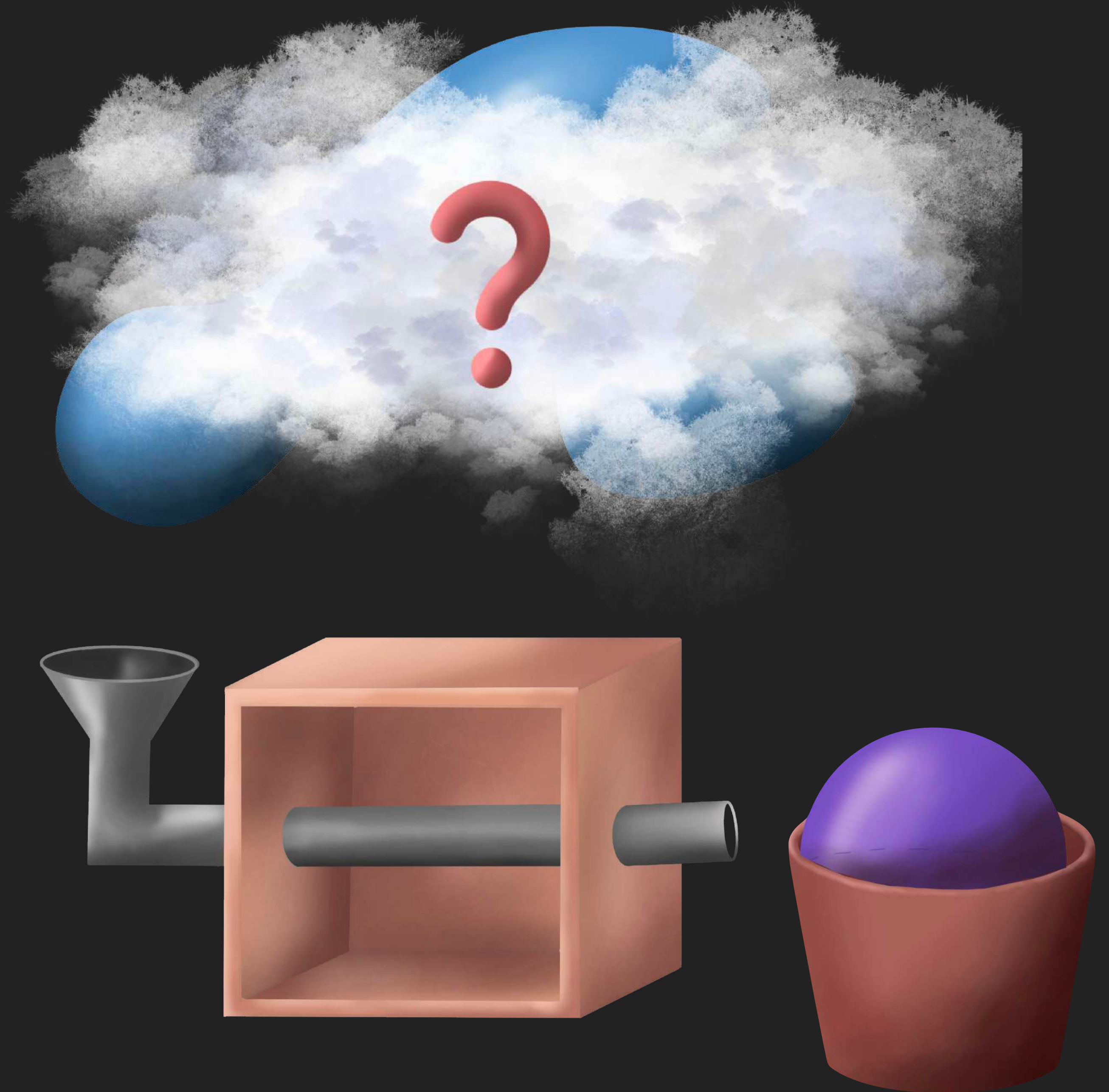


Geometry,
Topology
and the
**Poincaré
Conjecture**

Steve Trettel
University of San Francisco



Math is a conversation Spanning Centuries



Euclid, 300BCE

*“Is my 5th axiom actually a
consequence of the other 4?”*

2100 years



Bolyai, Lobachevsky, Gauss, 1800CE

*“No! There is more to geometry than
you could have imagined”*

Math is a conversation Spanning Centuries



Archimedes 250 BCE

“Is π irrational?”

2010 years



Johann Lambert, 1760CE

“Yes!”

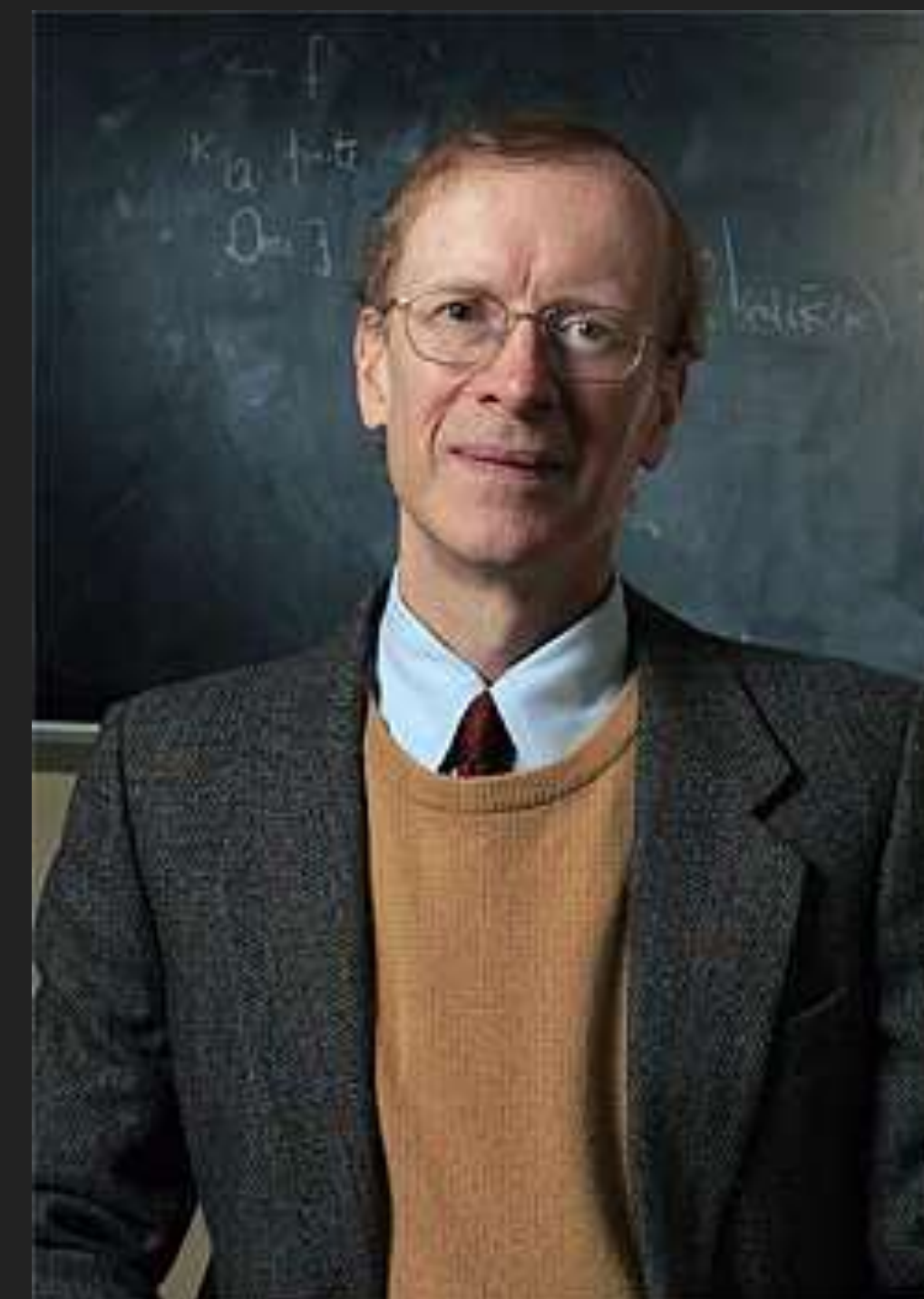
Math is a conversation Spanning Centuries



Fermat, 1637

*“Are there nontrivial
solutions to $x^n + y^n = z^n$?”*

358 years



Andrew Wiles, 1994

“There are not!”

Math is a conversation Spanning Centuries

Modern math holds a conference every century to lay out its biggest problems...



1900: Hilbert's List

100 years



2000: Millennium Problems





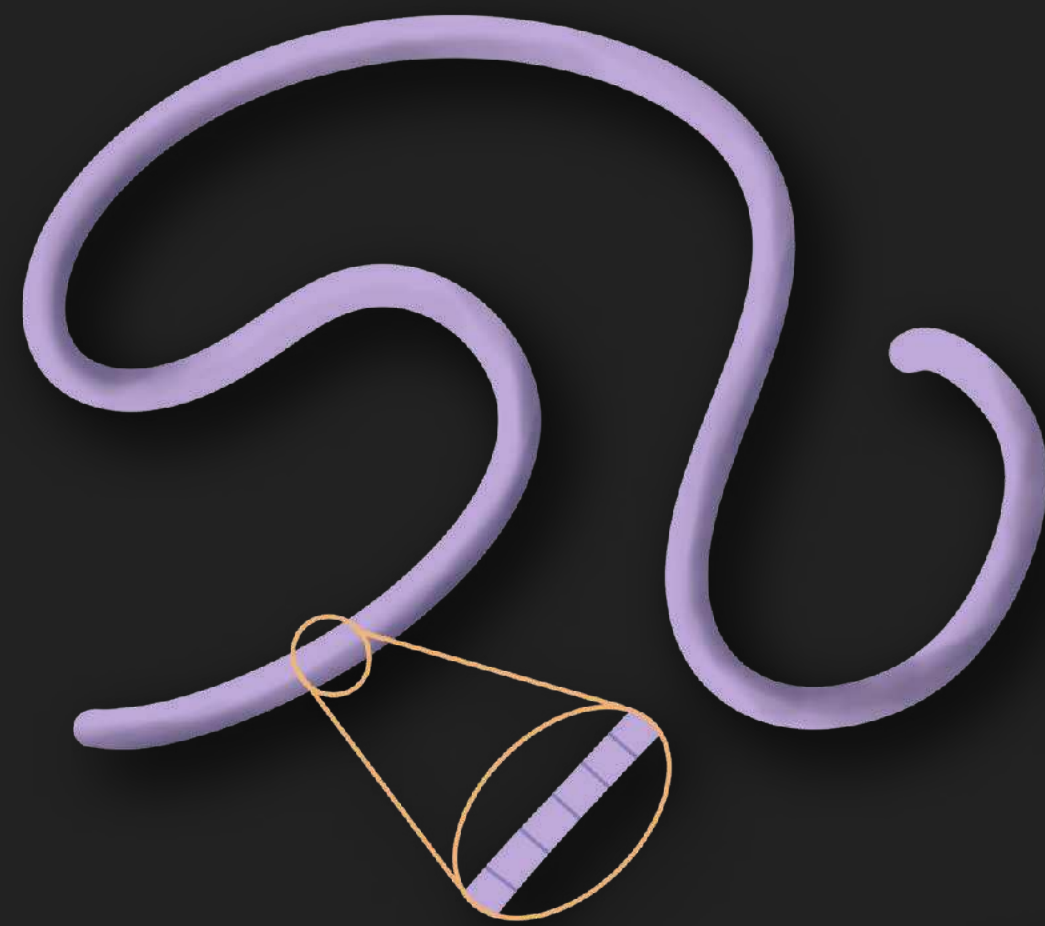
- **Birch and Swinnerton-Dyer conjecture:**
How well do we understand geometry with modular arithmetic?
- **Hodge Conjecture:**
How do topology and algebra interact when solving equations?
- **Navier Stokes:**
How well do we understand difficult differential equations?
- **P vs NP**
How well do we understand computation?
- **The Riemann Hypothesis**
How much does calculus know about prime numbers?
- **Yang Mills Mass Gap**
Can we rigorously understand modern quantum theory?
- **The Poincare Conjecture**
How well do we understand shapes (manifolds)?



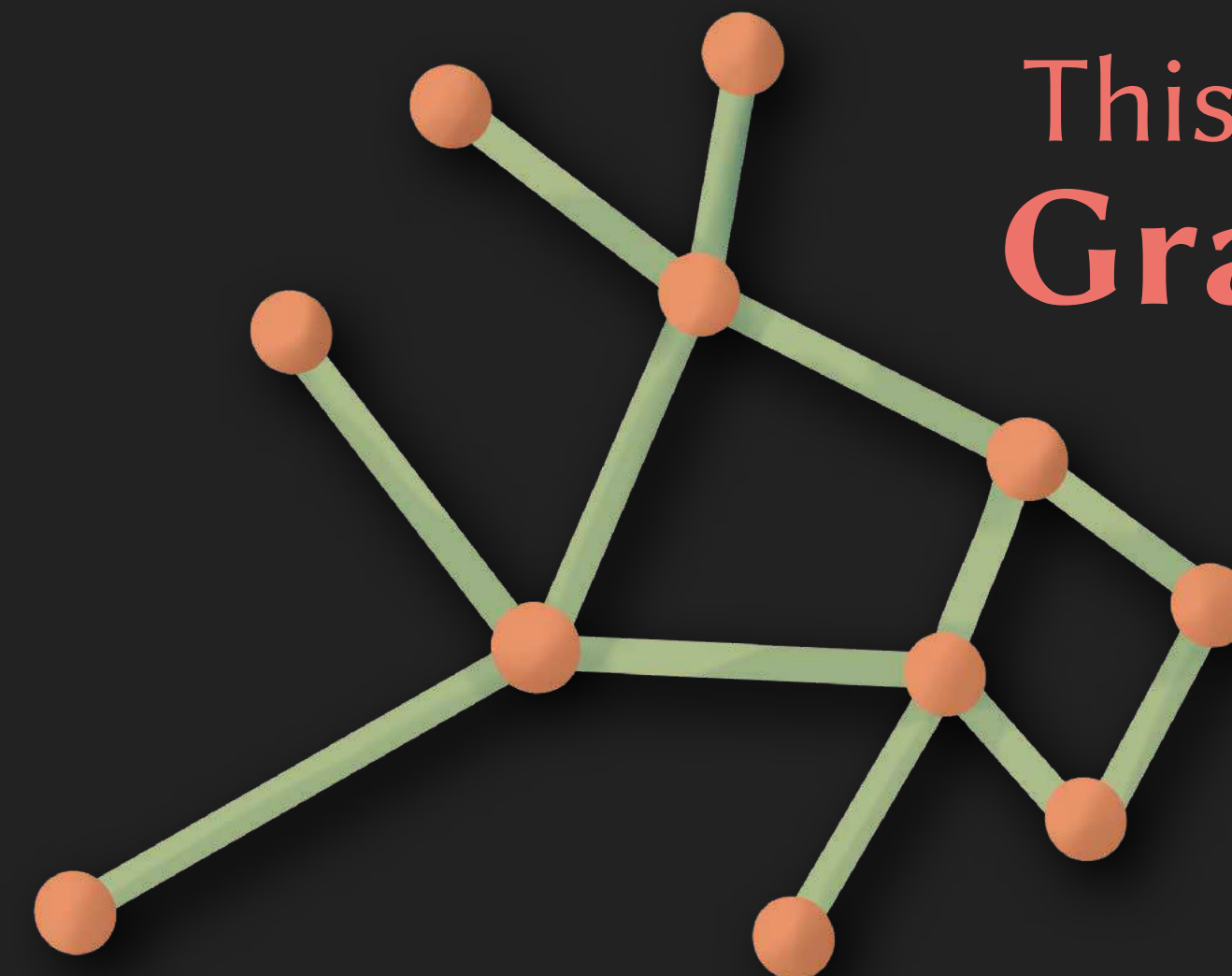
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How well do we understand shapes (manifolds)?

Manifolds

Spaces that look like $\mathbb{R}^n$ everywhere



This
Curve
is a
Manifold

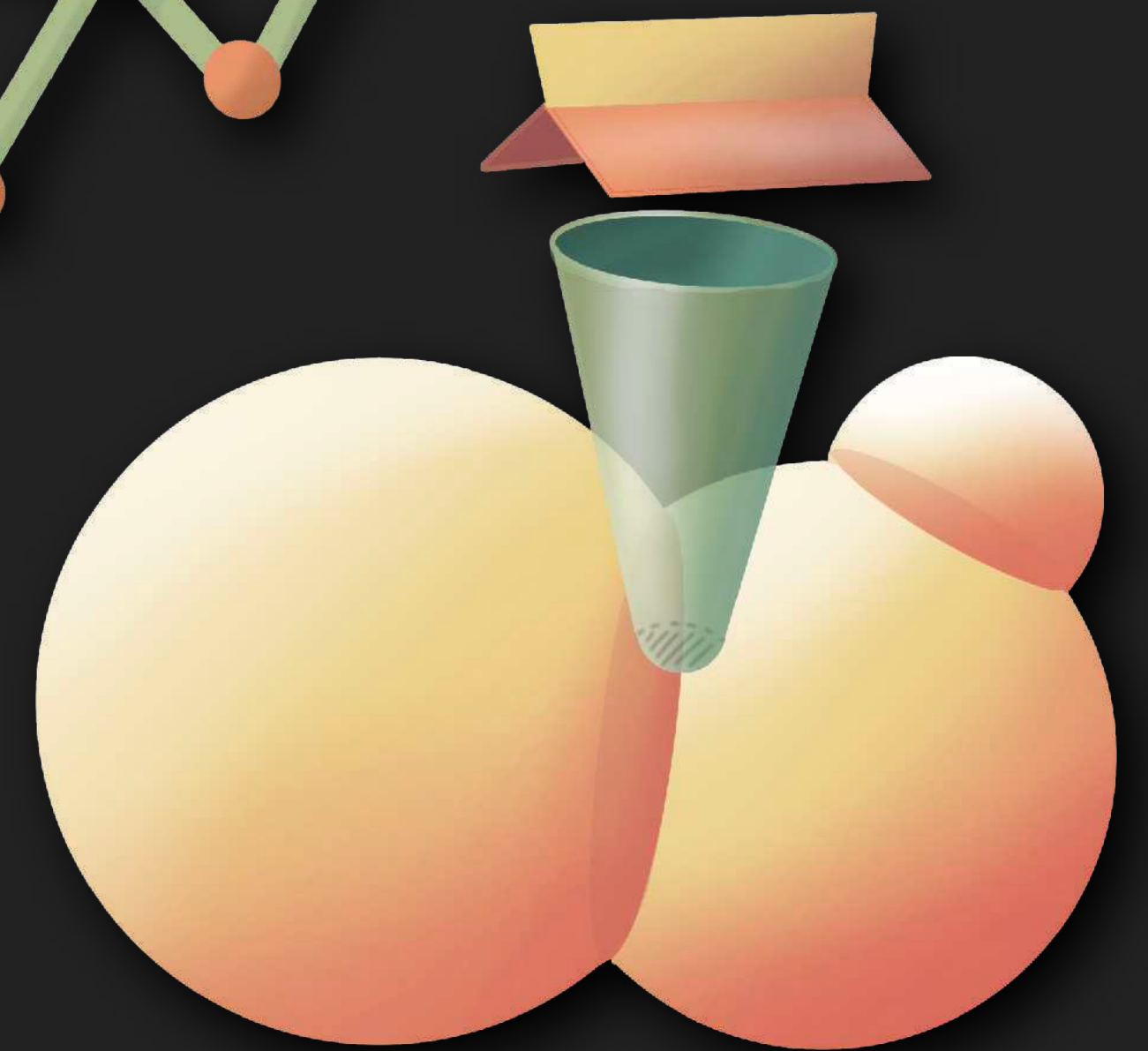


This
Graph is
Not



This
Surface
is a
Manifold

Soap
Bubbles
are
Not

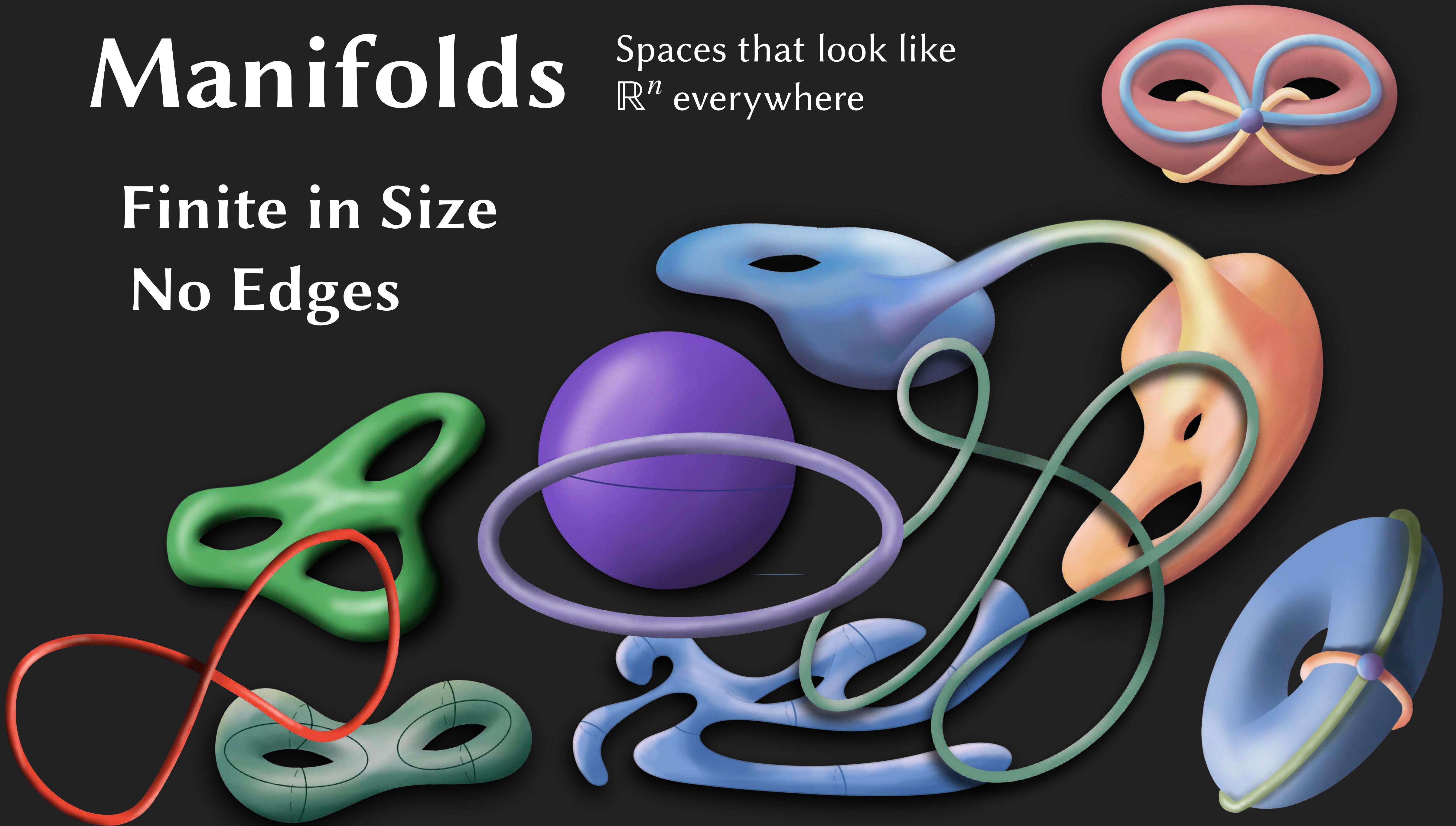


Manifolds

Spaces that look like $\mathbb{R}^n$ everywhere

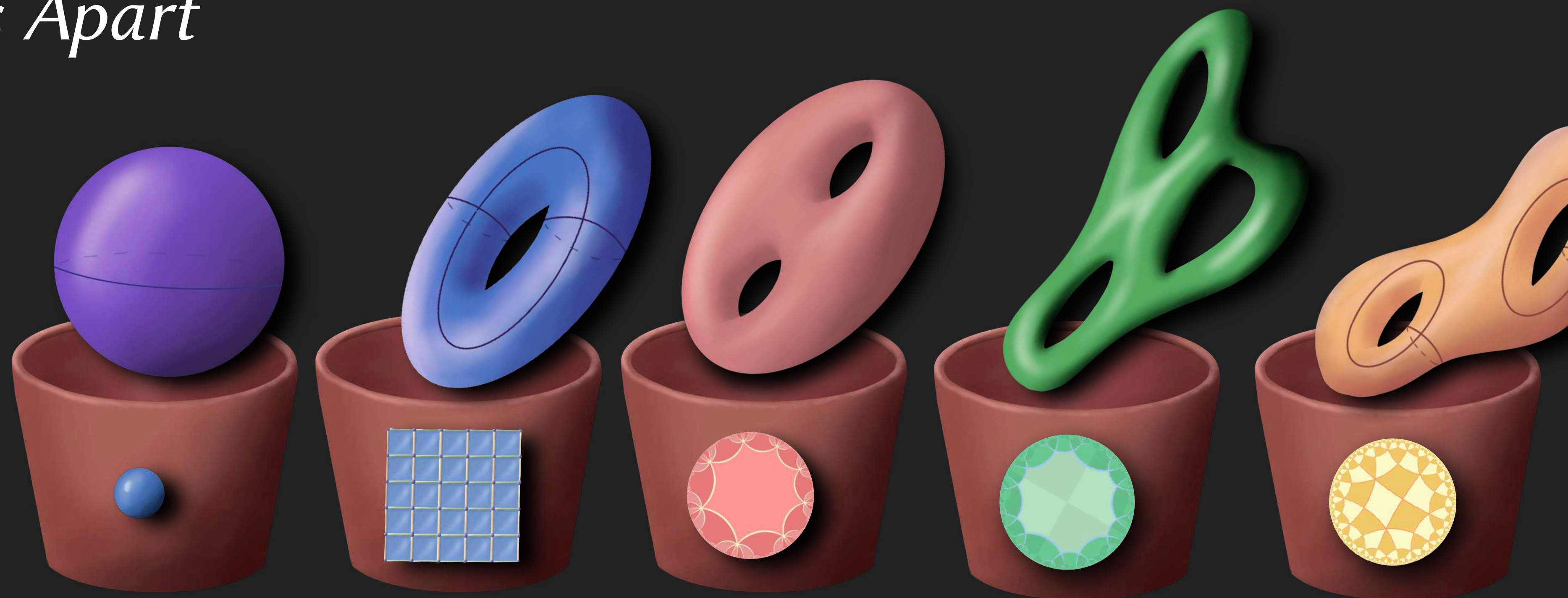
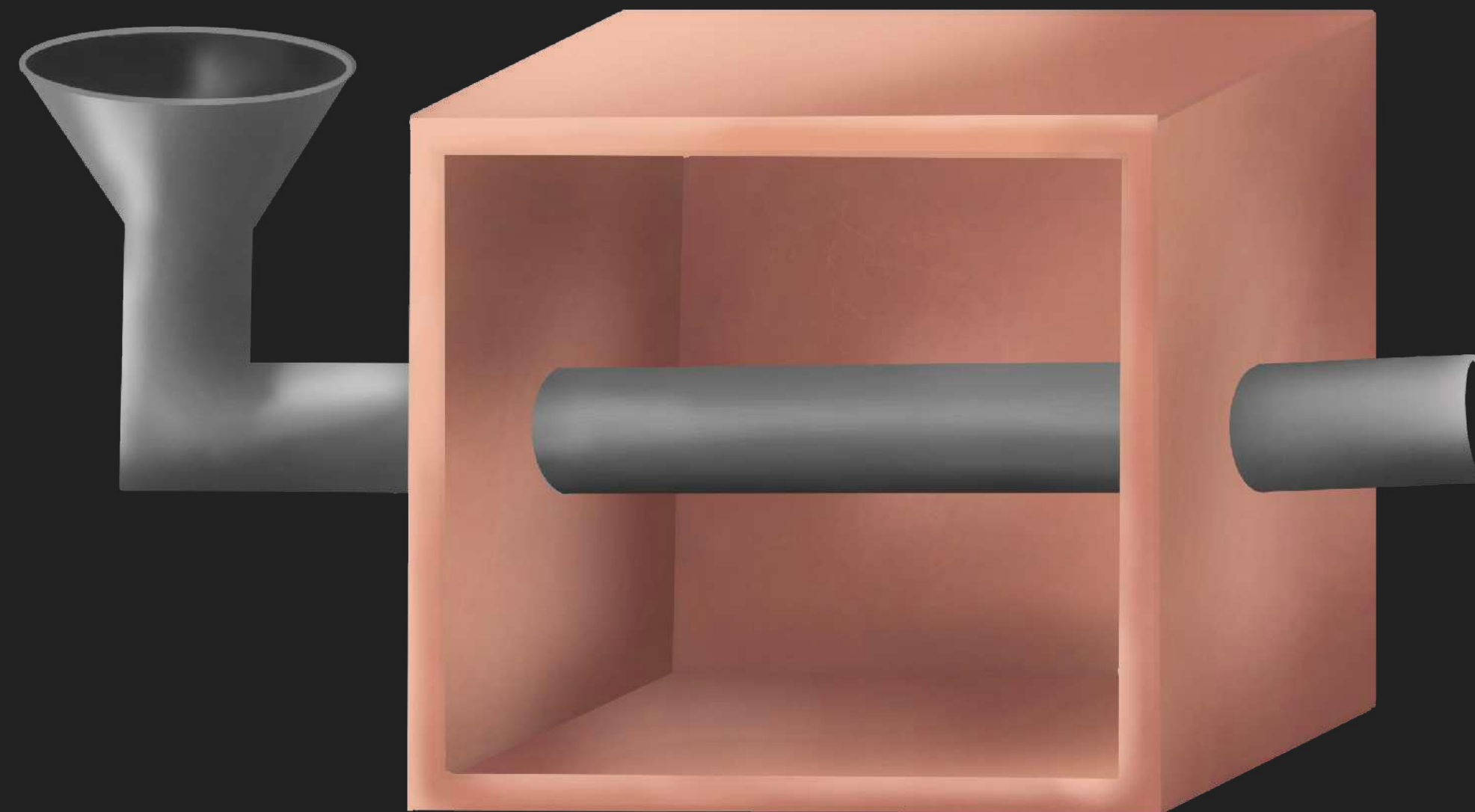
Finite in Size

No Edges



Part I

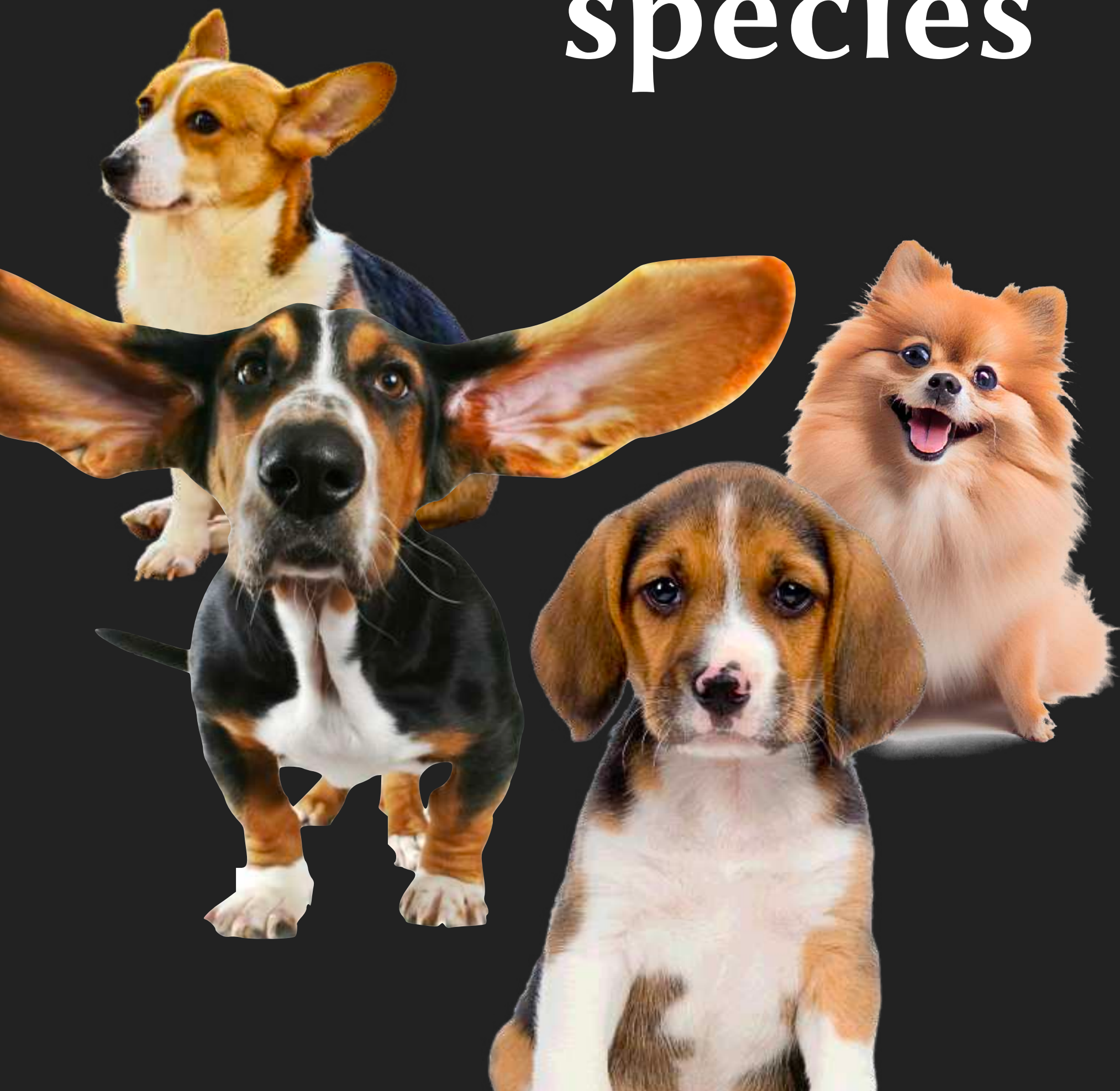
Telling Shapes Apart



When are
Two Things ?
the **Same** •



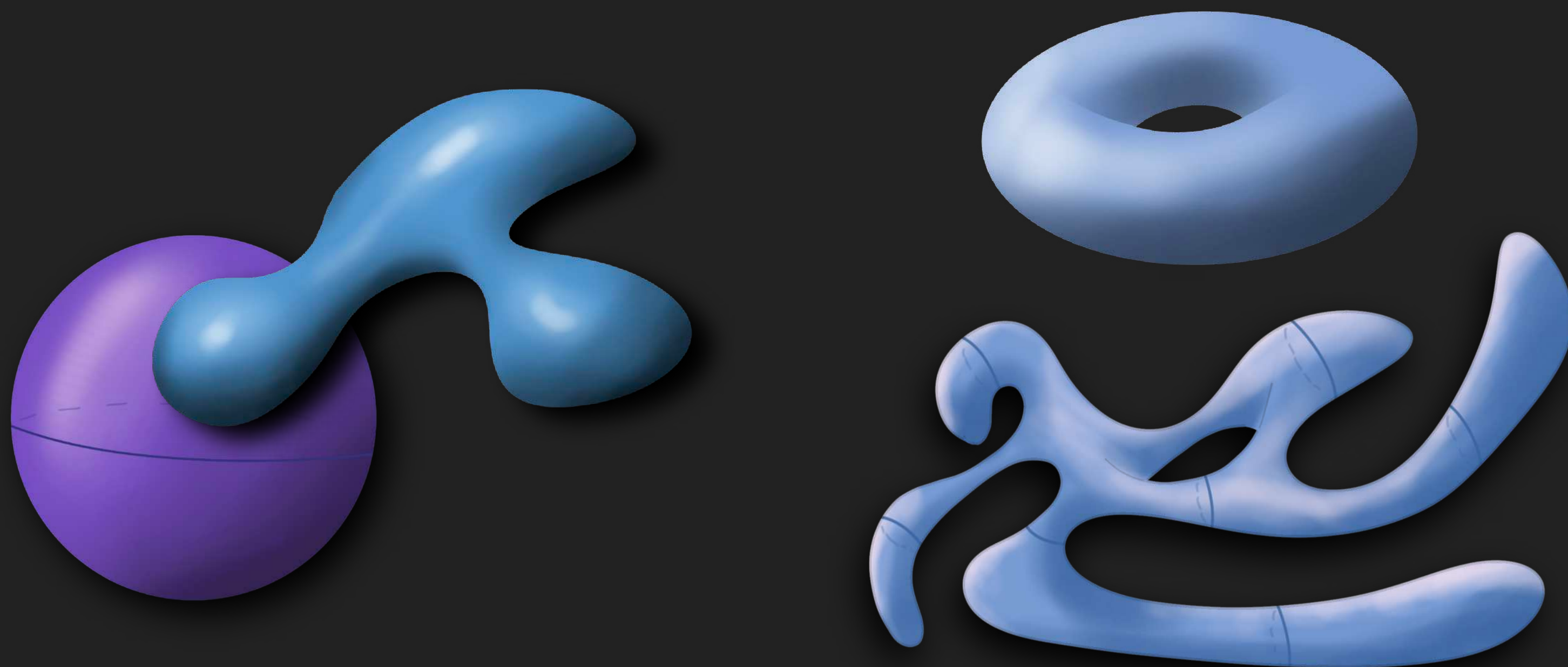
For **animals**,
individuals are
classified by
species



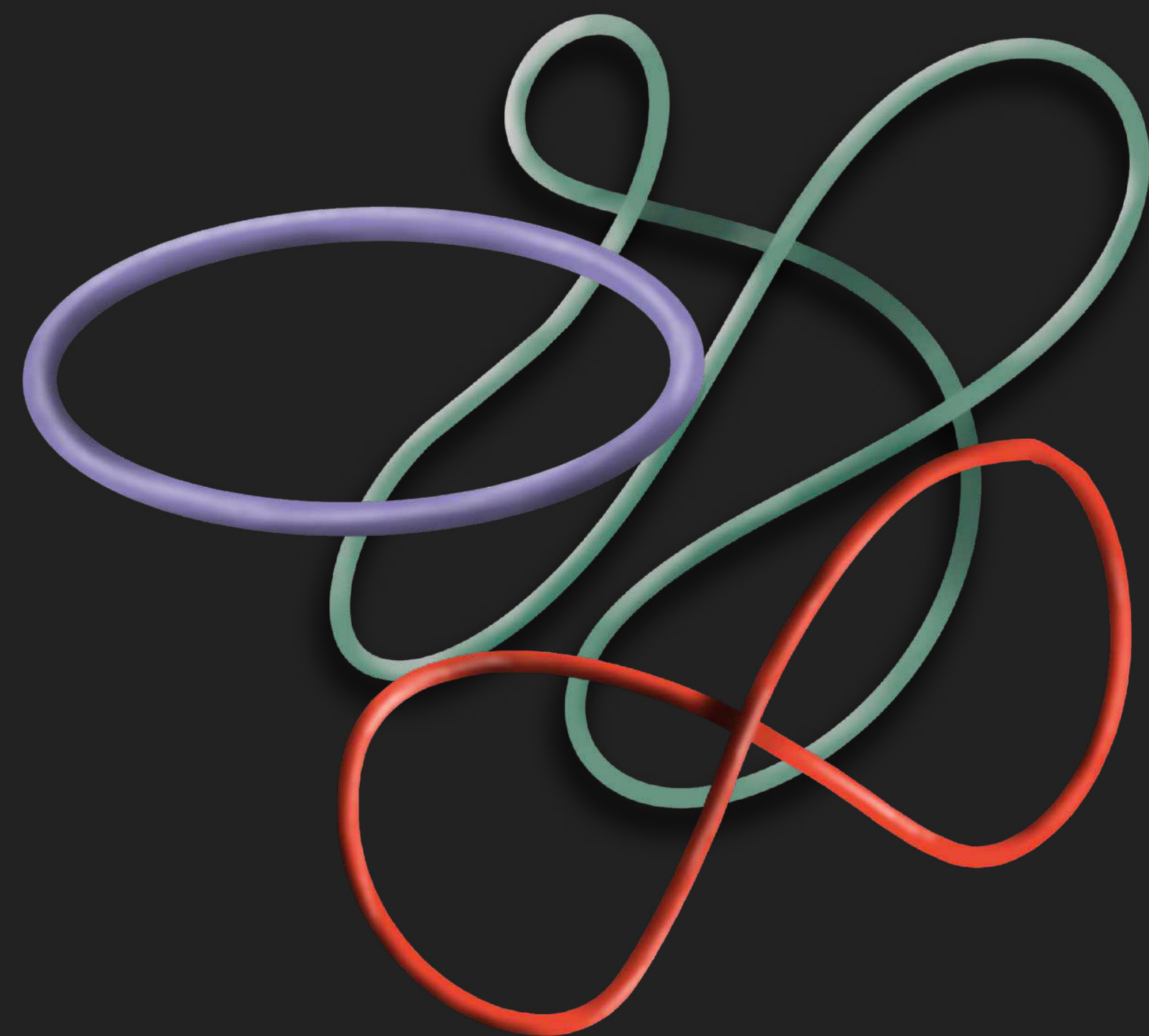
When are
Two Things ?
the **Same** •



For **shapes**,
individuals are
classified by
homeomorphism

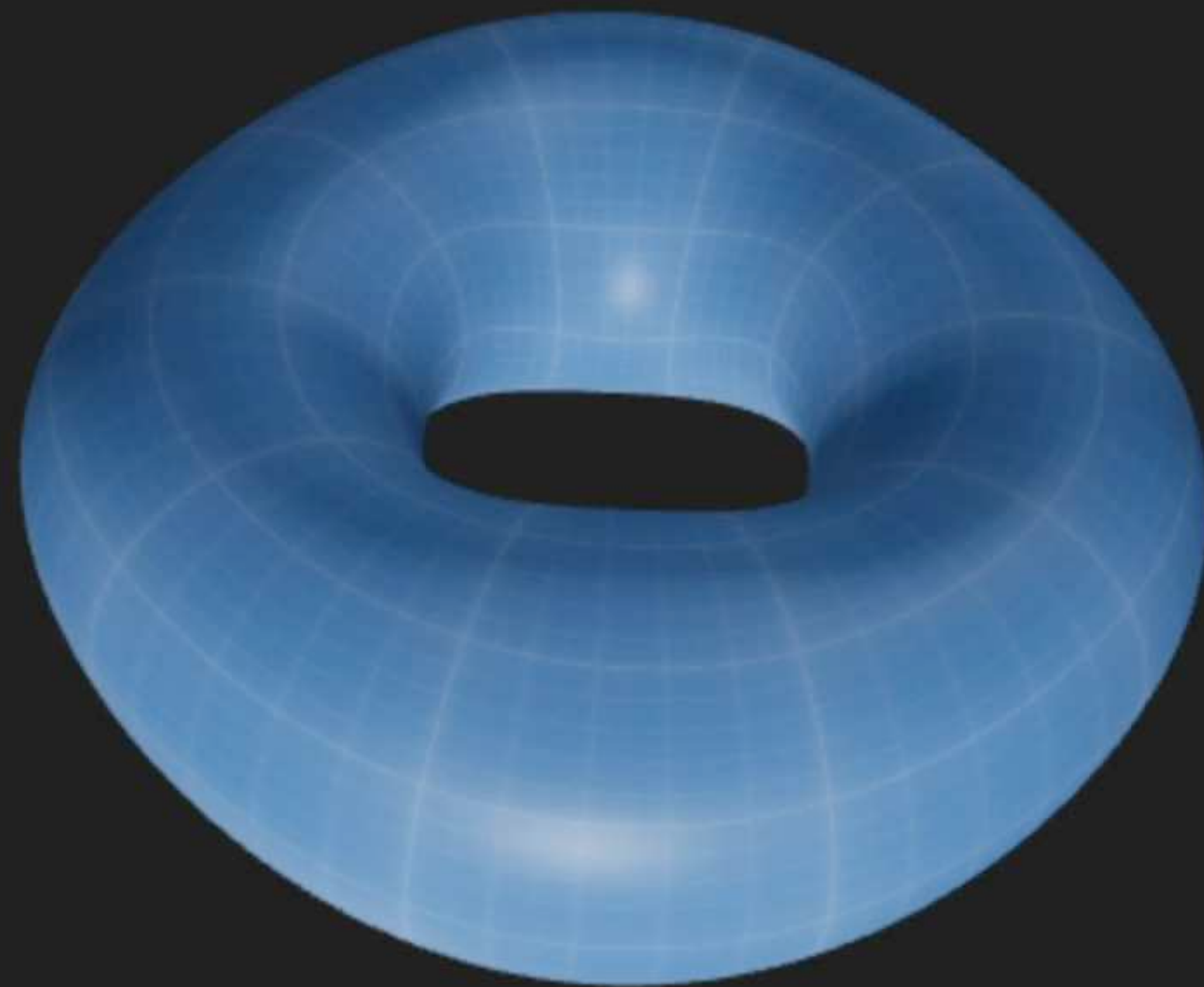


When are
Two Things ?
the **Same** •



When are
Two Things
the **Same** ?

Can be
continuously
deformed
into one another,
without
tearing



*This is always
a torus to
topologists!*

Sometimes it's hard
to tell by looking!

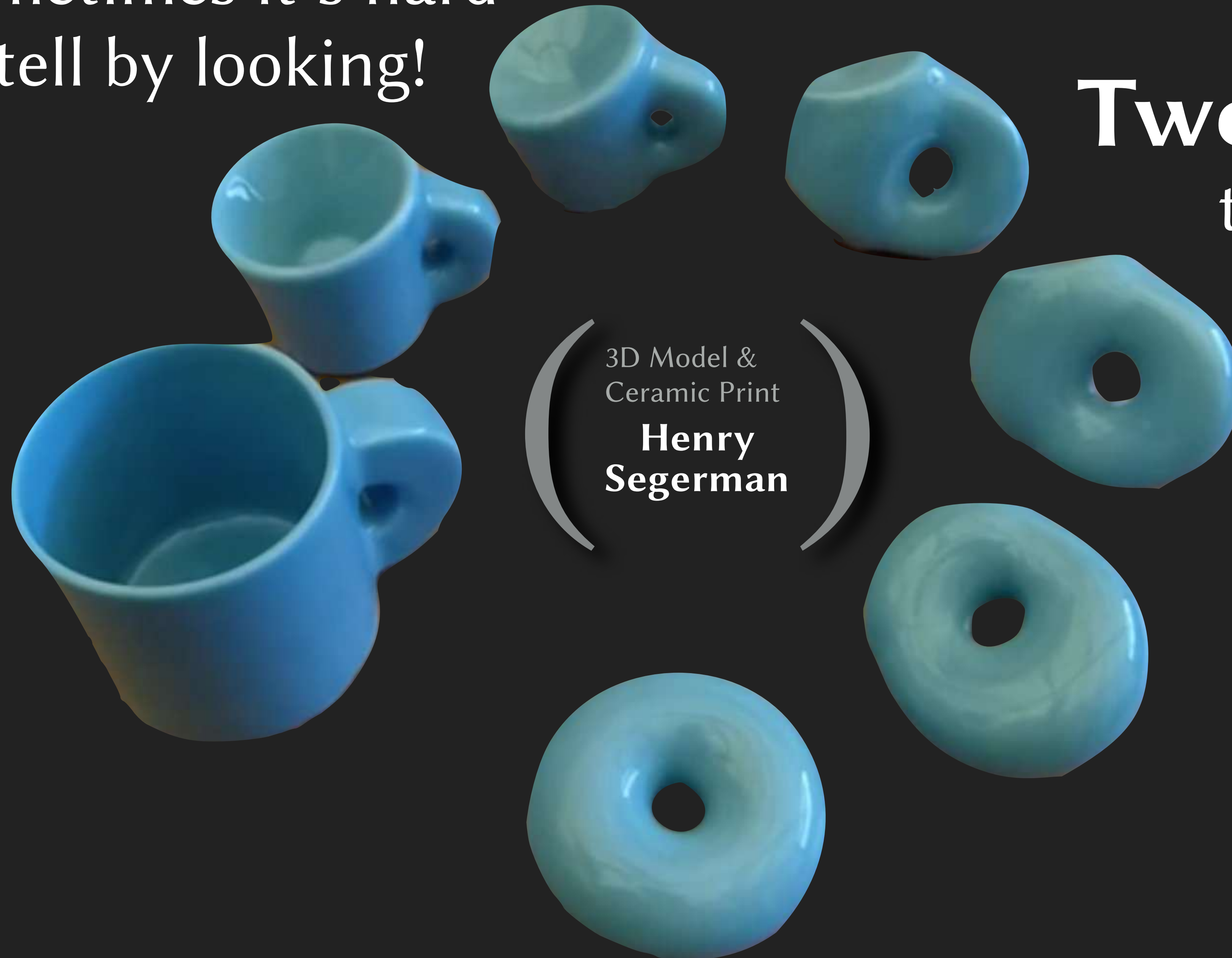


When are
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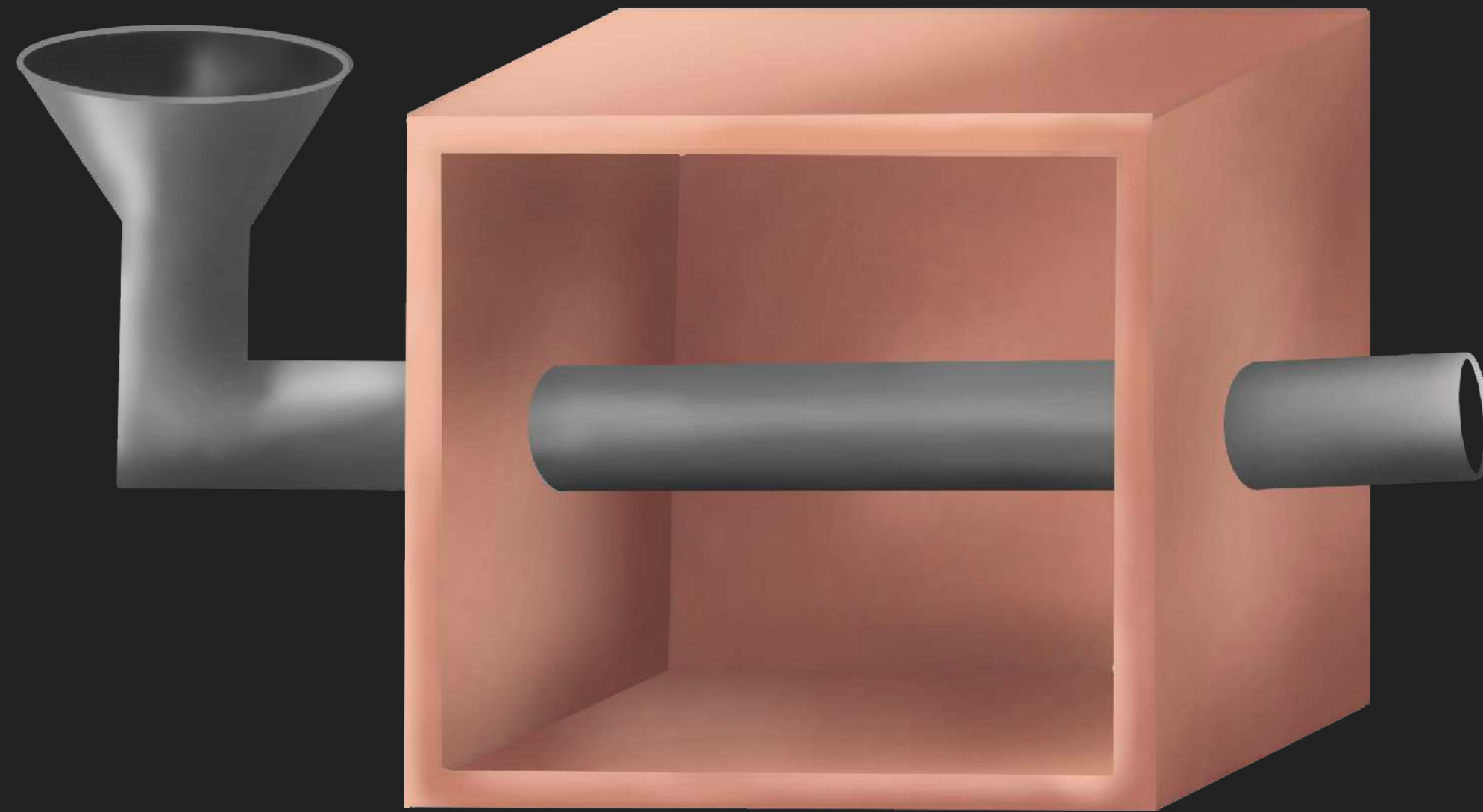
Sometimes it's hard
to tell by looking!

When are
Two Things
the **Same** ?



The coffee
cup & donut
are the same
topological
'species'

The idea of **Invariants**



*A quantity that is constant
on objects of the same
type is called an invariant.*

When are
Two Things
the **Same** ?

*The value of an invariant can
be used to sort objects.*



Biological Invariants

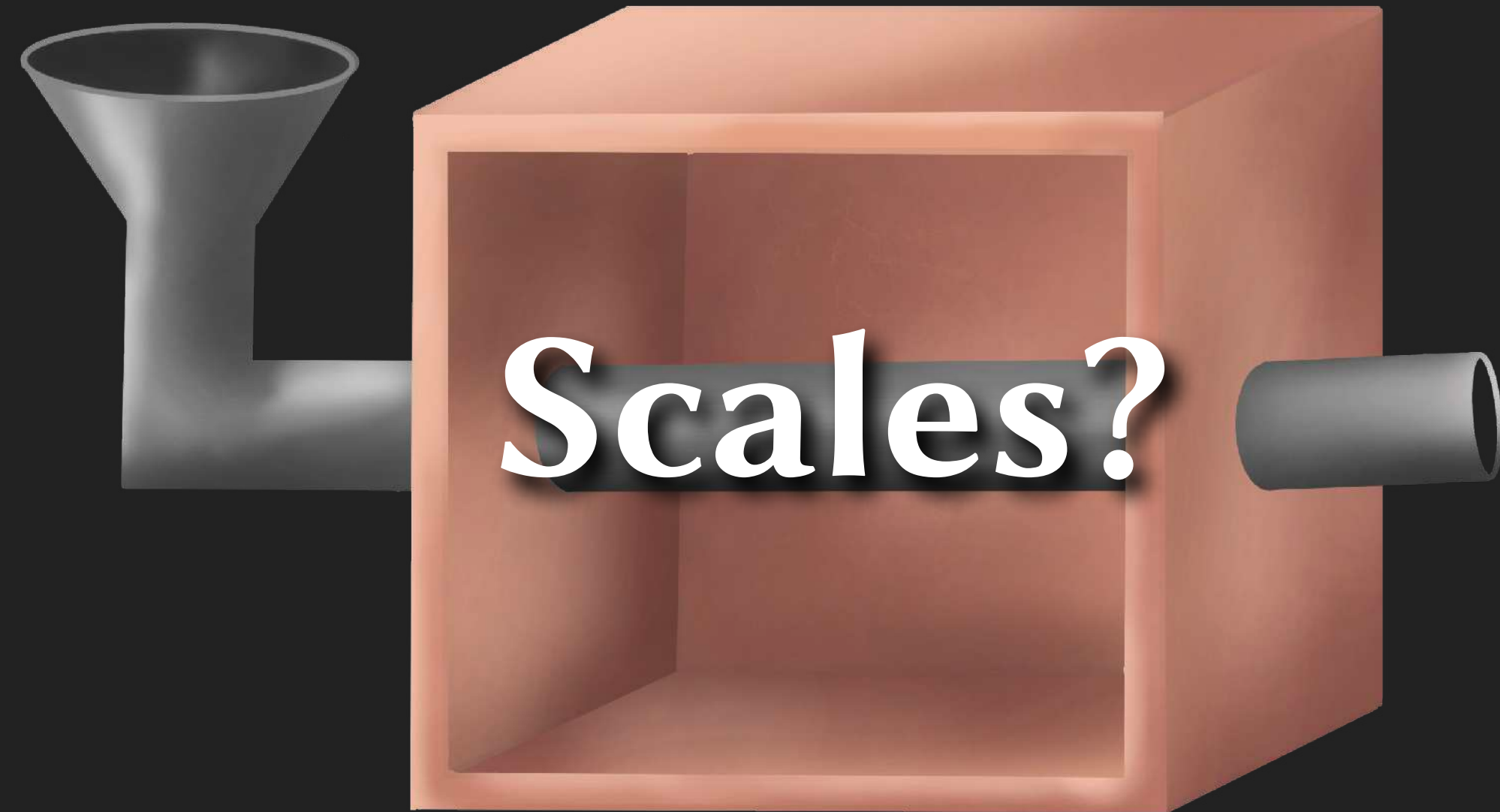


Biological **Invariants**

Gills?



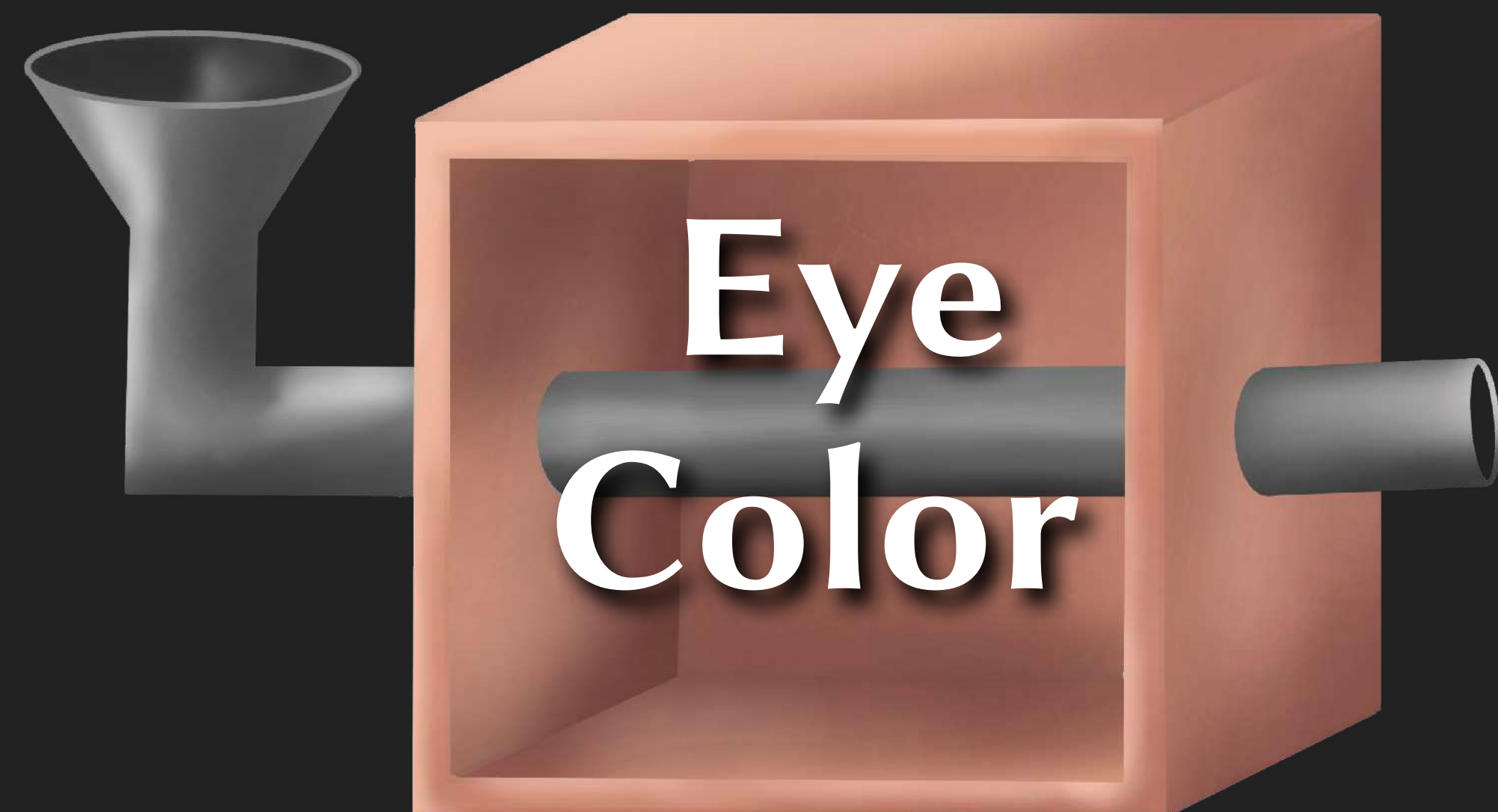
Biological **Invariants**



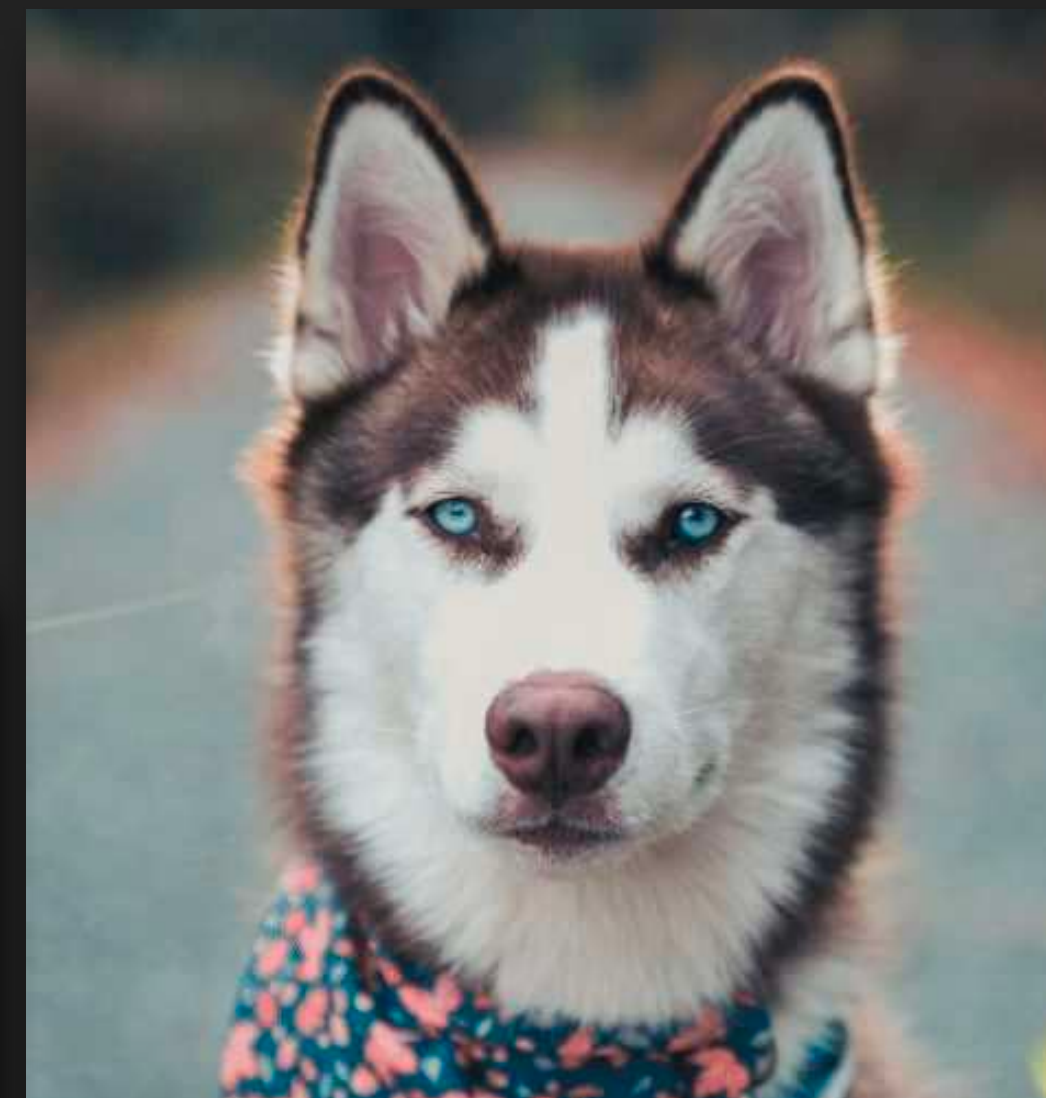
Biological Invariants

Class?





Biological Non-Invariants

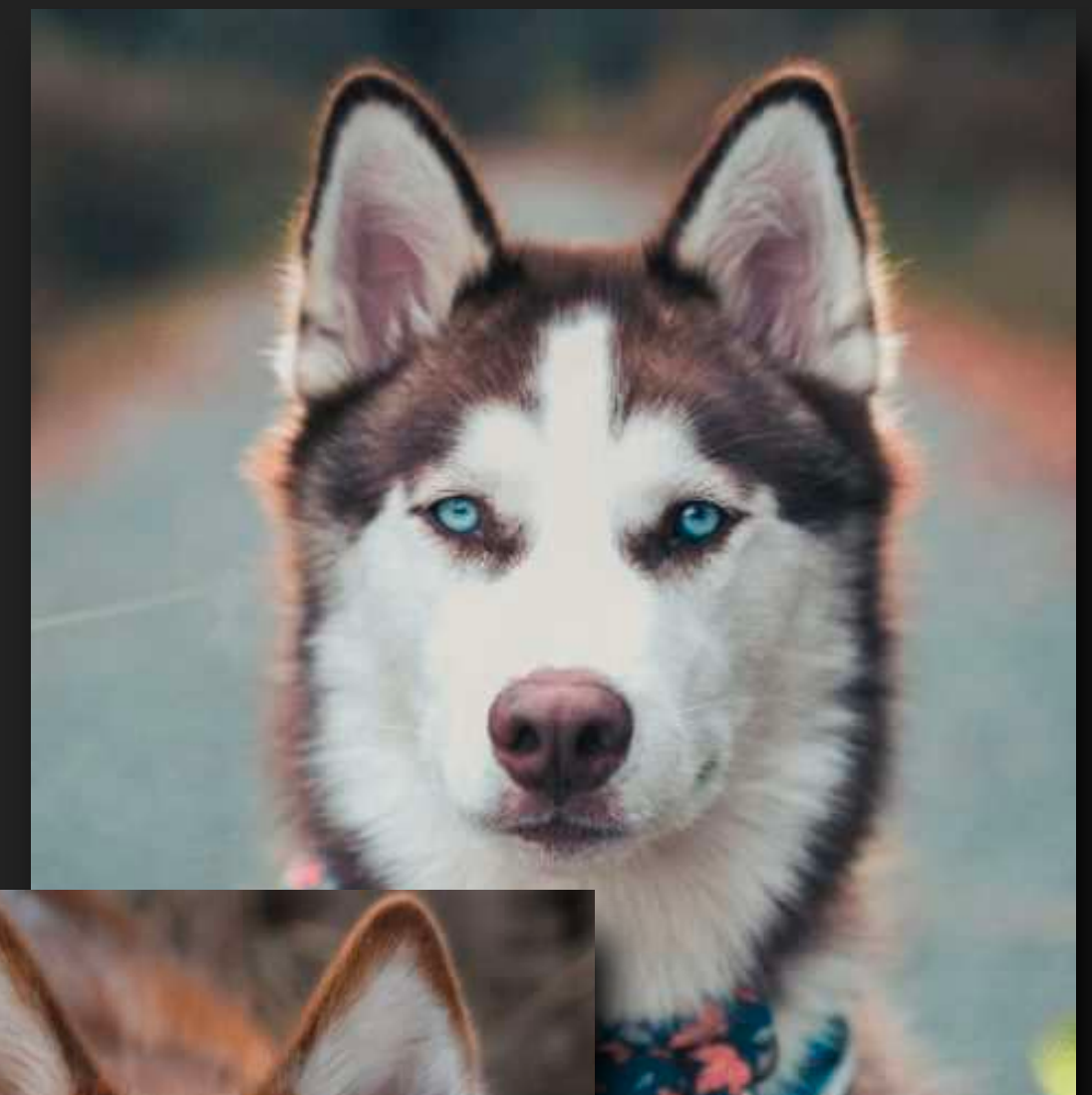


Biological

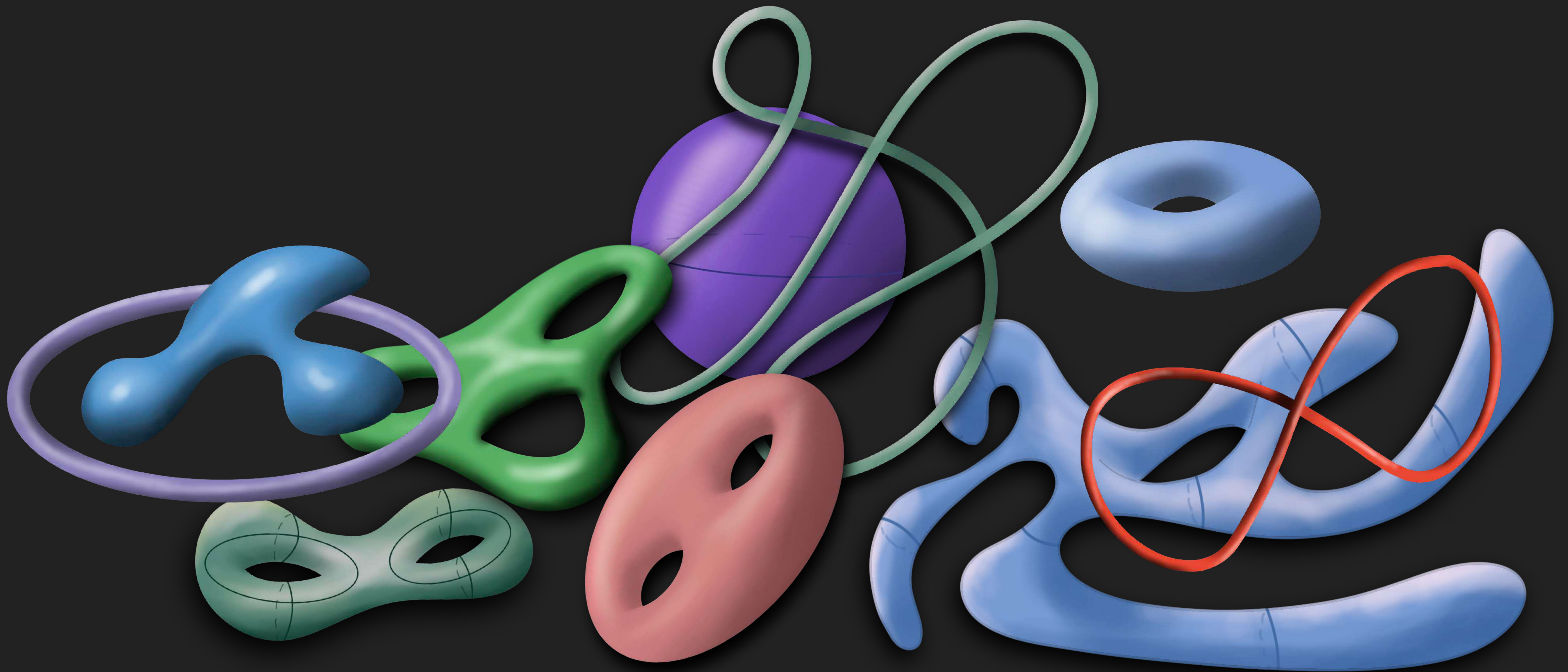
Non-Invariants

Eye
Color

?

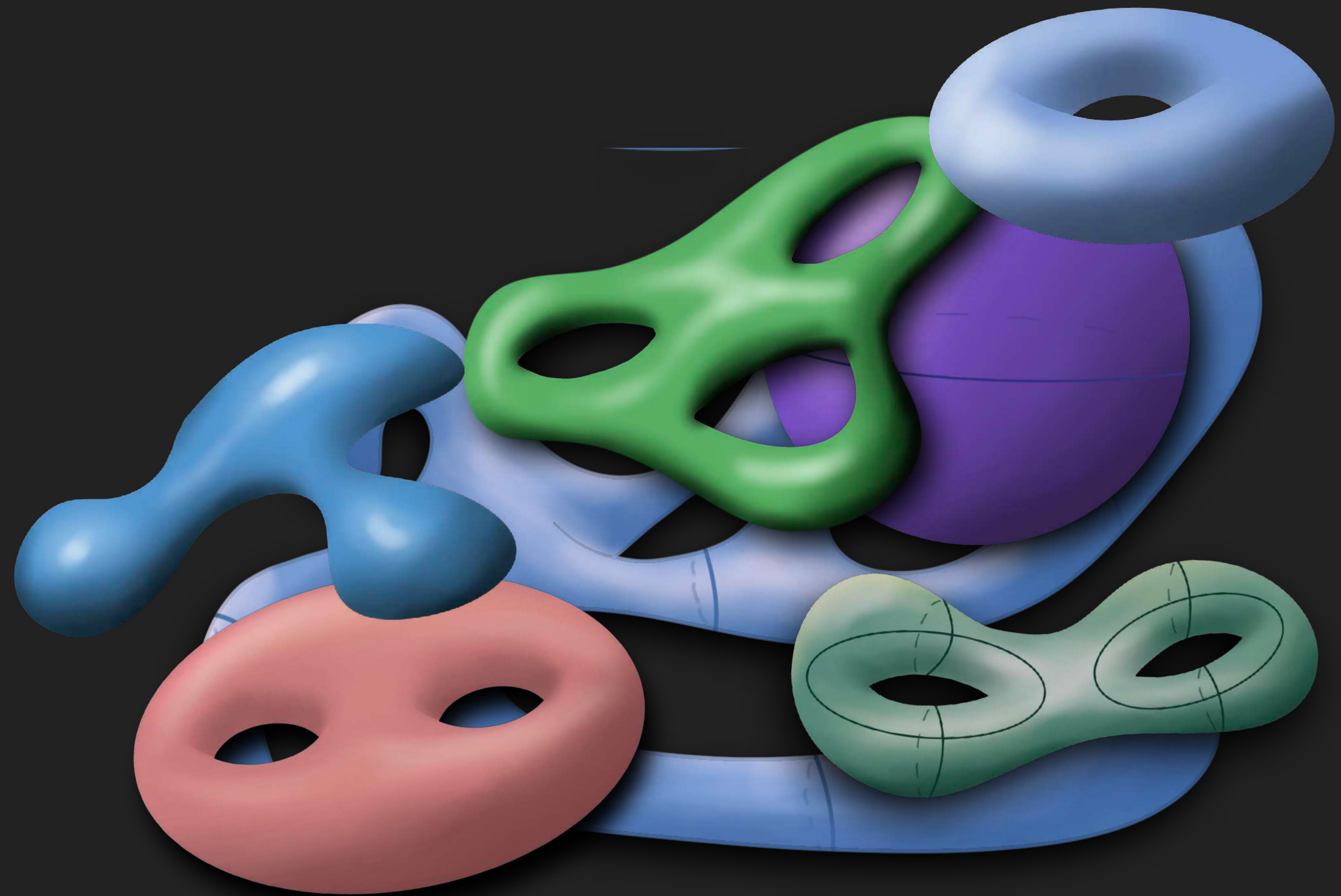
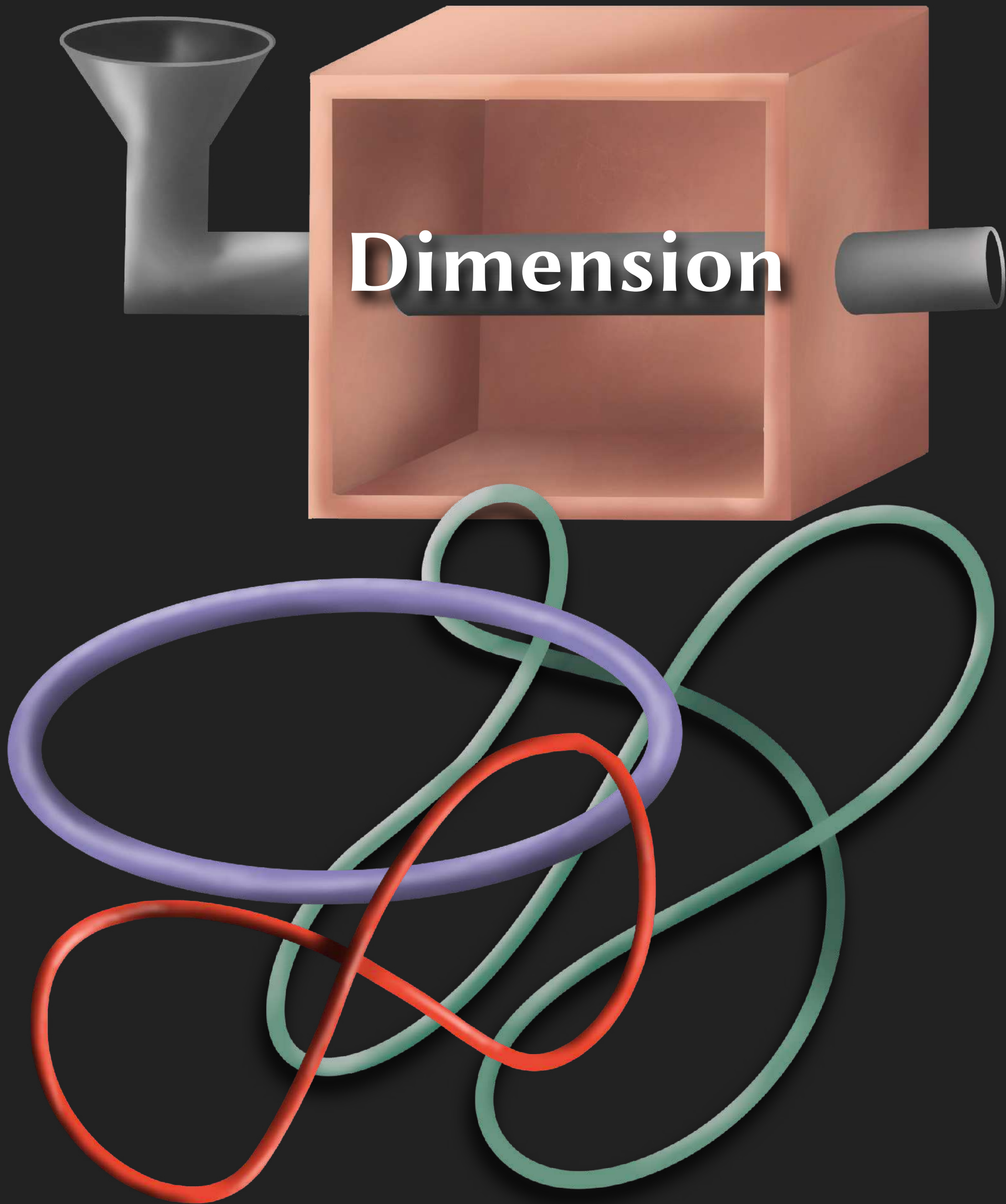


Topological **Invariants**

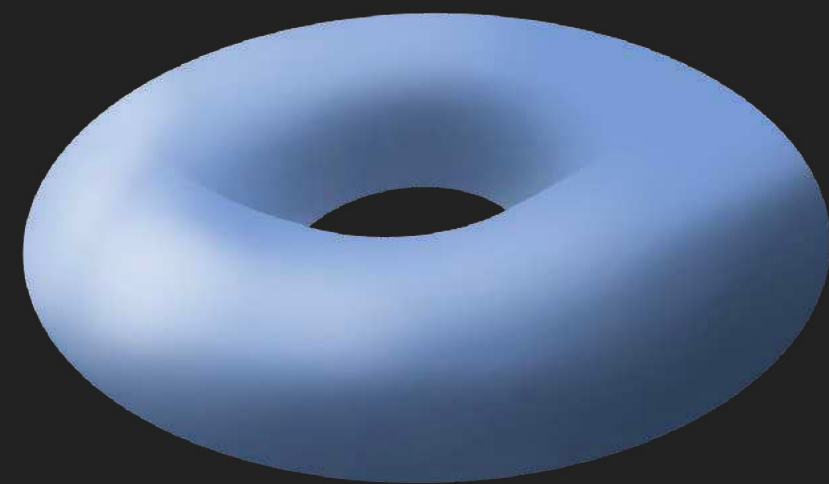
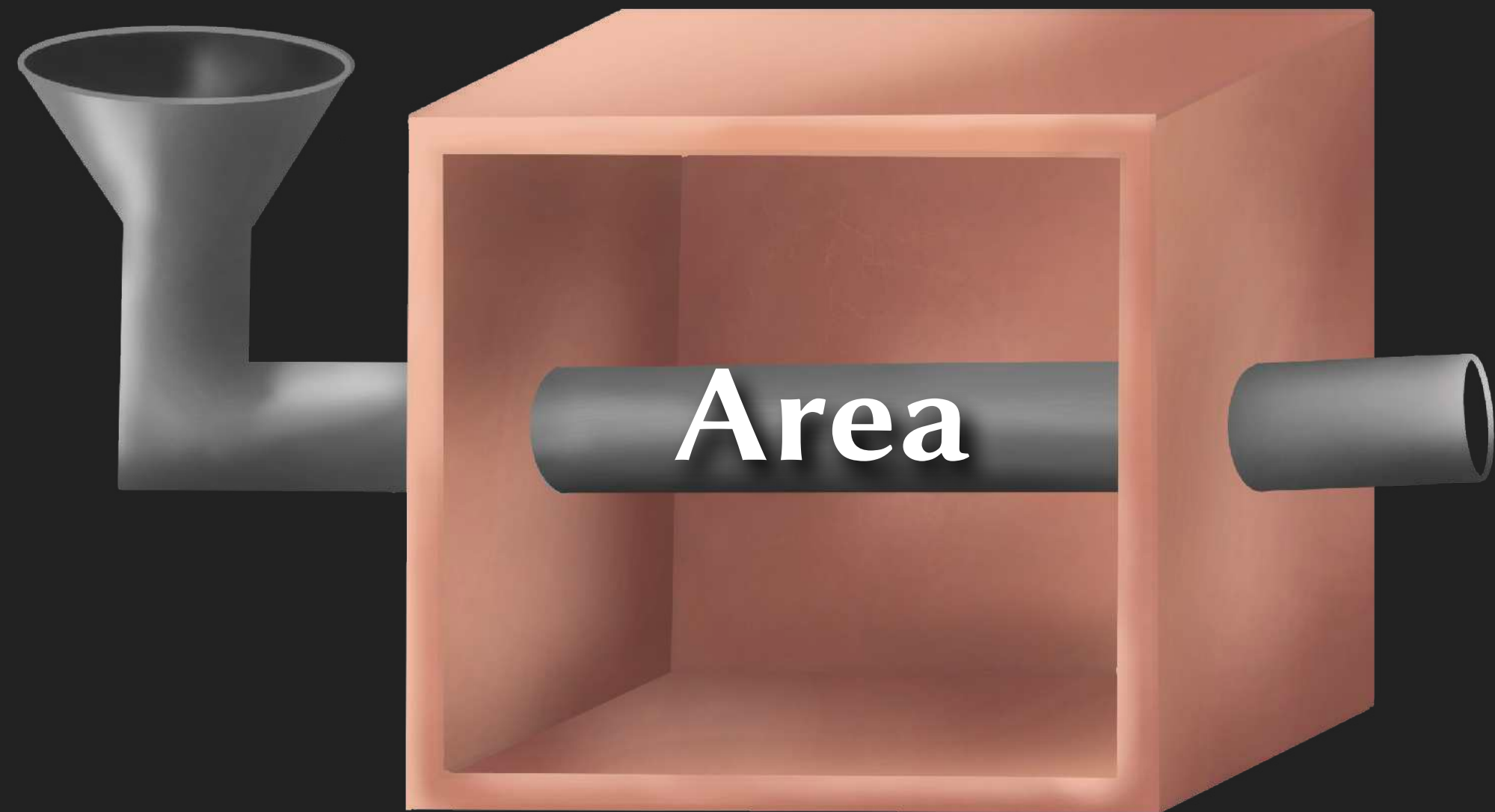


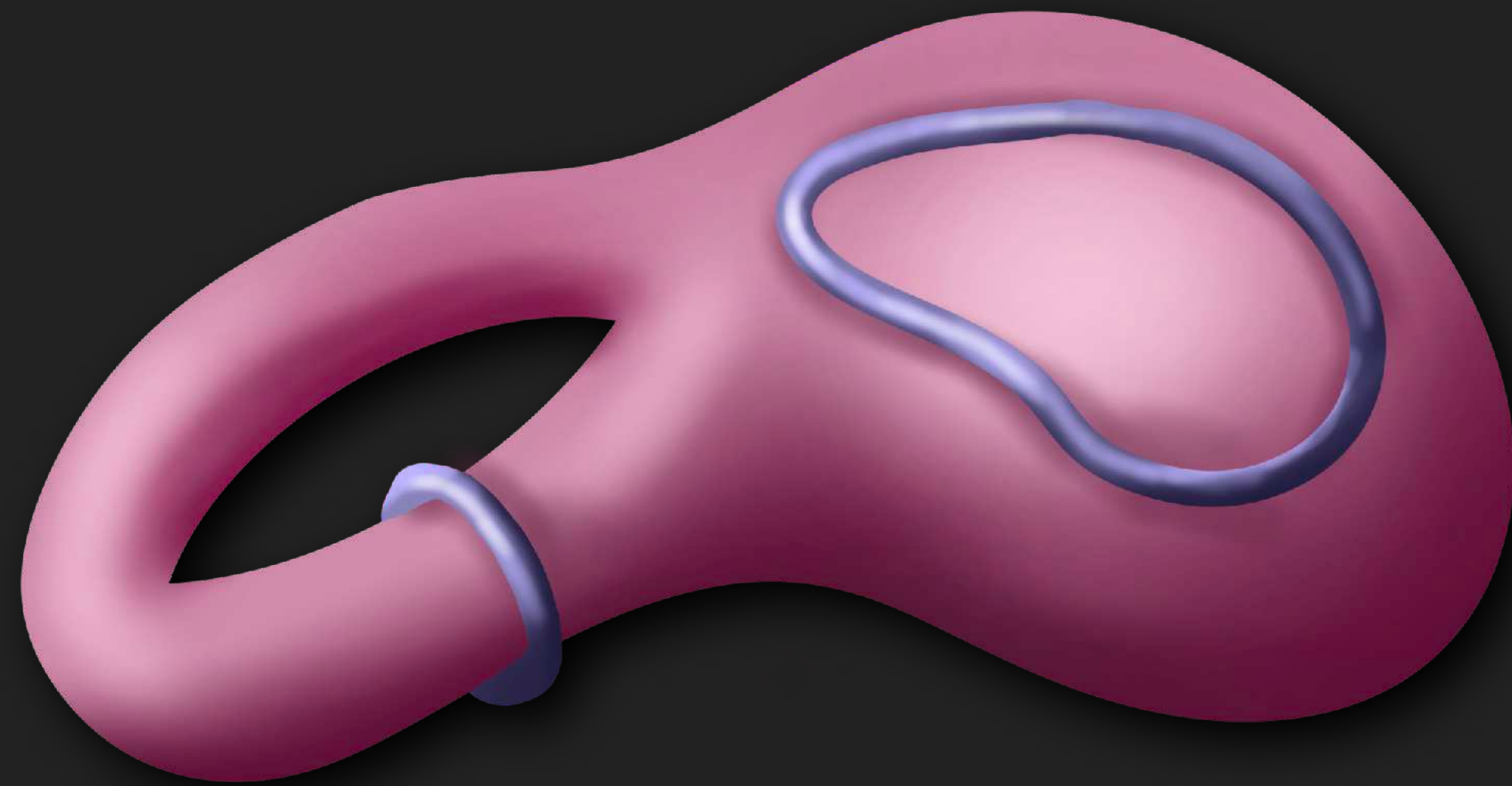
Topological **Invariants**

Dimension



Topological Non-Invariants





A loop is “stuck” if you can’t shrink it to a point, or disconnect the surface with it.

This property of a loop is unchanged by continuous deformation.

The Fundamental Group (1892)

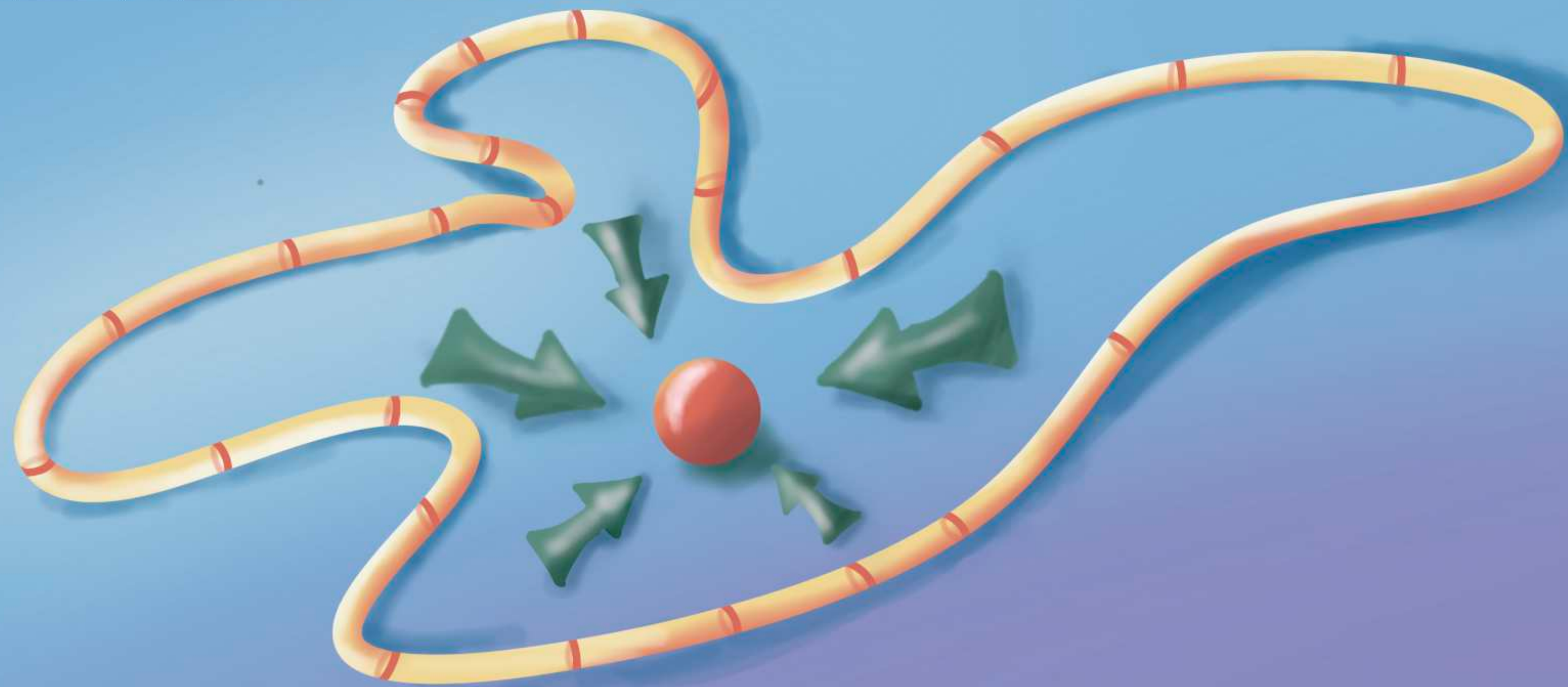


Henry Poincare

The collection of all stuck loops is a topological invariant.

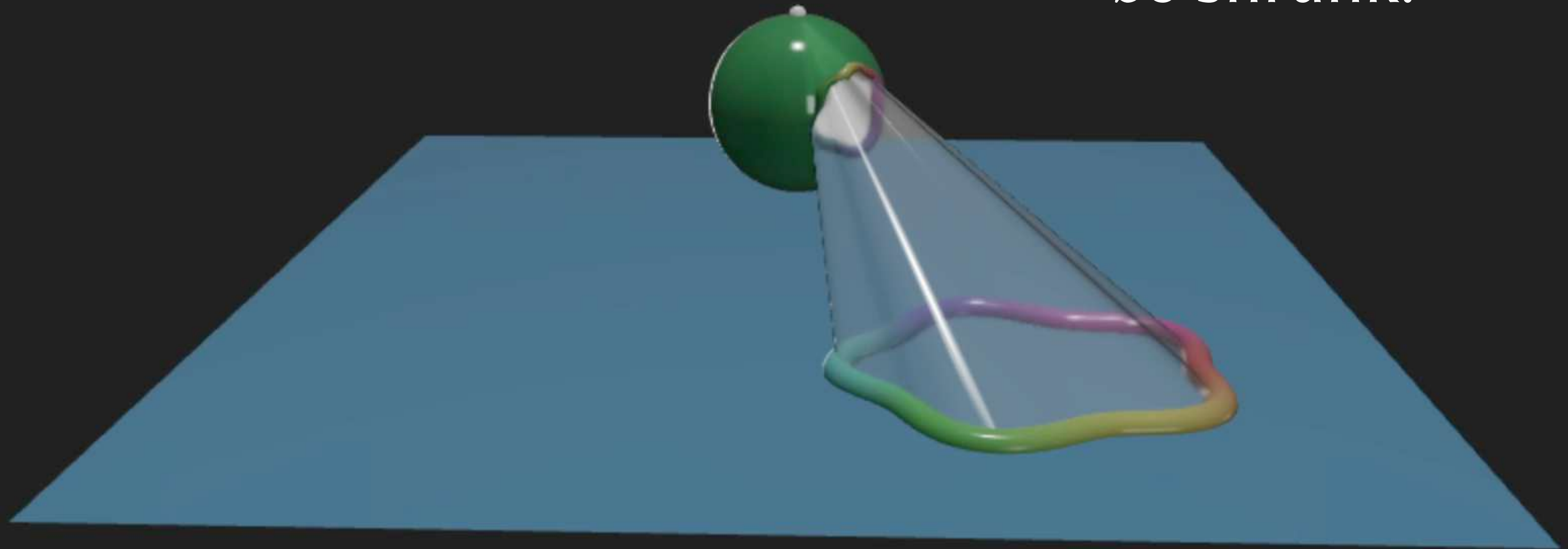
The **Plane** has **No “Stuck” Loops**

$$f(s) \mapsto t \cdot f(s)$$



The **Sphere** has
No “Stuck” Loops

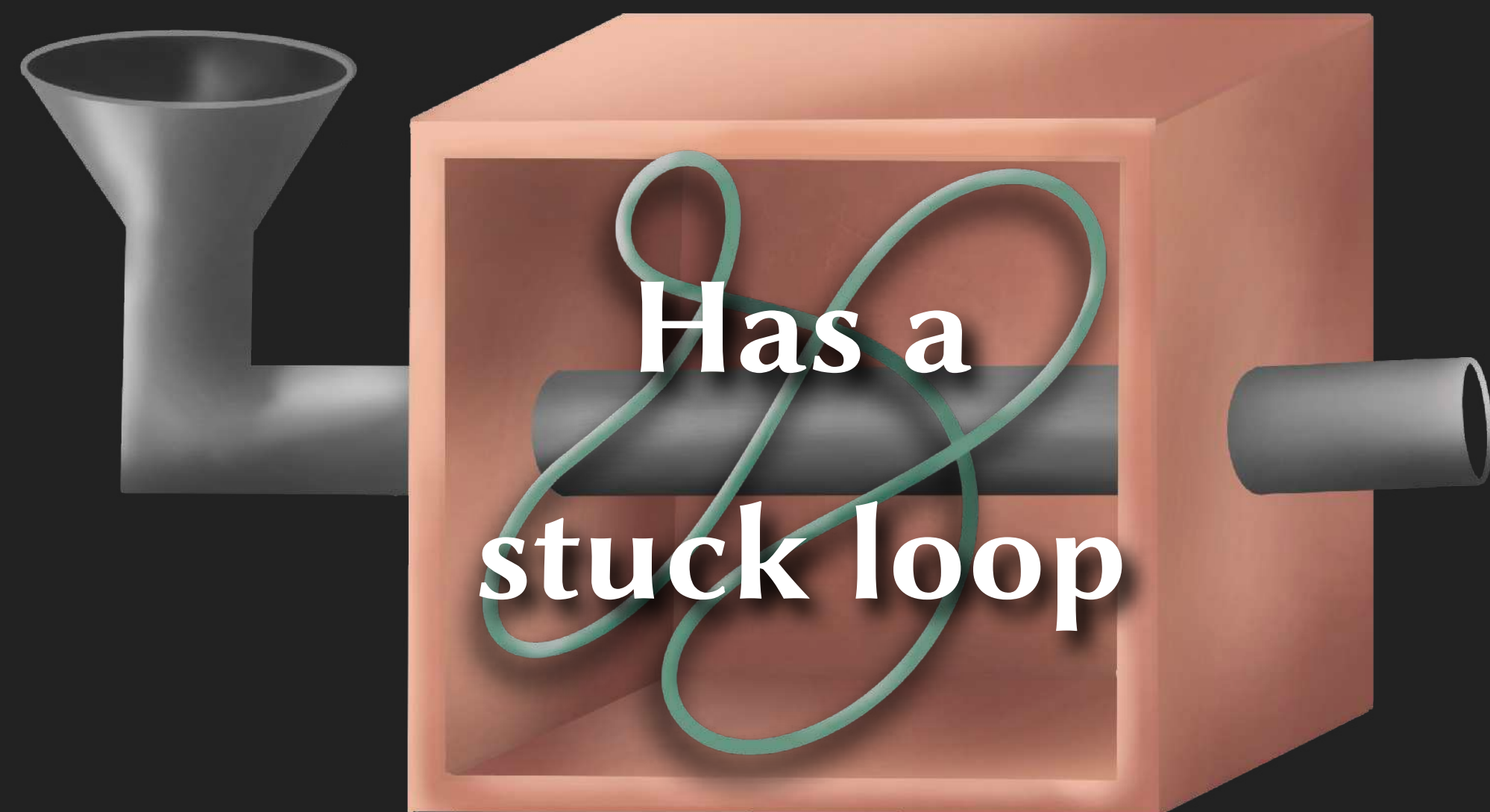
Look at planar
shadow, where
everything can
be shrunk!



The **Torus** has a
“**Stuck**” Loop!

*No amount of
wiggling will
shrink this loop
to a point!*

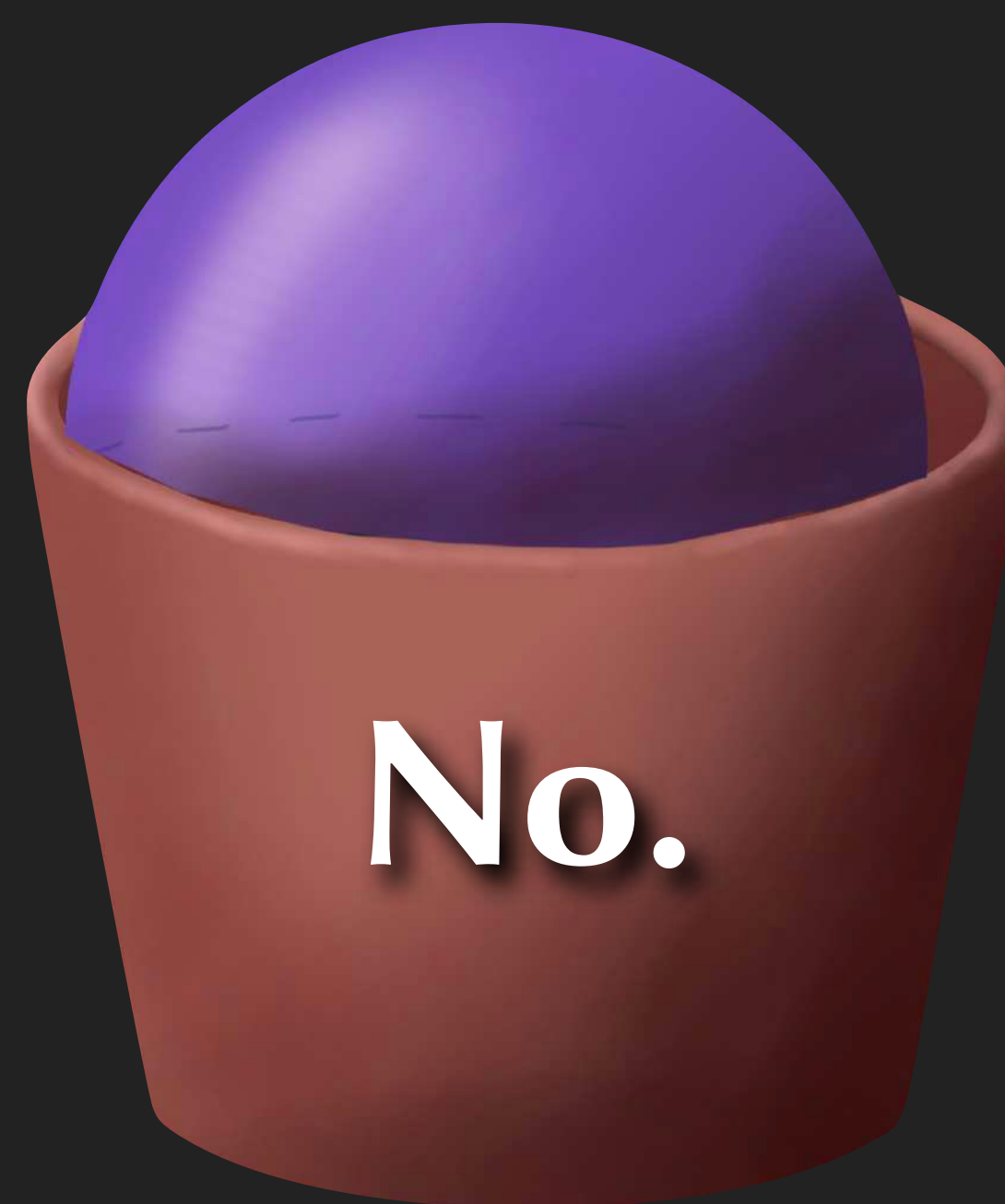




How good is
**The Loop
Invariant**



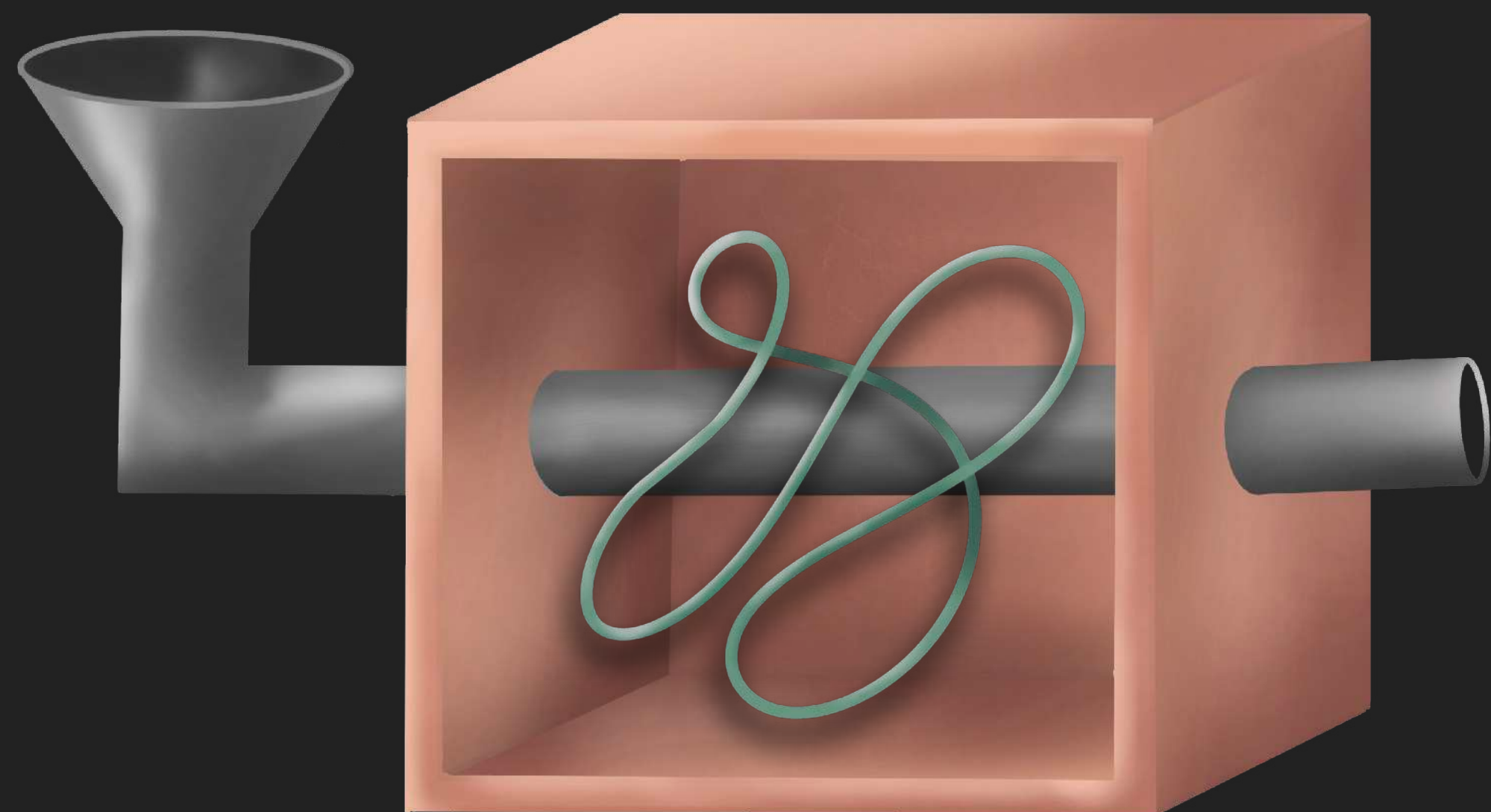
Loops can tell
apart the sphere
and the torus.



No.



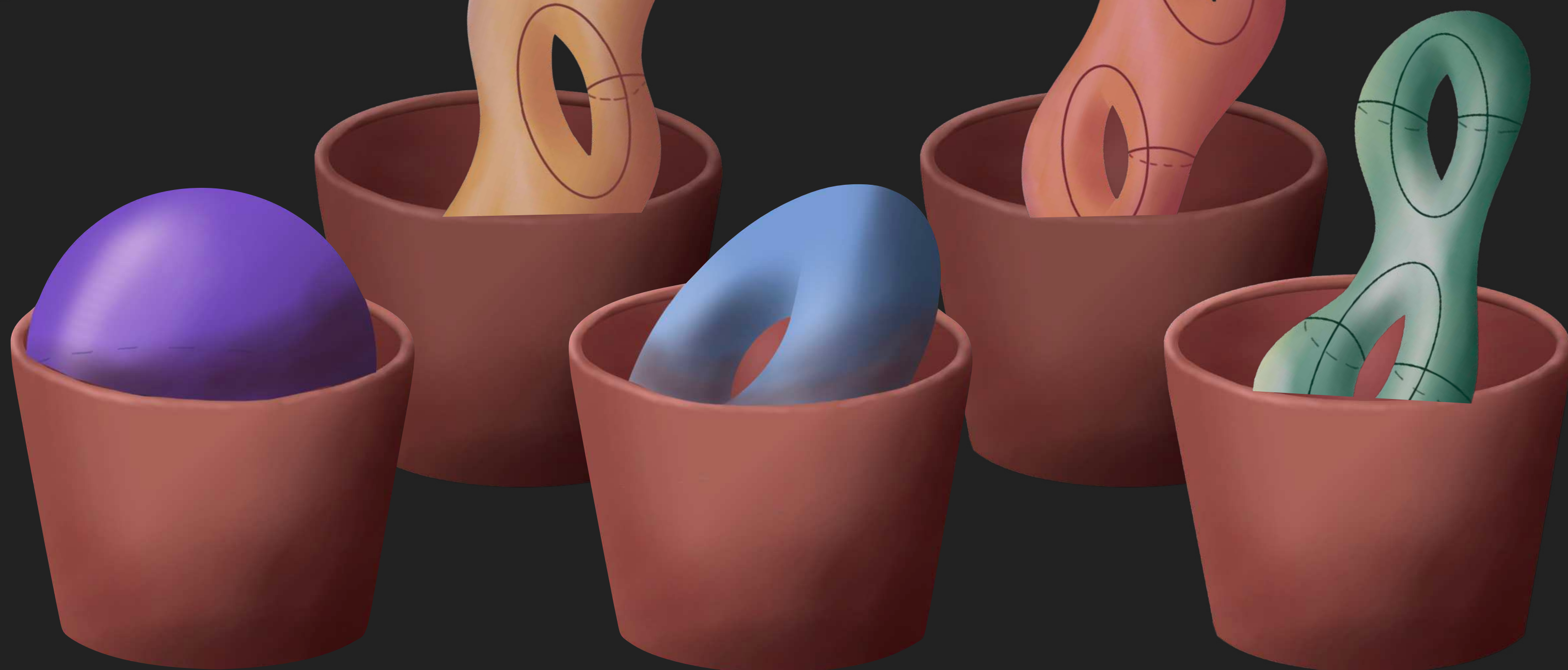
Yes.



How good is
**The Loop
Invariant**



Loops are a
Helpful
Invariant

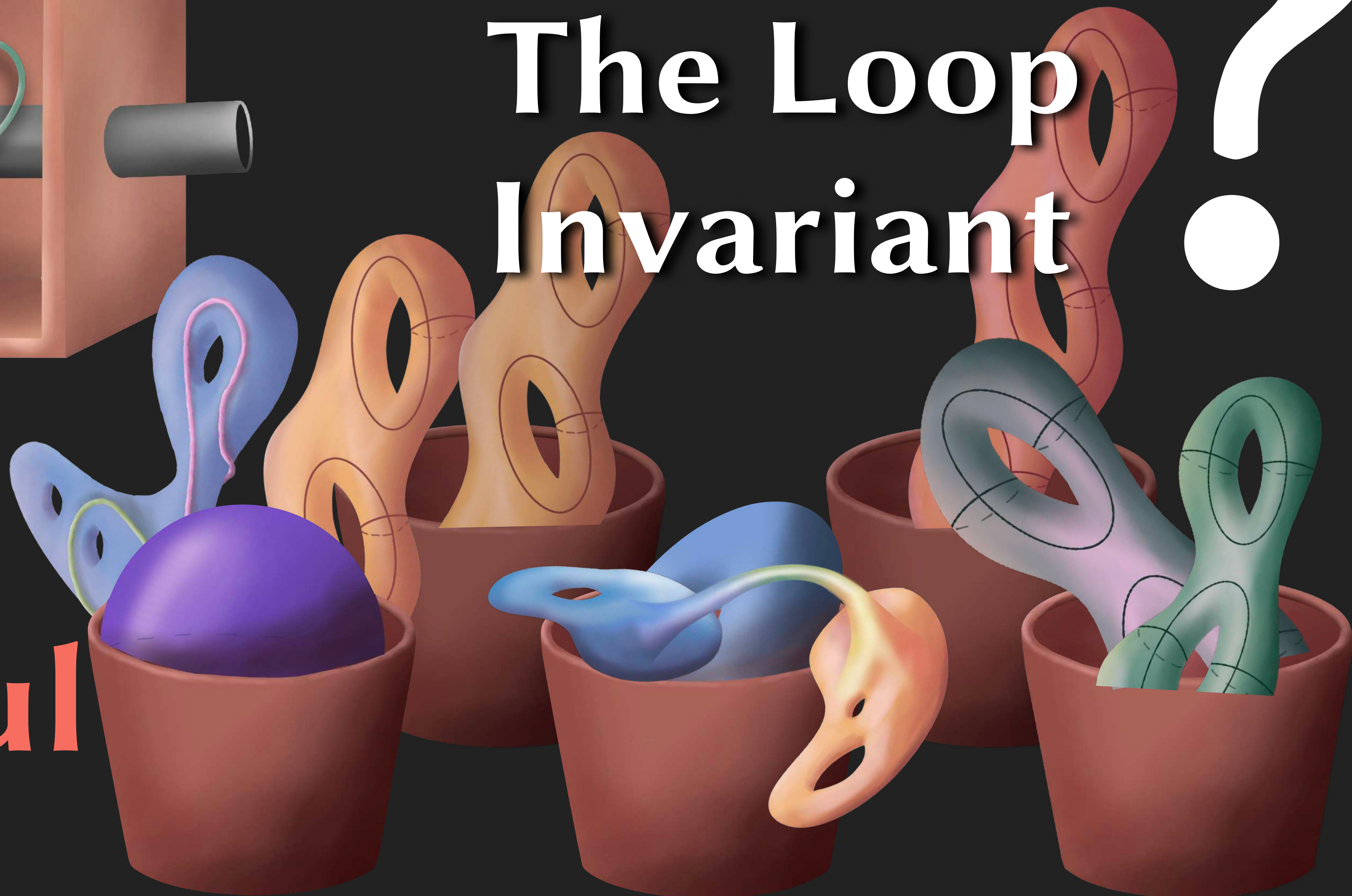
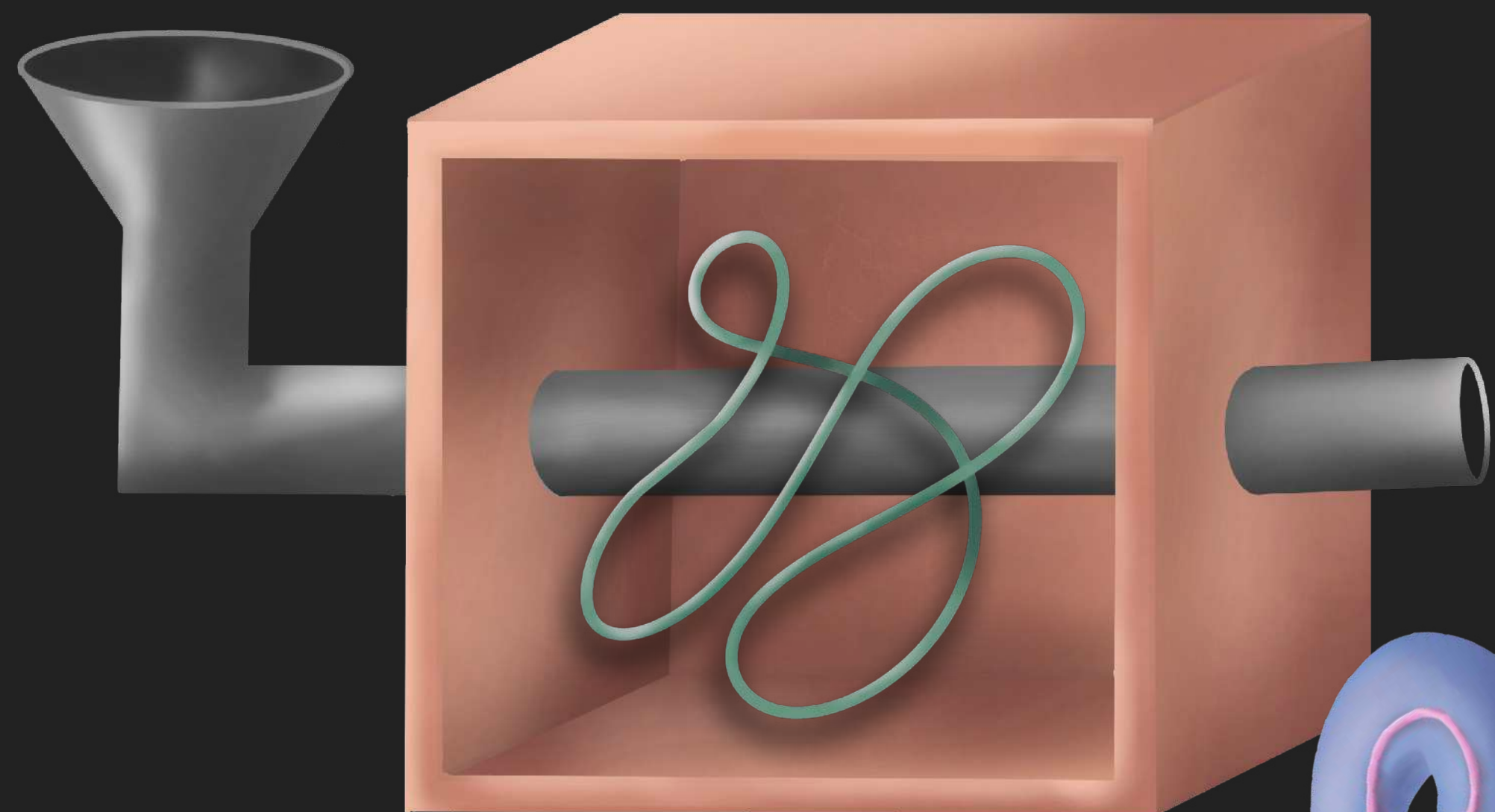


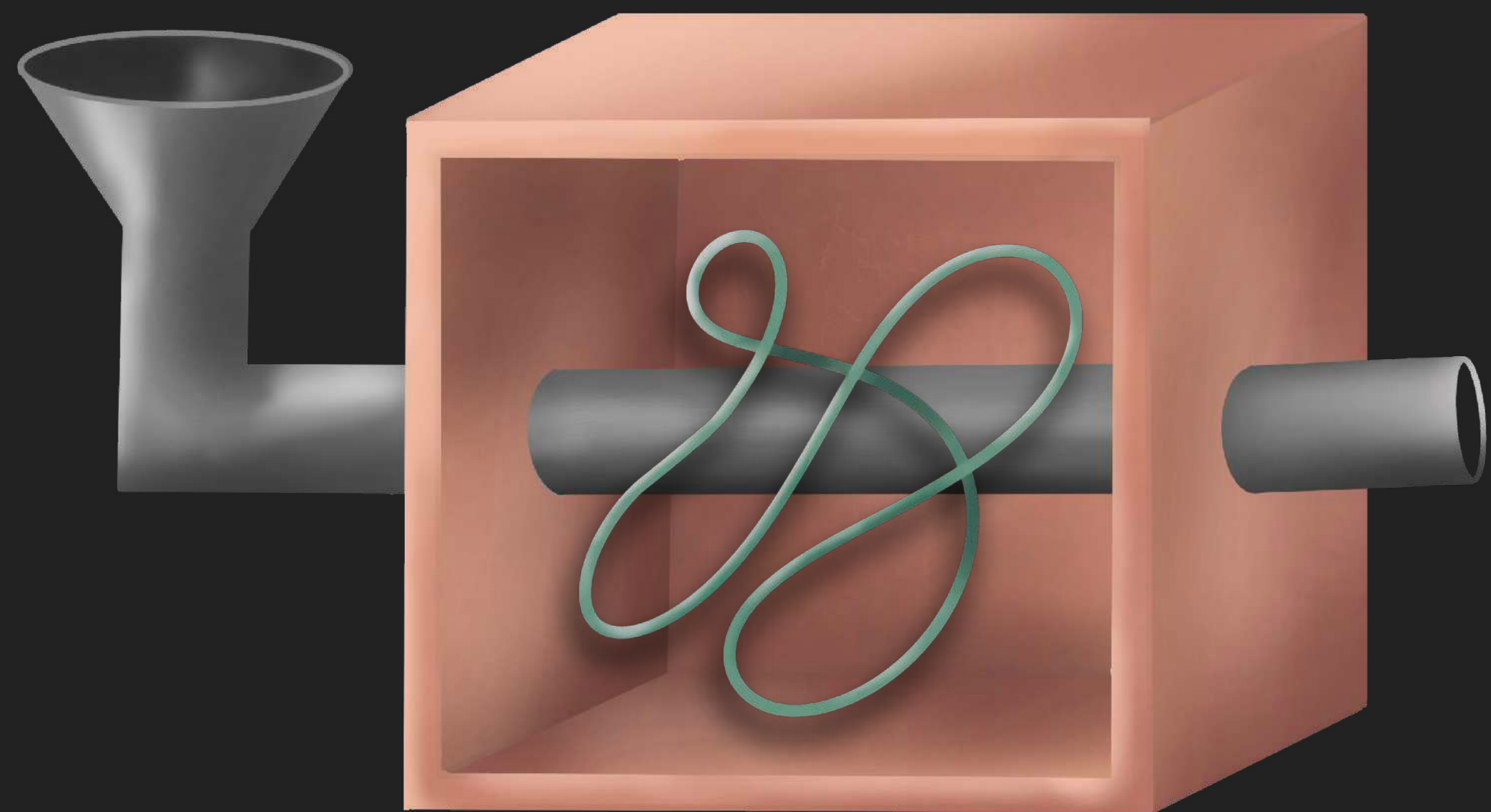
How good is

**The Loop
Invariant**



Loops are an
Unhelpful
Invariant

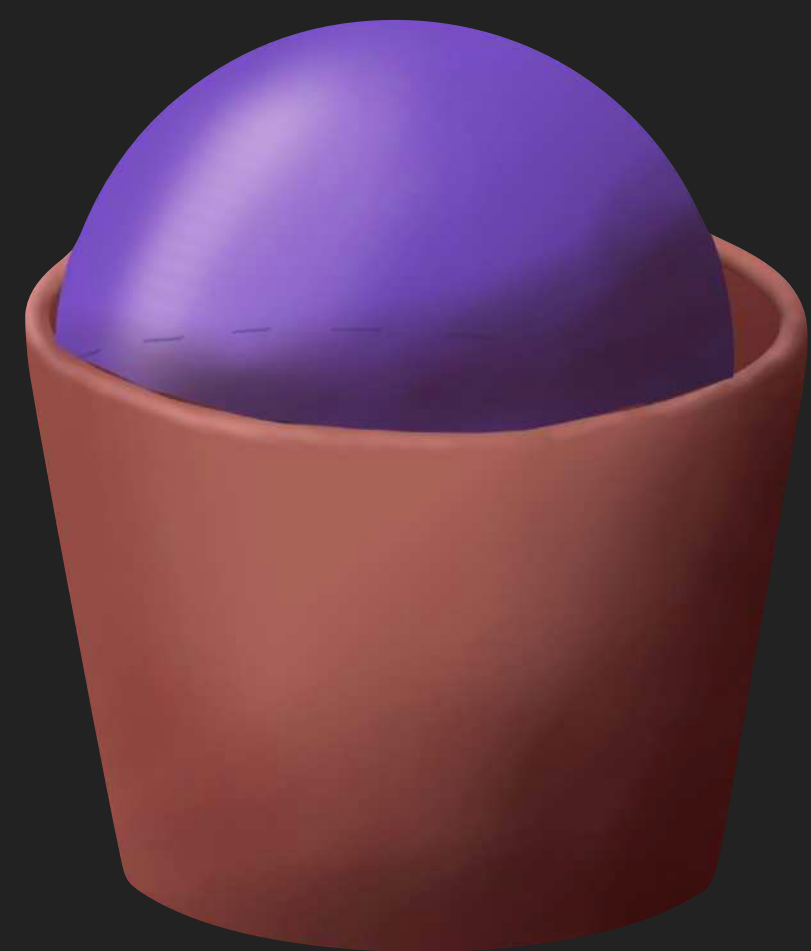




How good is
**The Loop
Invariant**



Can loops find
**The
Sphere?**

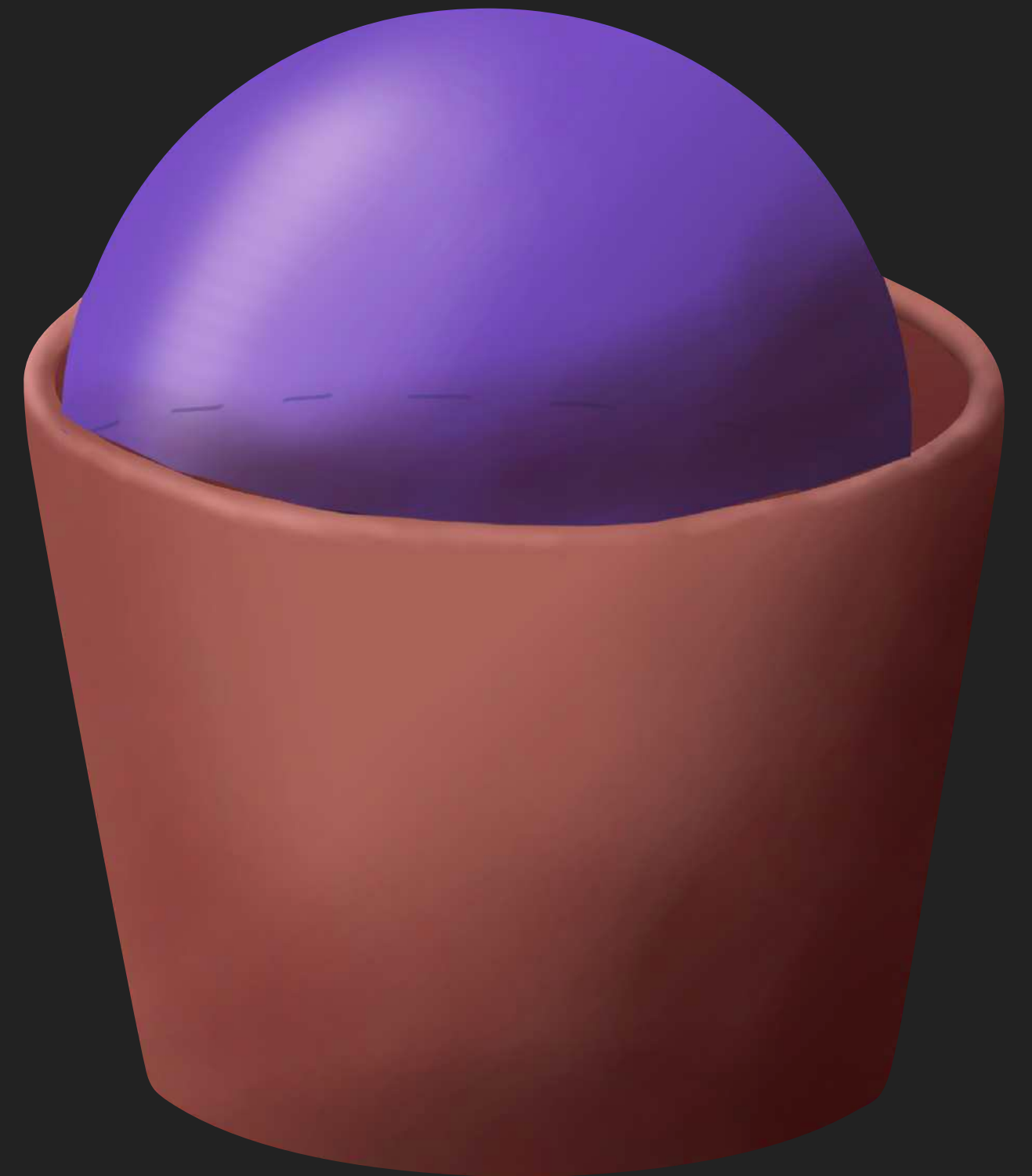


(Generalized) The Poincaré Conjecture



*“I think loops are a
good invariant.
They can tell if a
shape is a sphere.”*

-Poincaré, 1904



The Poincare Conjecture

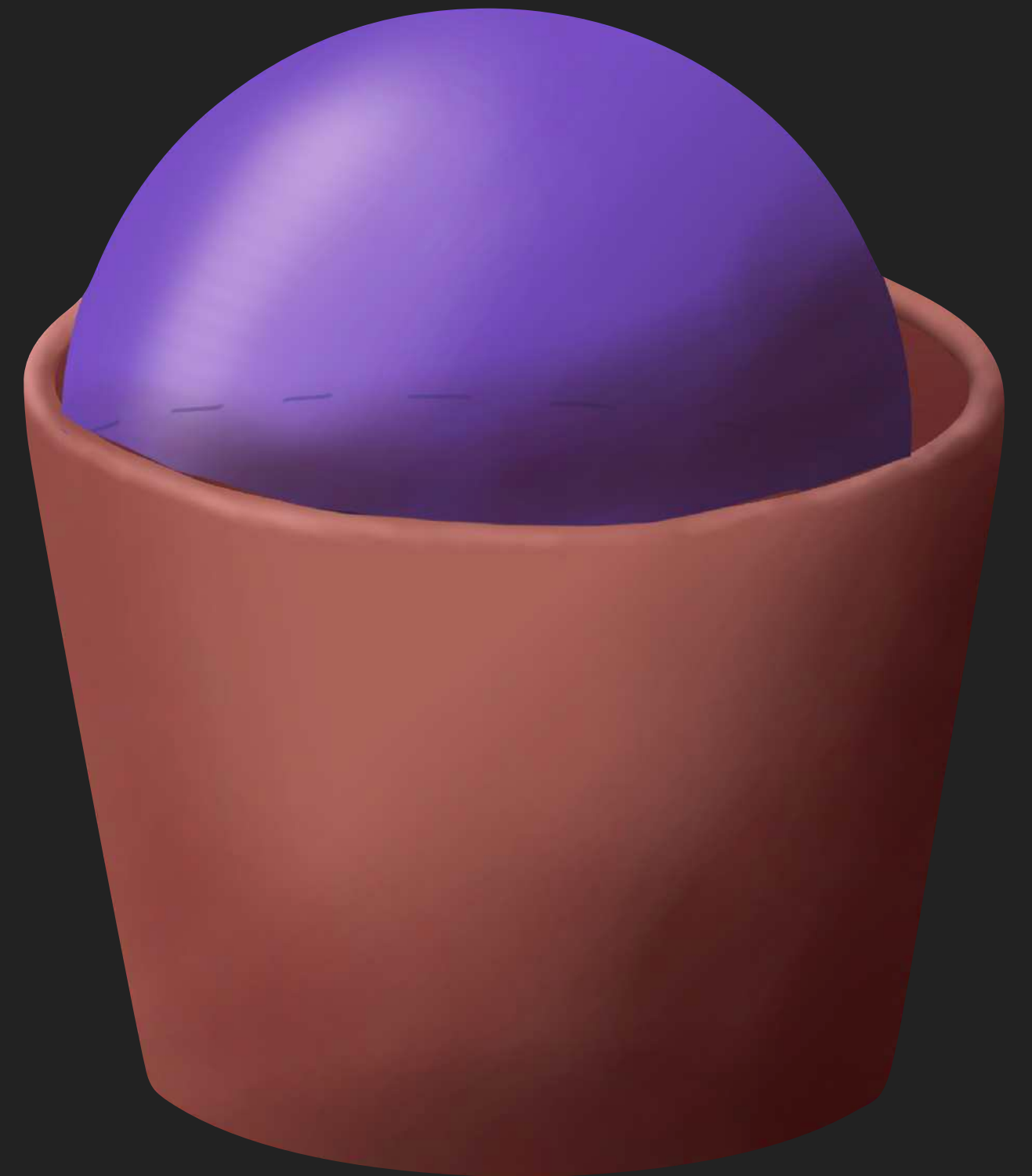
If a shape has the same

Loop Invariants

as the sphere, then

it is the sphere

-Poincaré, 1904



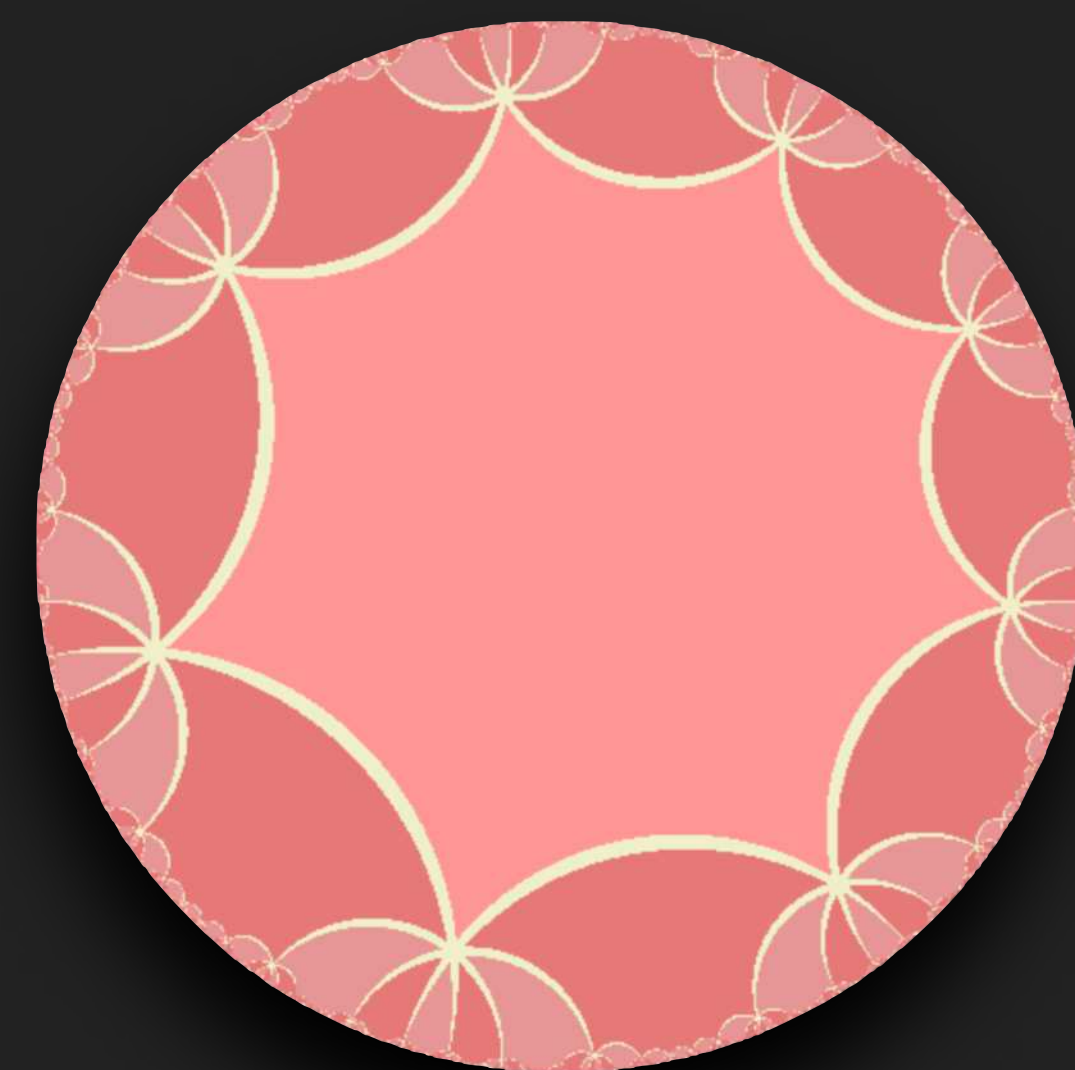
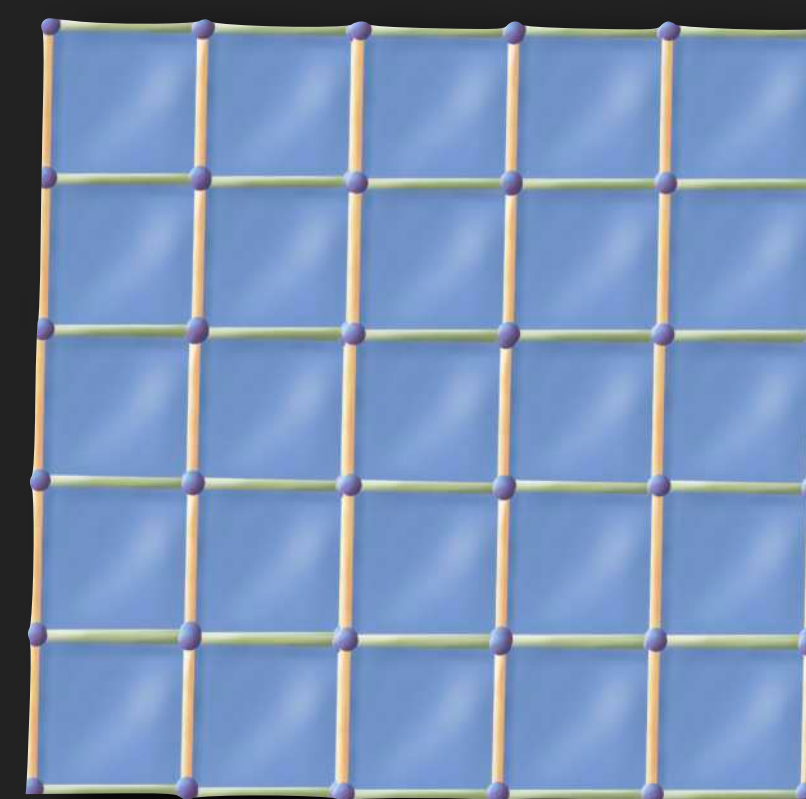
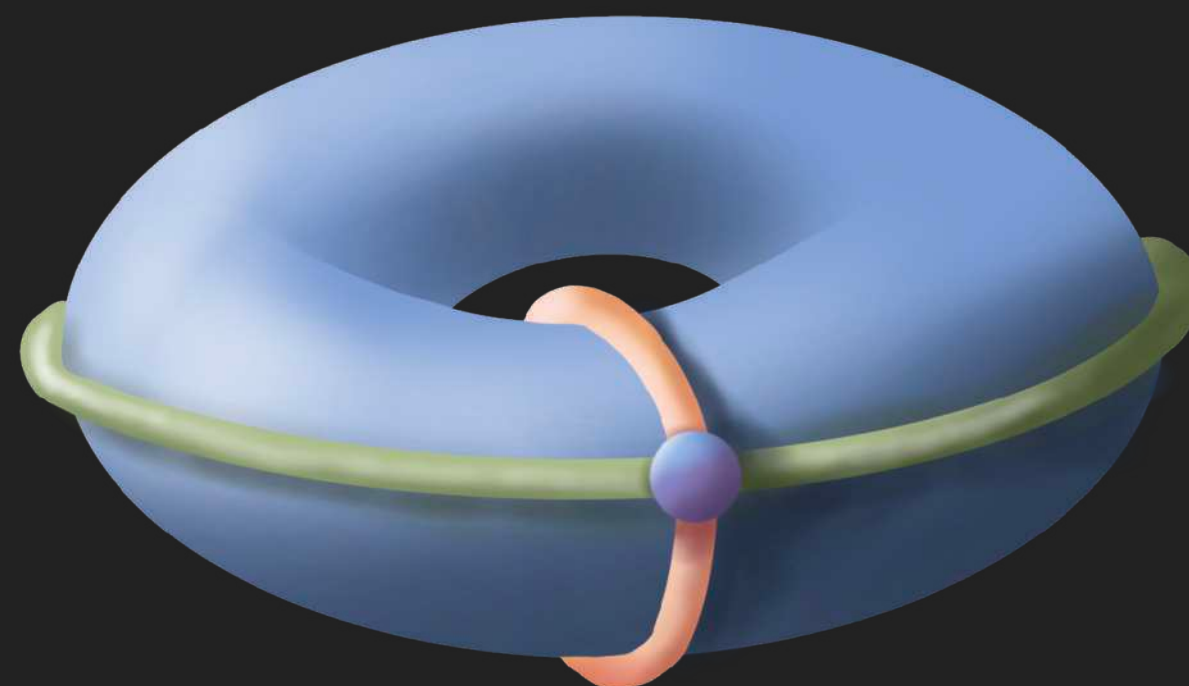
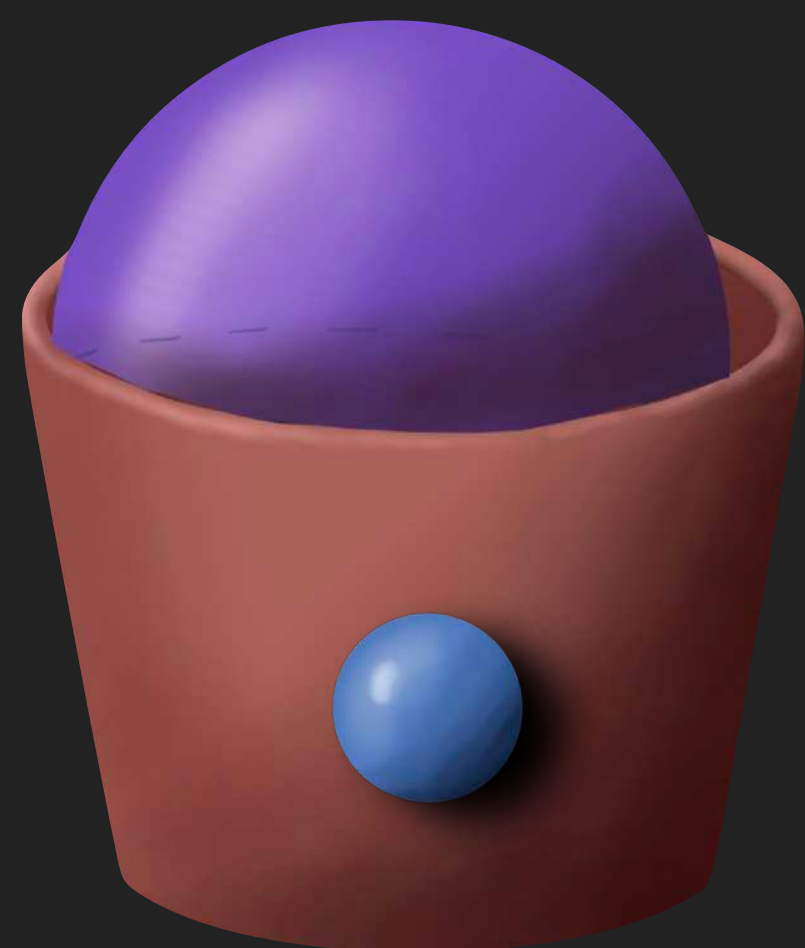
The Poincare Conjecture

If it
walks & quacks
like a duck, then
it is a duck

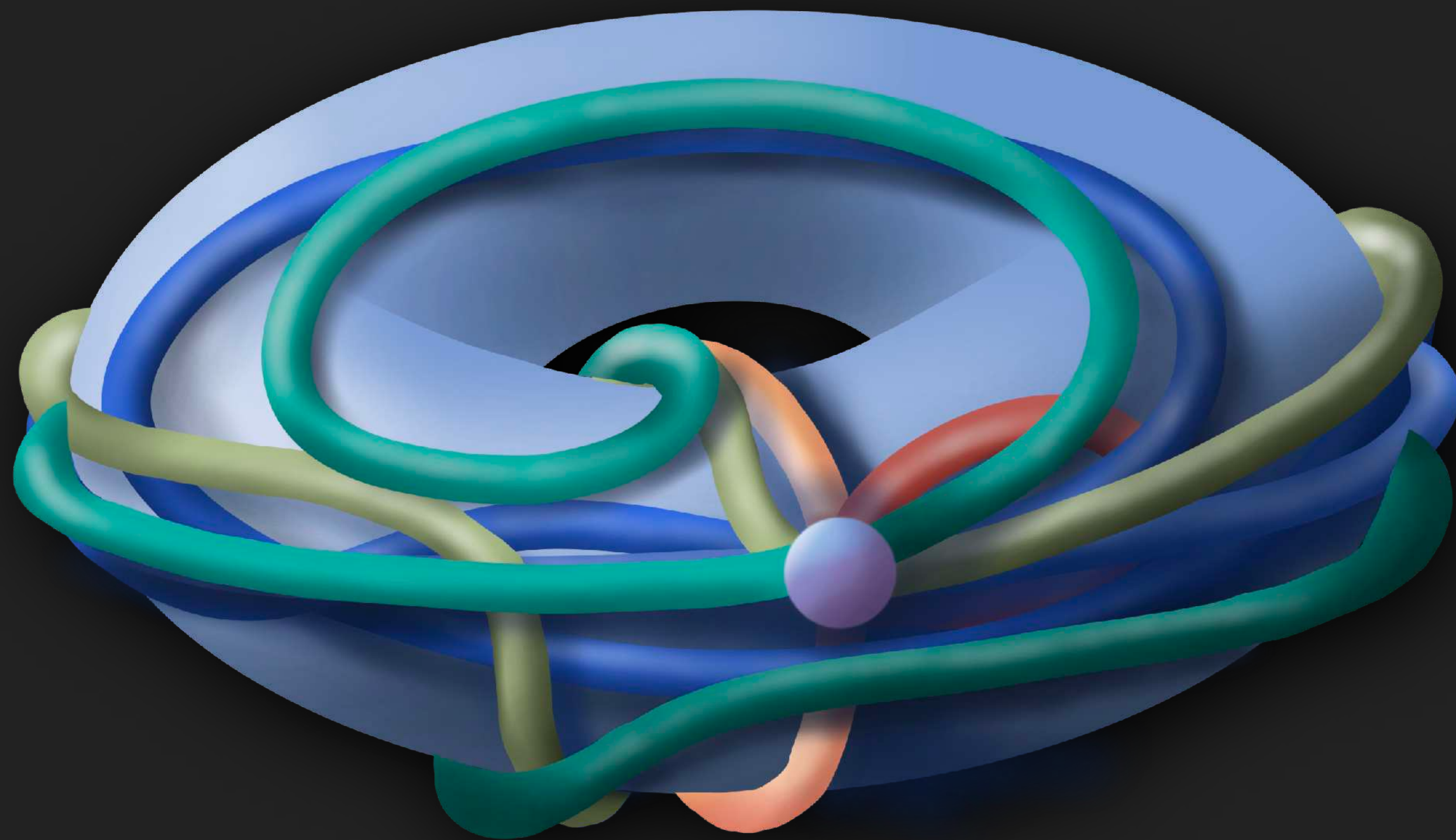


Part II

*Loops and
Tilings*



Computing The Loop Invariant

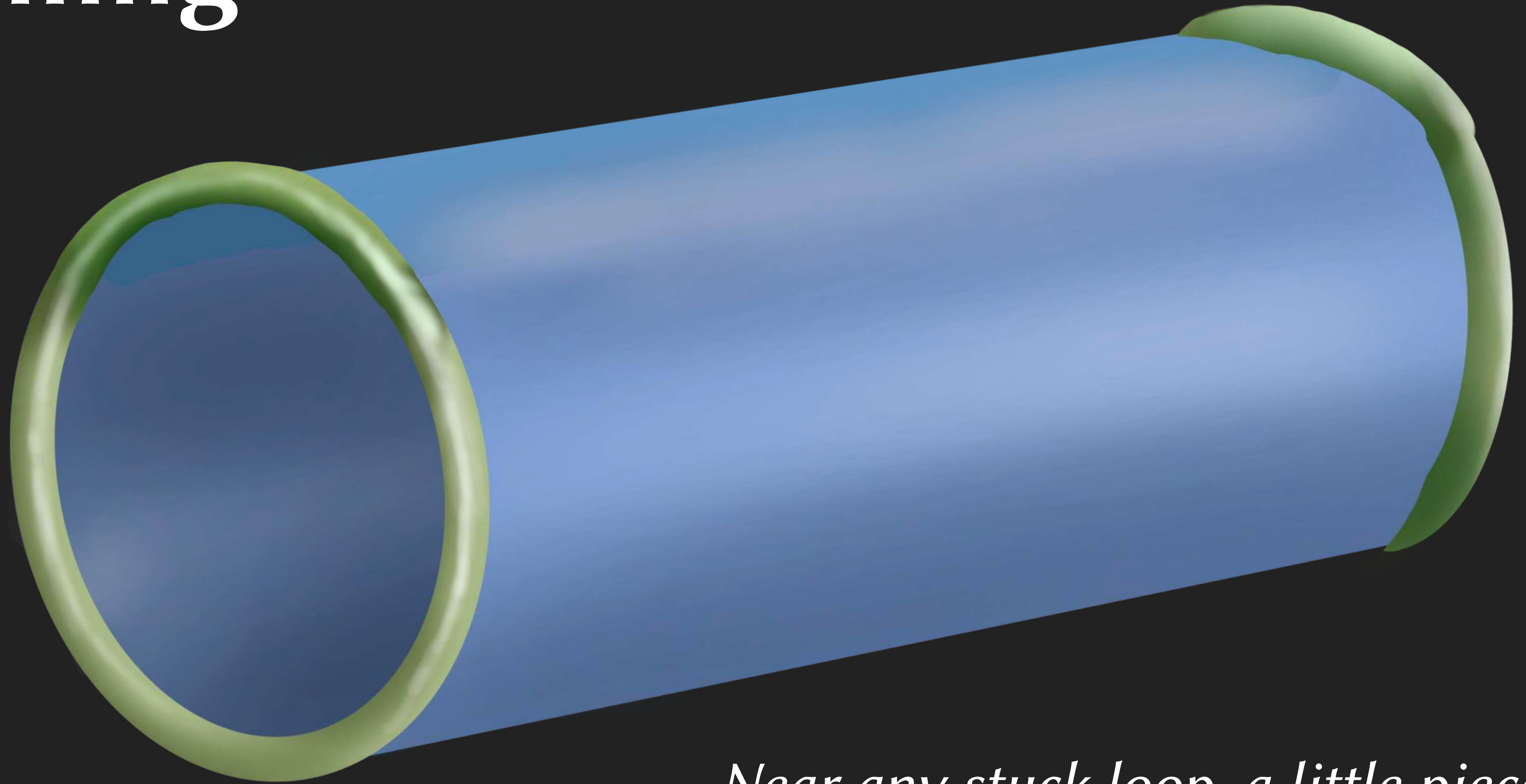


How to
understand the
collection of
**all stuck
loops**
on a manifold?

Unrolling

along a

Loop

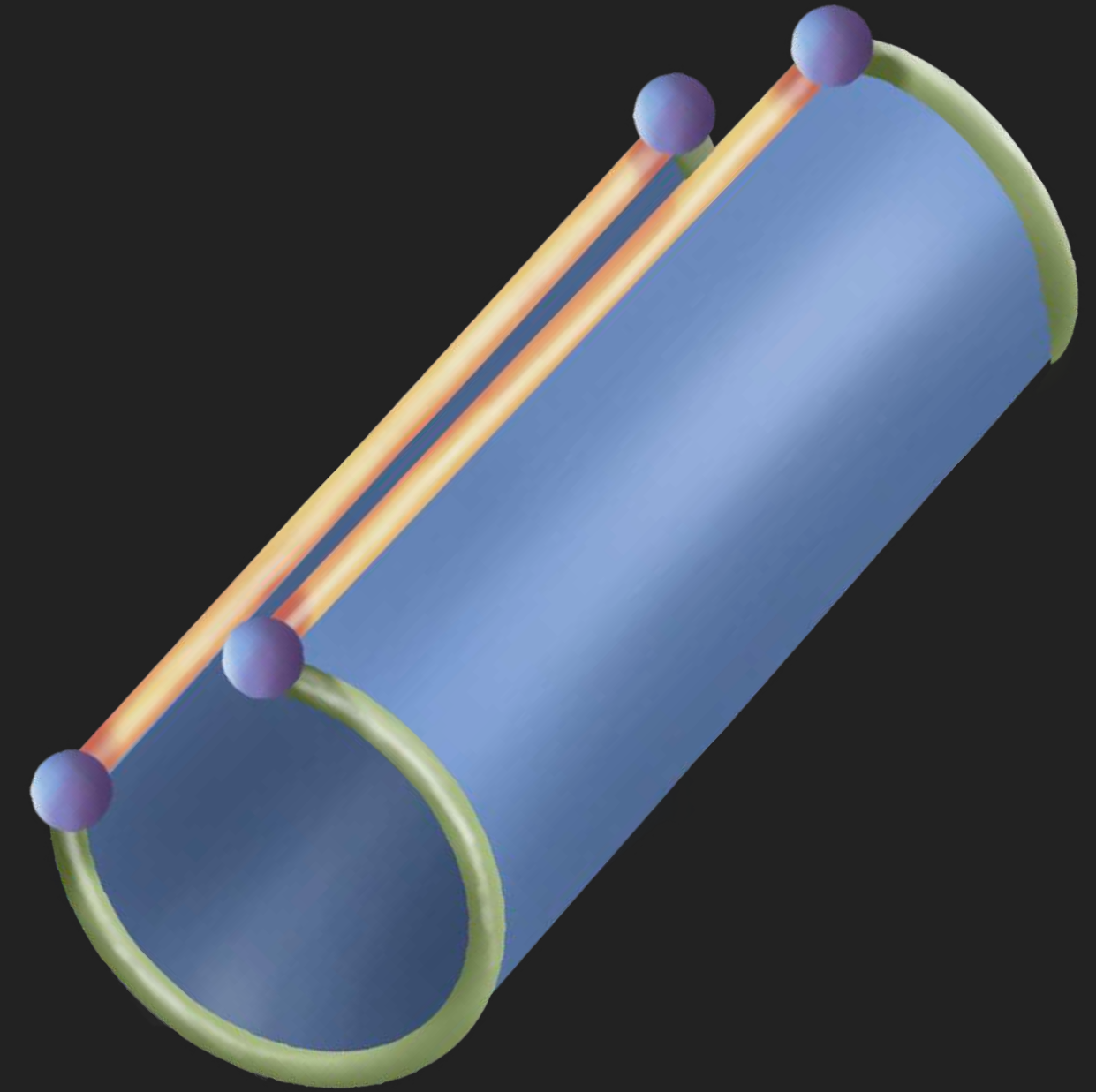
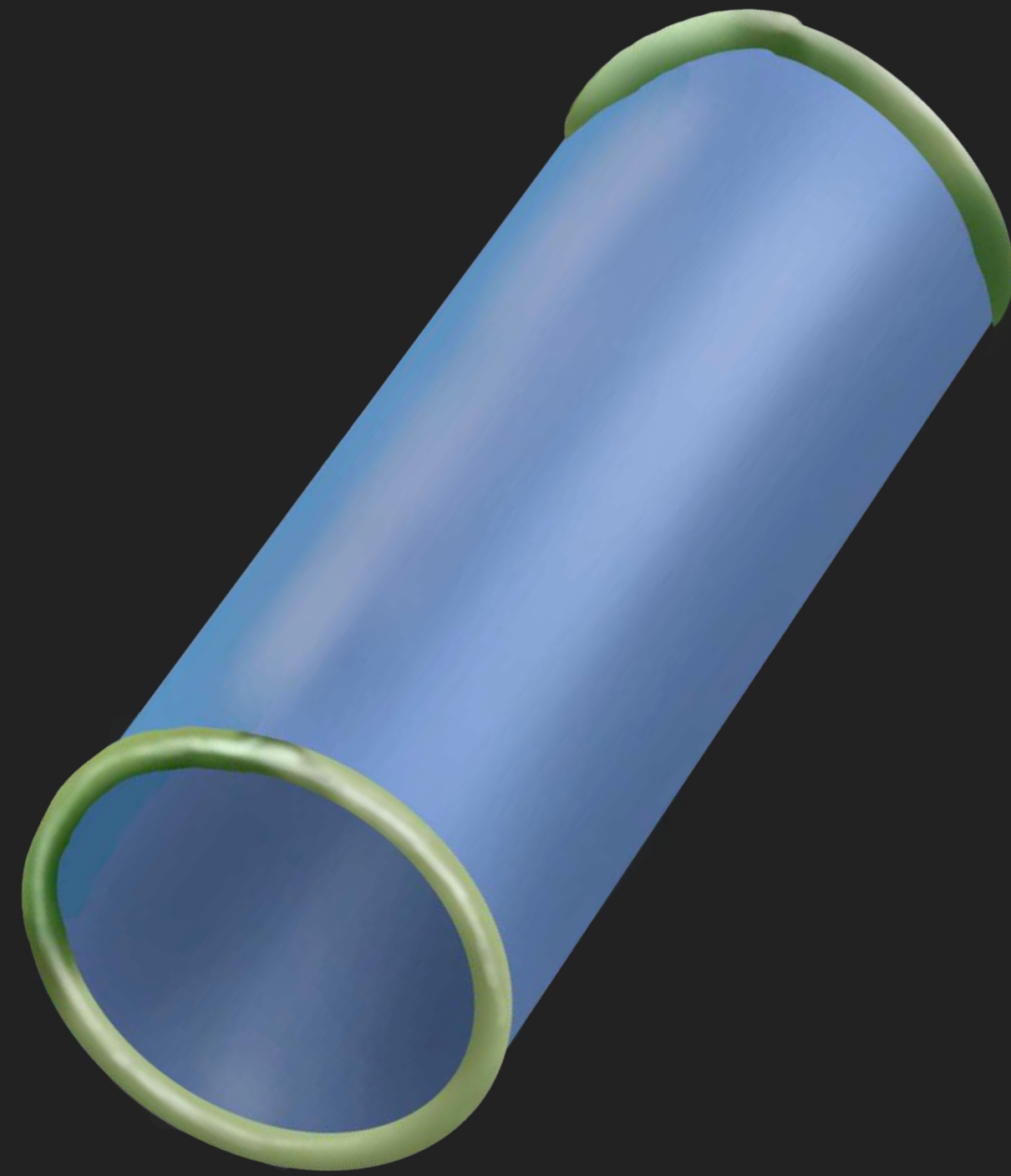


Near any stuck loop, a little piece of the surface looks like a cylinder.

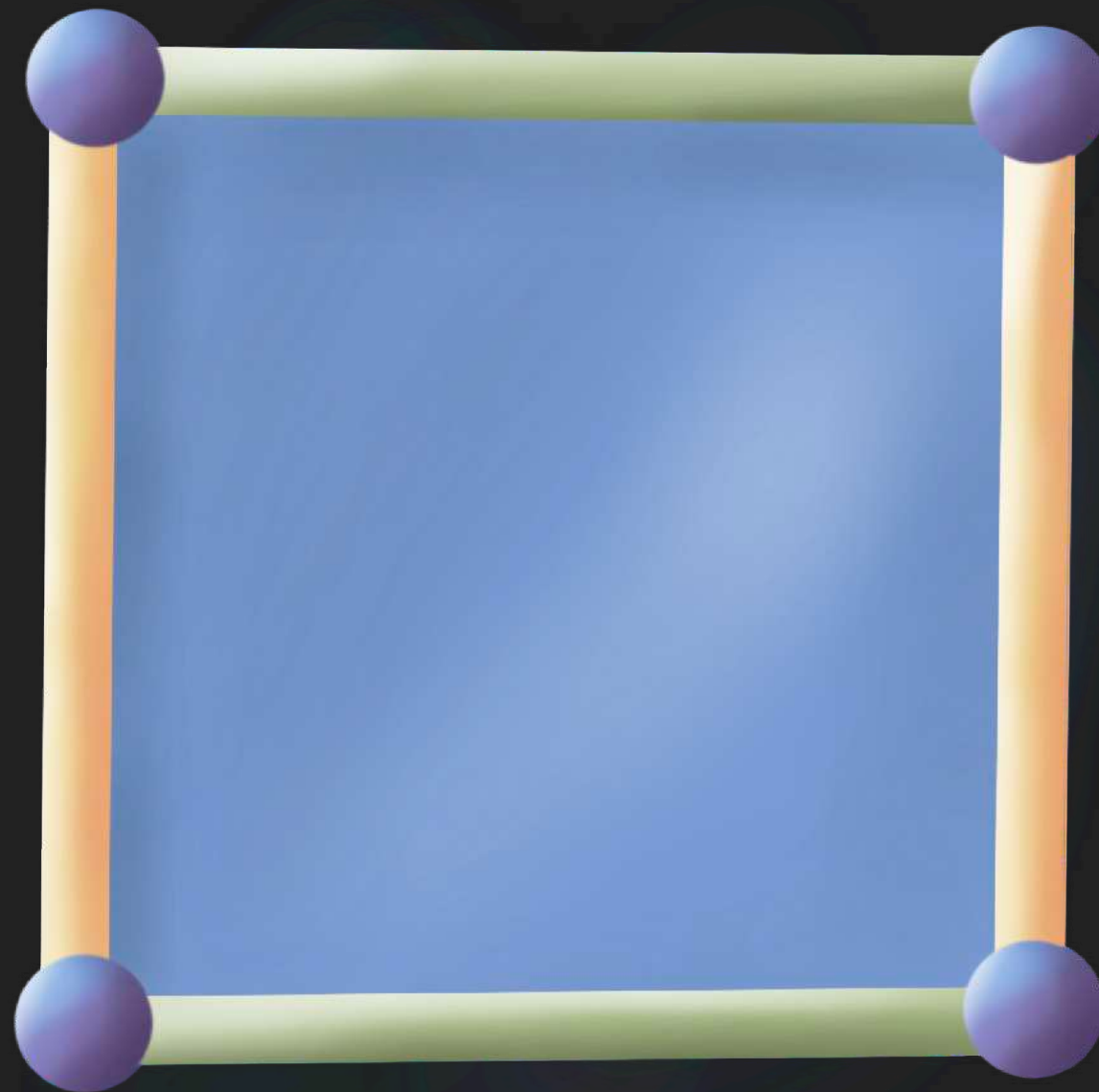
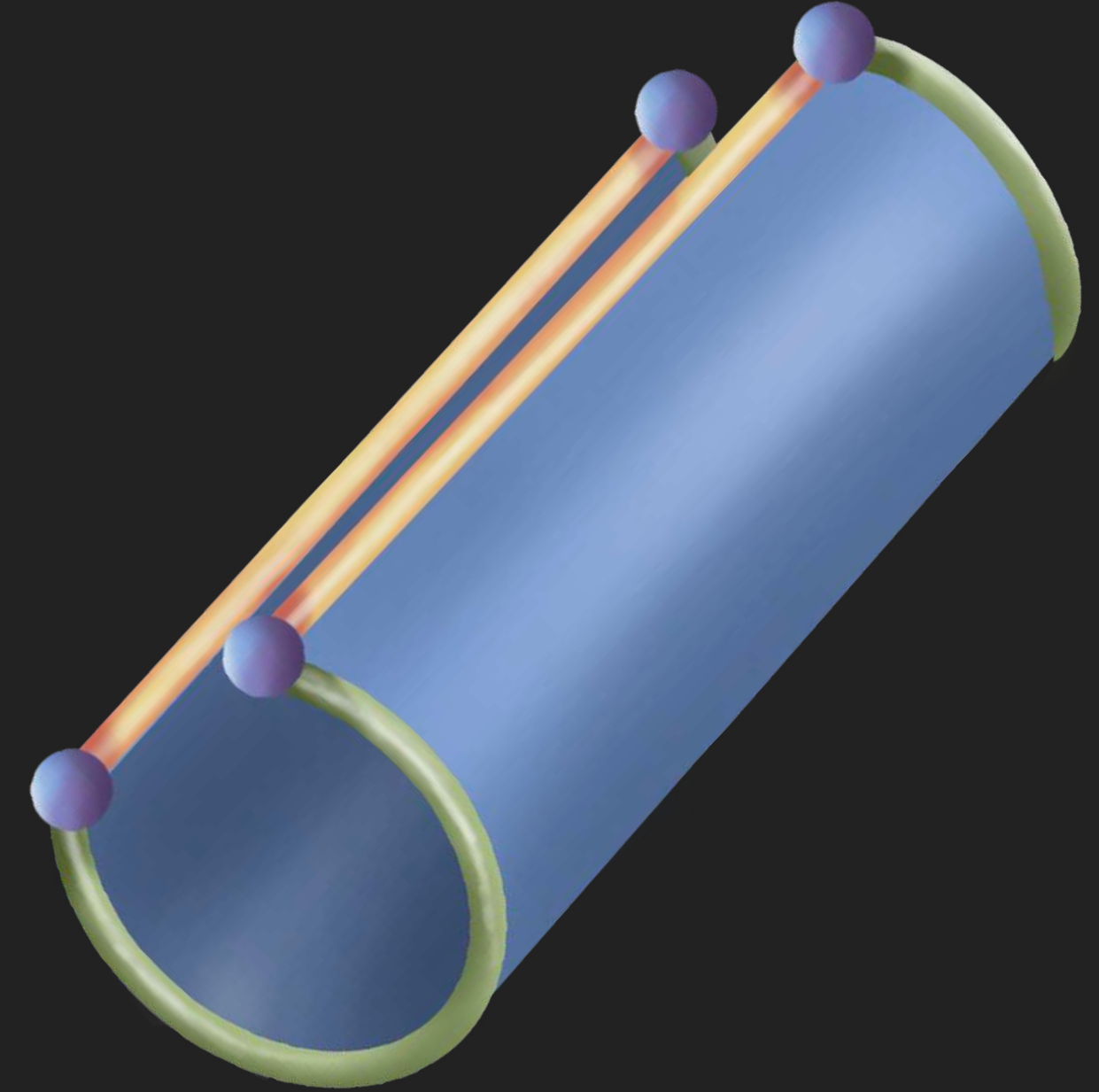
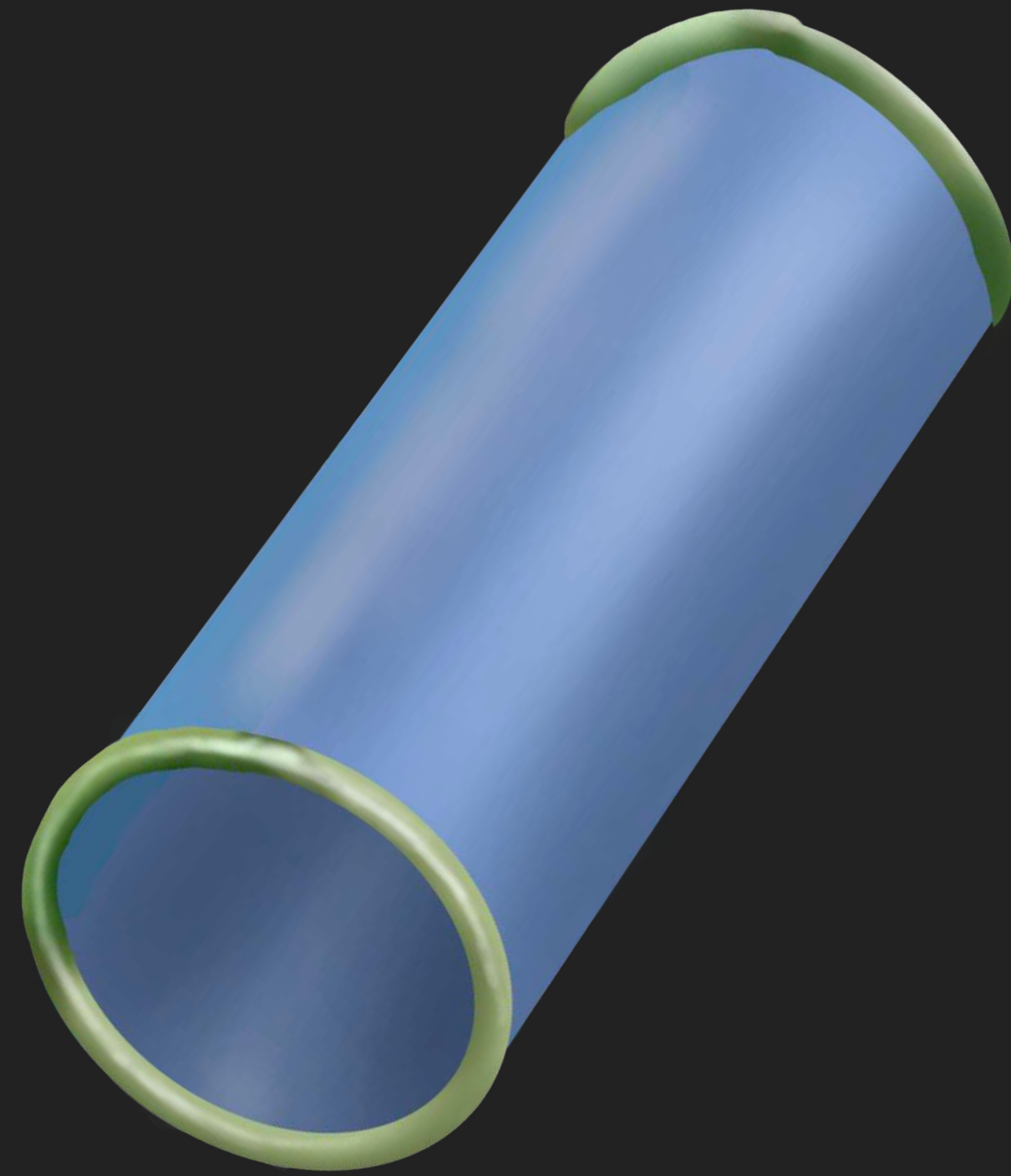
Unrolling

along a

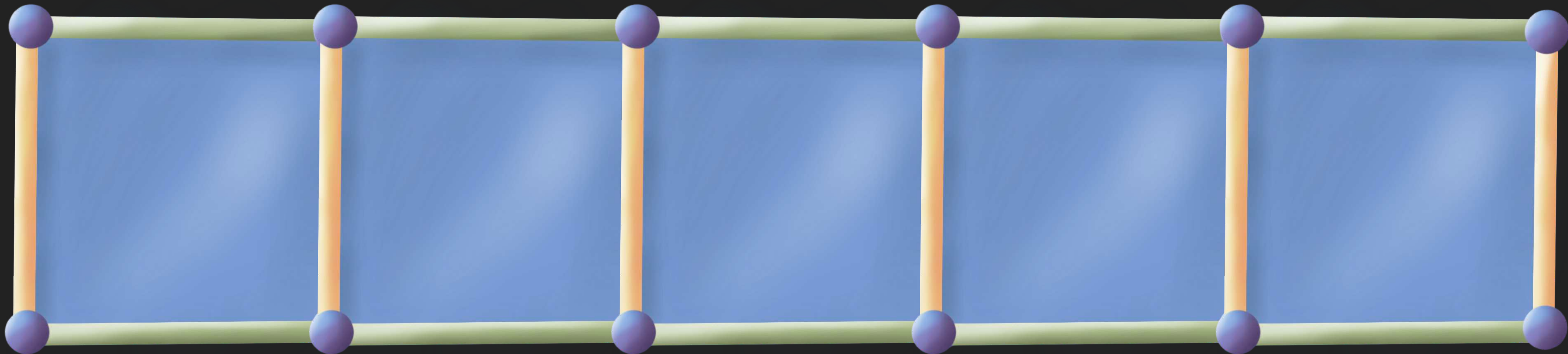
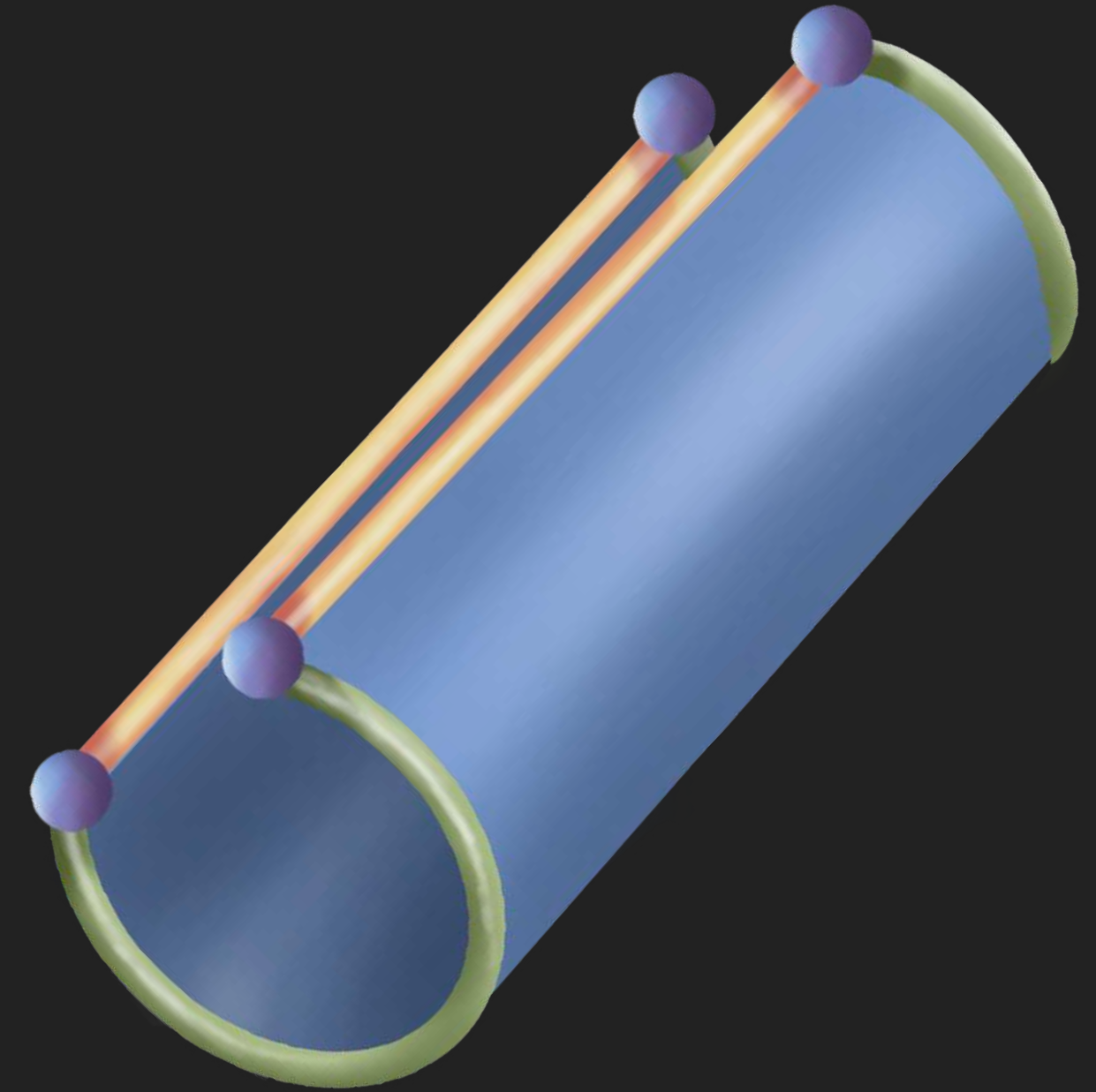
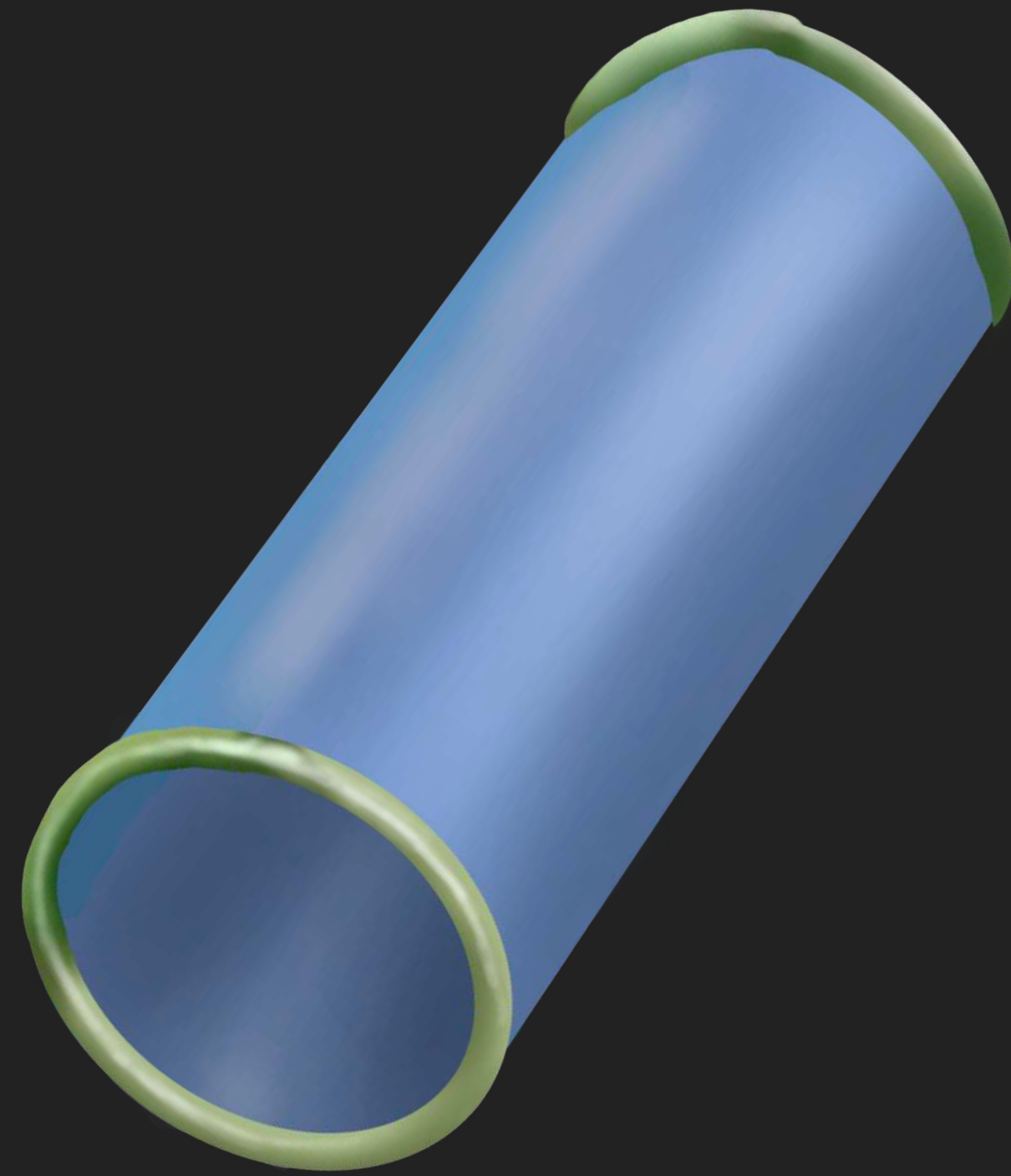
Loop

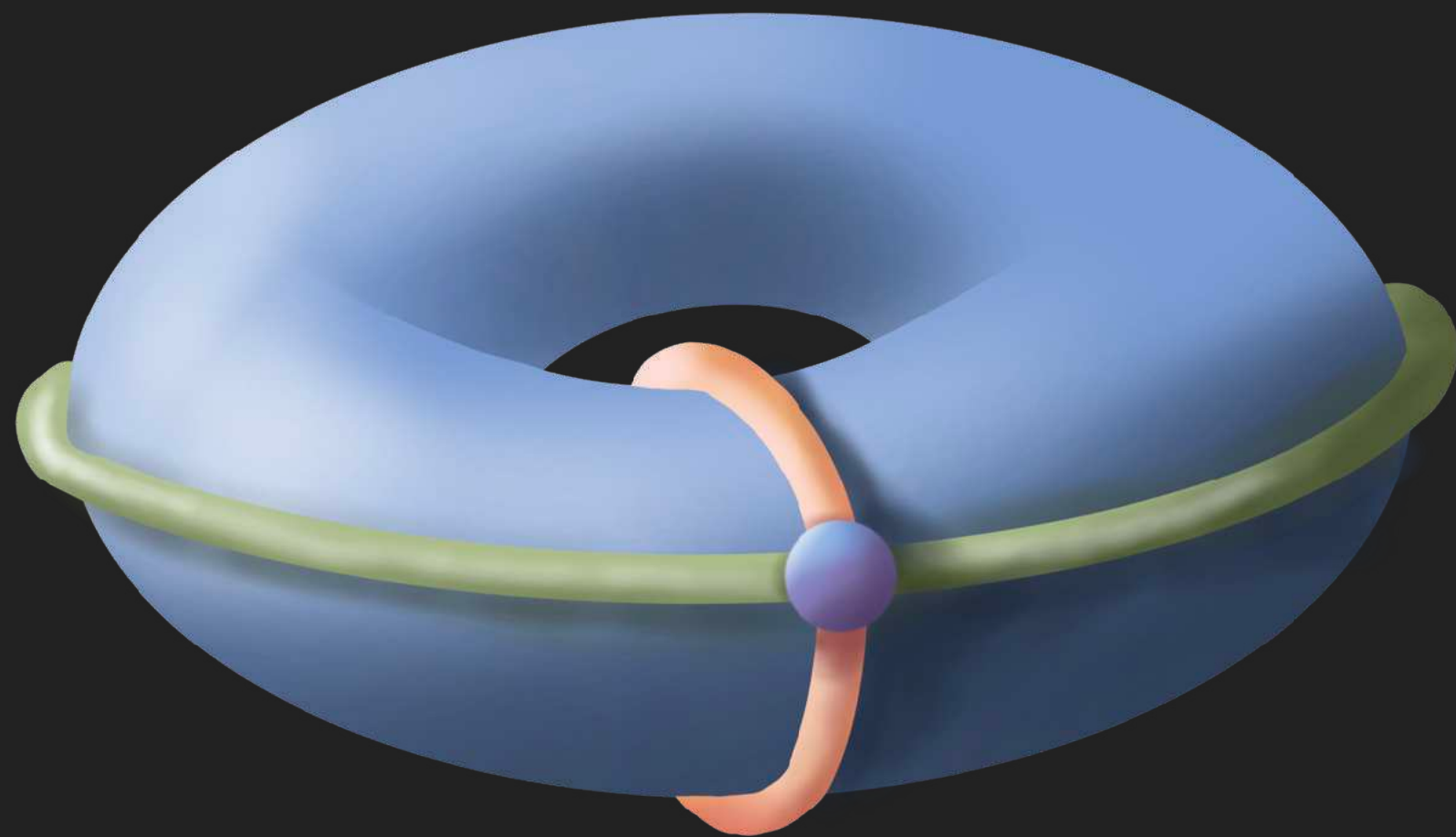


Unrolling *along a* Loop



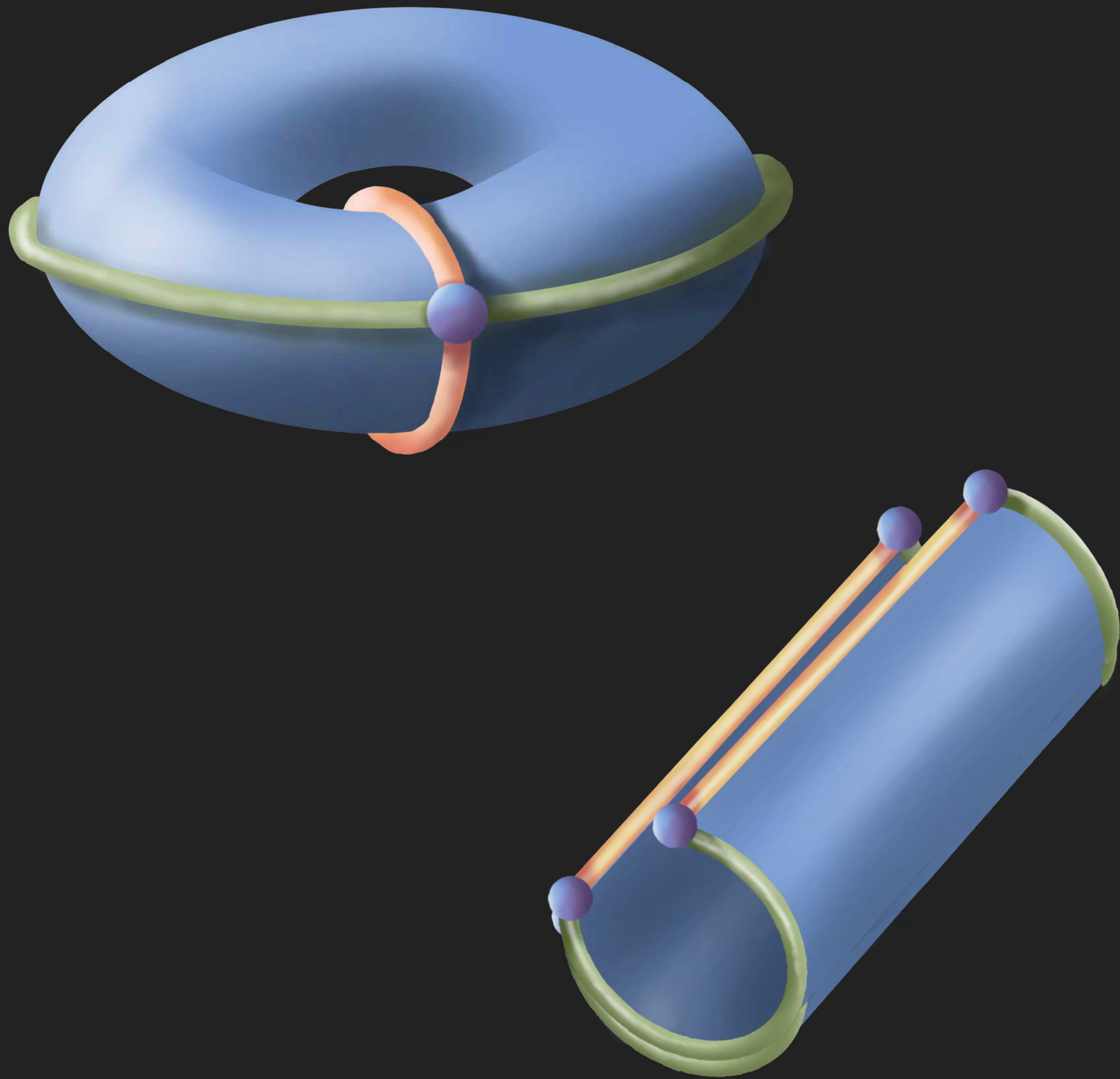
Unrolling *along a* Loop





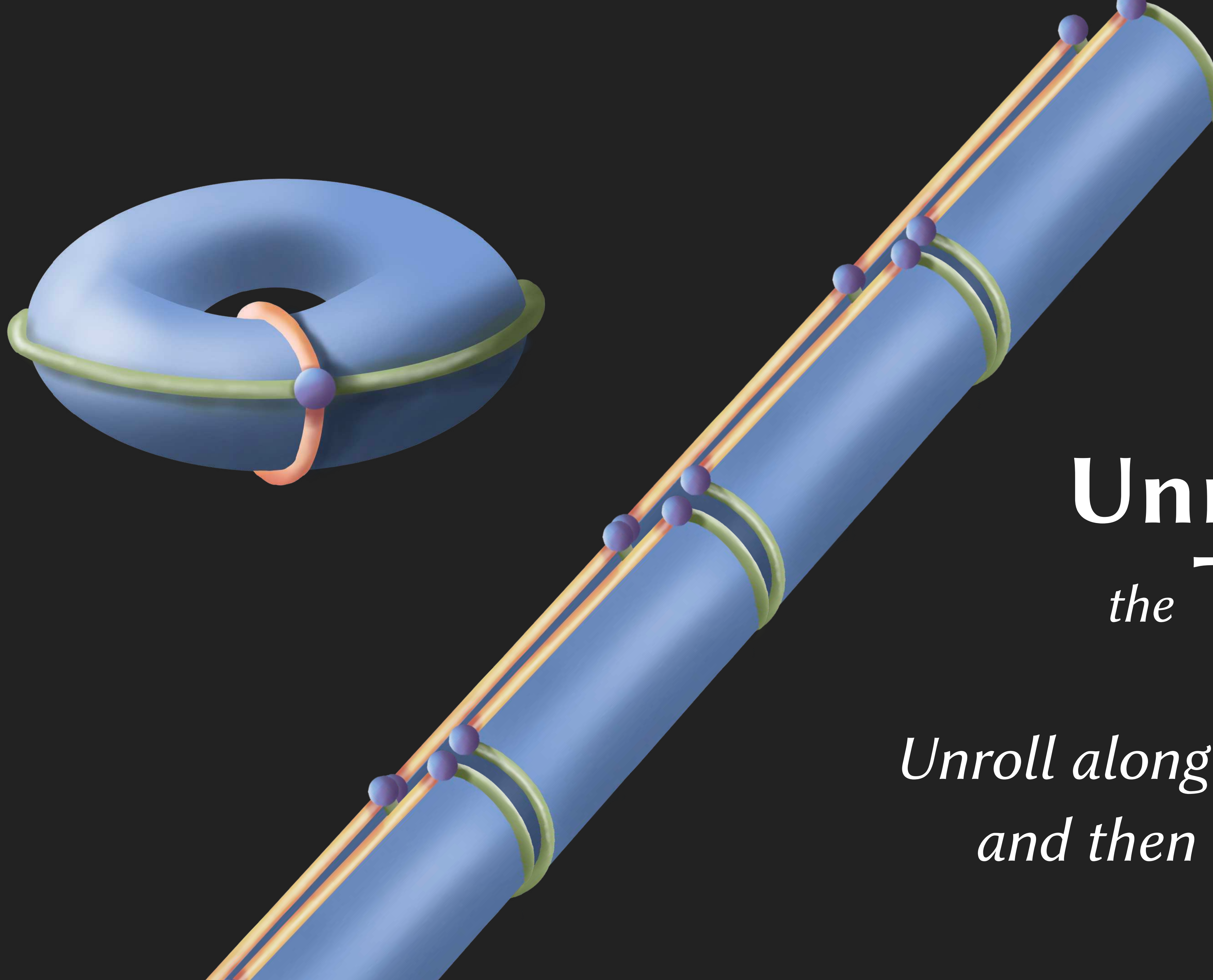
Unrolling *the* Torus

Unroll along a stuck loop...



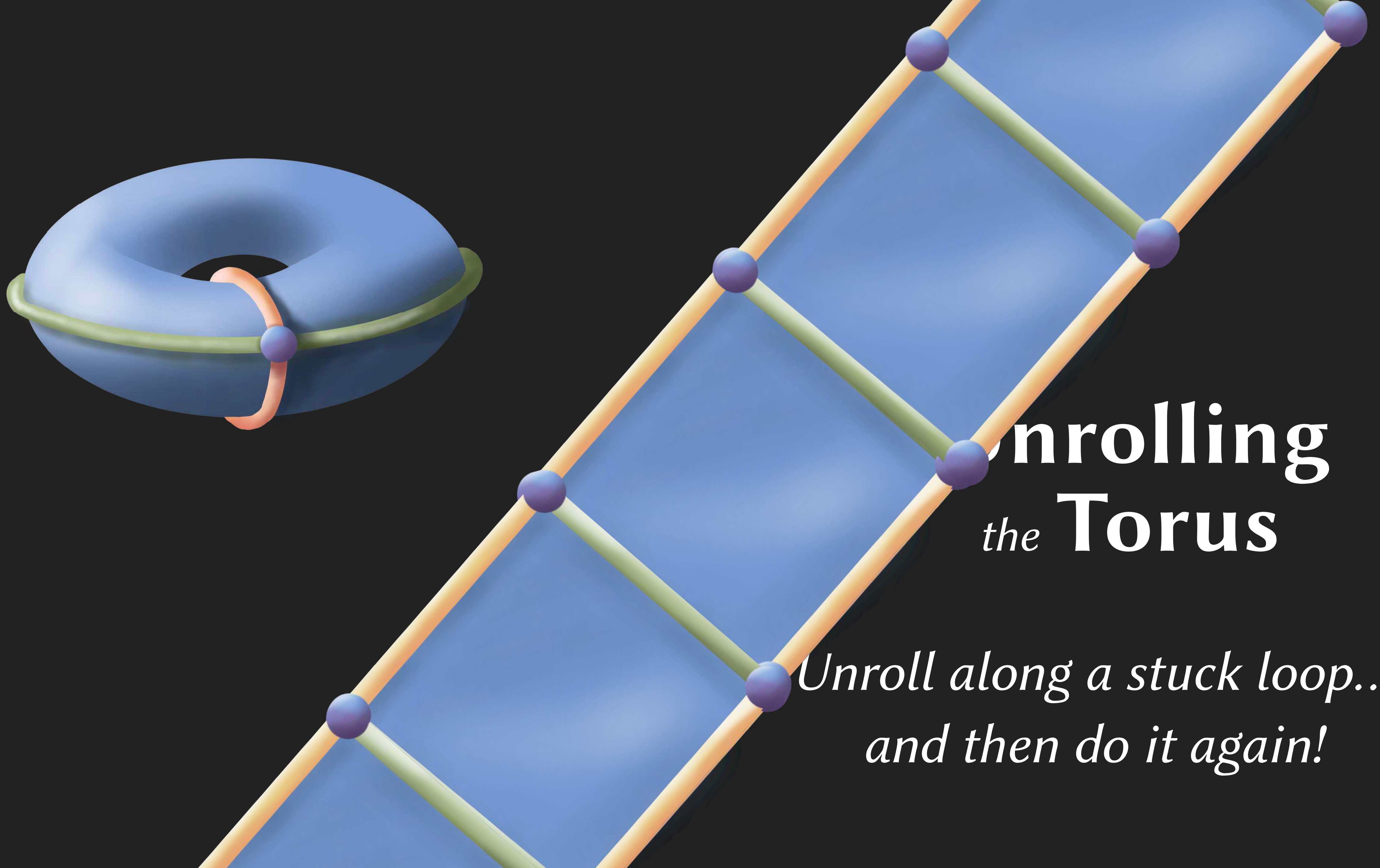
Unrolling *the* Torus

Unroll along a stuck loop...



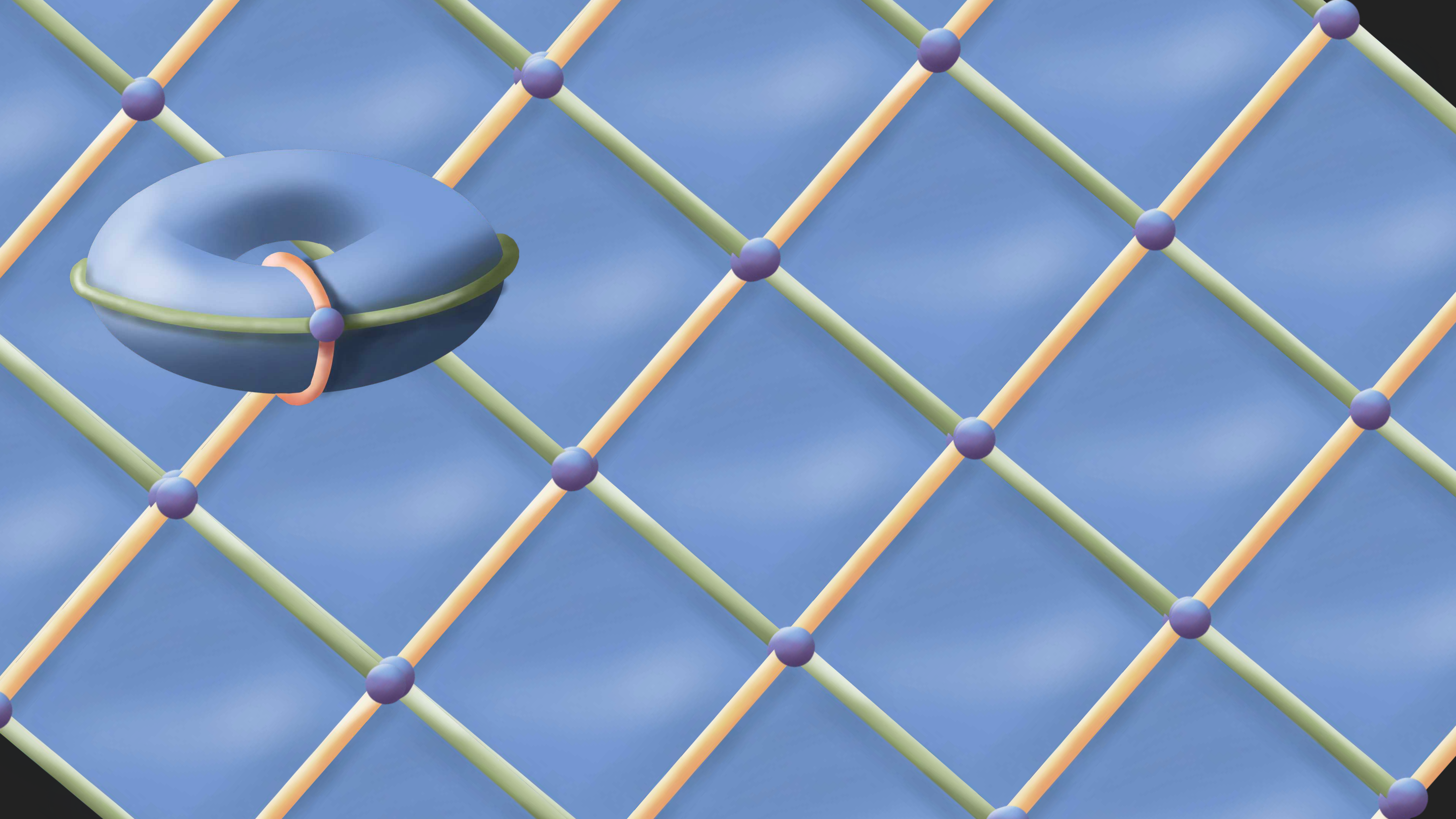
Unrolling *the* Torus

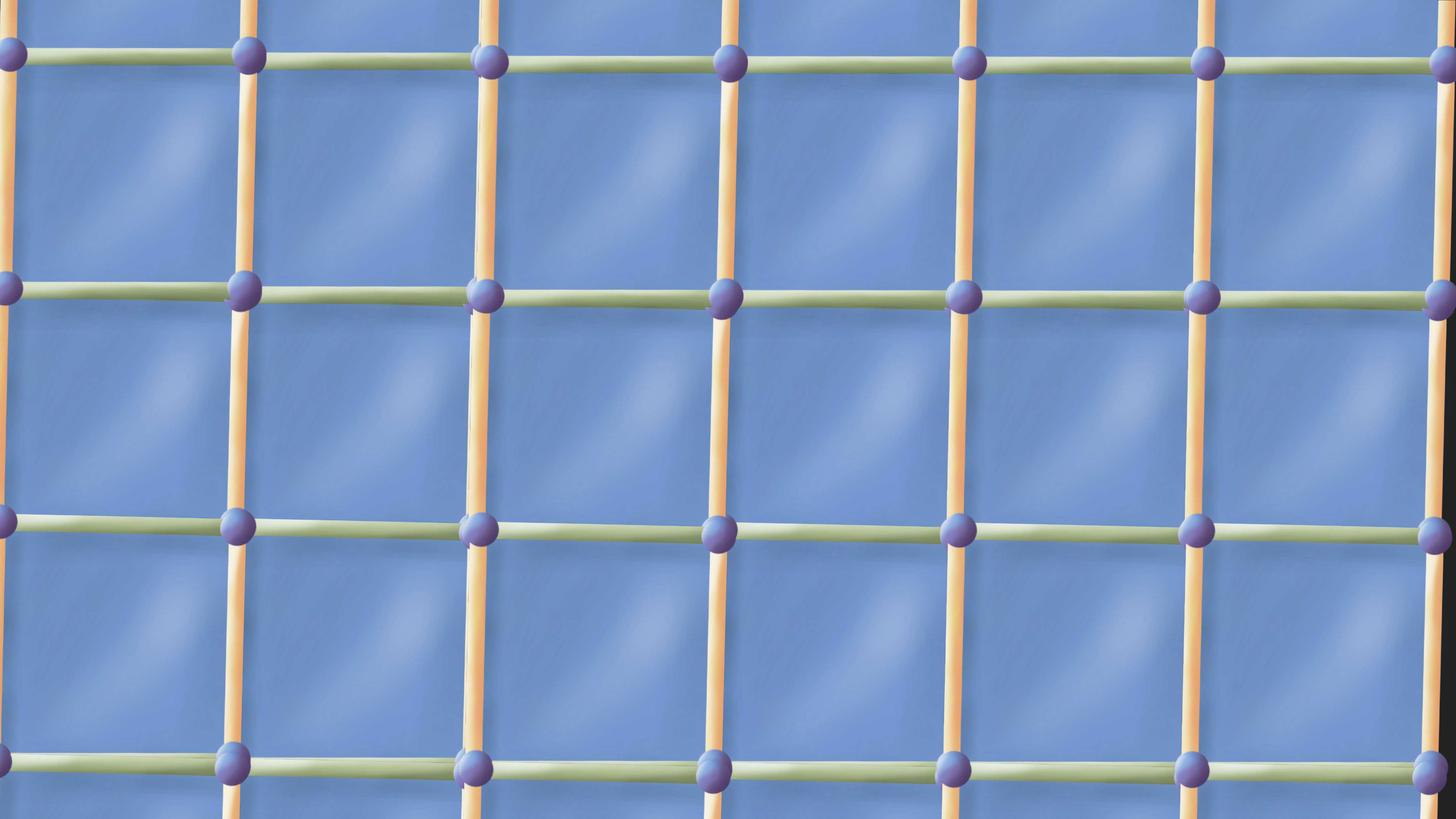
*Unroll along a stuck loop...
and then do it again!*

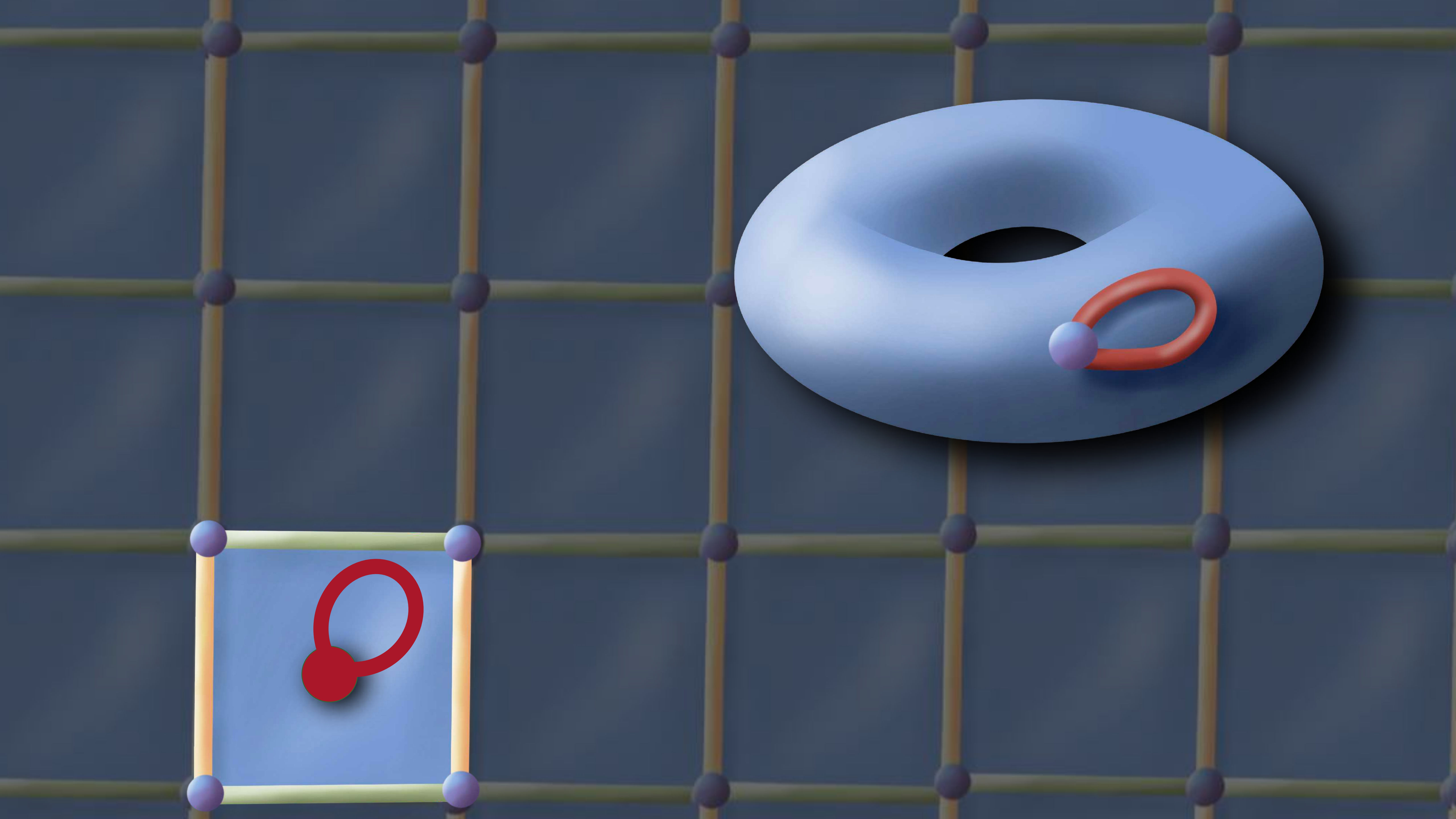


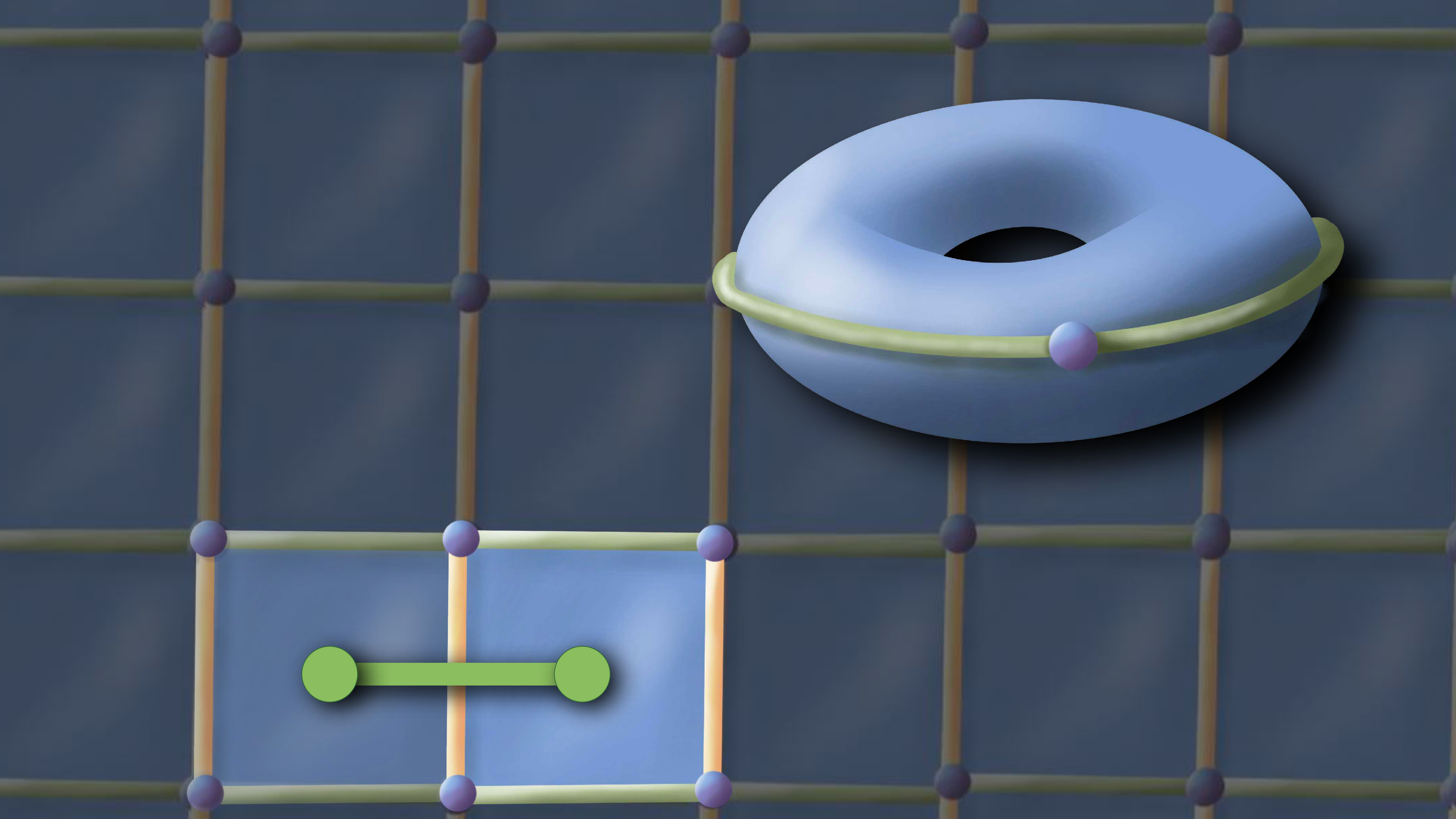
Unrolling *the* Torus

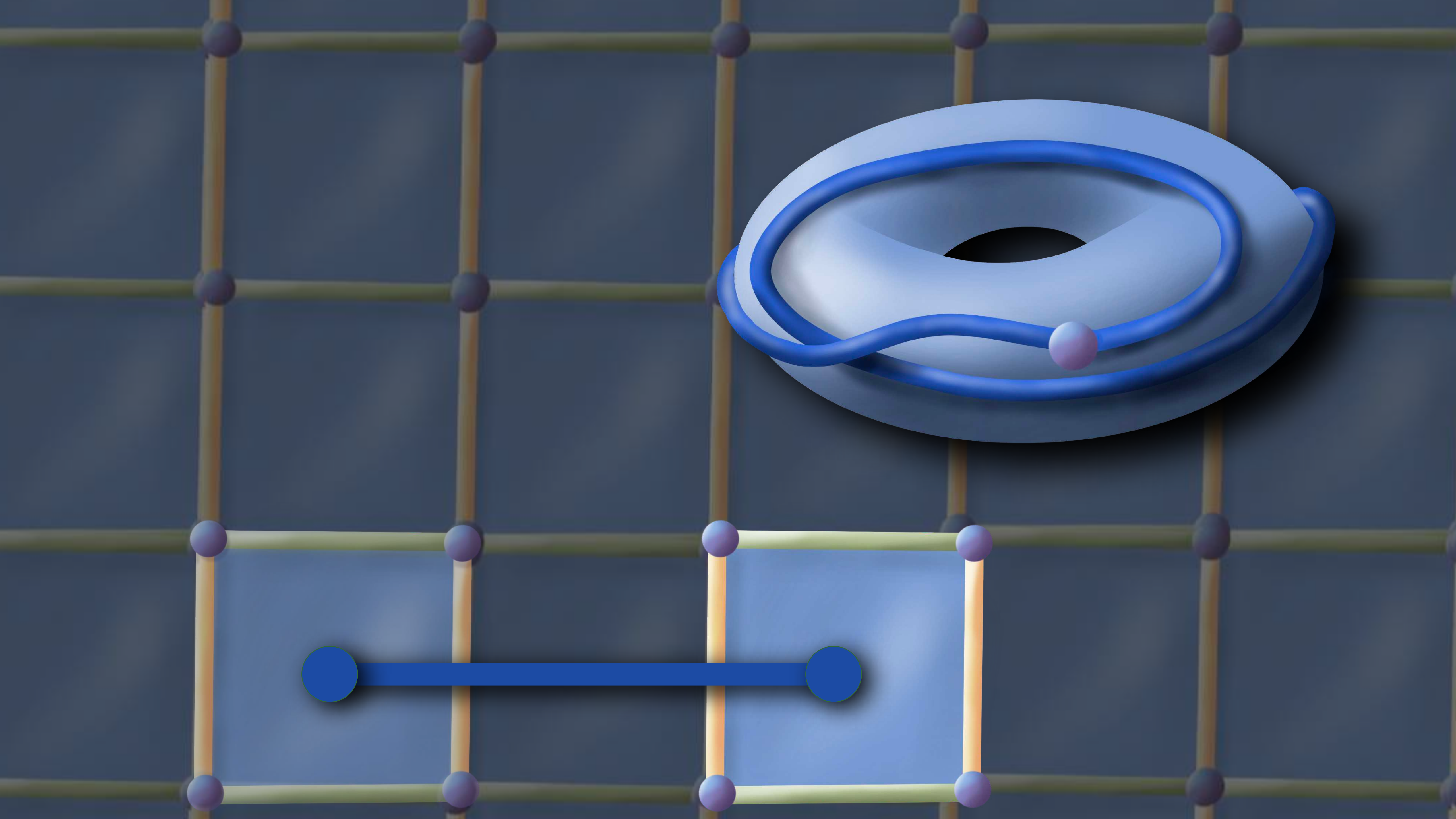
*Unroll along a stuck loop...
and then do it again!*

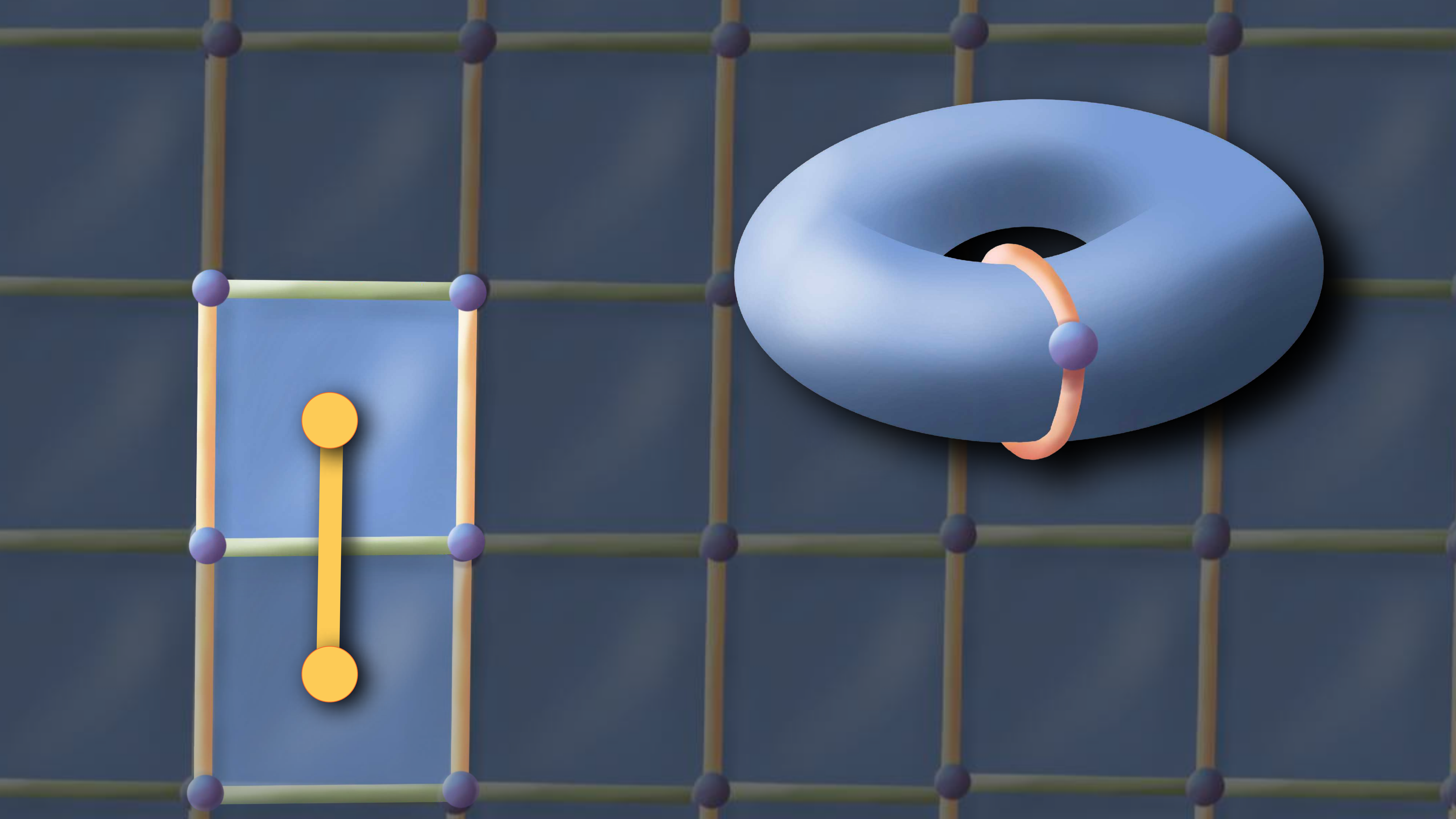


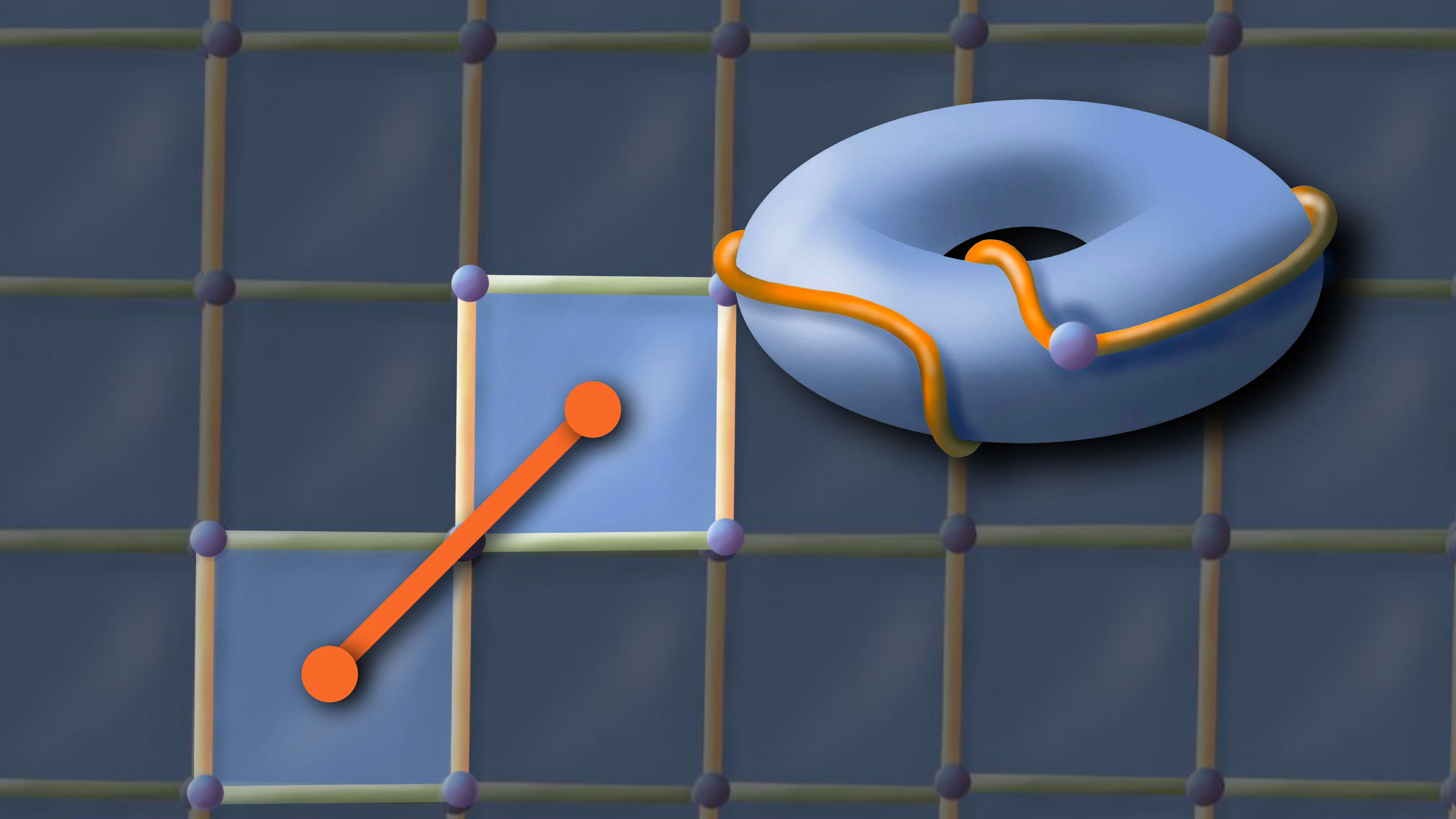


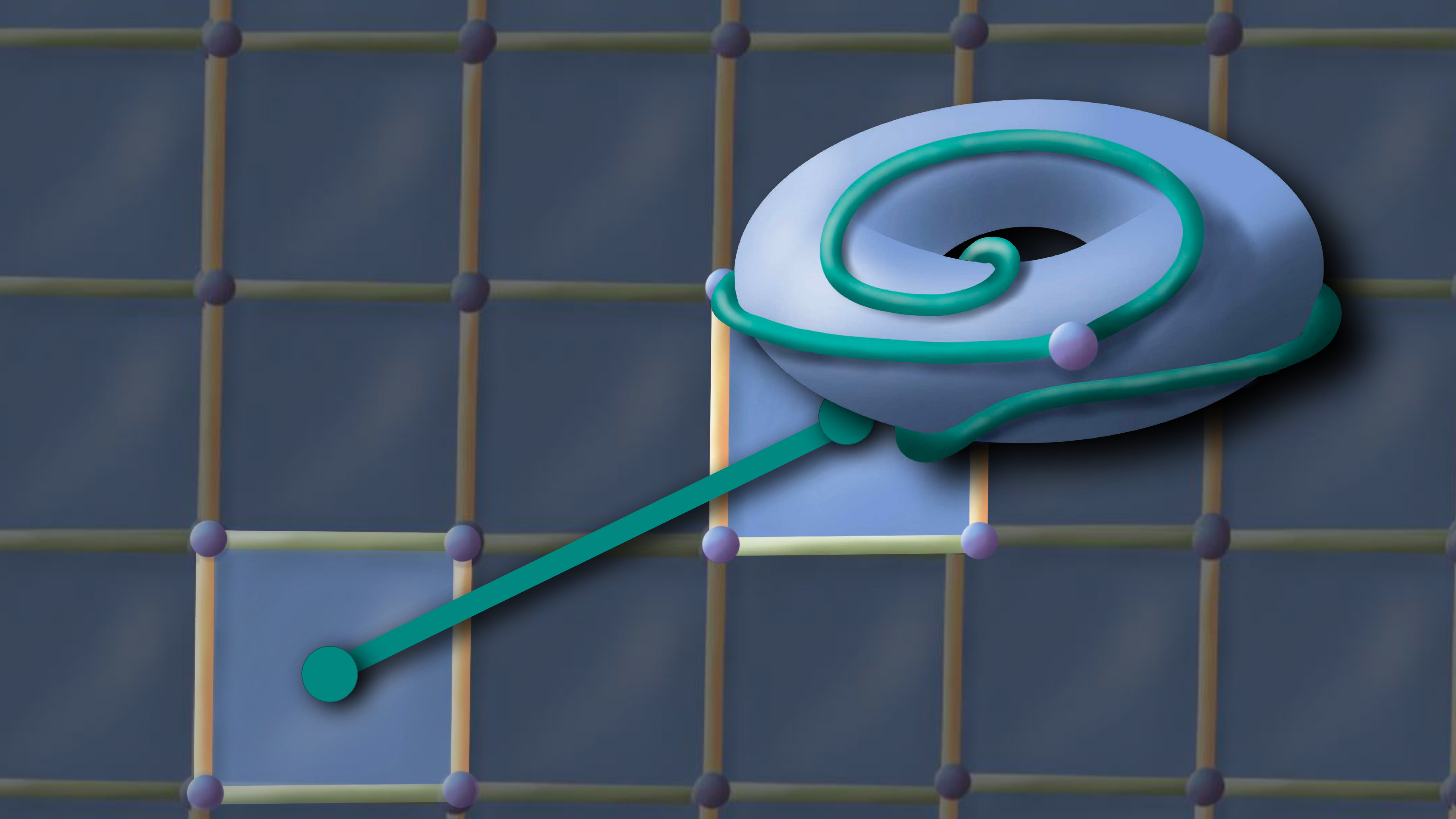






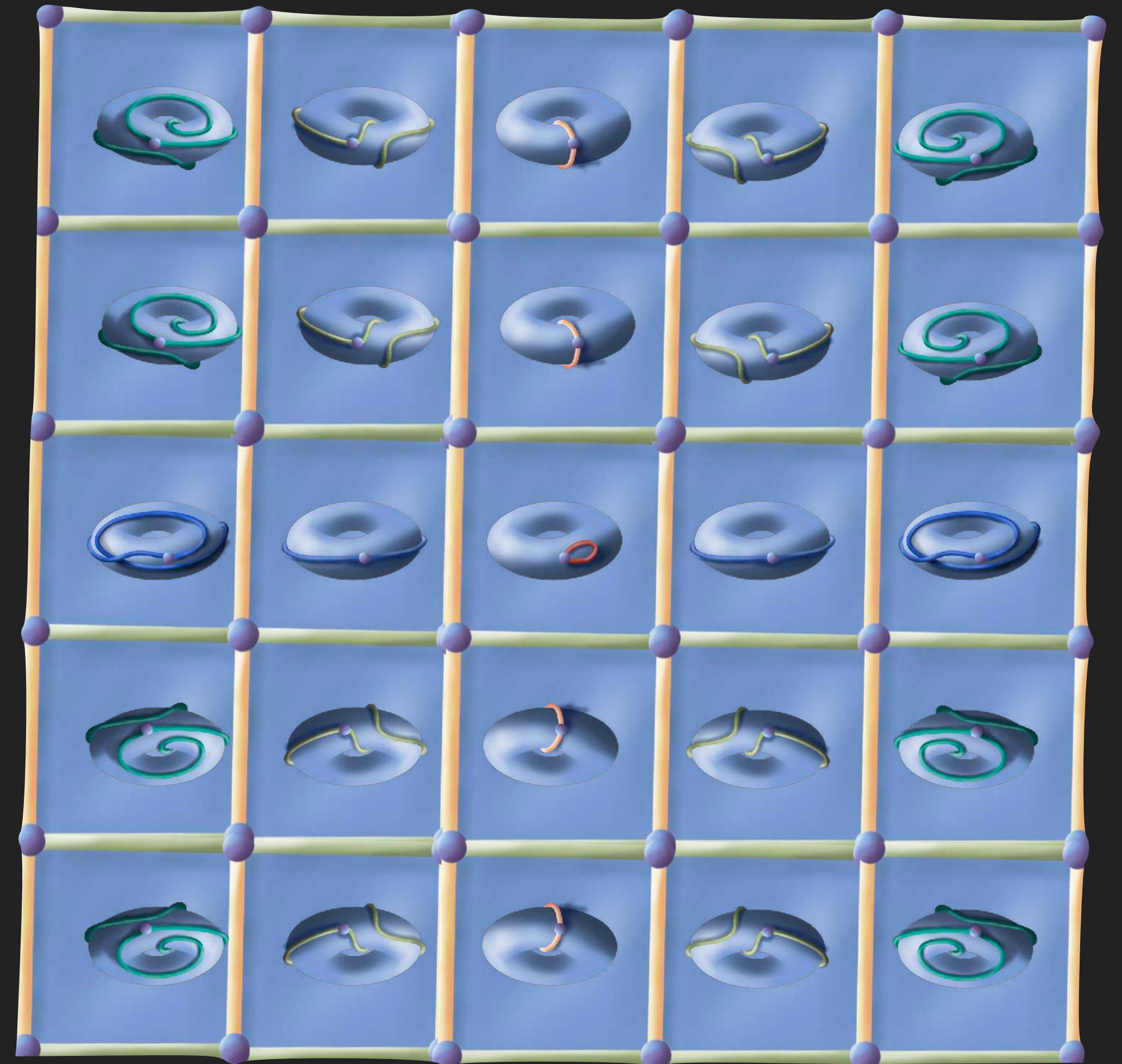
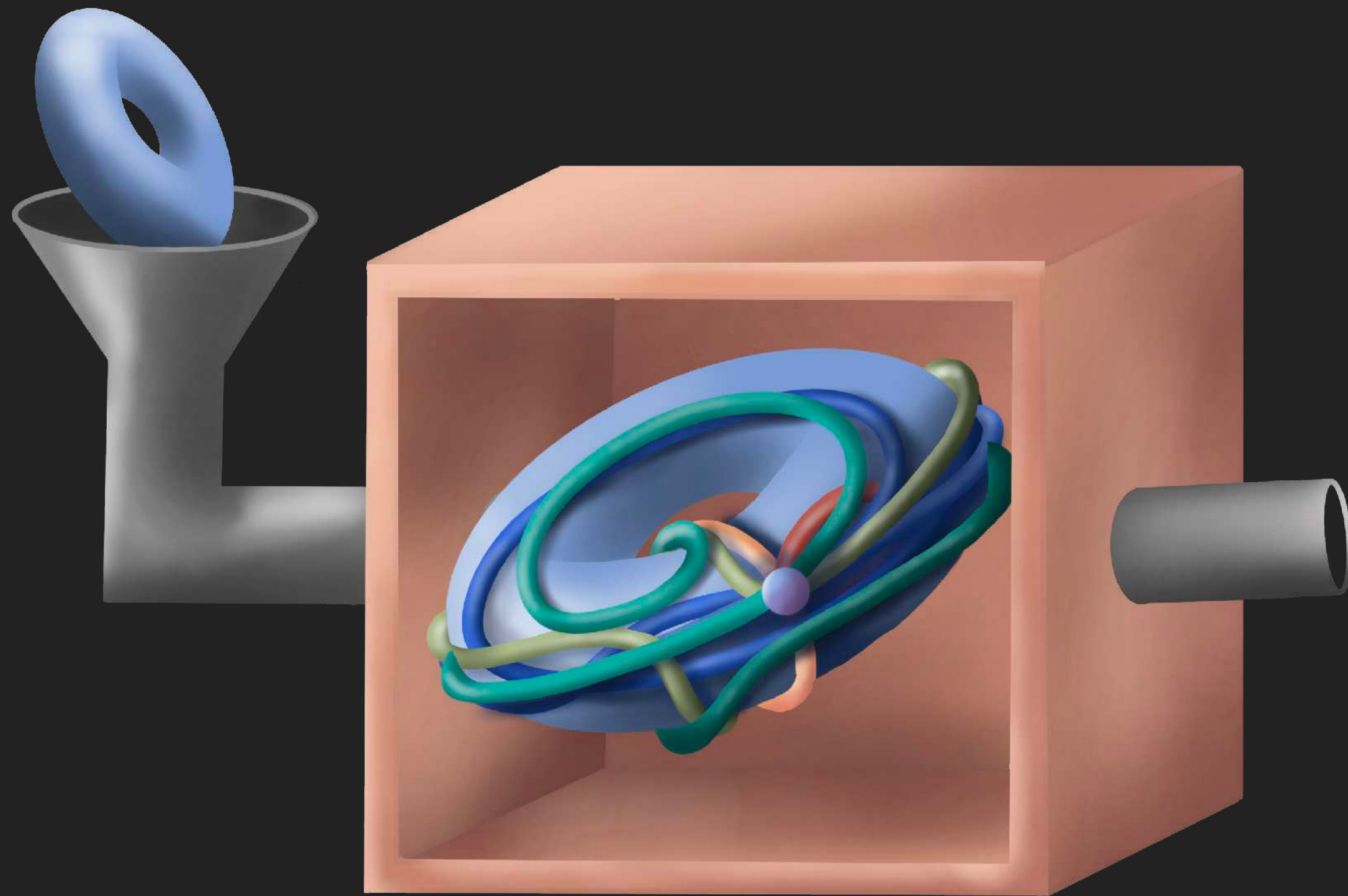






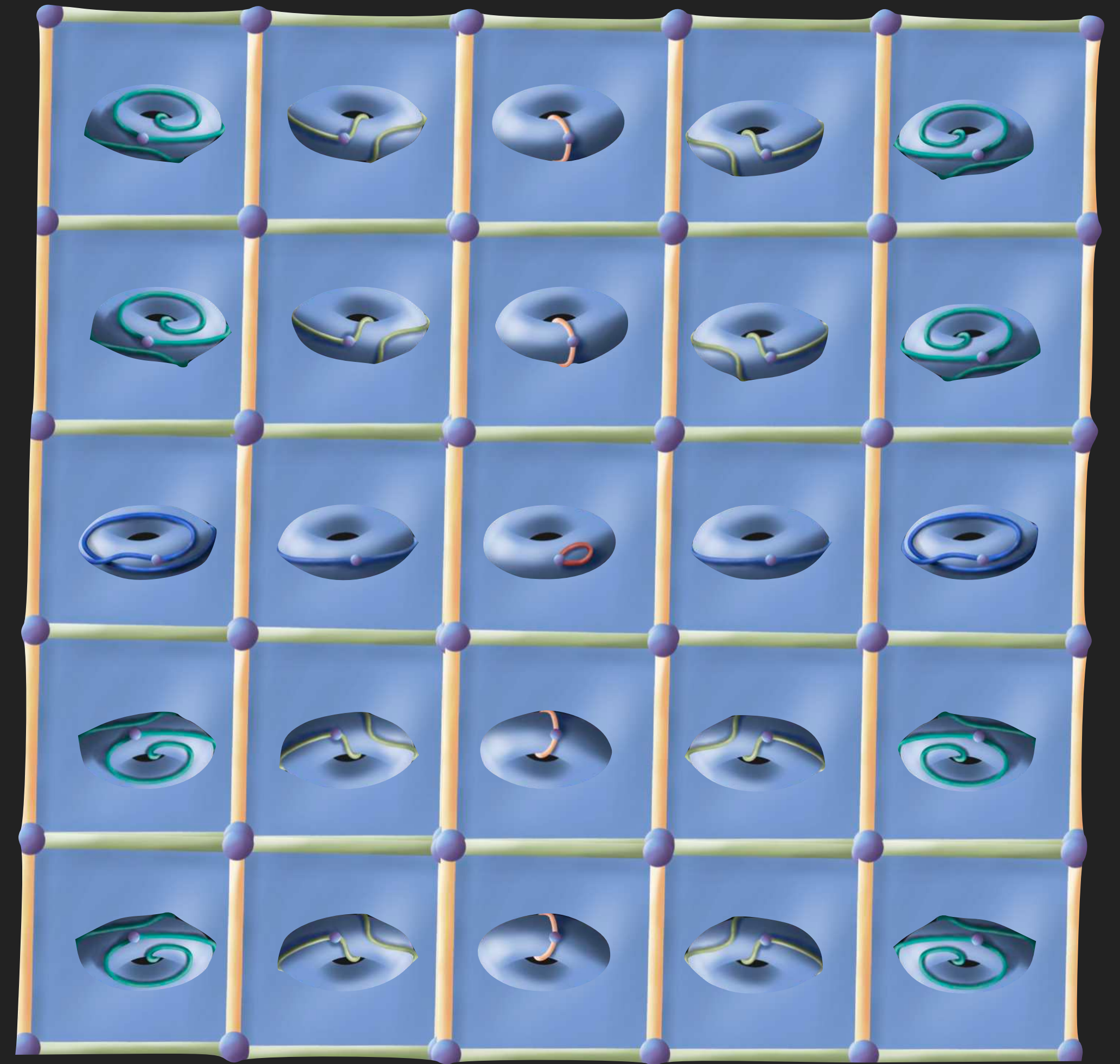
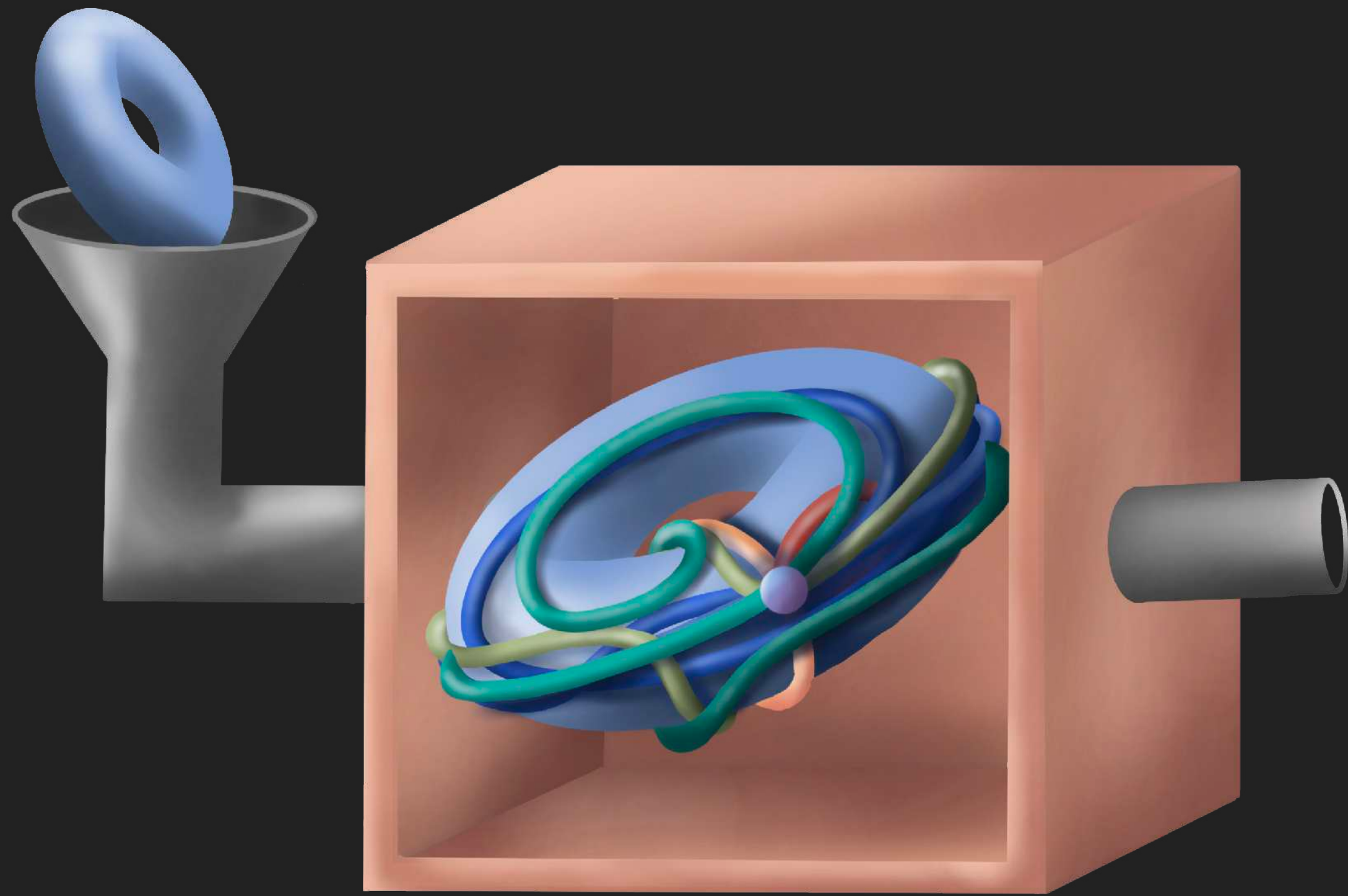
The Loop Invariant

The collection of loops & their structure is the fundamental group.



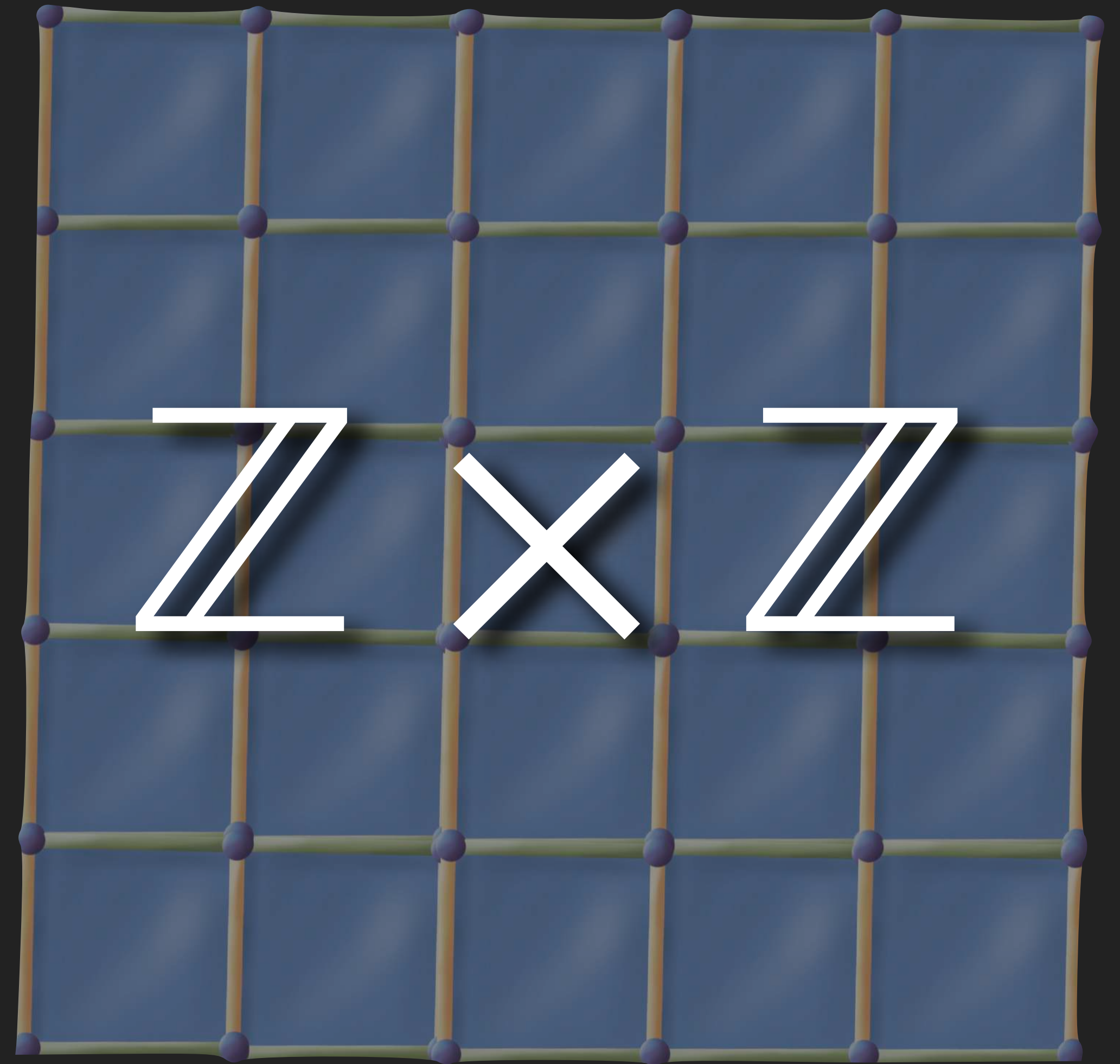
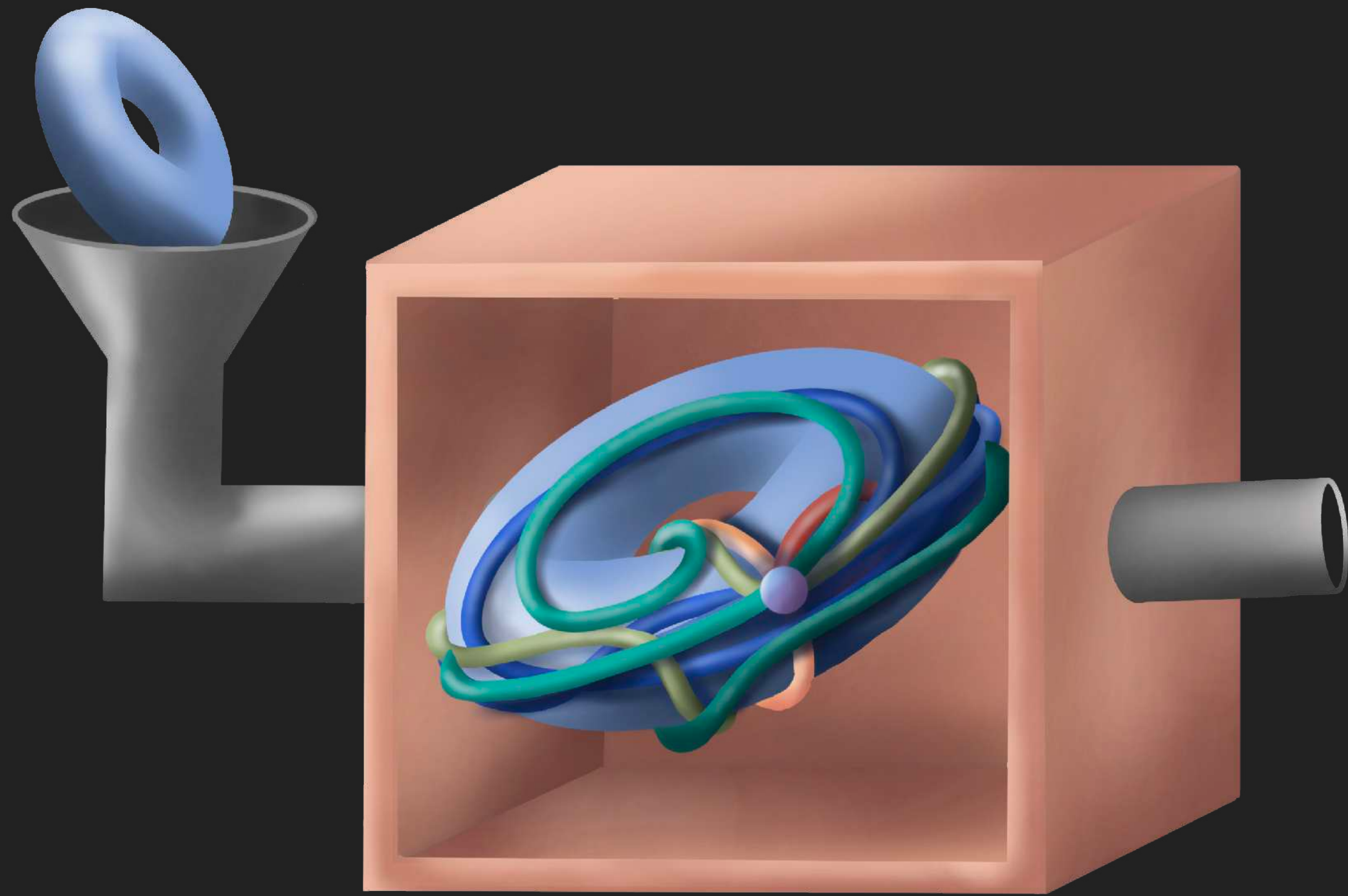
The Loop Invariant

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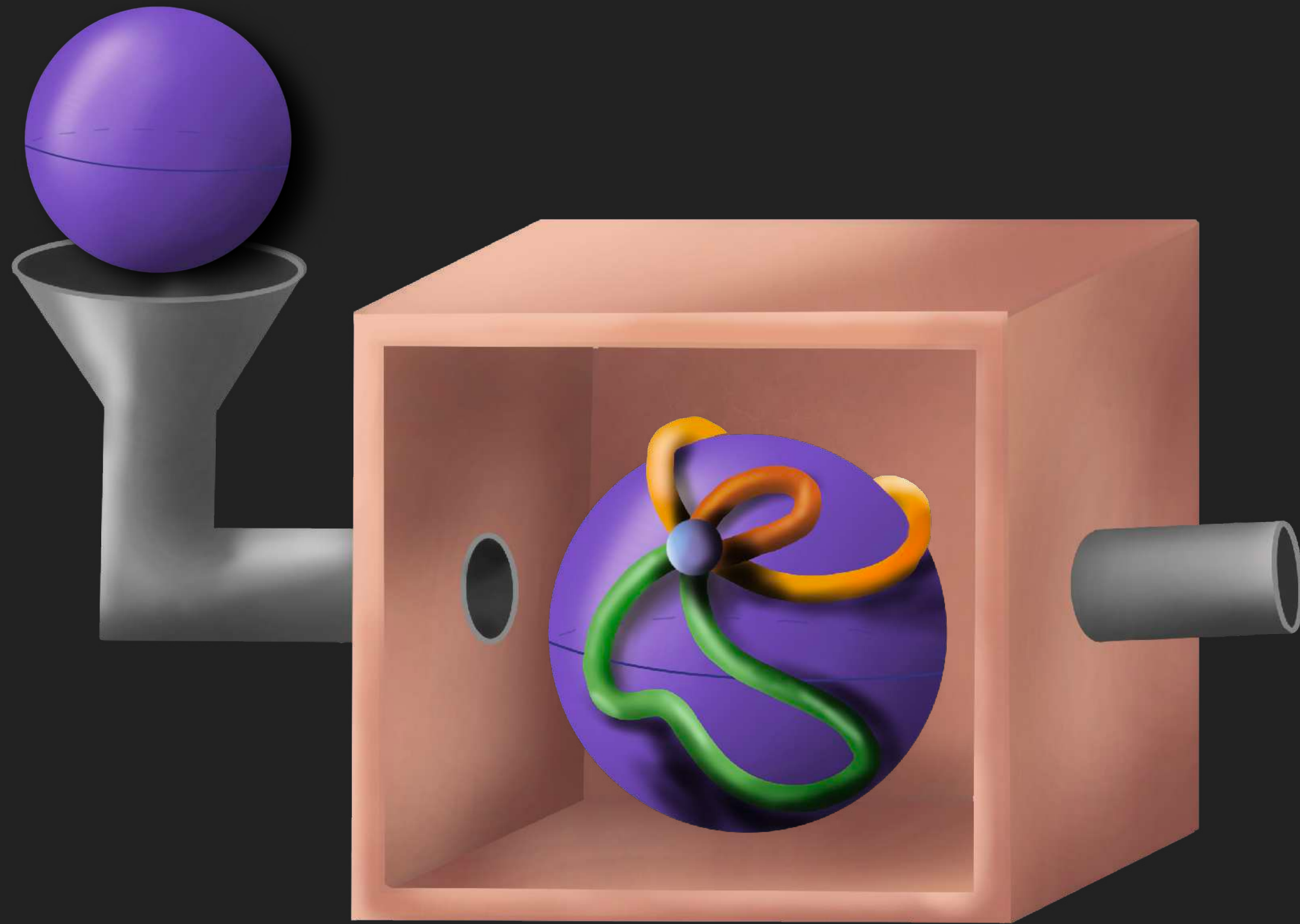
The Loop Invariant

The collection of loops & their structure is the fundamental group.



The Loop Invariant

The collection of loops & their structure is the fundamental group.



The sphere has no stuck loops, so it doesn't unwrap.



$\{0\}$

Manifolds have DNA:

A discrete code, built out of loops.

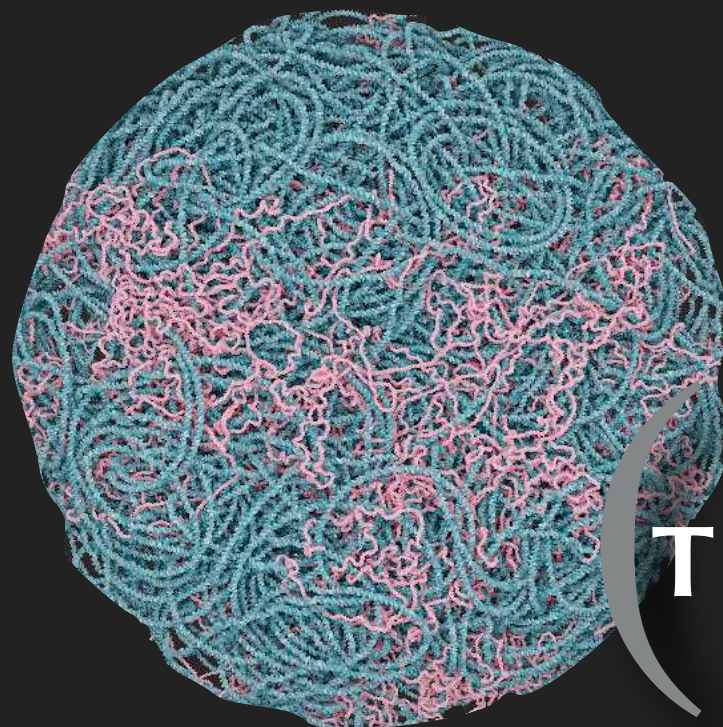
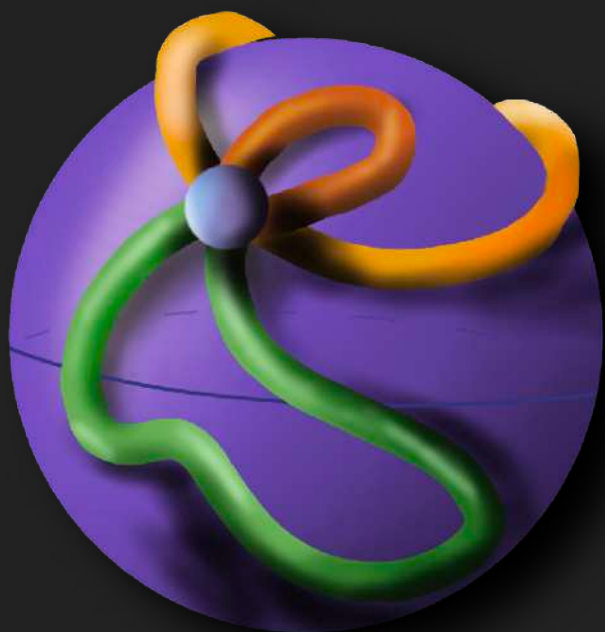
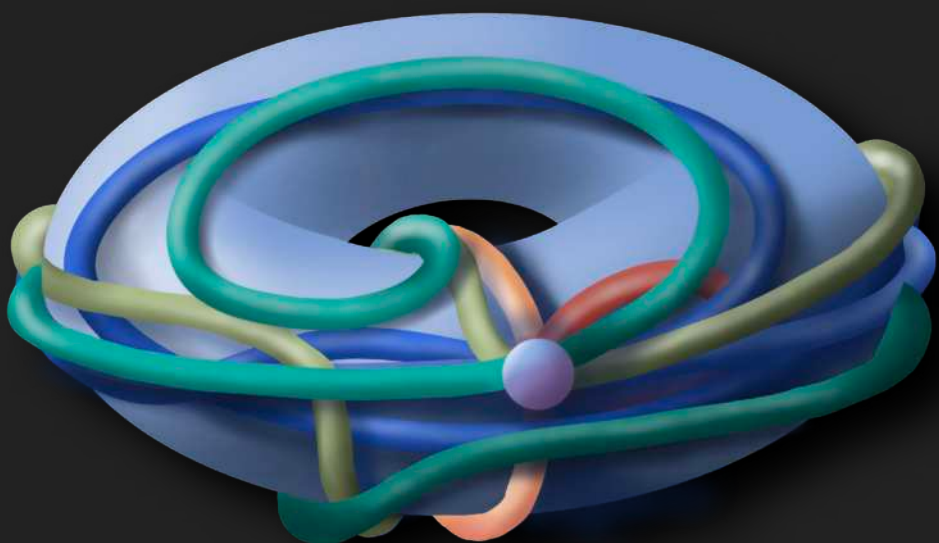
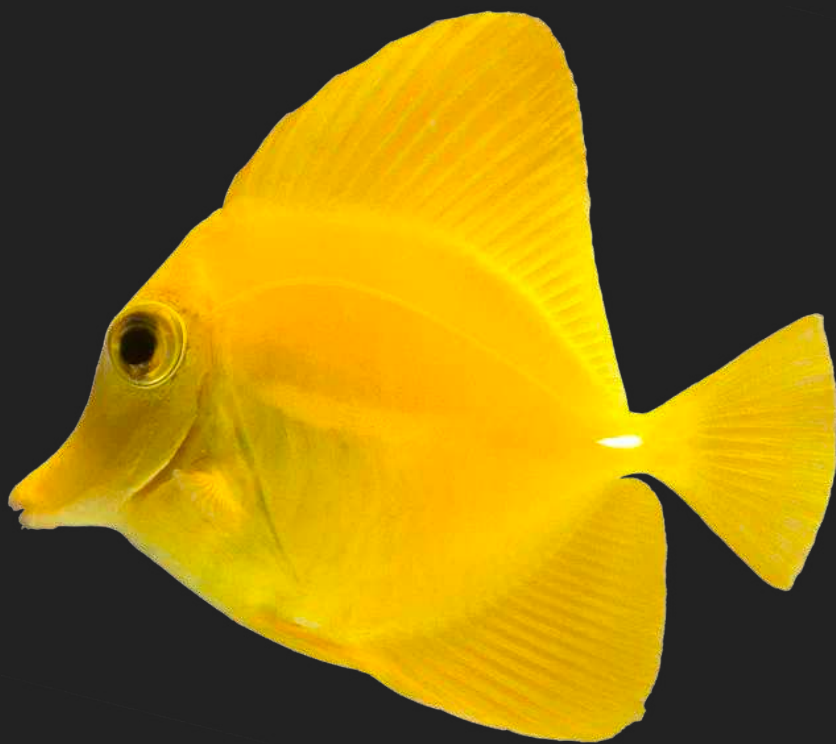
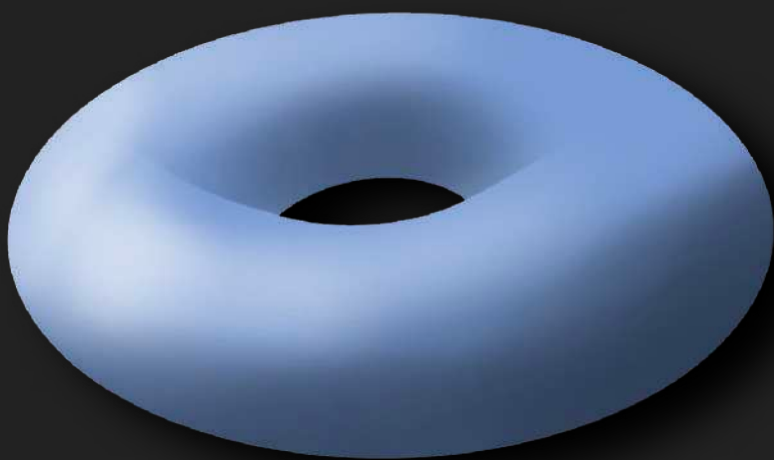
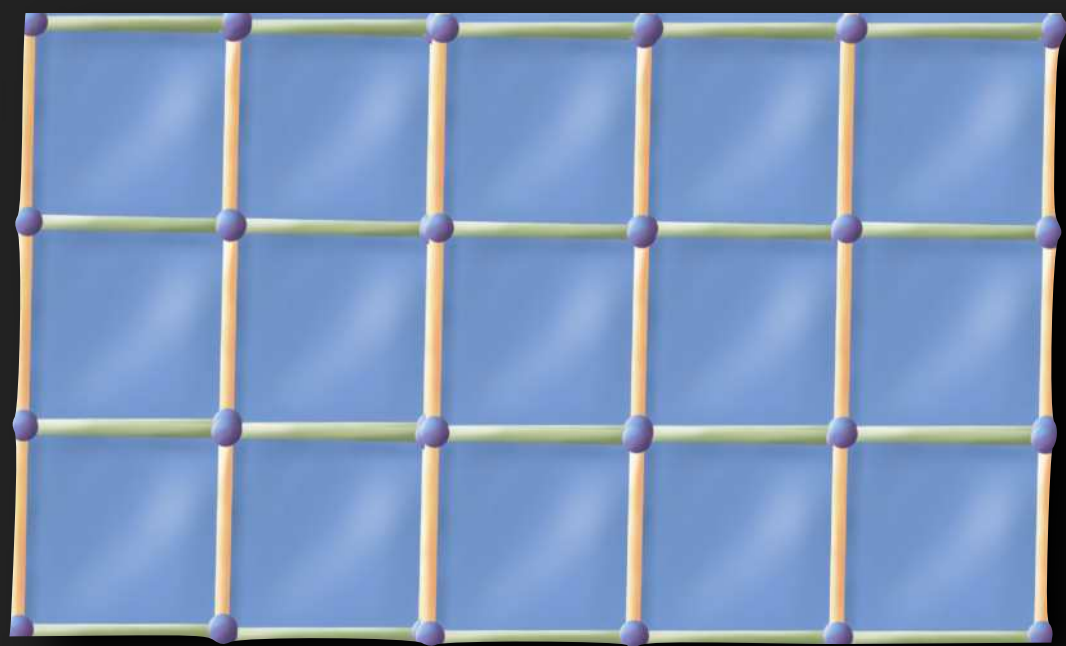


Image Credit:
Tim Davidson

$$\mathbb{Z} \times \mathbb{Z}$$

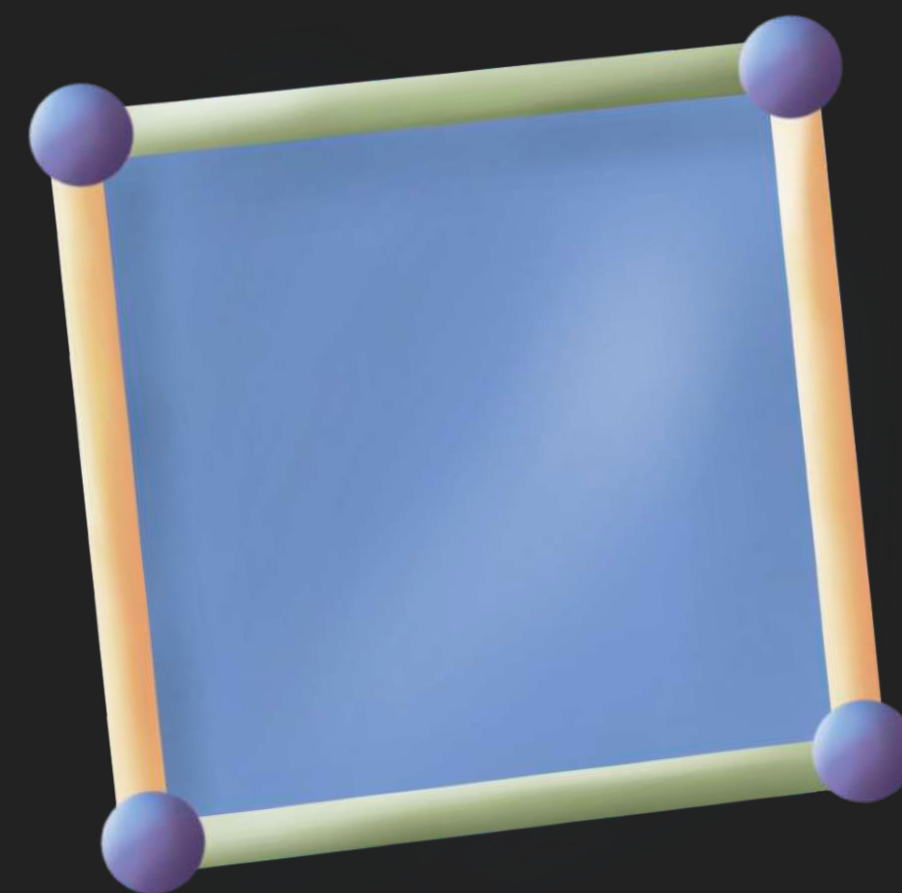
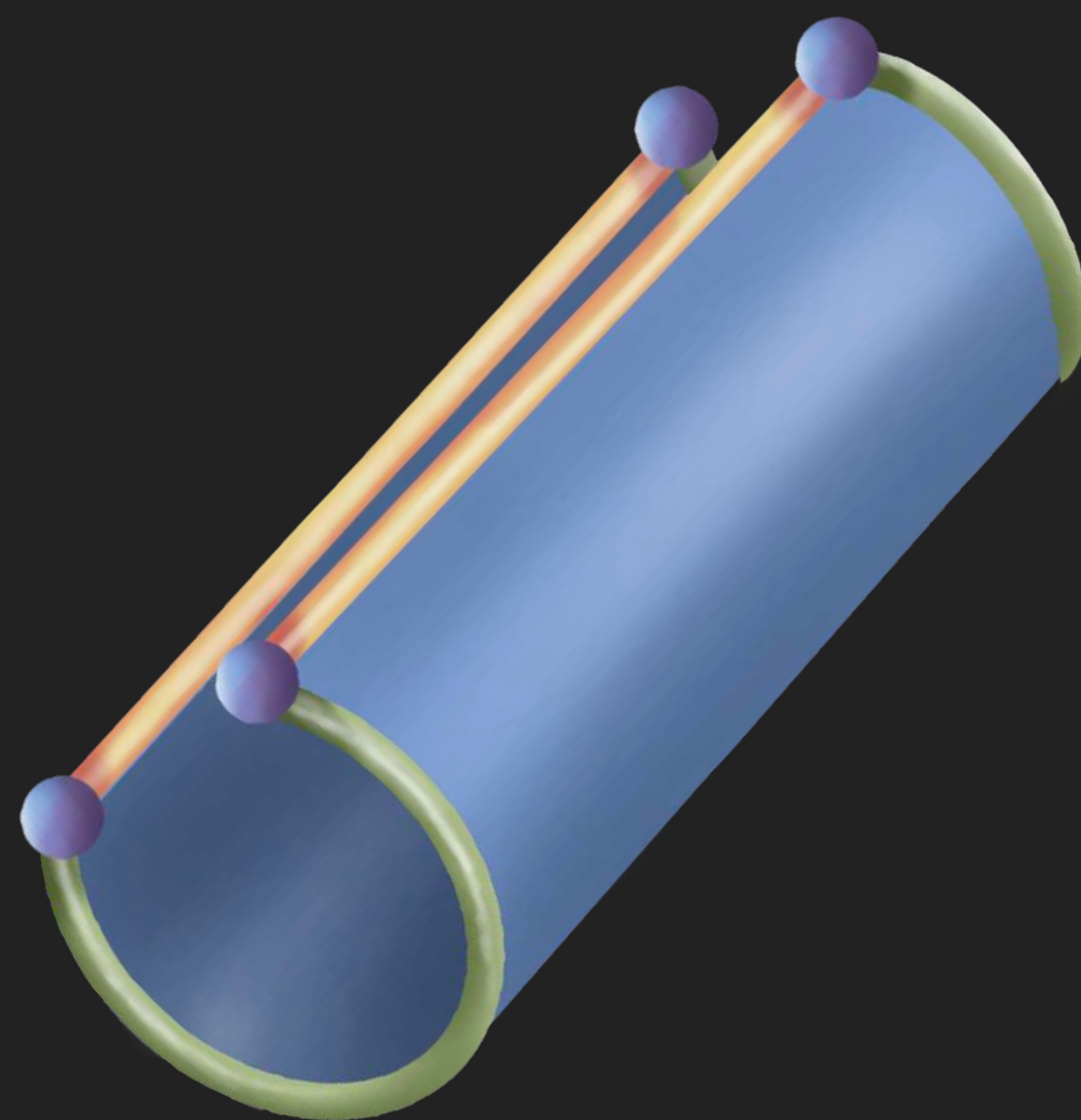
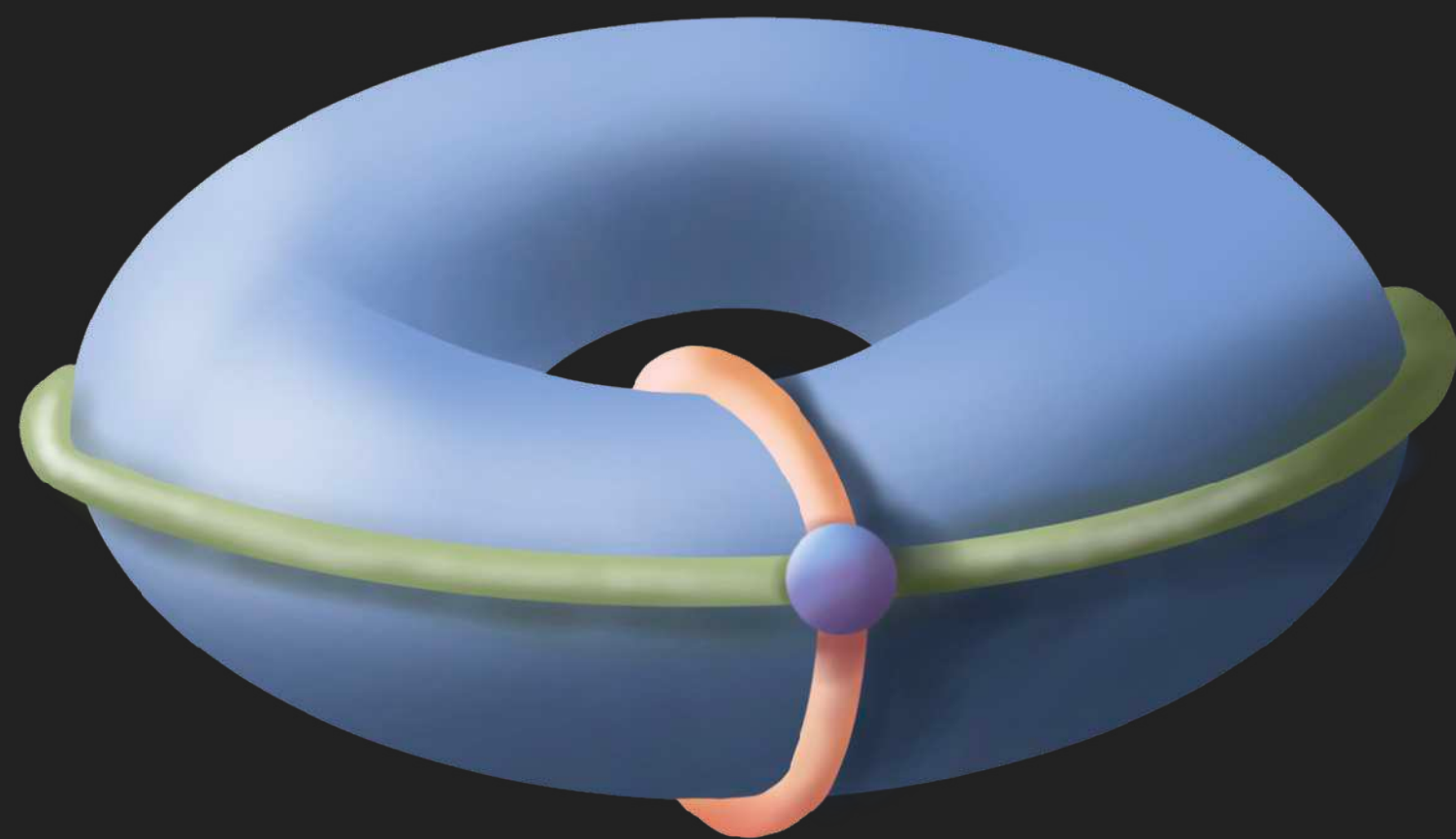


$$\{0\}$$



GCCTAGCCAAAGCTCTT
CCAAGGTGACTCTCAGT
TCAAGCTGCCTAGCCCG
CAGTTCAAGCTGCCTAG

The
Torus
unrolls to
Squares
on the
Plane



But

How

do we

Unroll

this

Surface?



But

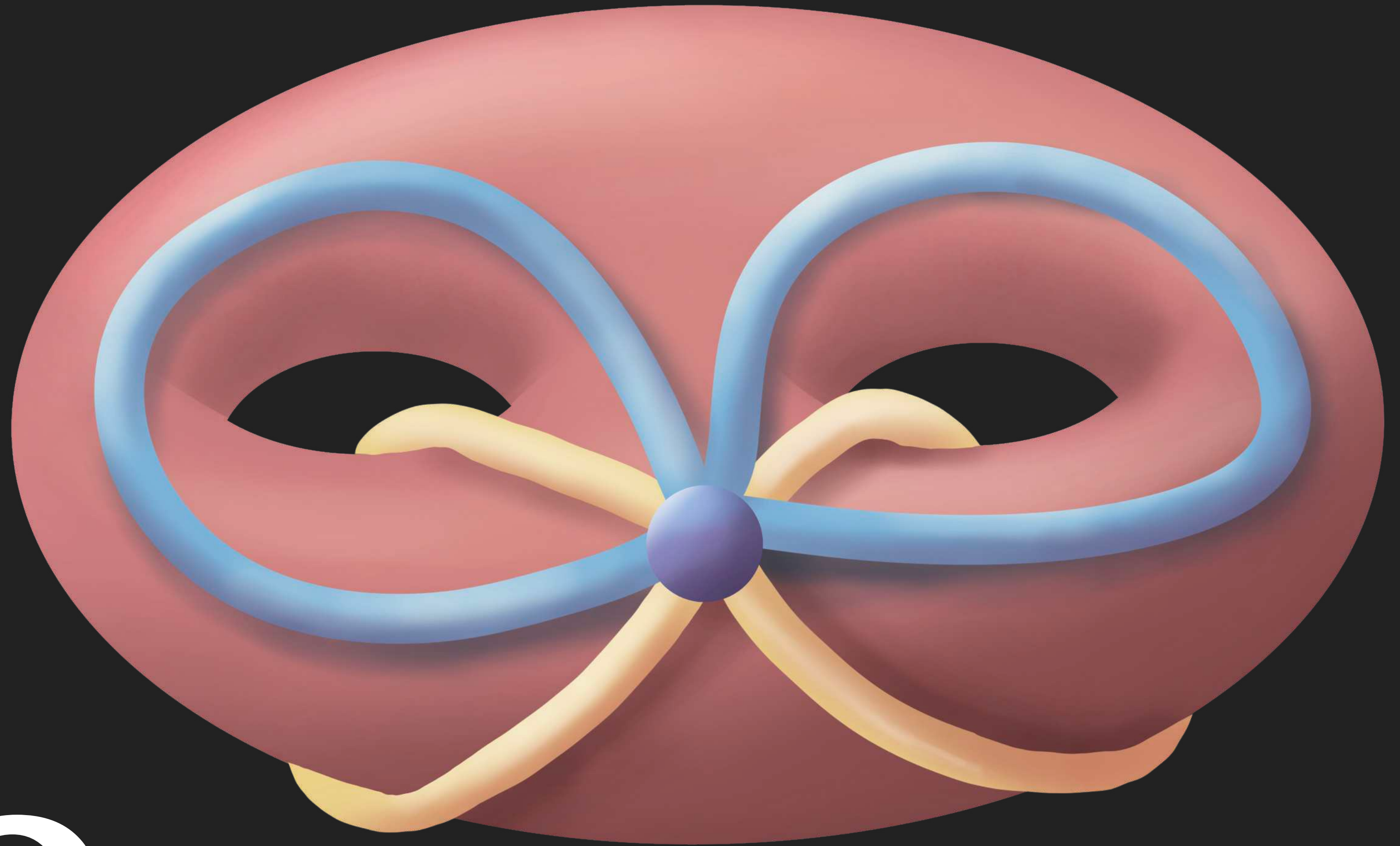
How

do we

Unroll

this

Surface?



But

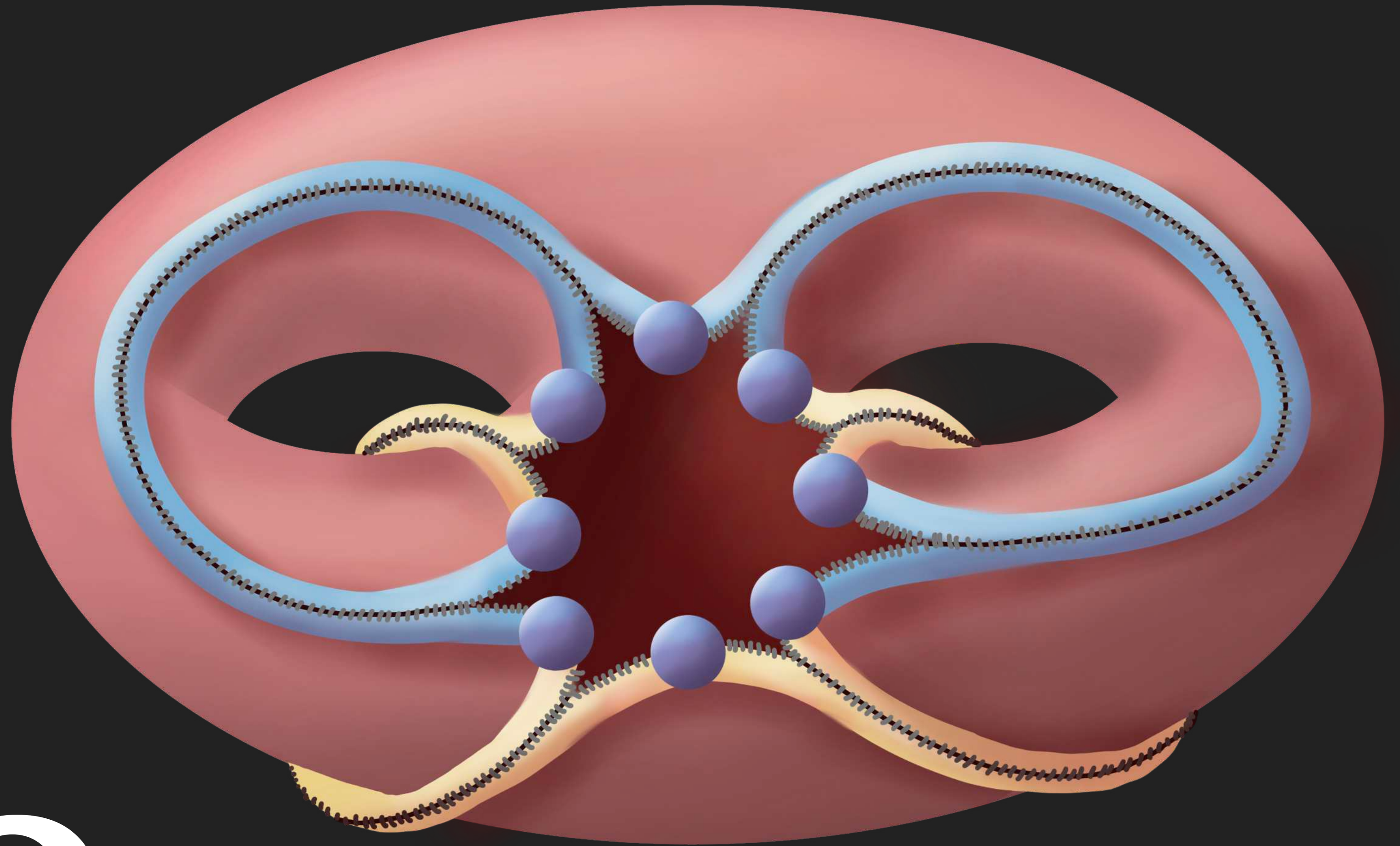
How

do we

Unroll

this

Surface?



But

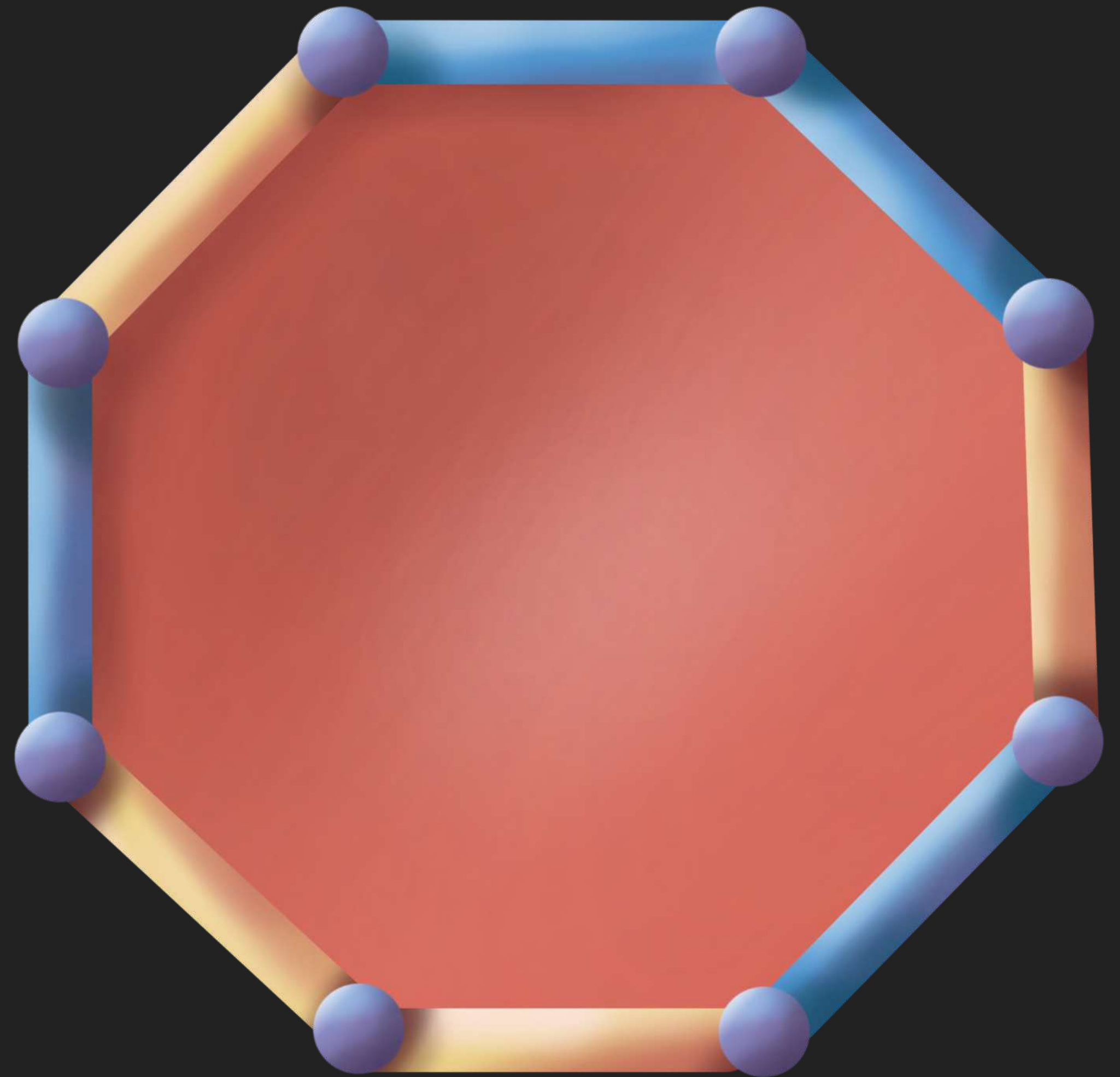
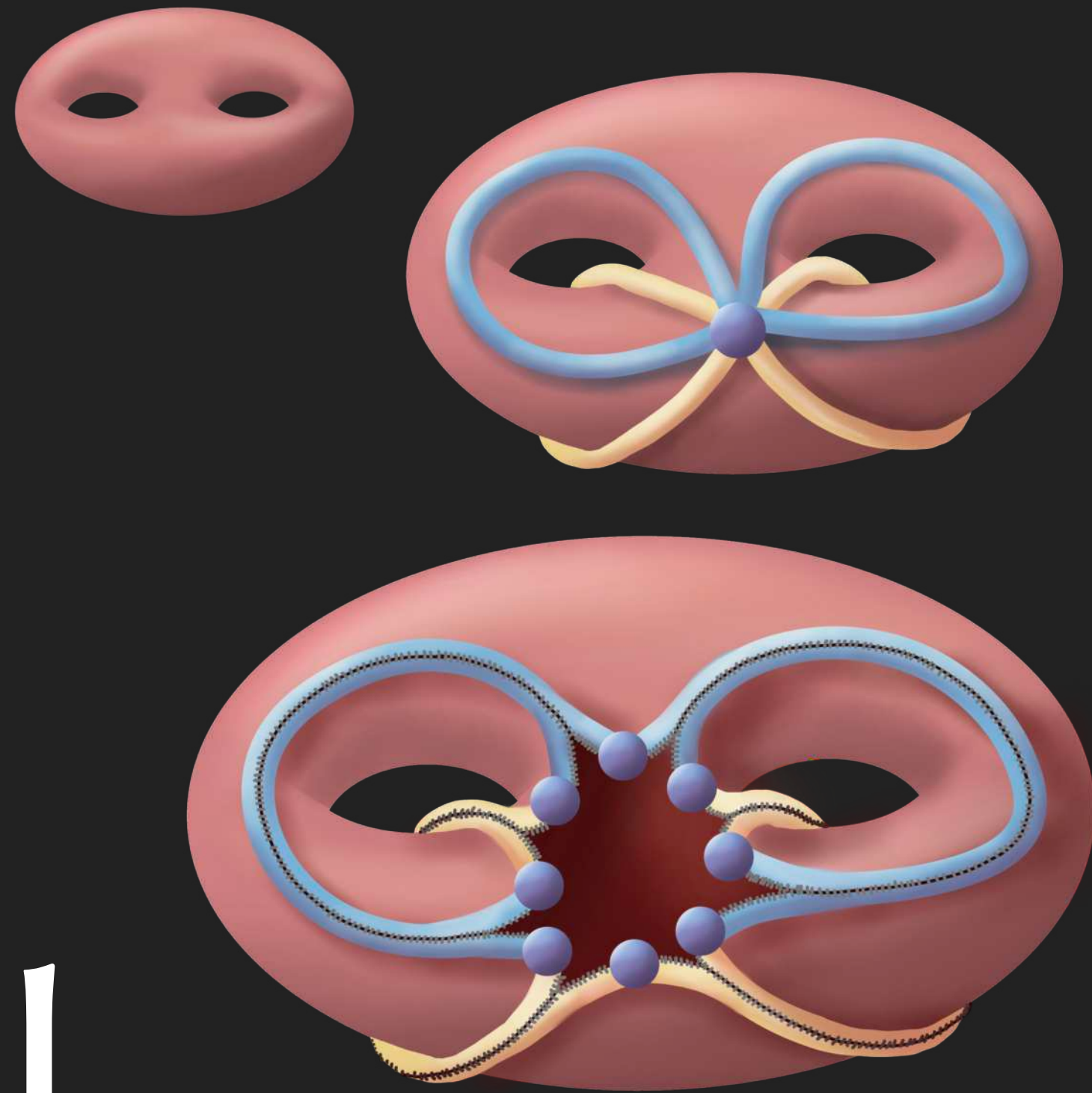
How

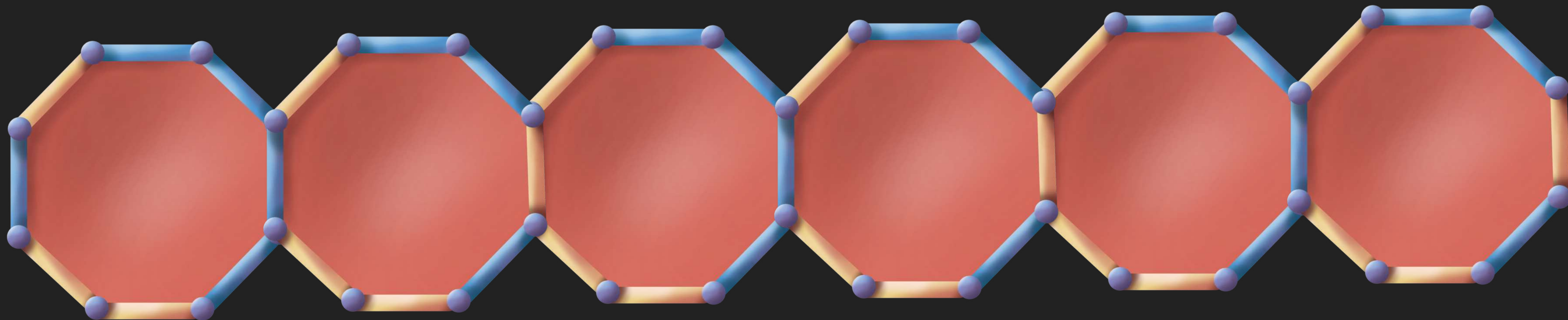
do we

Unroll

this

Surface?





Unrolling Loops

should produce a

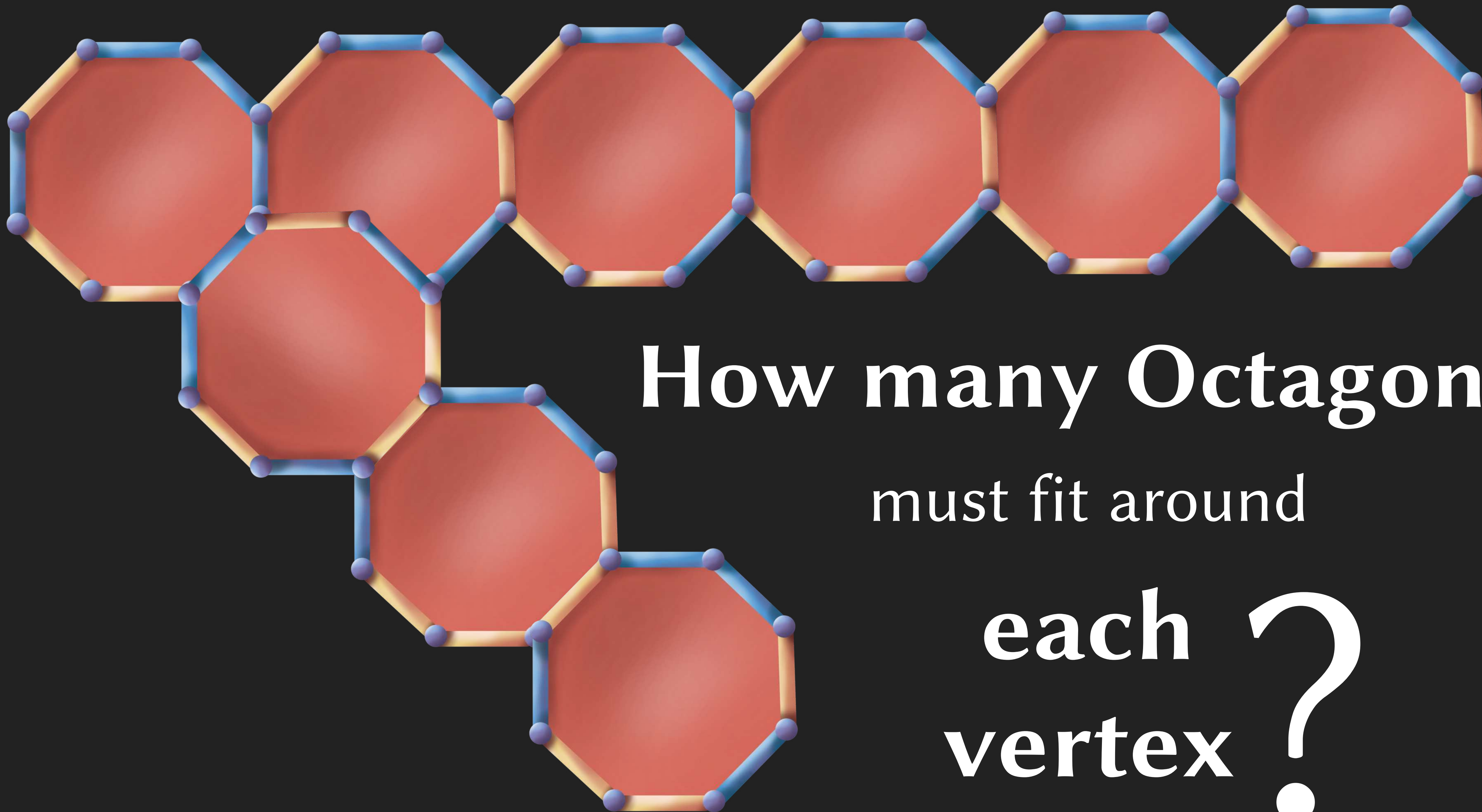
Tiling by Octagons



Unrolling Loops

should produce a

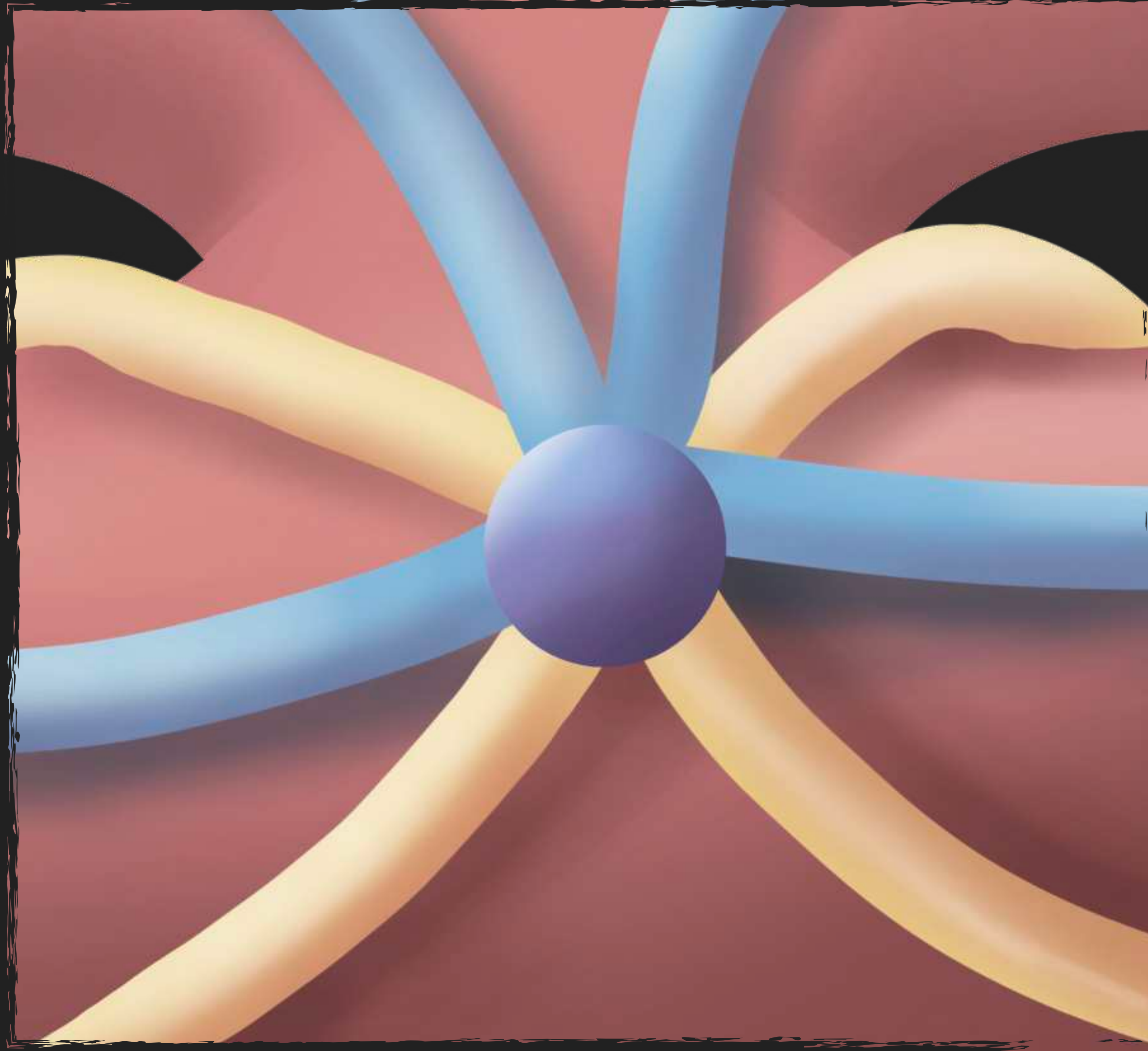
**Tiling by
Octagons**



How many Octagons

must fit around

**each
vertex?**

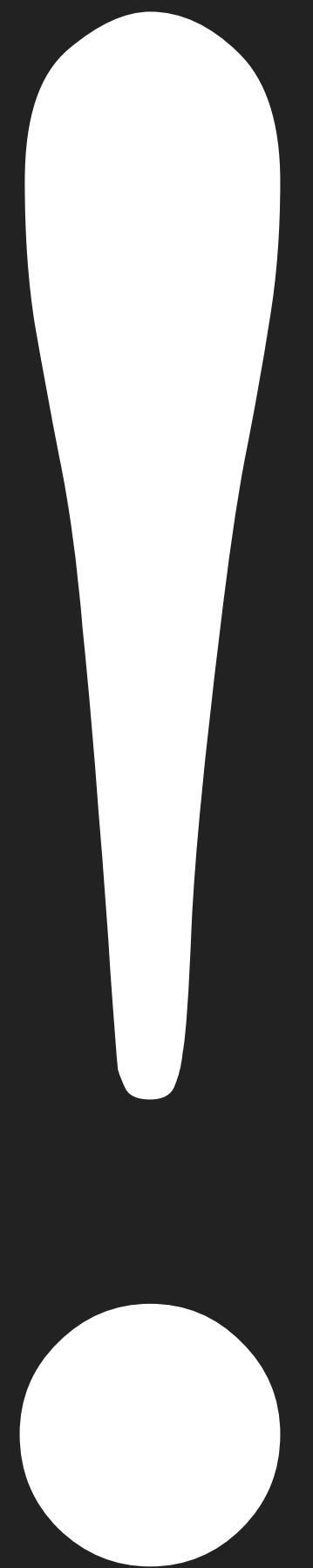


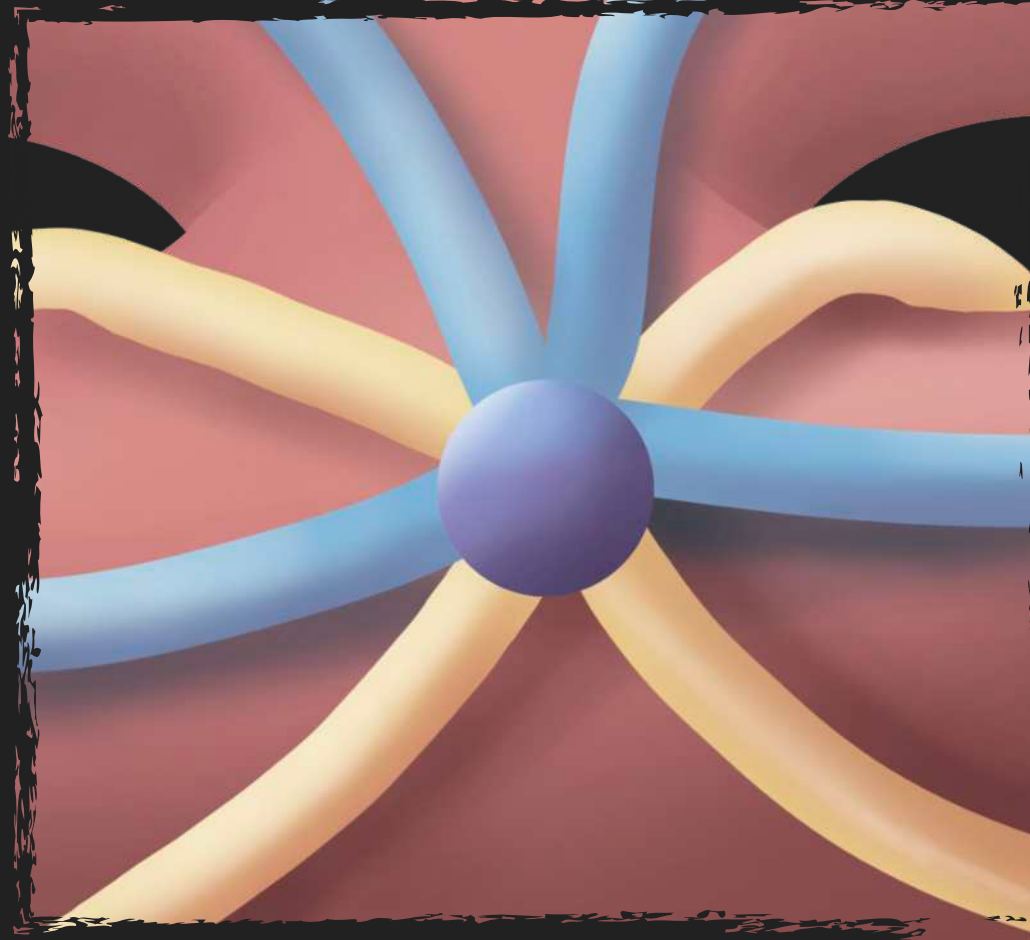
Eight

Octagons

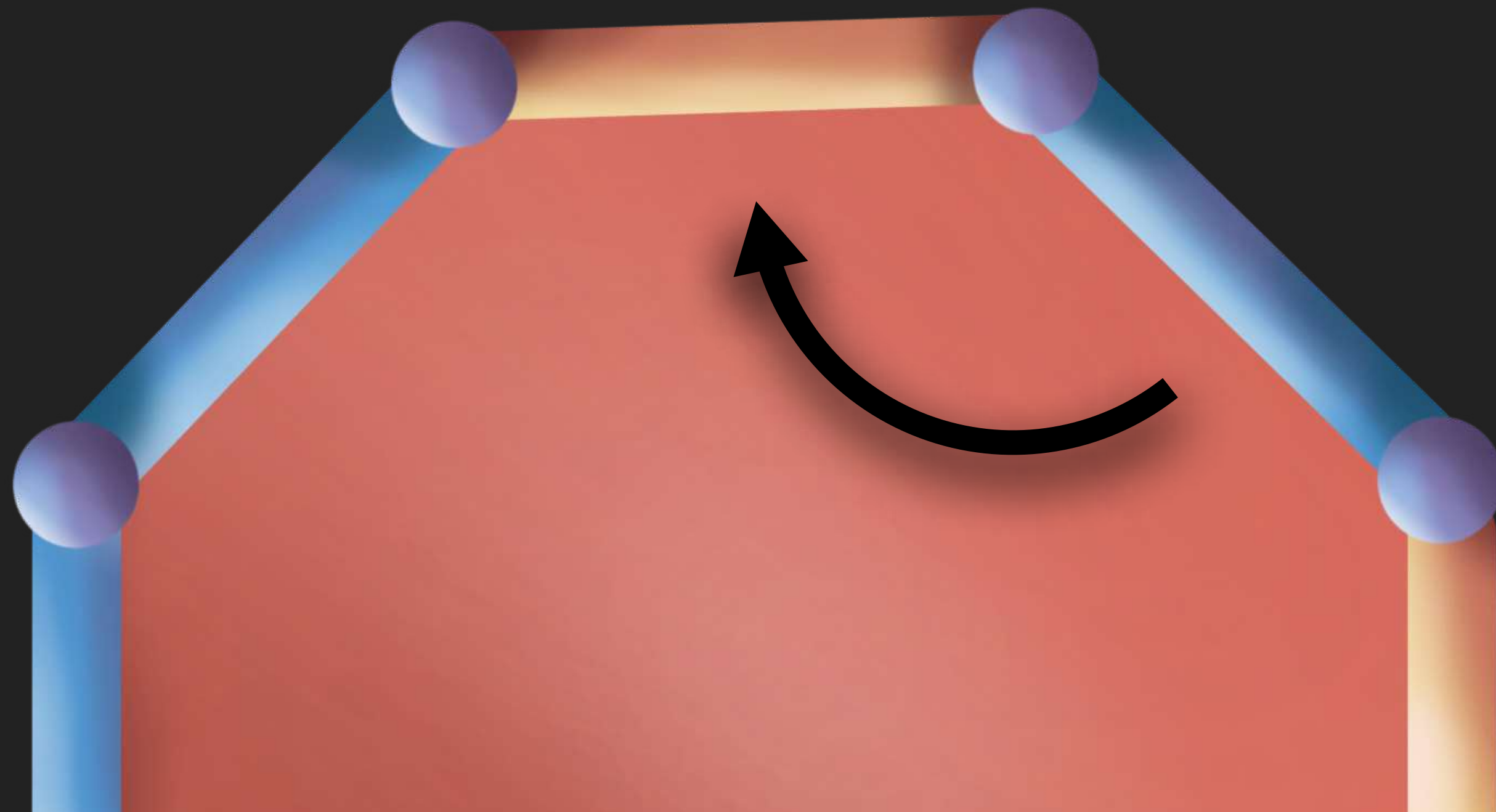
must fit around

each
vertex ?





*This angle
isn't 45°* 🤔



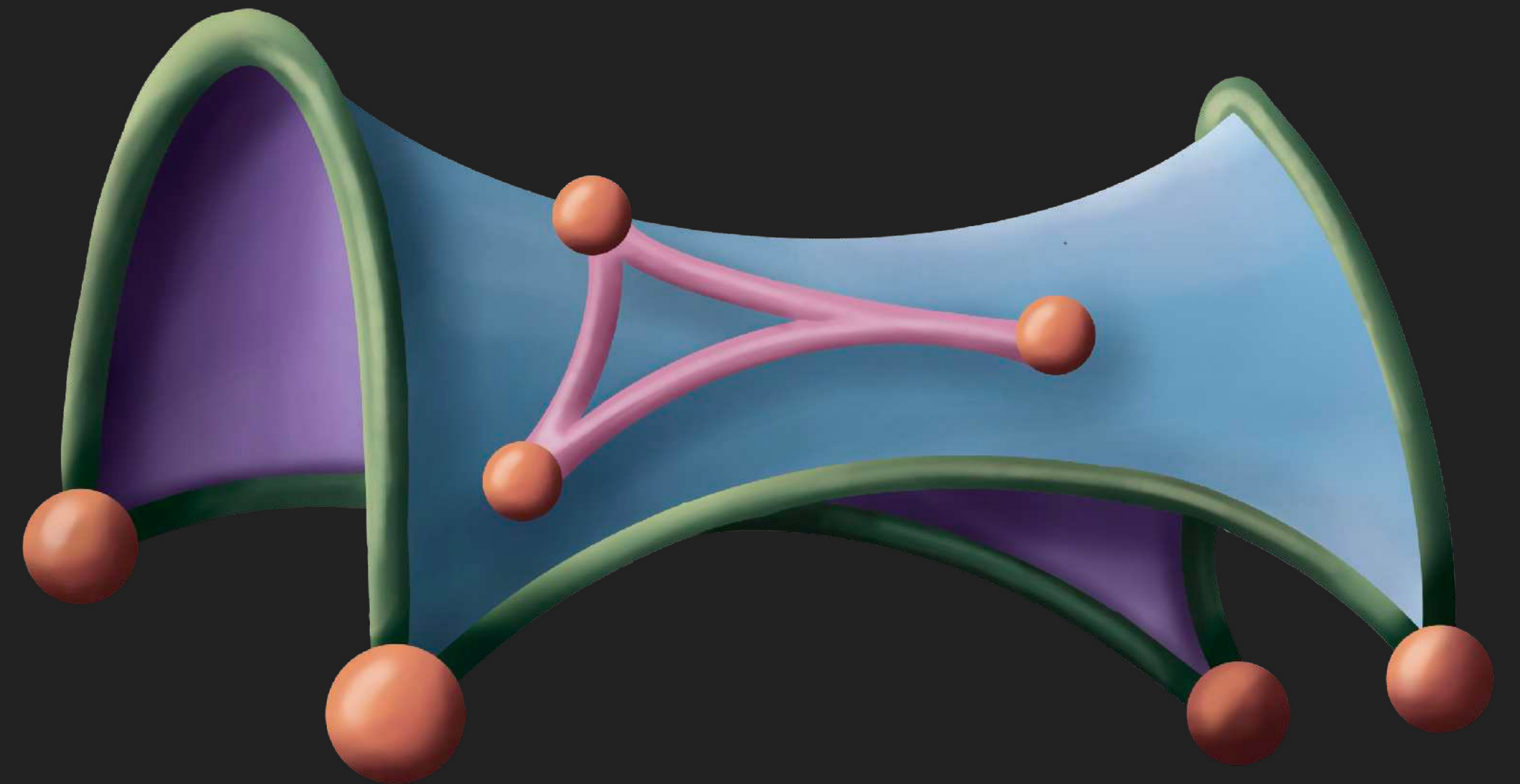
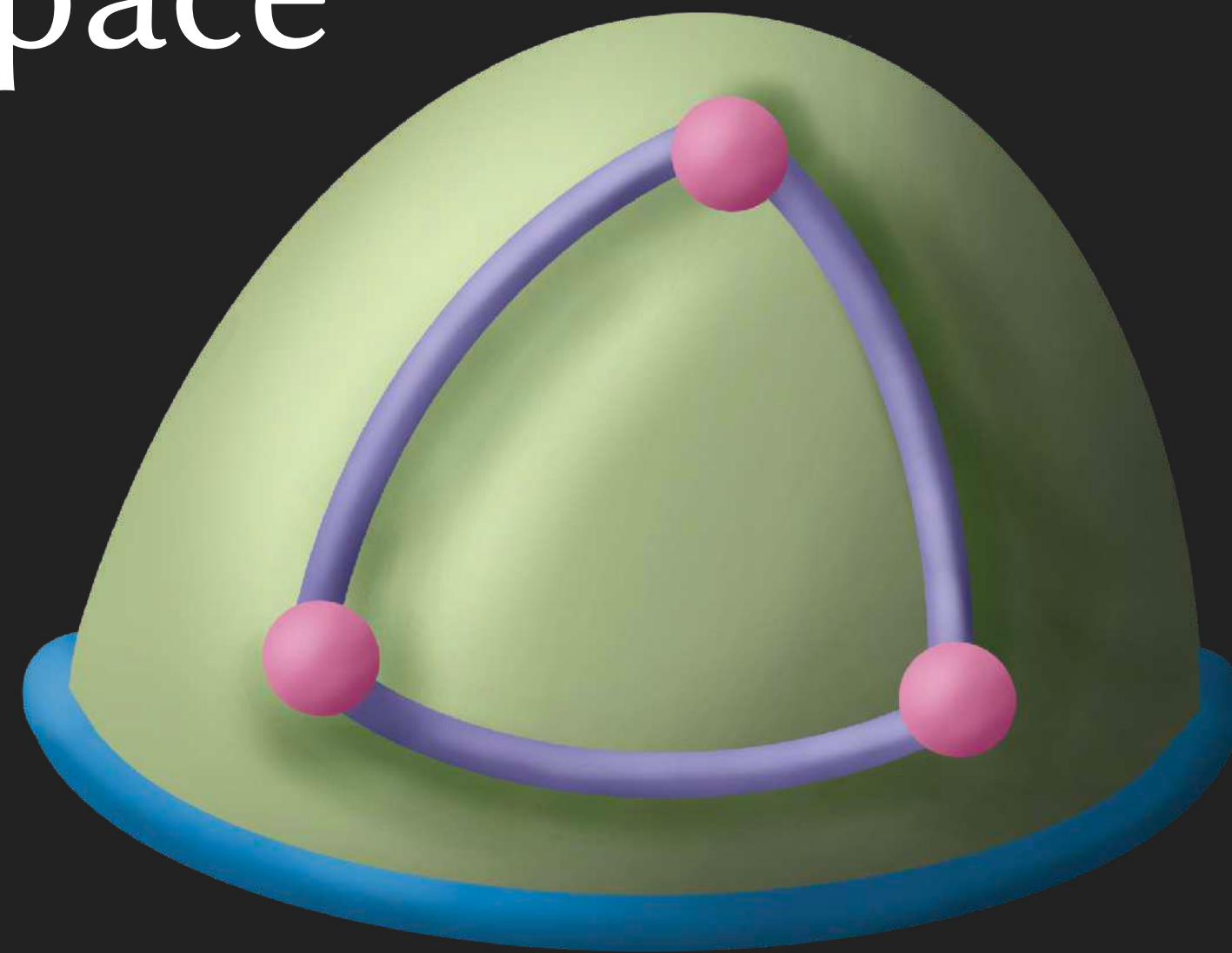
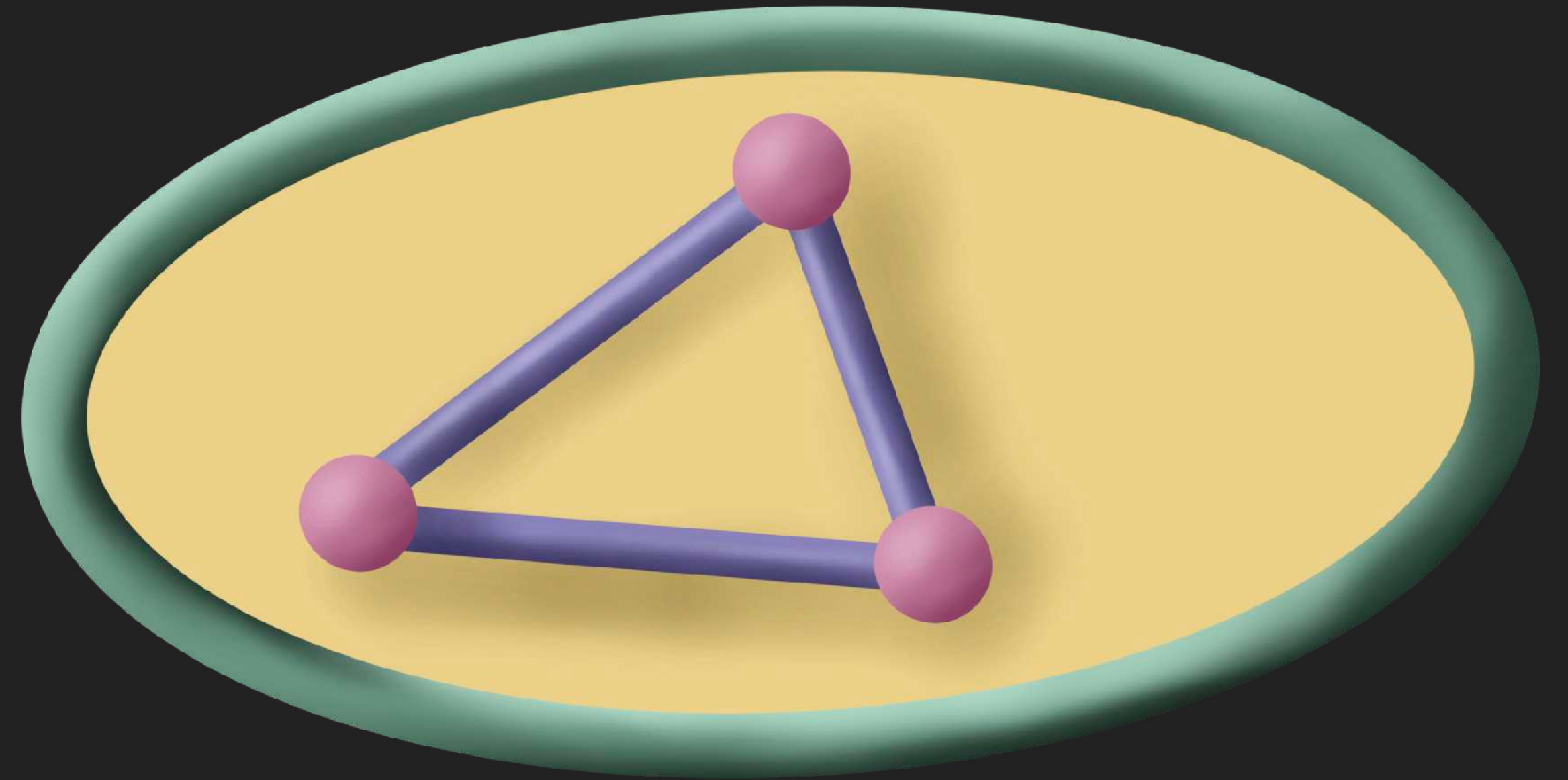
Eight Octagons

must fit around

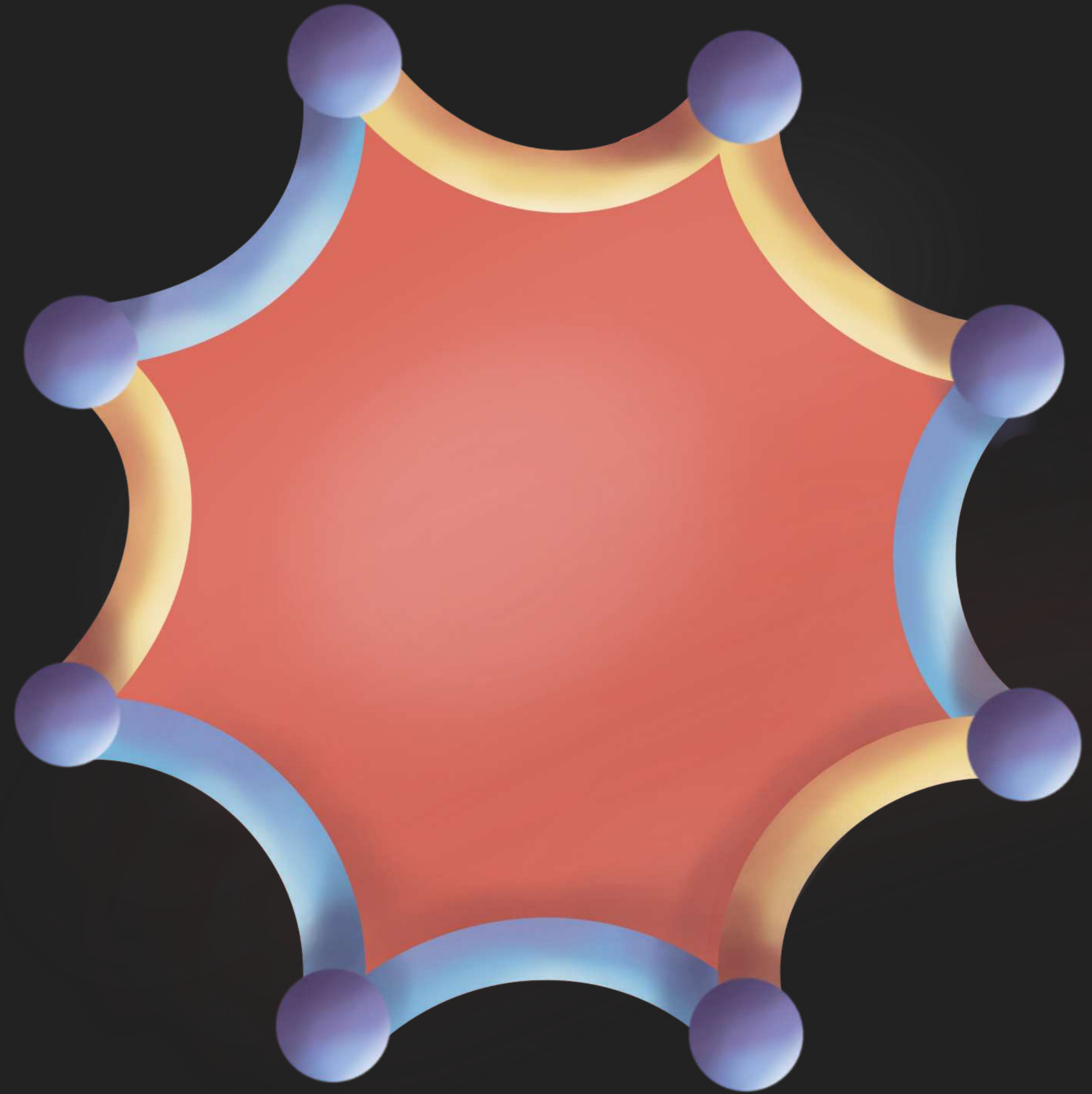
each
vertex ? !

But...

Angles change
if you let space
Curve

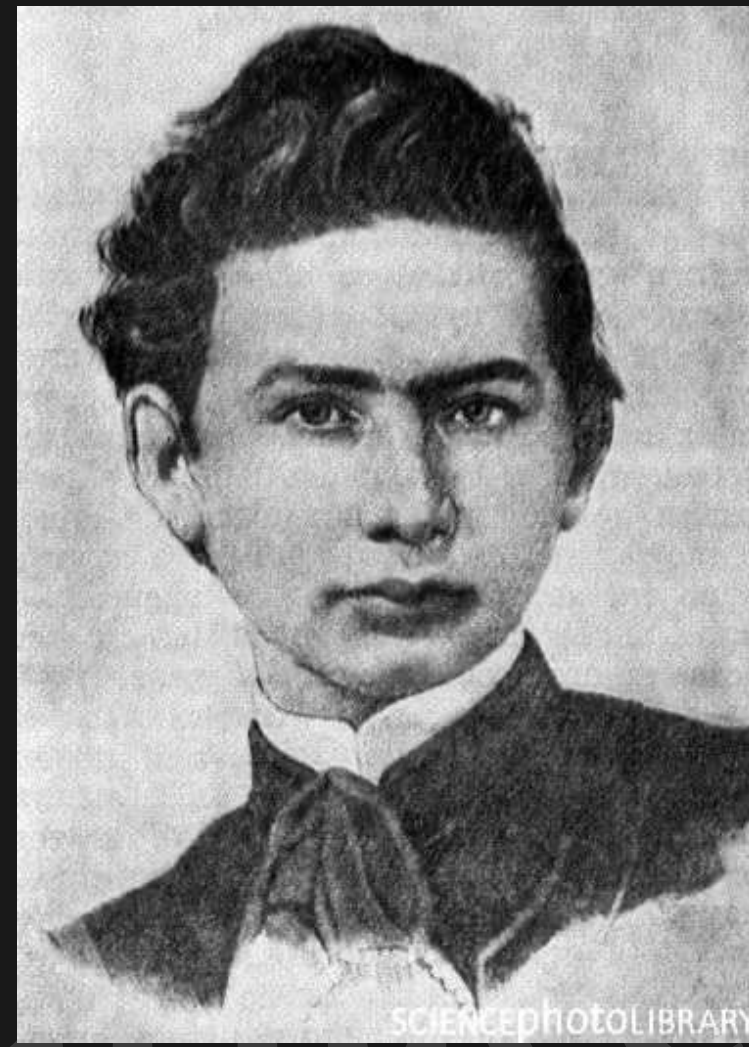


If you curve space
Just Right
you can make an
Octagon
with
 45° angles





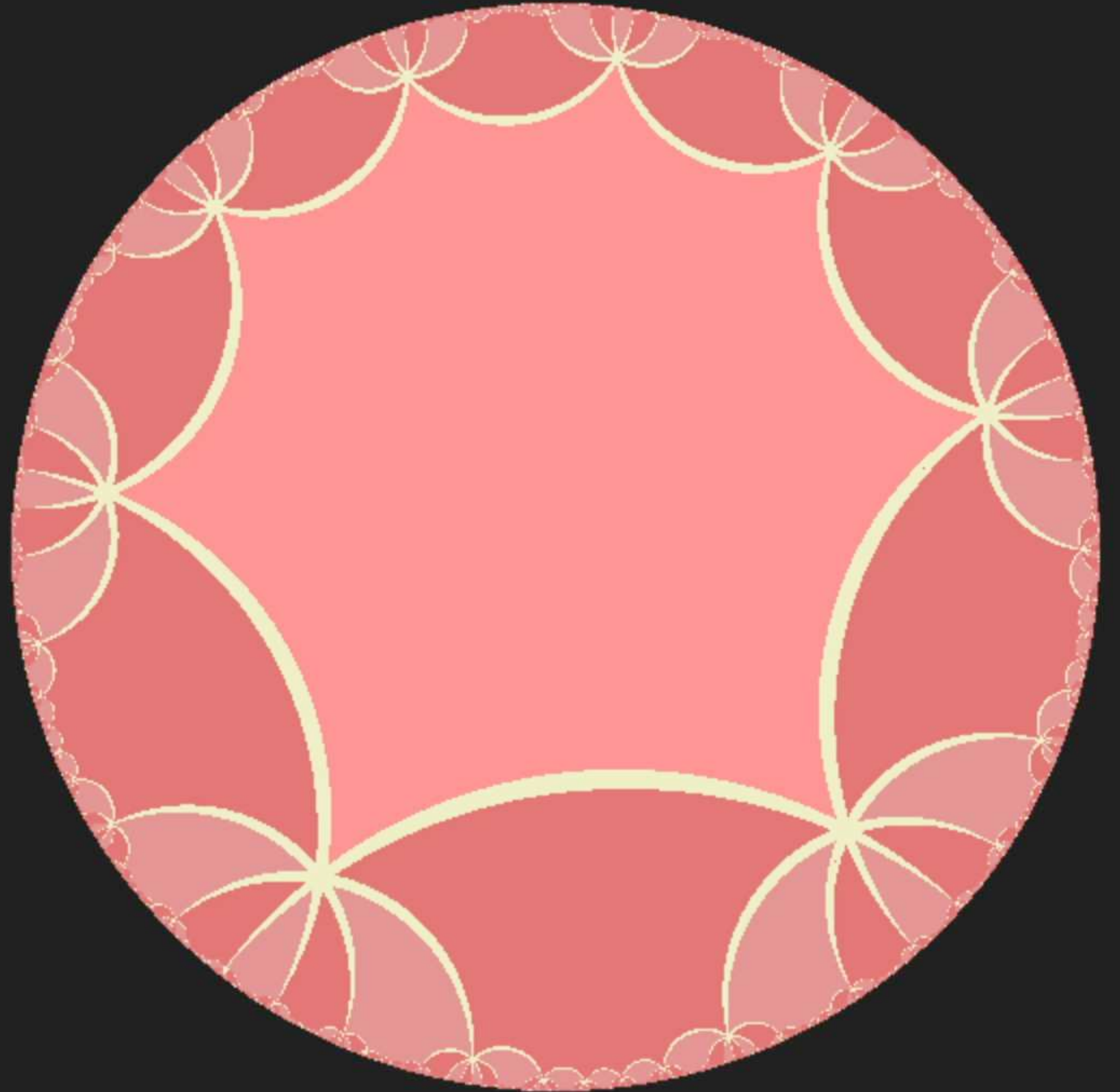
Nikolai Lobachevsky
1829 CE



Janos Bolyai
1832 CE



Carl Friedrich Gauss
Early 1800s CE



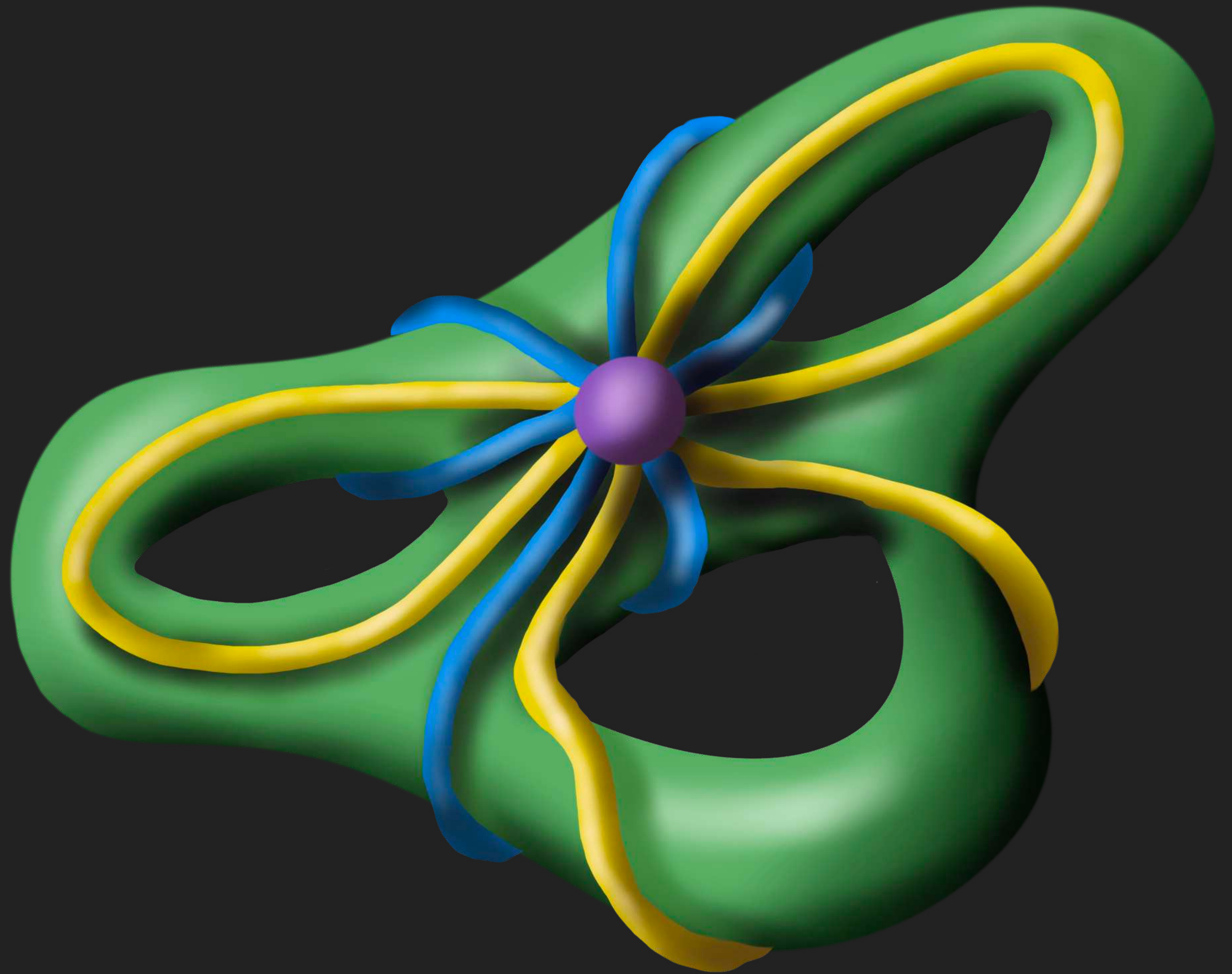
How about
other?
surfaces

*More complicated
surfaces cut into
polygons with more
sides.*

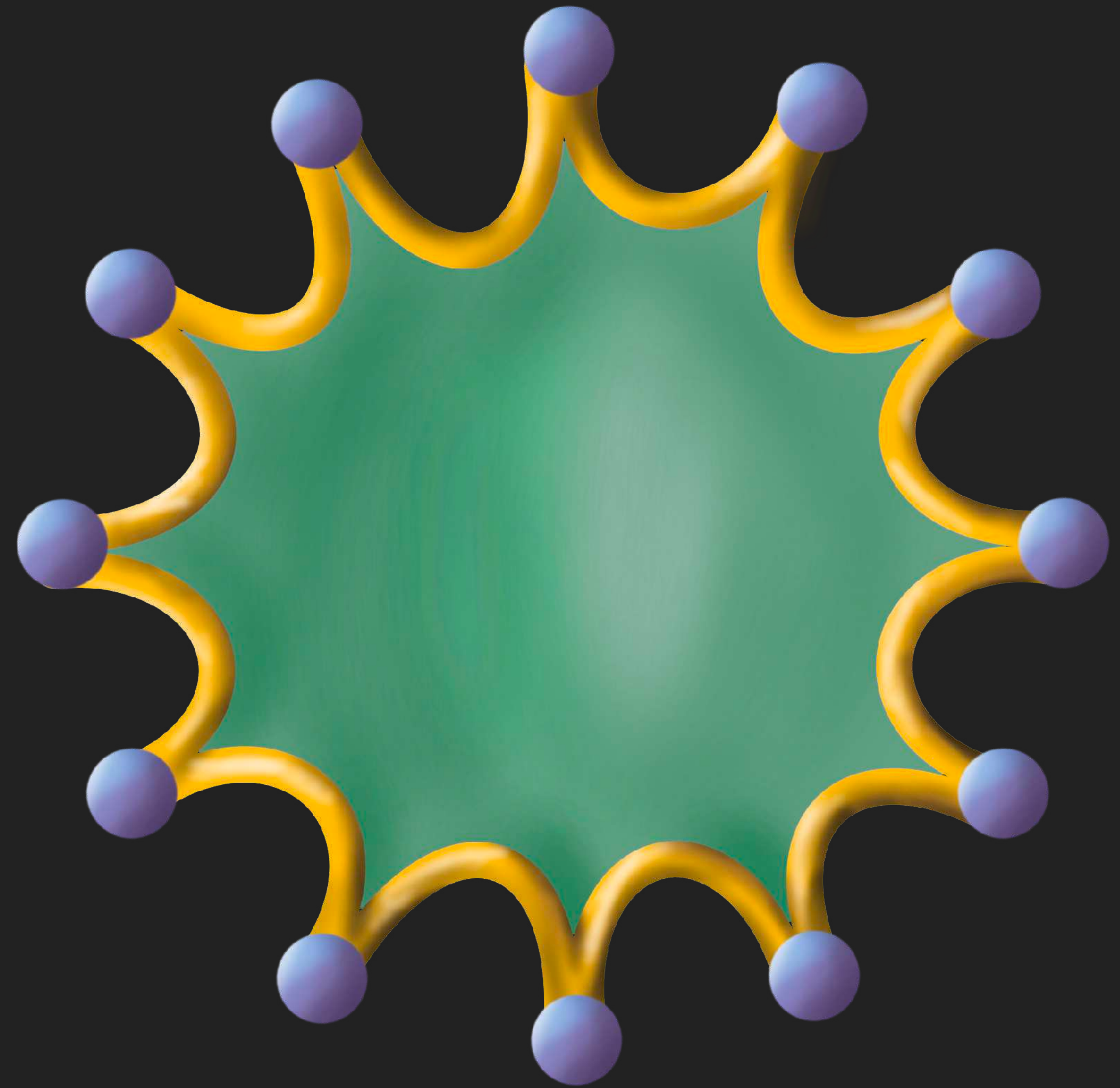
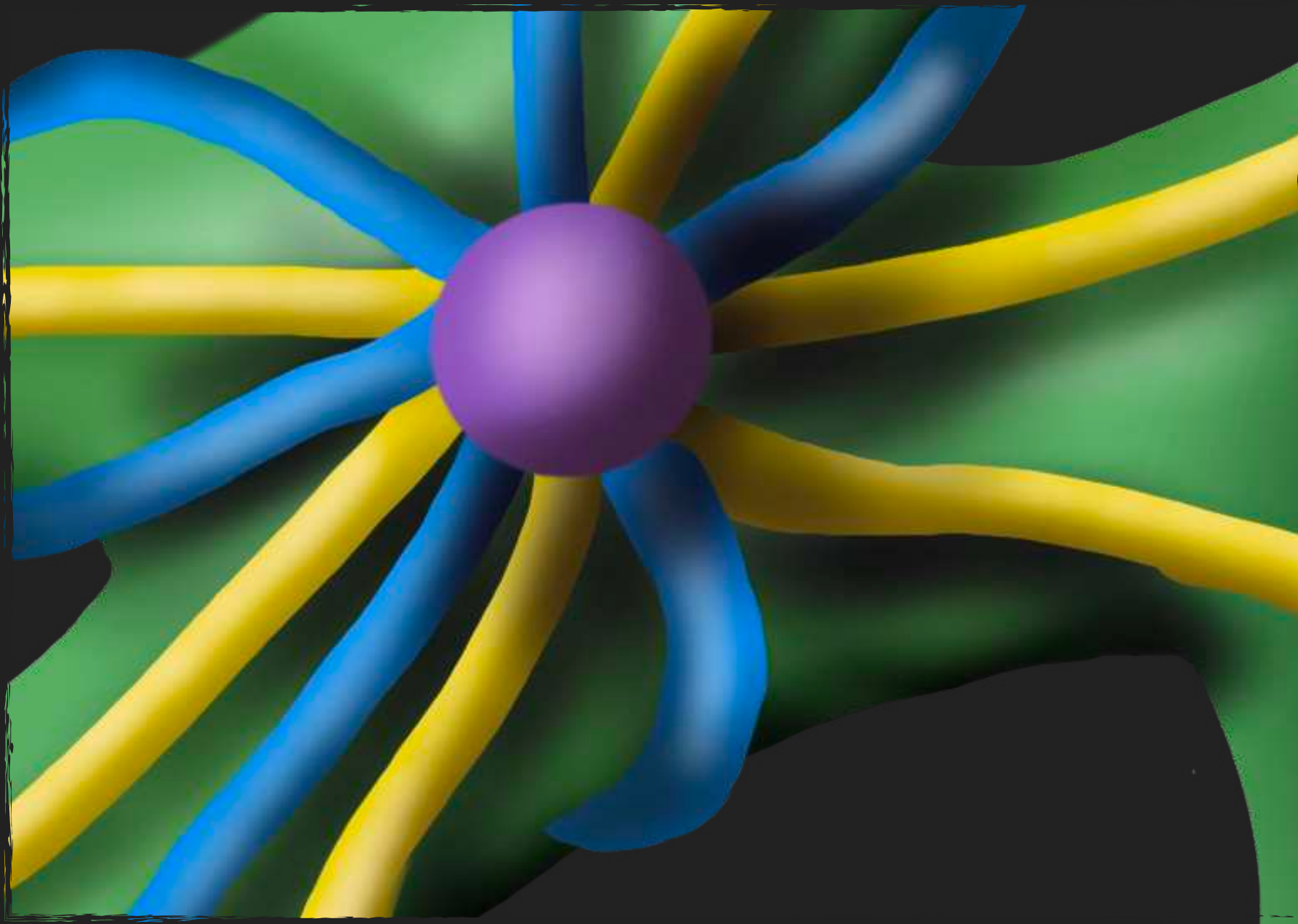


How about
other?
surfaces

*More complicated
surfaces cut into
polygons with more
sides.*



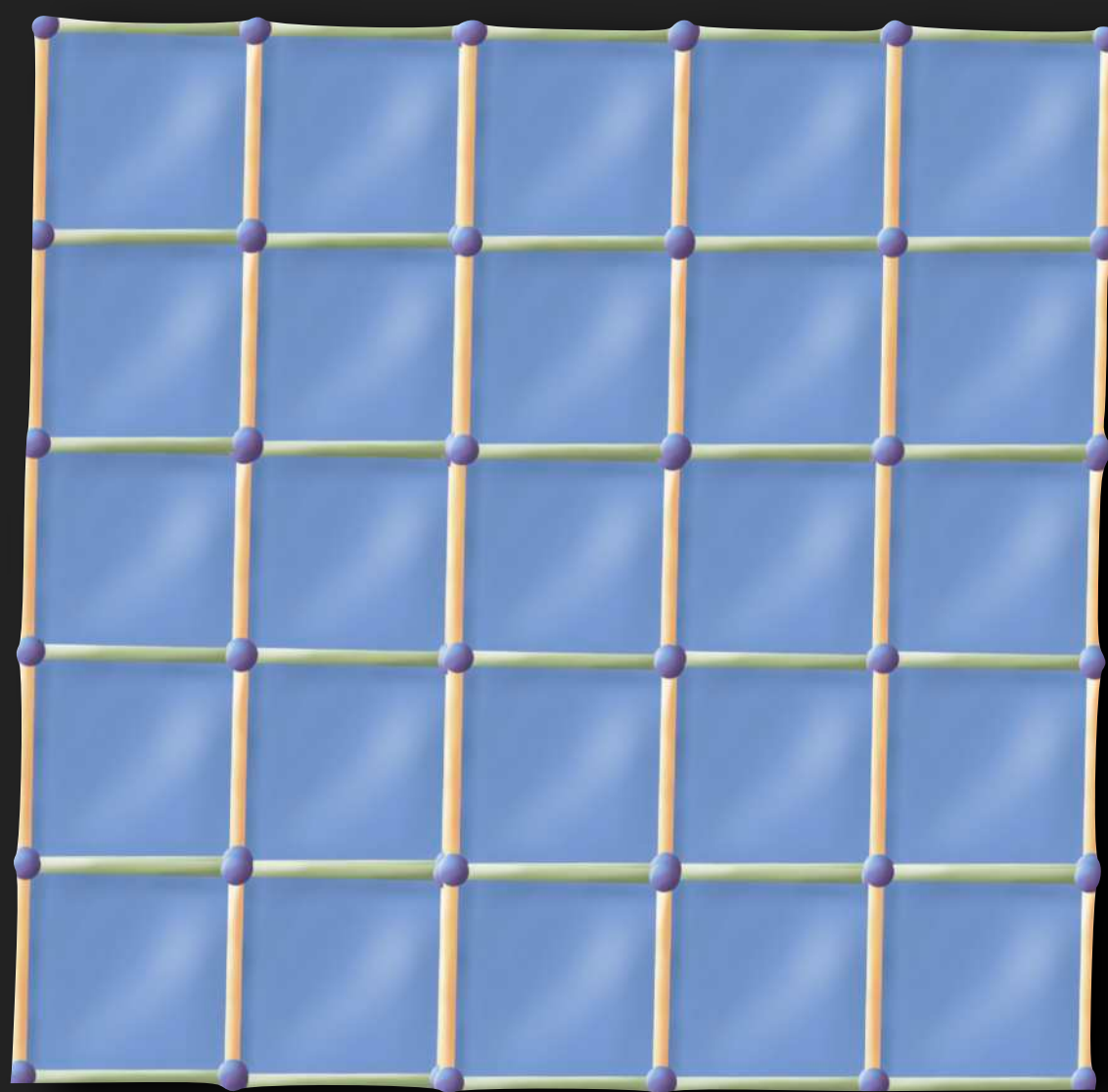
How about
other?
surfaces



*With negative curvature, its still possible
to make these have small enough angles!*

Every surface *can be unrolled onto*
the sphere, the Euclidean plane, or
the hyperbolic disk.

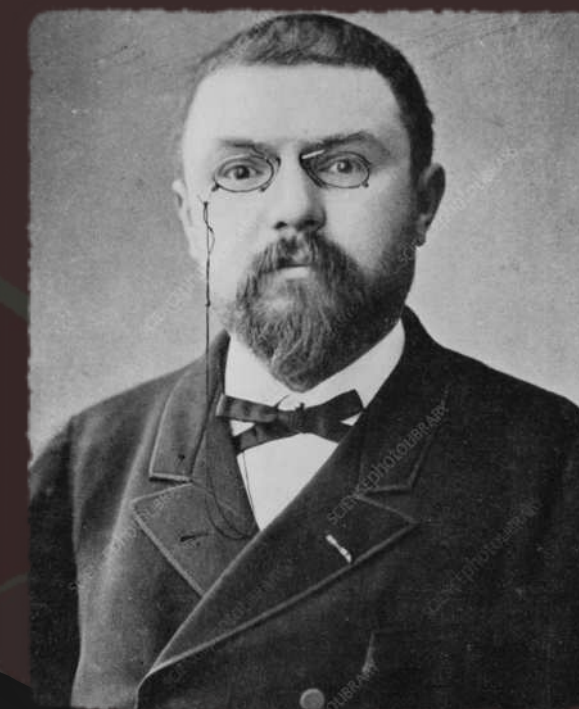
Uniformization
Theorem (1907)



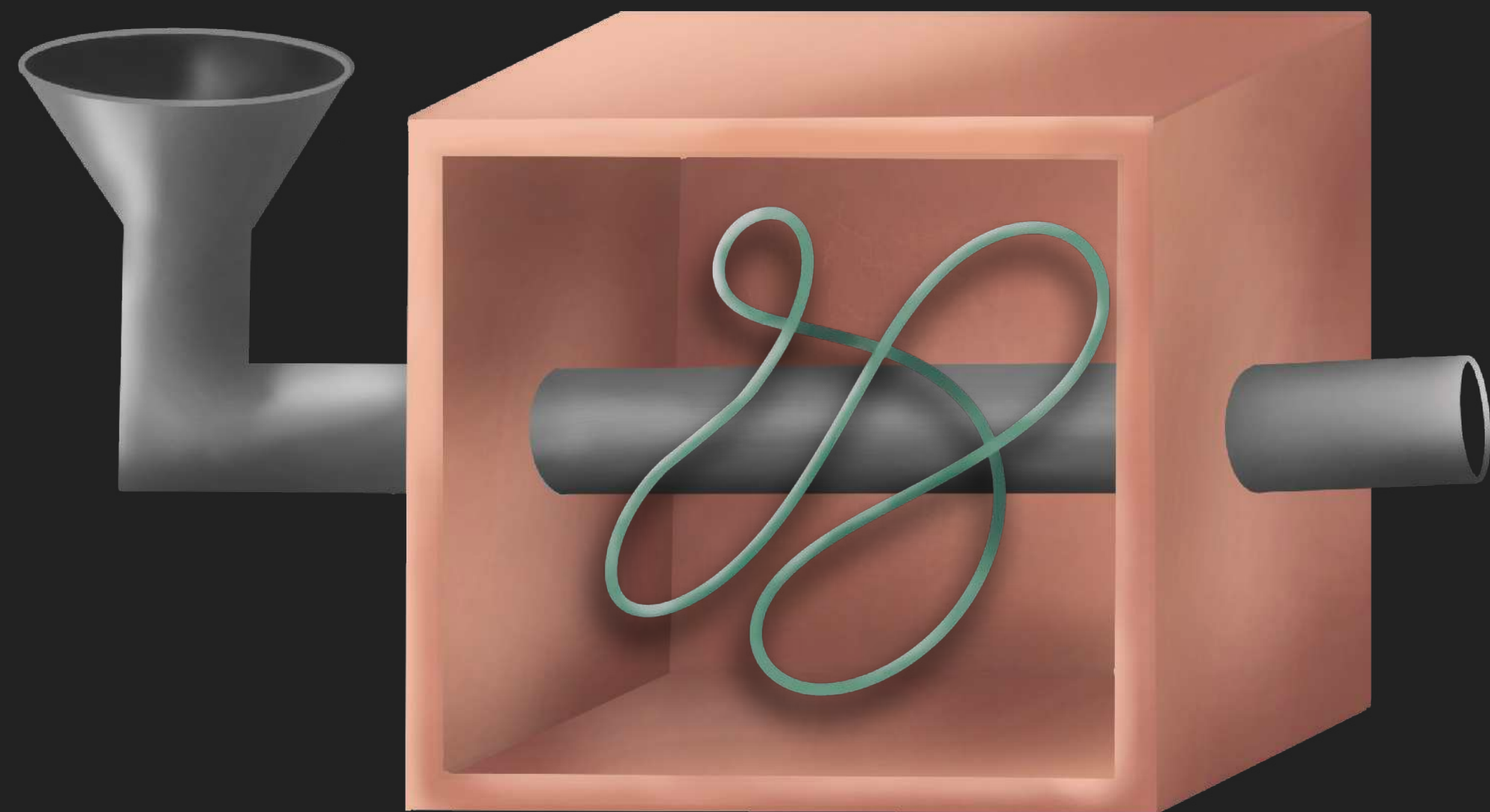
Felix Klein



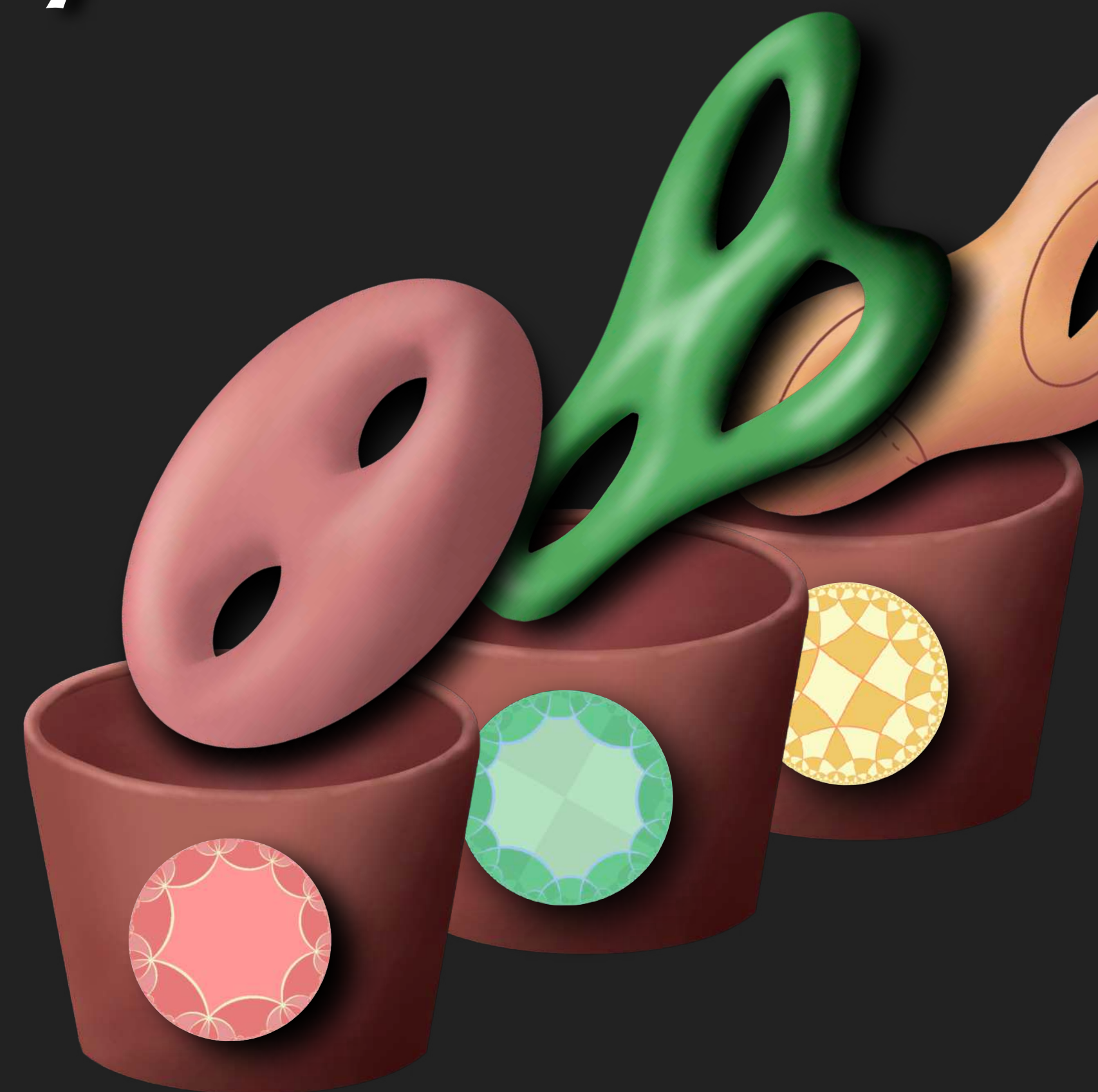
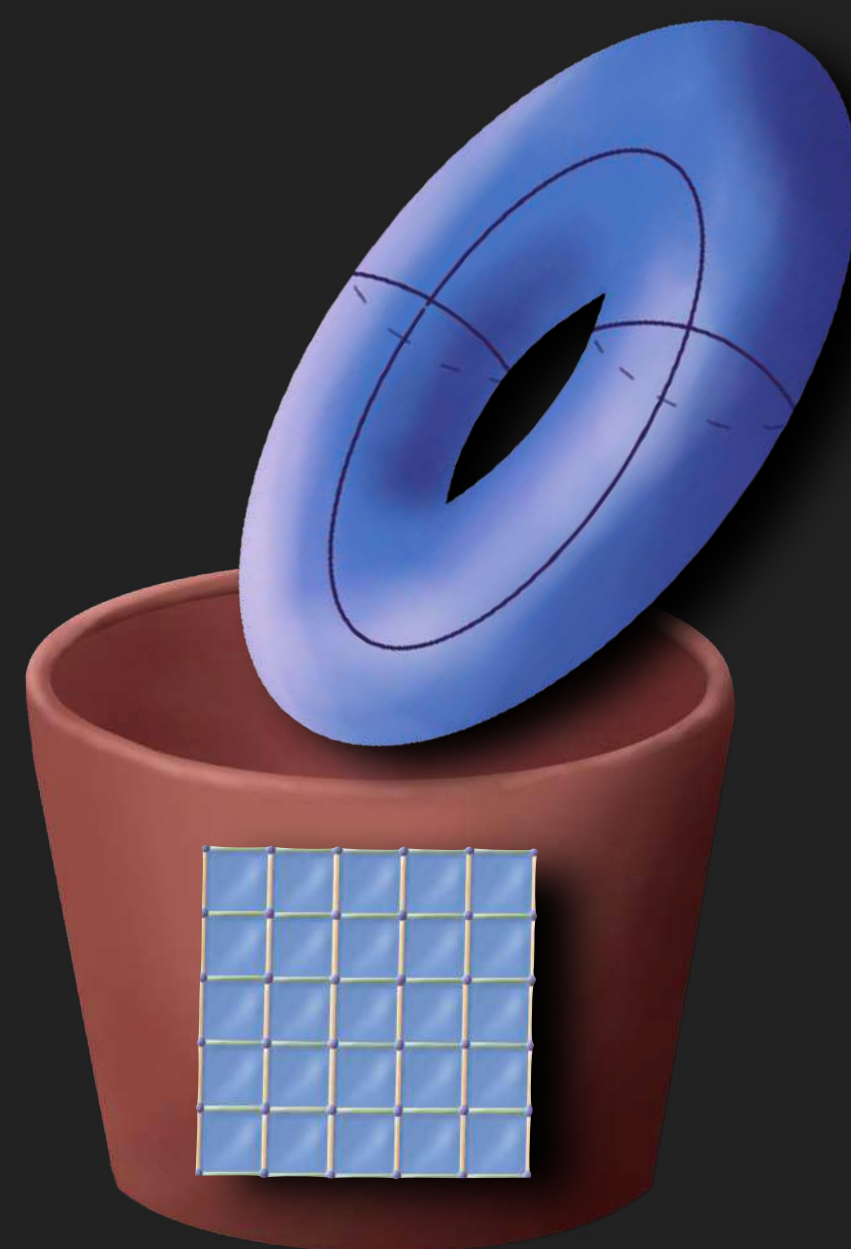
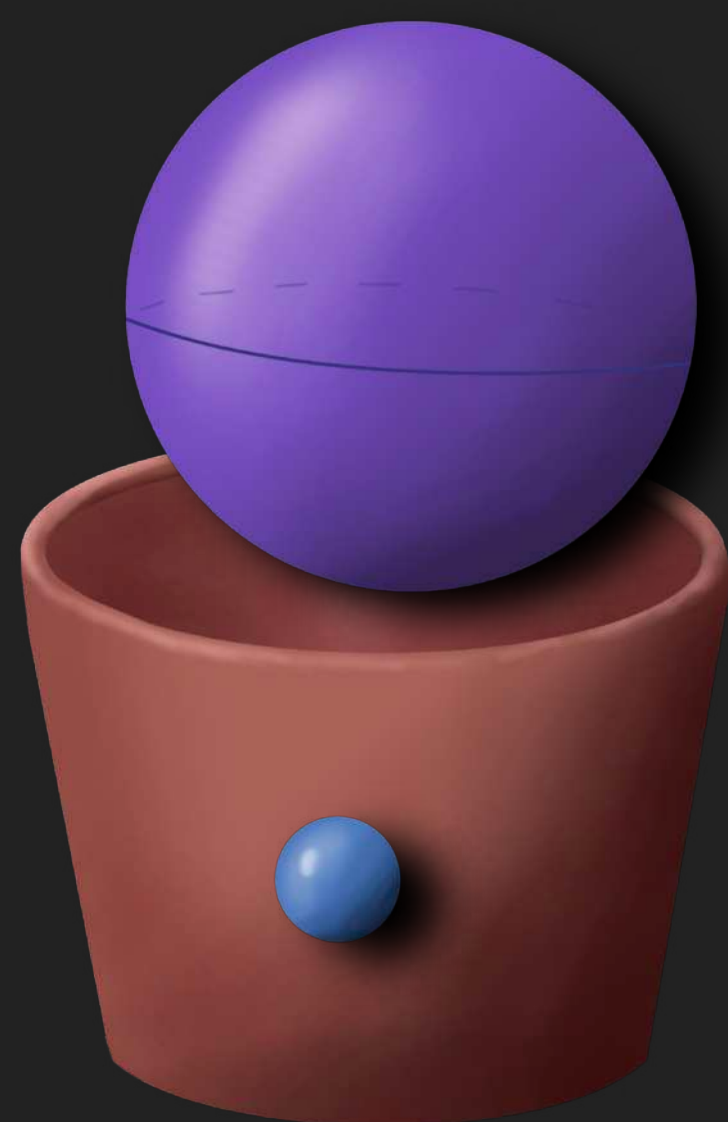
Paul Koebe

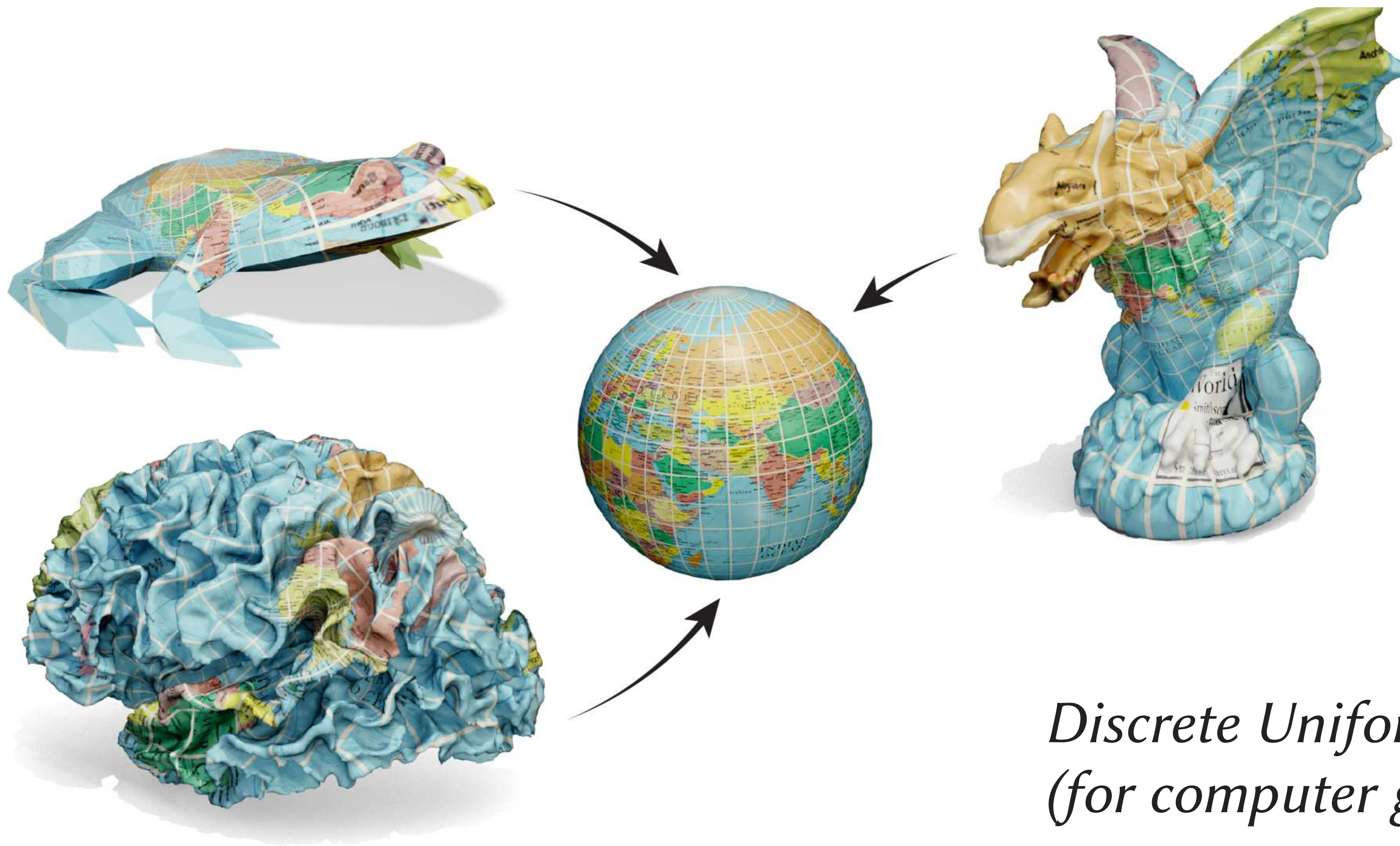


Henry Poincaré



The loop invariant
sorts surfaces by
geometry



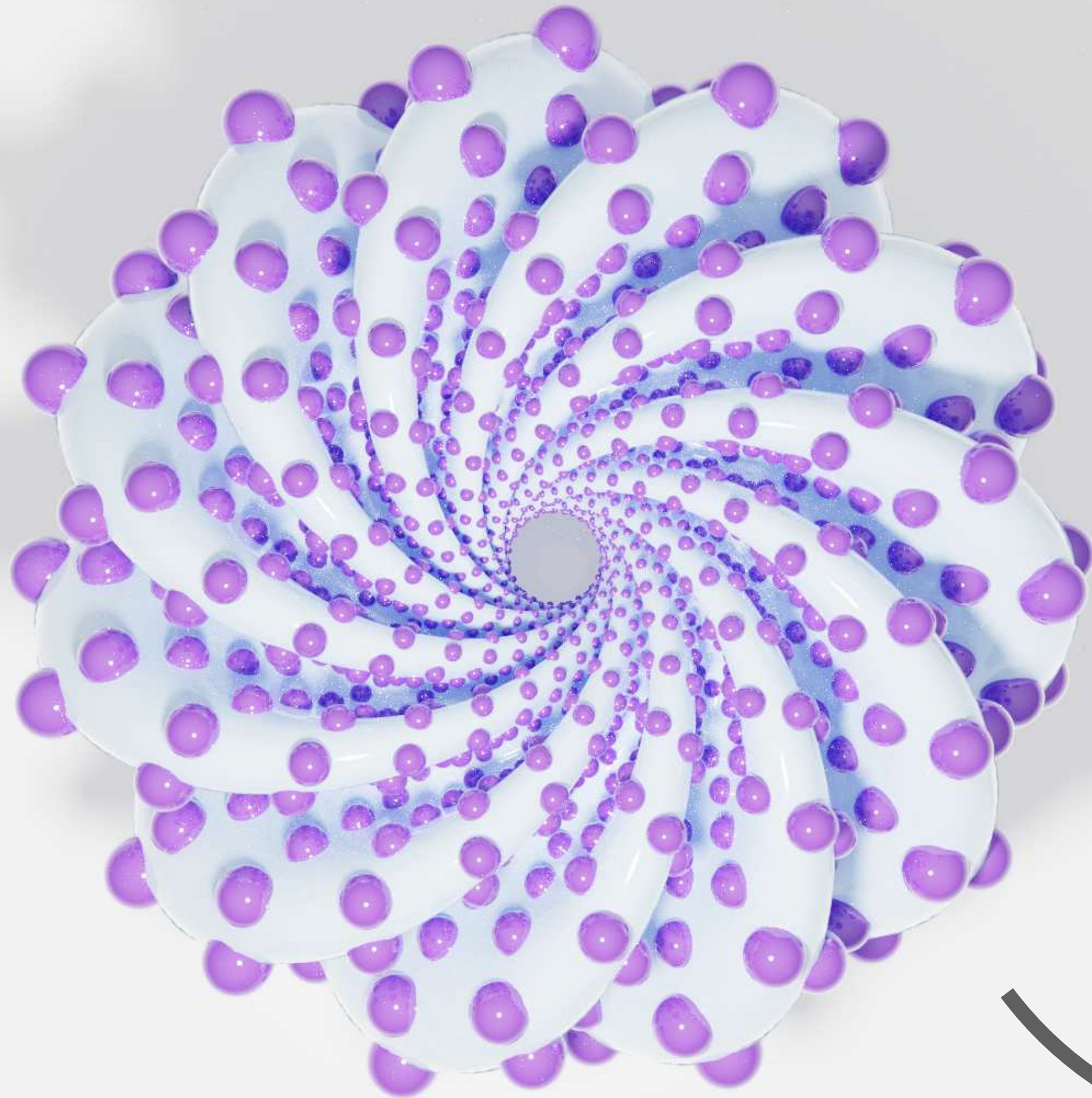


*Discrete Uniformization
(for computer graphics)*

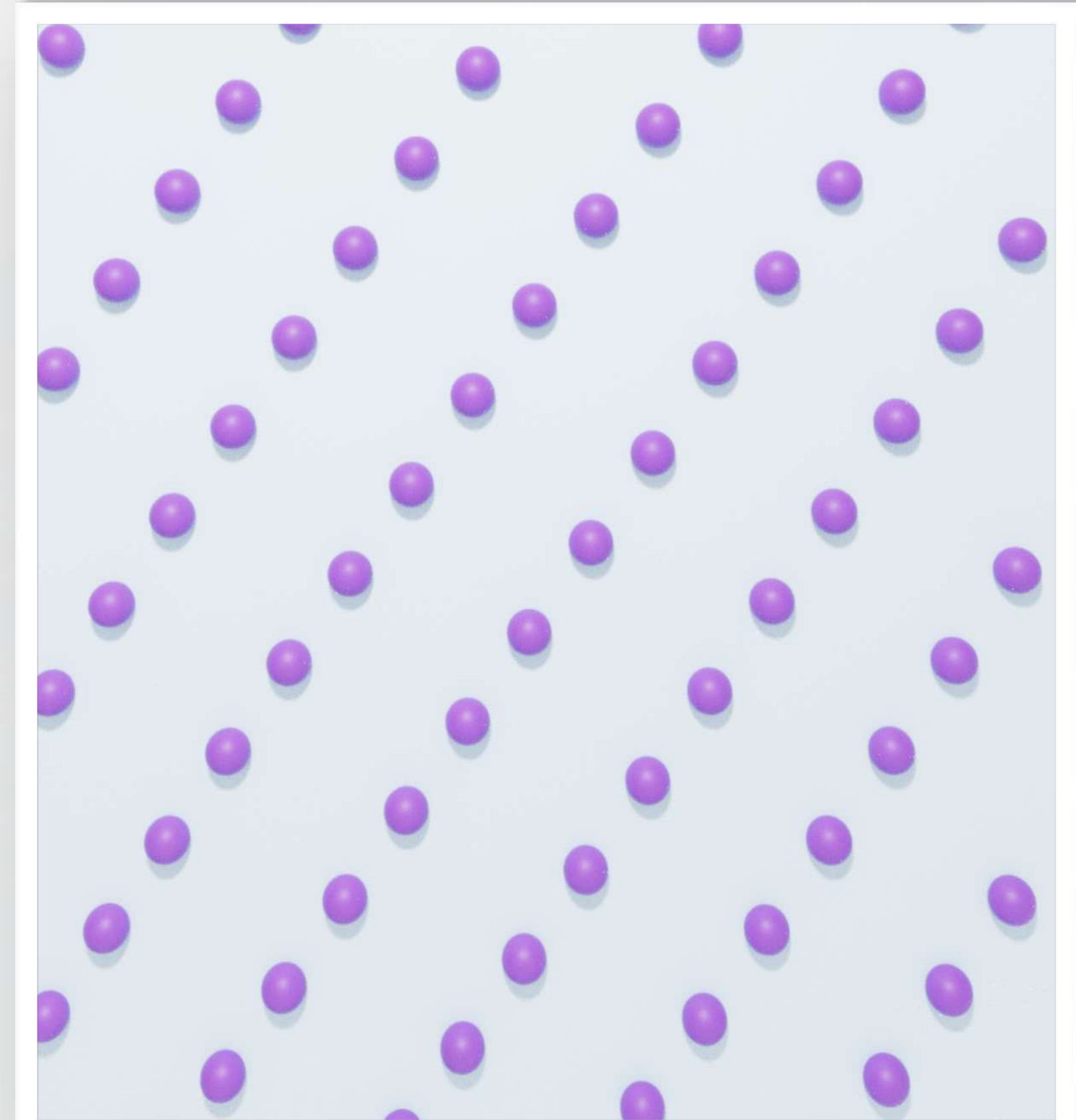
Keenan Crane, 2022

Visualizing Elliptic Curves over Finite Fields

joint with Nadir Hajouji, 2025



Uniformization



Part III

*Proving the
Poincare Conjecture
(In 2 dimensions)*



The Poincare Conjecture

If it
walks & quacks
like a duck, then
it is a duck



The Poincare Conjecture

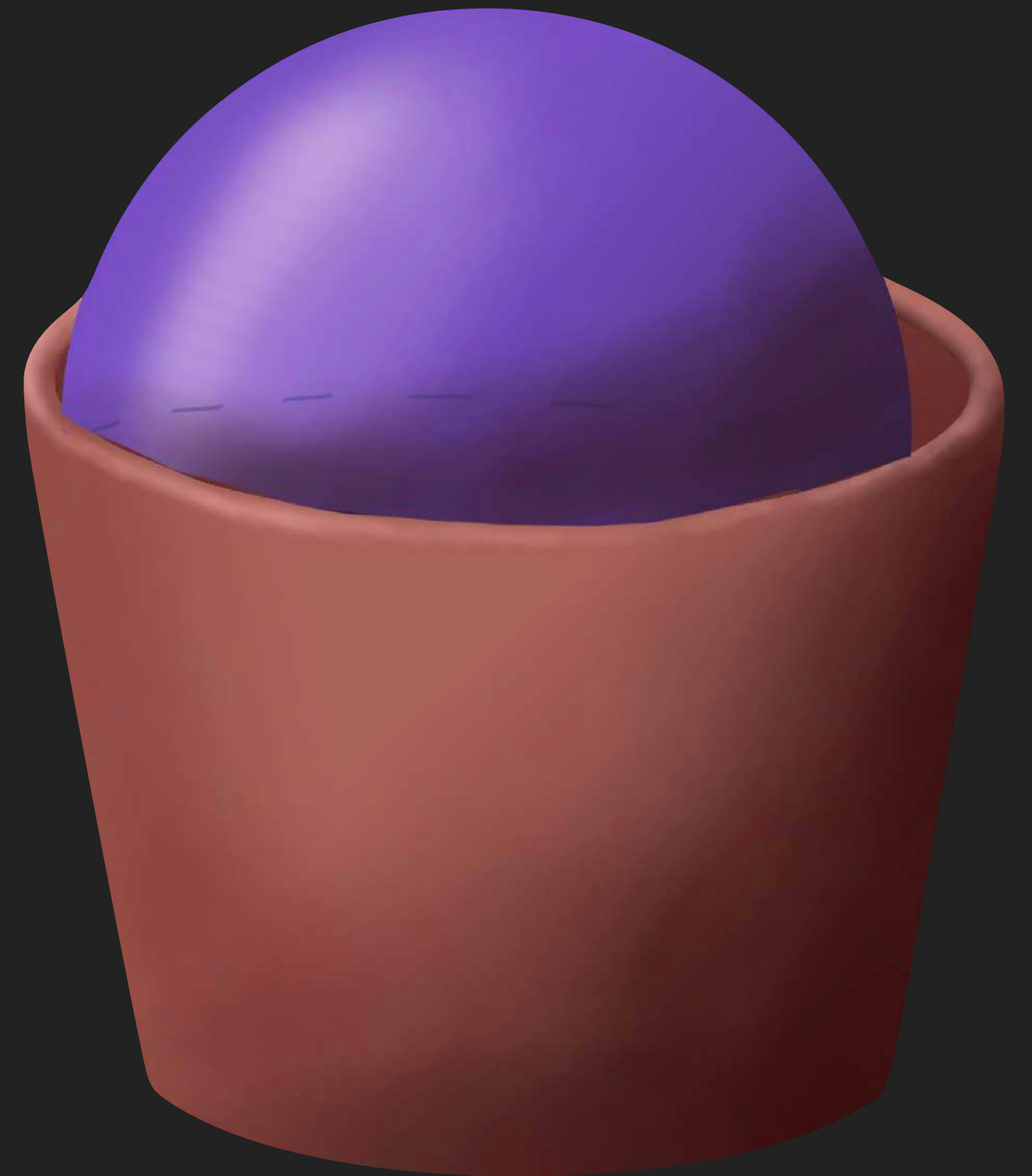
If a shape has the same

Loop Invariants

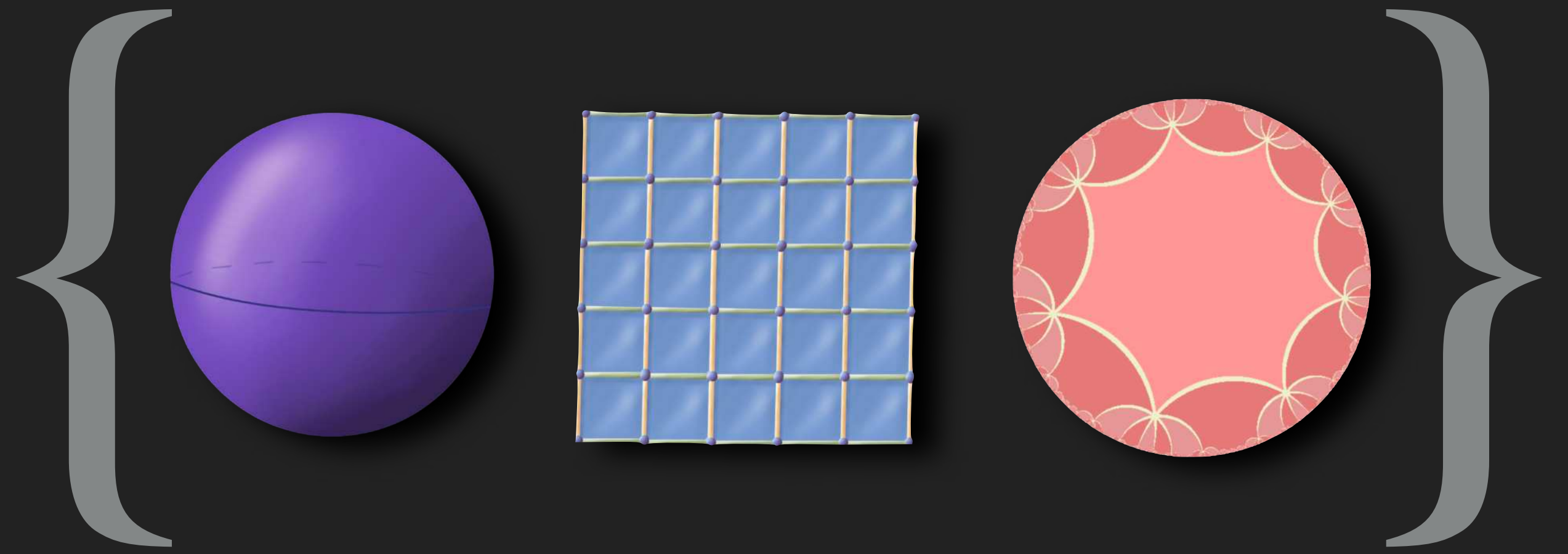
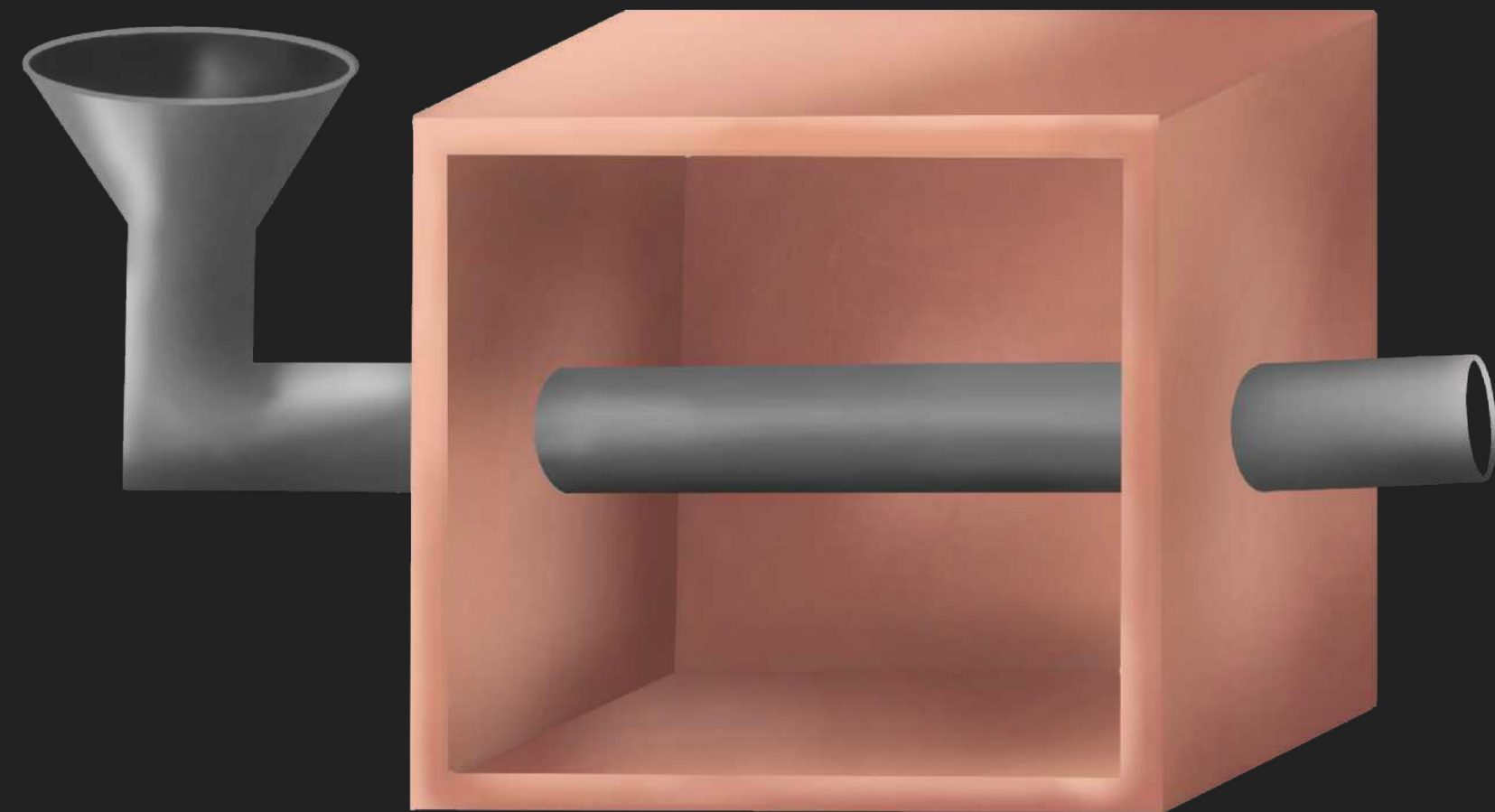
as the sphere, then

it is the sphere

-Poincaré, 1904



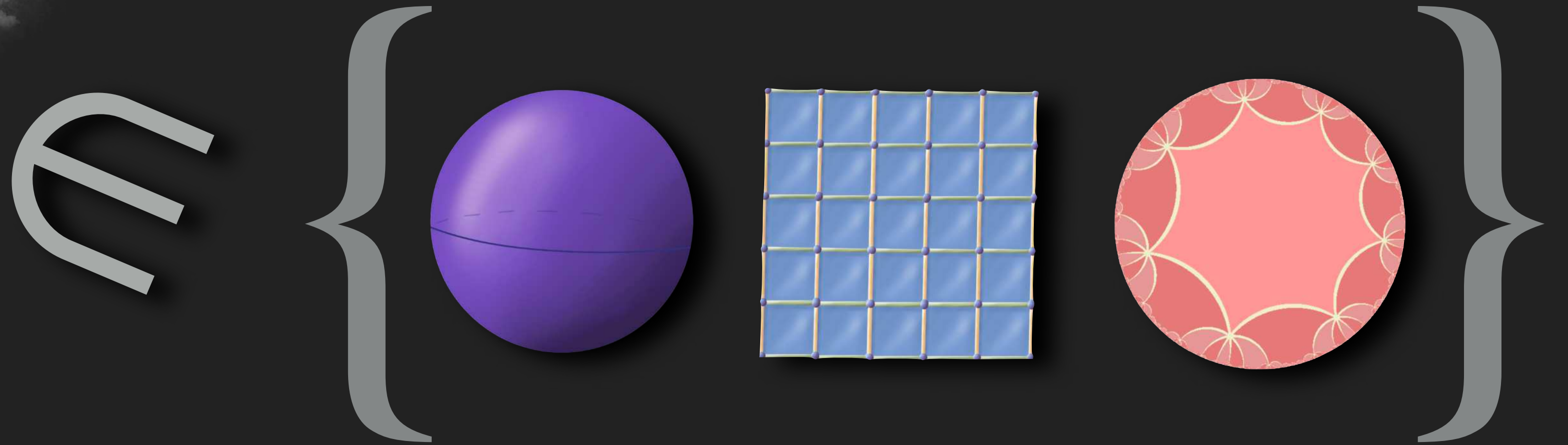
Proving the Poincaré Conj. for Surfaces



*Start with a mystery surface.
With the same loop
invariants as the sphere.*

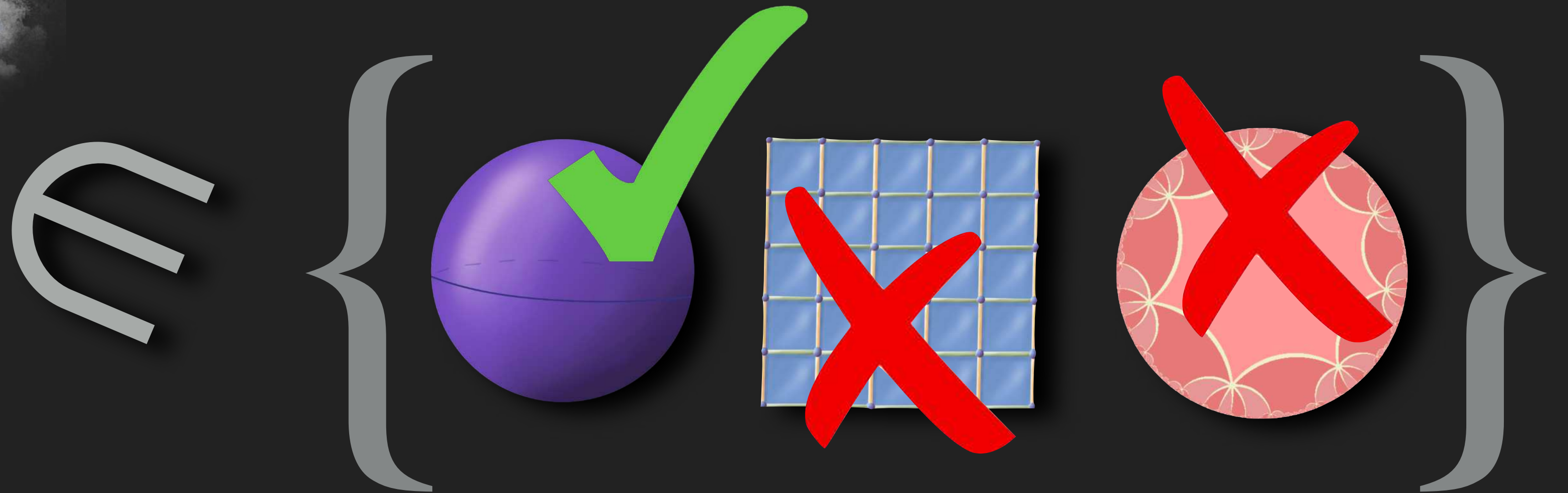
*Feeding it into the uniformization machine
unrolls it into one of the three geometries.*

Proving the Poincaré Conj. for Surfaces



Our surface is simply connected, so it didn't need any unrolling! Thus it IS one of the geometries.

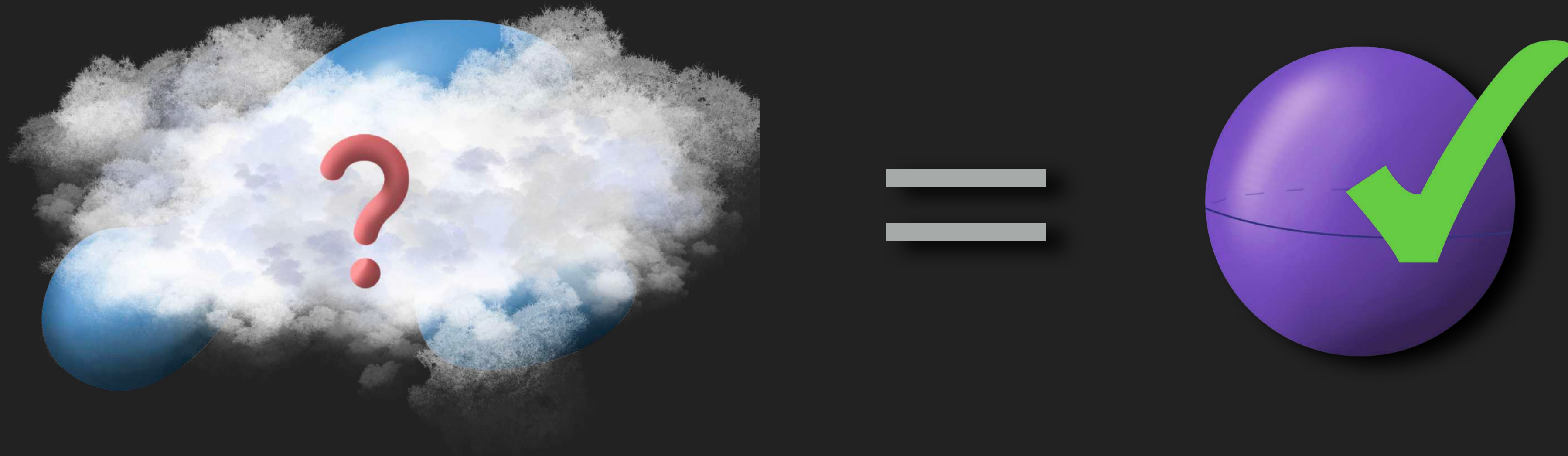
Proving the Poincaré Conj. for Surfaces



*Our surface is **finite**, so it cannot equal one
of the infinite geometries.*

*Knowing only that the mystery surface
had the same **loop invariants** as the
sphere, told us it was the sphere!*

Proving the Poincaré Conj. for Surfaces



Thus, our surface must be the sphere.

*Knowing only that the mystery surface
had the same loop in
sphere, told us it w*

Proving the Poincaré Conj. for Surfaces



Th

here.

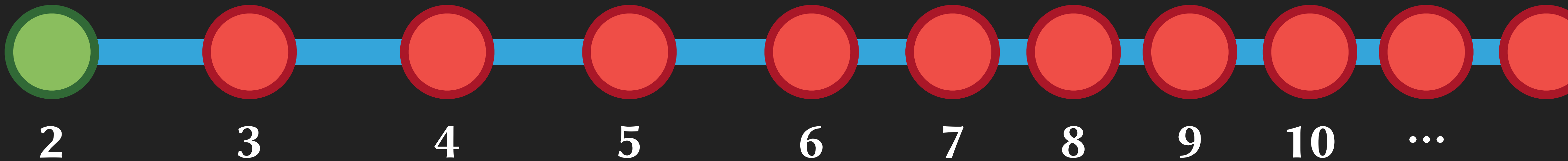
Where things go **From Here**

Poincare's original question was about *three dimensional spaces*, not two dimensional surfaces.



But the possibility of wormholes makes this difficult

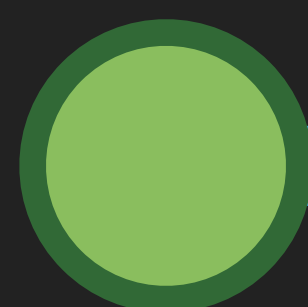
In Any Dimension...



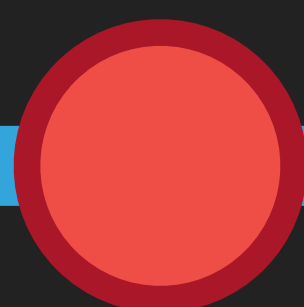


In Any Dimension...

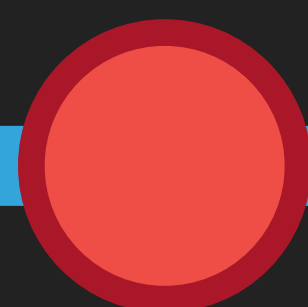
In 1960, Stephen Smale proved the Poincare Conjecture in dimension 7



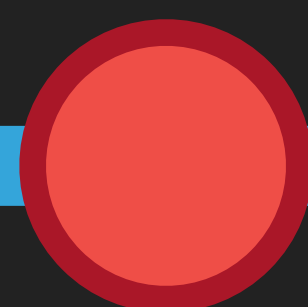
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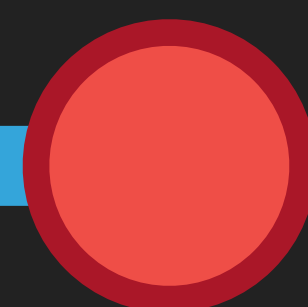
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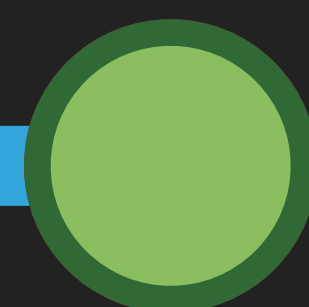
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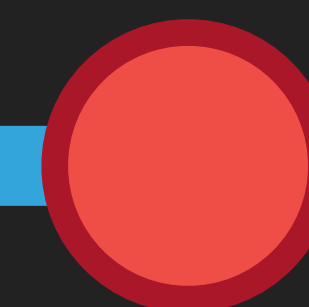
5



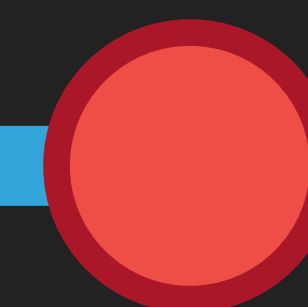
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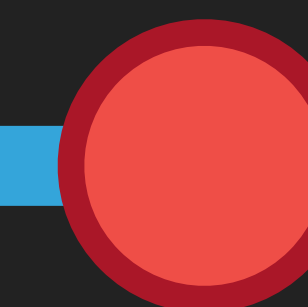
7



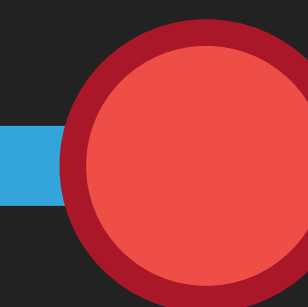
8



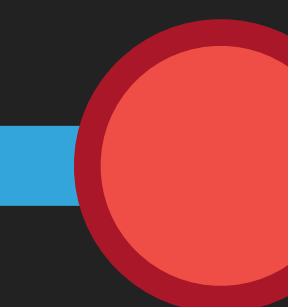
9



10



...

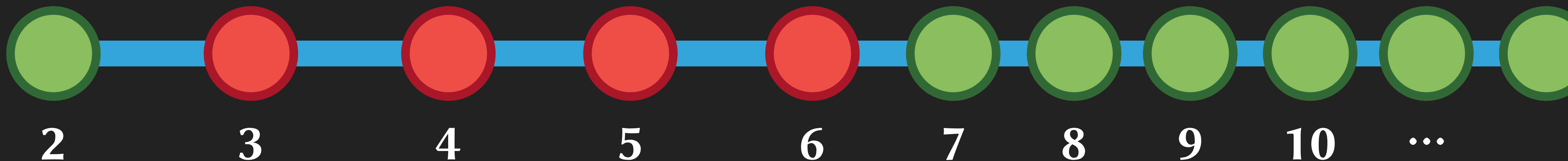




In Any Dimension...

In 1960, Stephen Smale proved the Poincare Conjecture in dimension 7

...and showed the same proof worked in all higher dimensions!



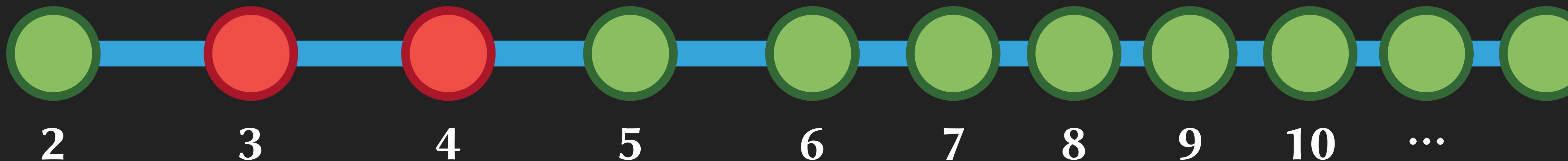


In Any Dimension...

In 1960, Stephen Smale proved the Poincare Conjecture in dimension 7

...and showed the same proof worked in all higher dimensions!

In 1961, Smale showed his techniques also work in dimension 5 and 6

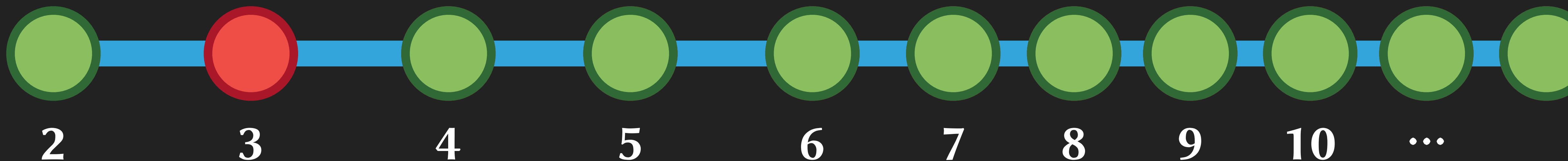
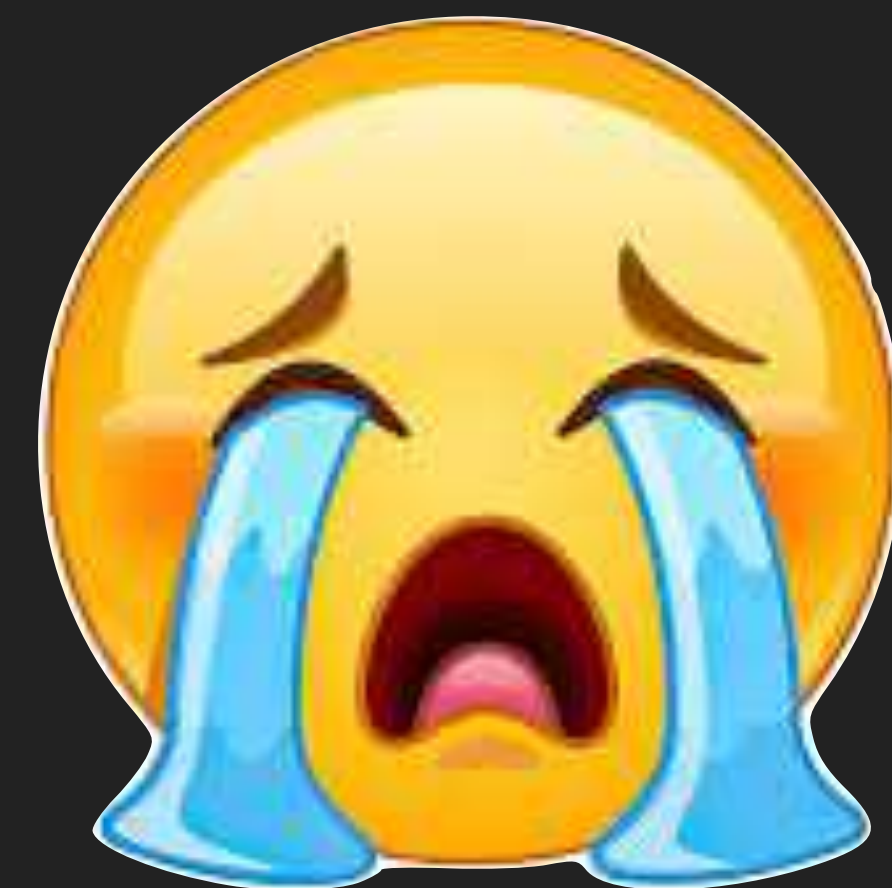


In Any Dimension...



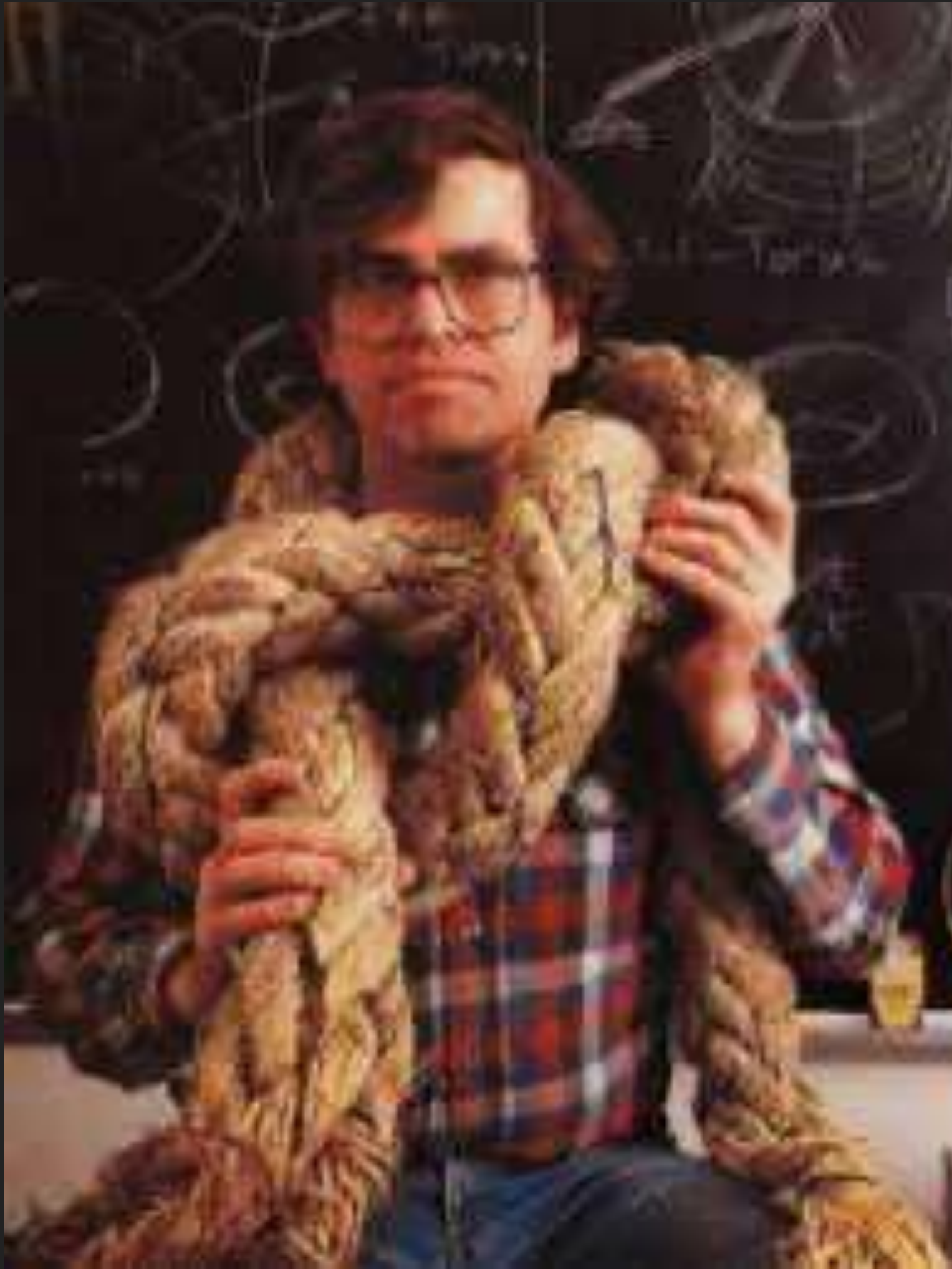
In 1982, Michael Freedman used *totally different techniques* to prove the Poincare Conjecture in dimension 4.

But none of the ideas from dimension 4 and above help at all in dimension 3.

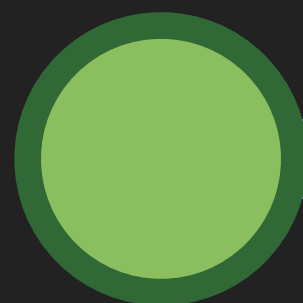


In Any Dimension...

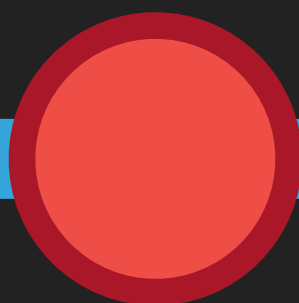
In the 1980s,
William Thurston
laid out a vision
for how to modify
the geometric
ideas from
dimension 2 to
accommodate
wormholes.



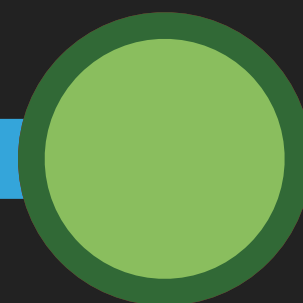
And Richard Hamilton
suggested a way to prove it
using differential equations



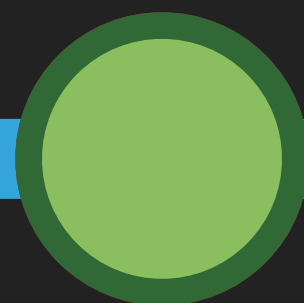
2



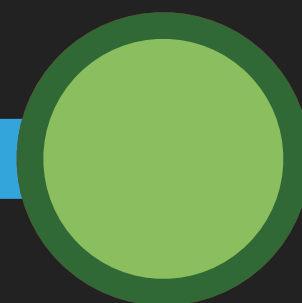
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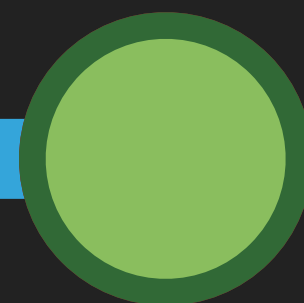
4



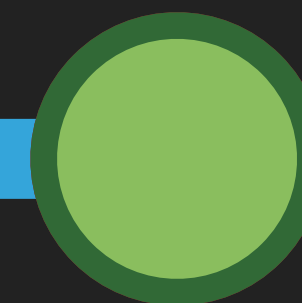
5



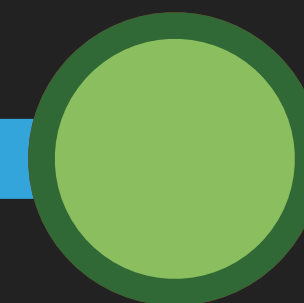
6



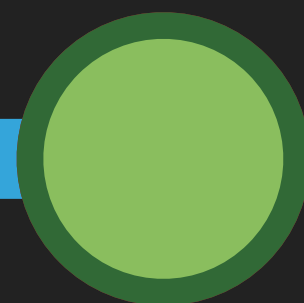
7



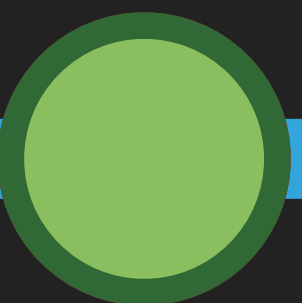
8



9



10



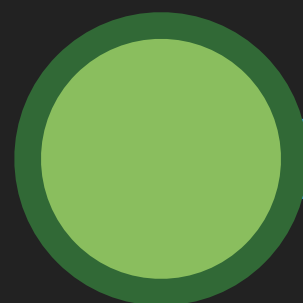
...



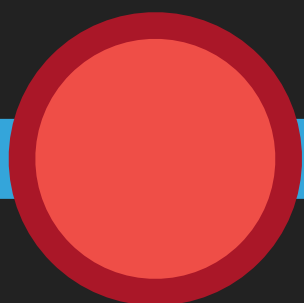
In Any Dimension...



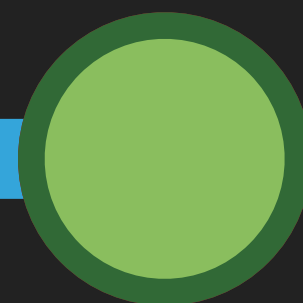
In 2002, Grigori Perelman succeeded in making this proof strategy work! **After nearly 100 years, the 3 Dimensional Poincare Conjecture is solved!**



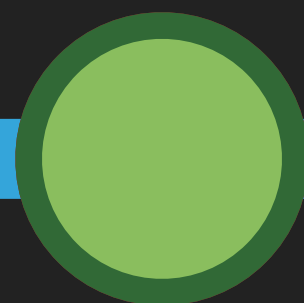
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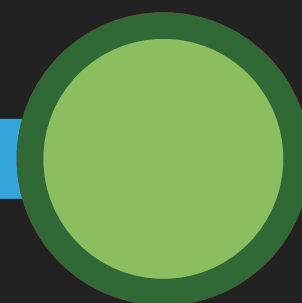
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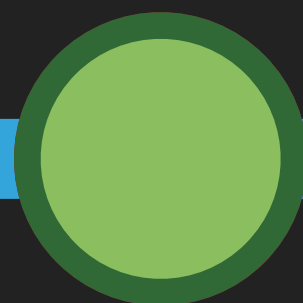
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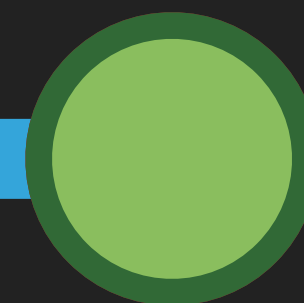
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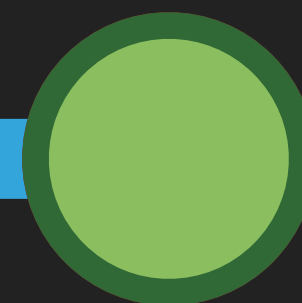
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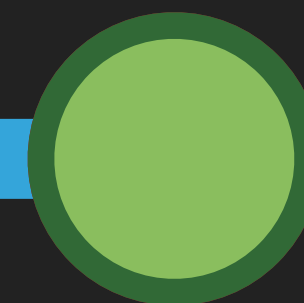
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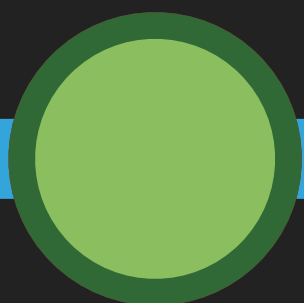
8



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