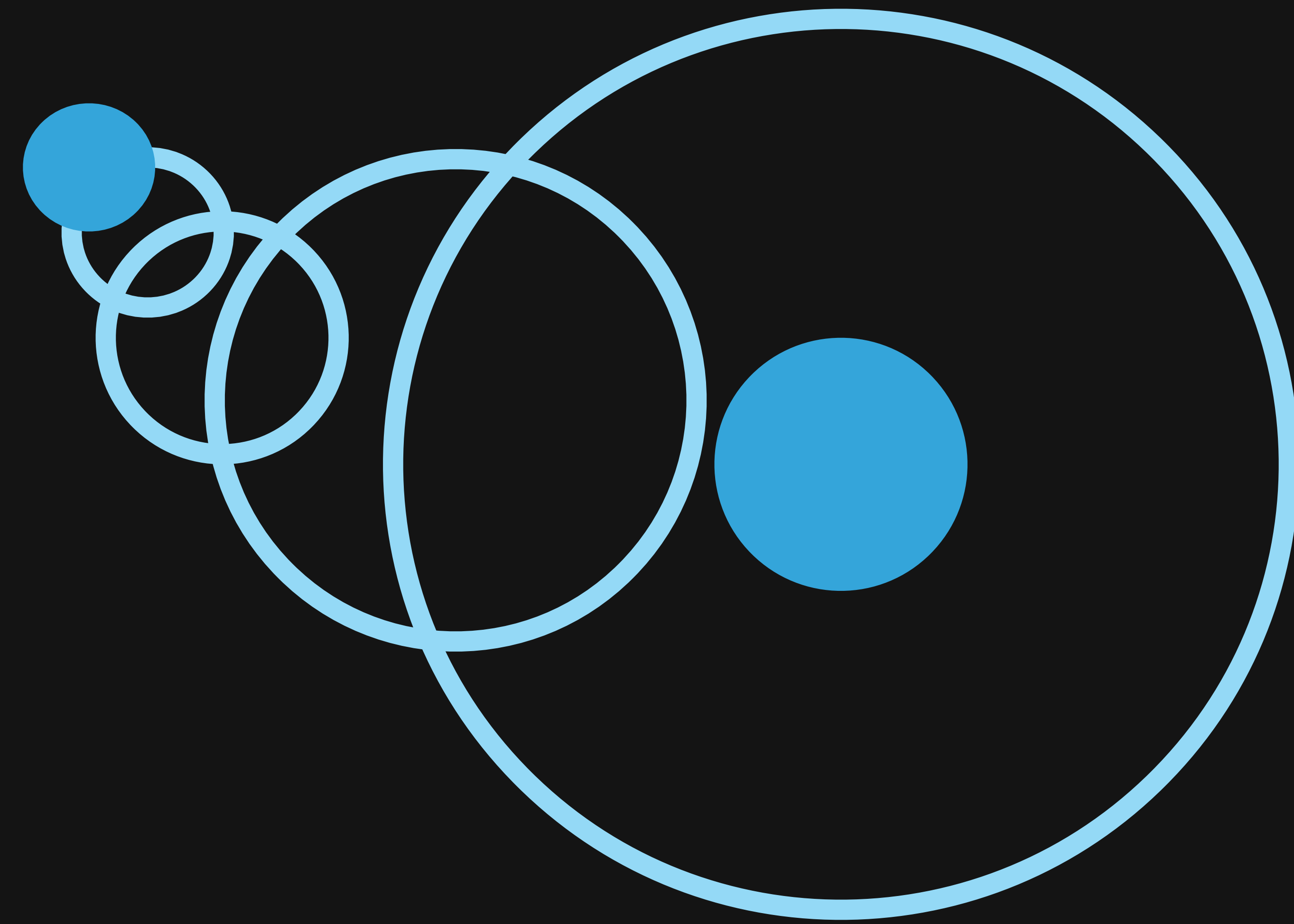


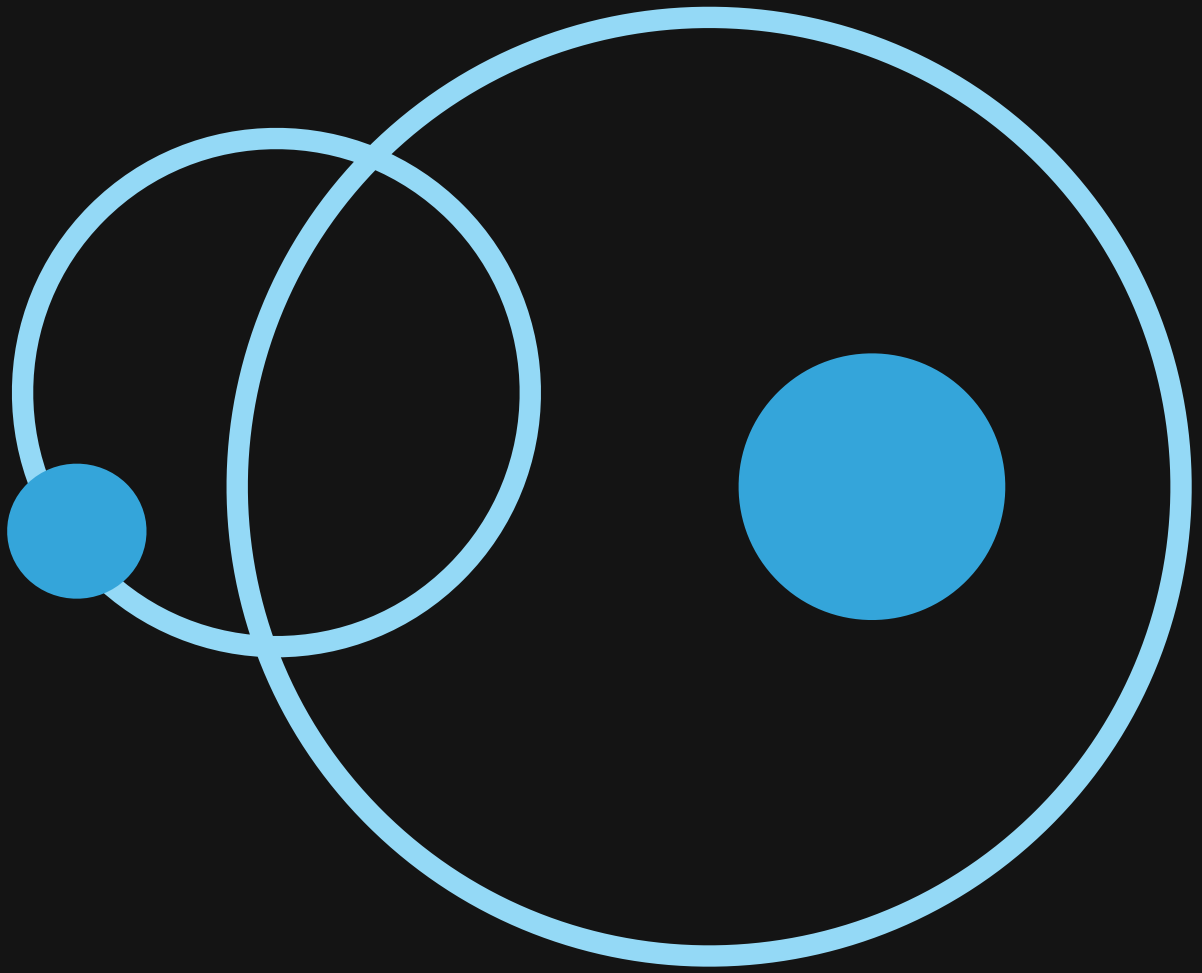
THE MATHEMATICS OF

PTOLEMY, IBN AL SHATIR AND FOURIER

Introduce some of the ideas preceding Fourier analysis

$$f(x) = \sum_{n} a_n \sin(nx) + b_n \cos(nx)$$

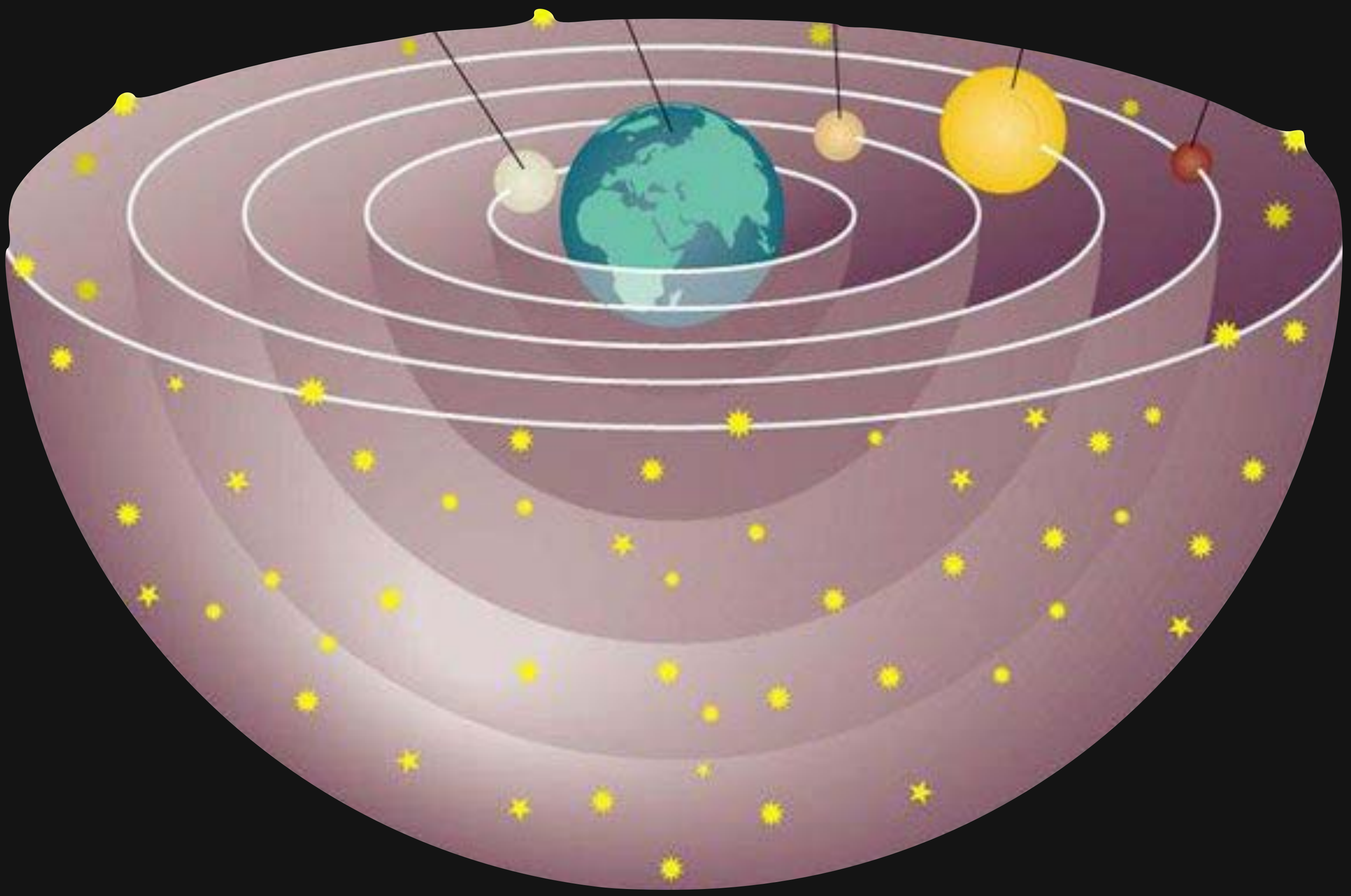
These ideas motivate modern analysis (Friday's topic).



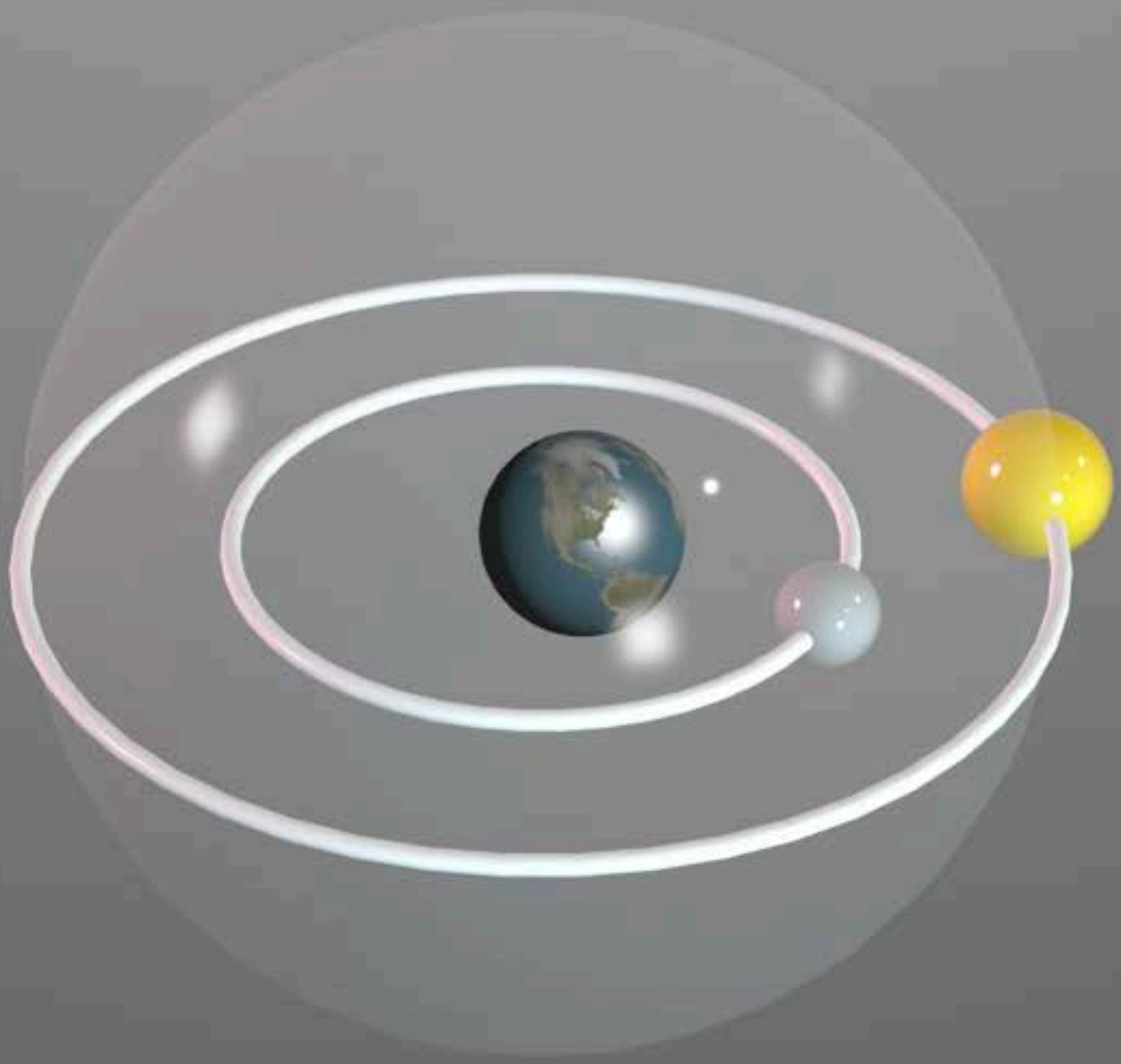
PLANETARY MOTION AND THE

PTOLEMAIC SYSTEM

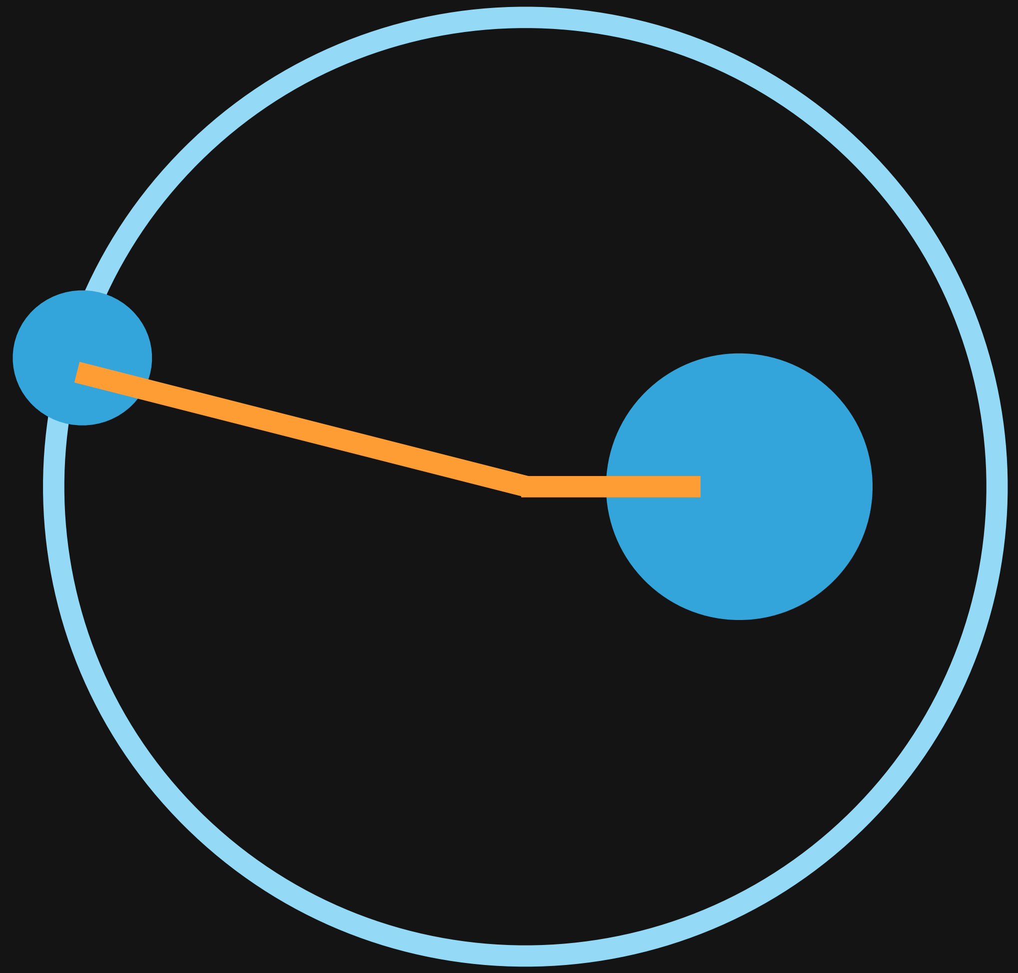
From a modern perspective, easy to denounce the earlier geocentric models of the cosmos as overly simplistic.



But, while ultimately incorrect, these were developed as rigorous scientific / mathematical models.



www.stevejtrettel.site/code/Ptolemy1/index.html



In modern notation:

$$\gamma(t) = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + R_0 \begin{pmatrix} \cos(\omega_0 t) \\ \sin(\omega_0 t) \end{pmatrix}$$

Offset **Orbit**

https://vimeo.com/182735230

This behavior of the planets is hard to describe if each orbits the earth on a (off-center) sphere.

But it is this property which gave them their name:

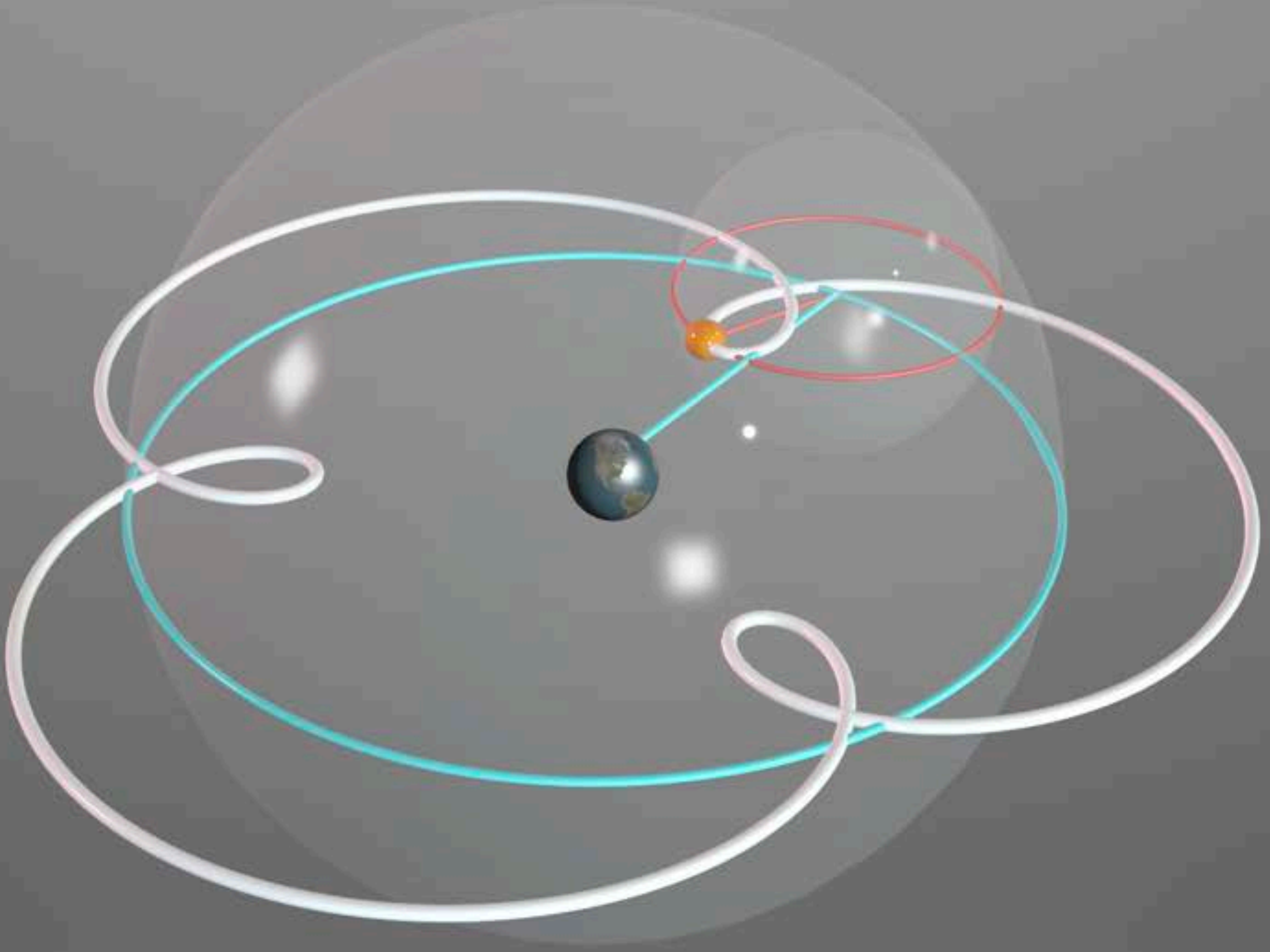
ἀστήρ πλανήτης

Wandering Star

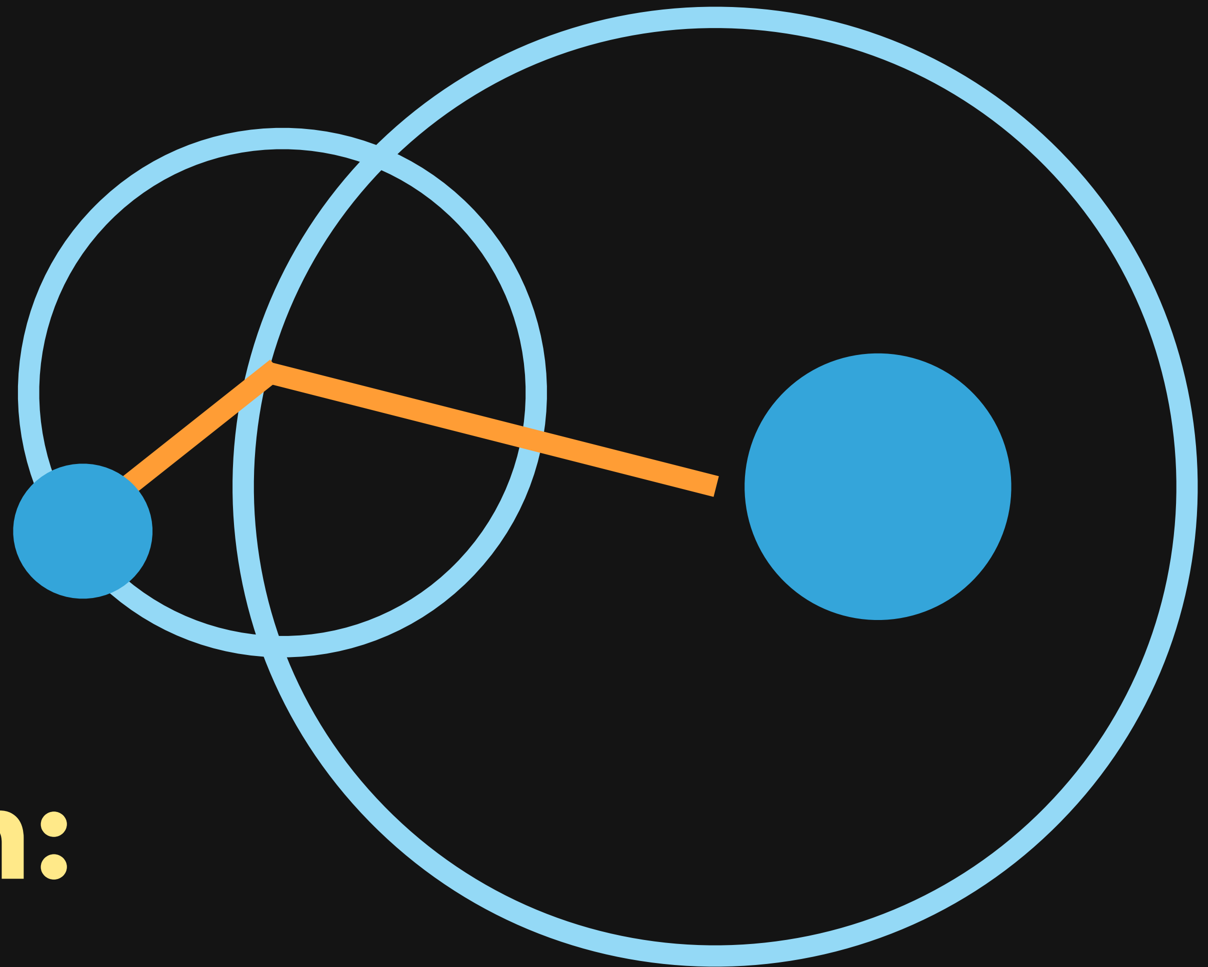
Ptolemy of Alexandria: 150CE

Assembled these ideas into a coherent *predictive* theory of the cosmos.

By carefully choosing the size and speed of epicycles, was able to explain planetary orbits to measured precision.



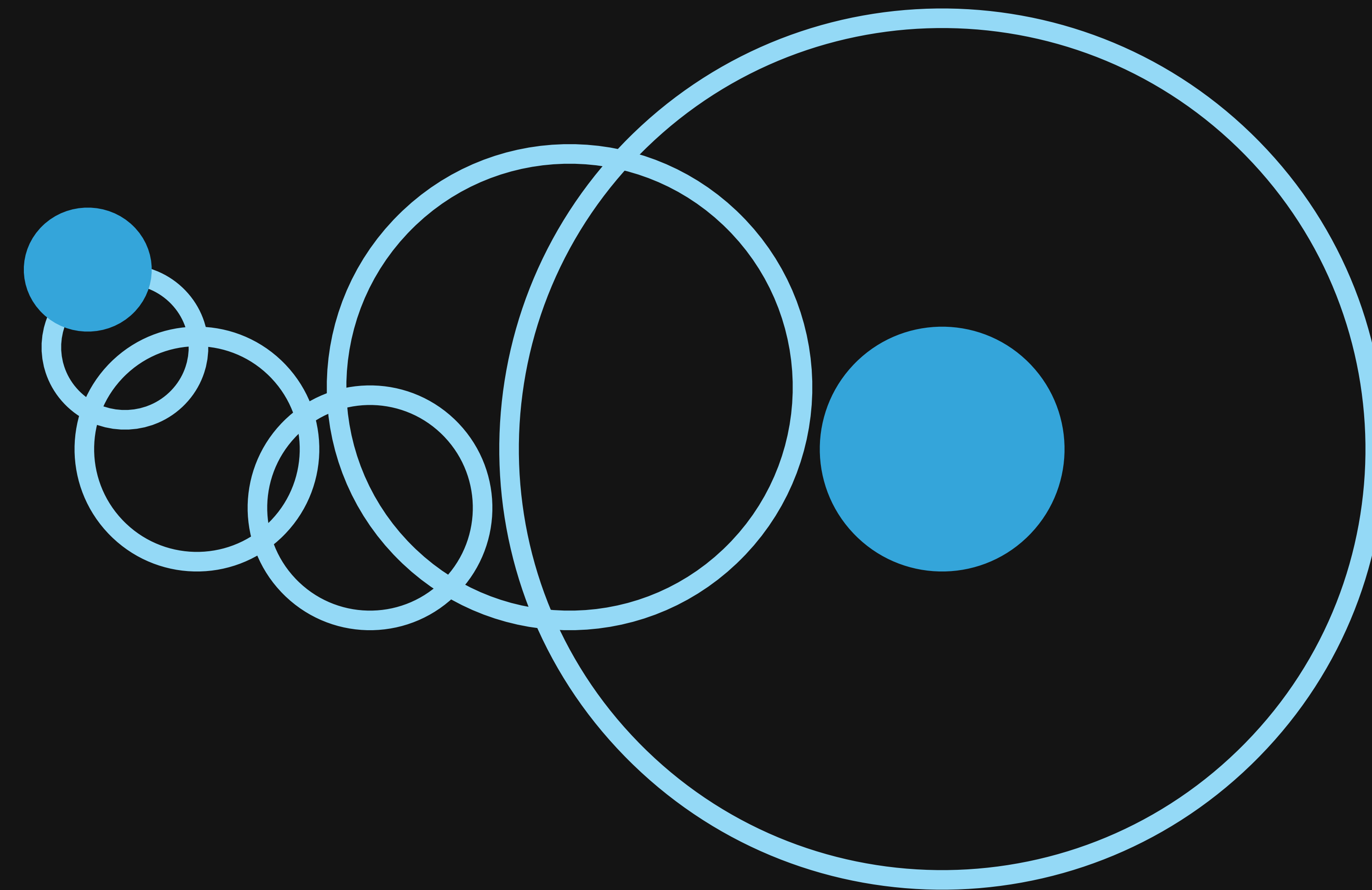
www.stevejtrettel.site/code/Ptolemy2/index.html



In modern notation:

$$\gamma(t) = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + R_0 \begin{pmatrix} \cos(\omega_0 t) \\ \sin(\omega_0 t) \end{pmatrix} + R_1 \begin{pmatrix} \cos(\omega_1 t) \\ \sin(\omega_1 t) \end{pmatrix}$$

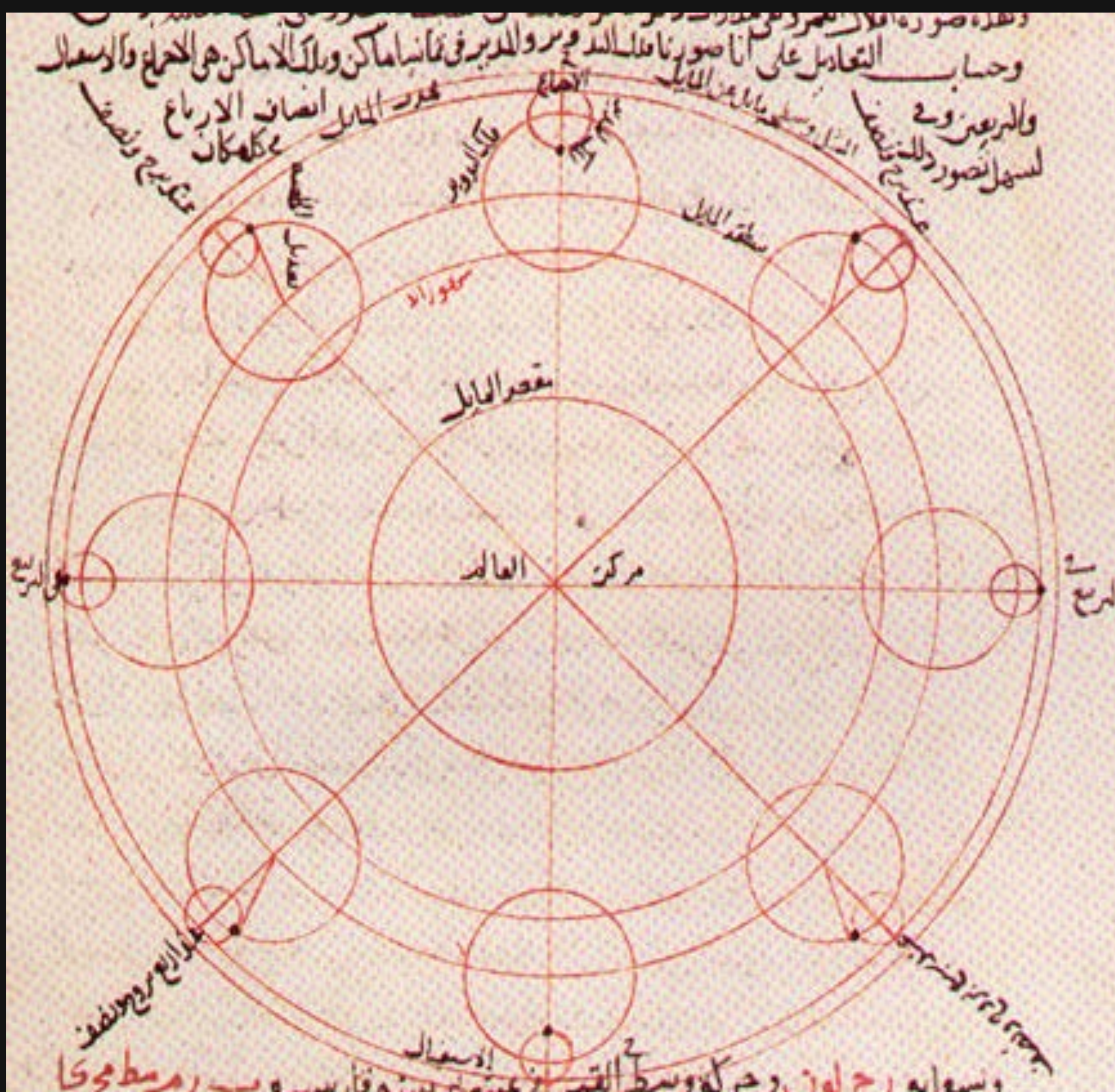
Offset **Orbit** **Epicycle**



IMPROVEMENTS BY

ابن الشاطر (IBN AL- SHATIR)

Astronomer, mathematician, engineer from Damascus, late 1300s.



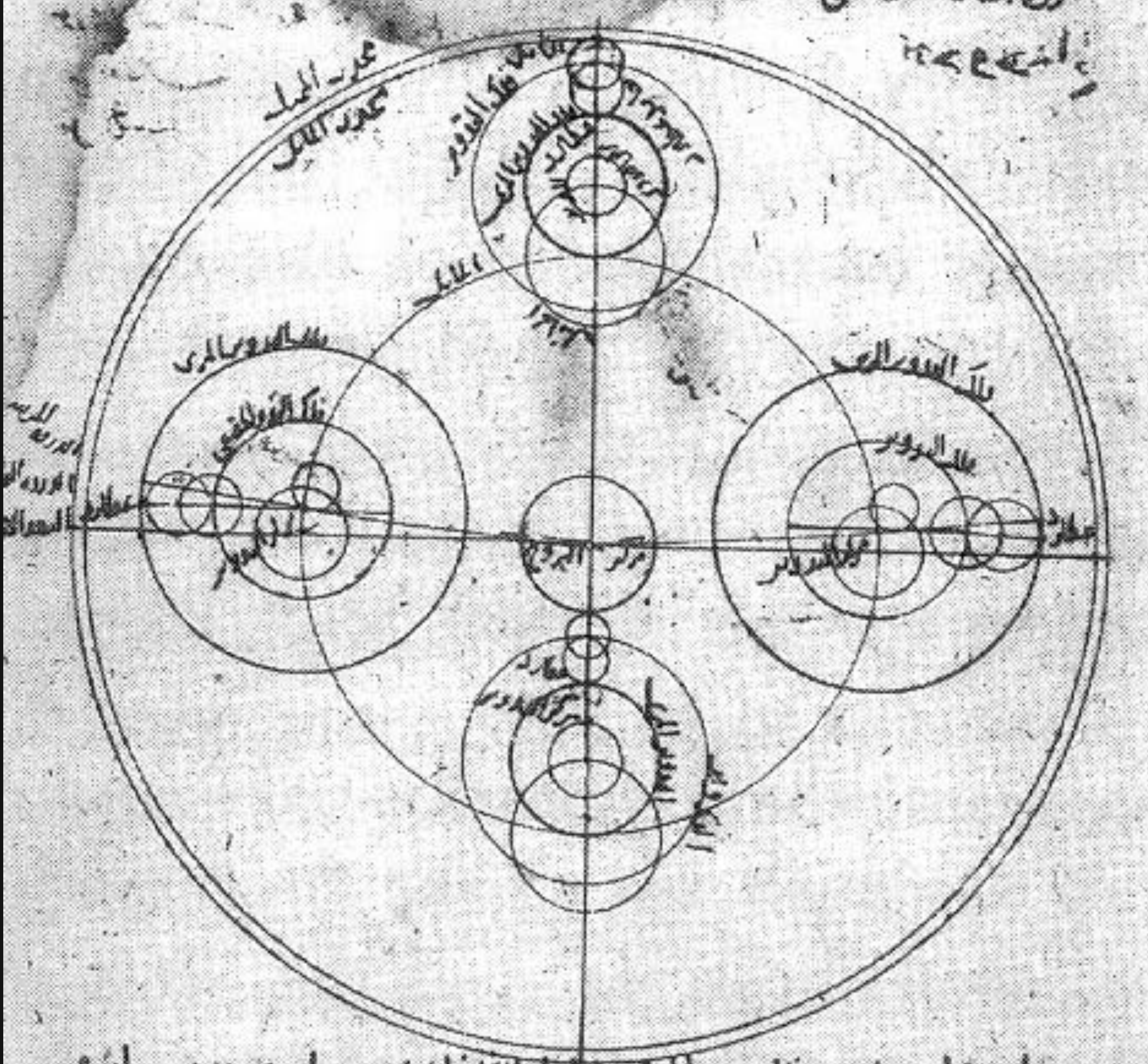
Was not concerned with the philosophy of the planetary model, so much as its empirical accuracy.

First: got rid of the offset by adding another epicycle to mimic its effect on the sky!

This relied on a trick for turning rotational motion into linear motion: discovery by Nasir al-Din al-Tusi in 1247.

But once you've had the idea to add a second epicycle…

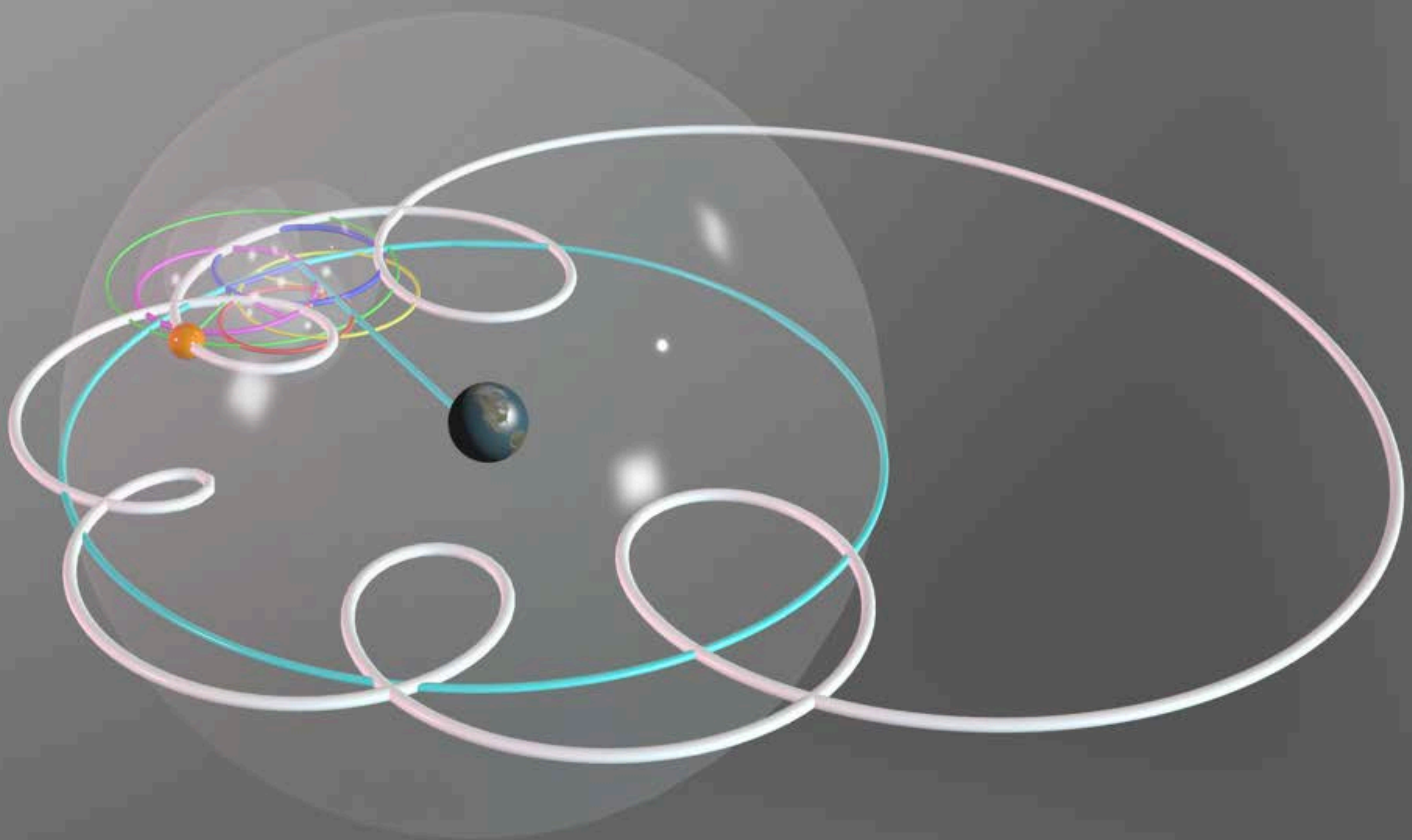
When should you stop? If your goal is to have a predictive theory - can you adjust for any small discrepancies with more epicycles?



في صفة حركات عطارد وتقشيرها لنصف نهار البلنيا اول سنة احدى وسبع مائة

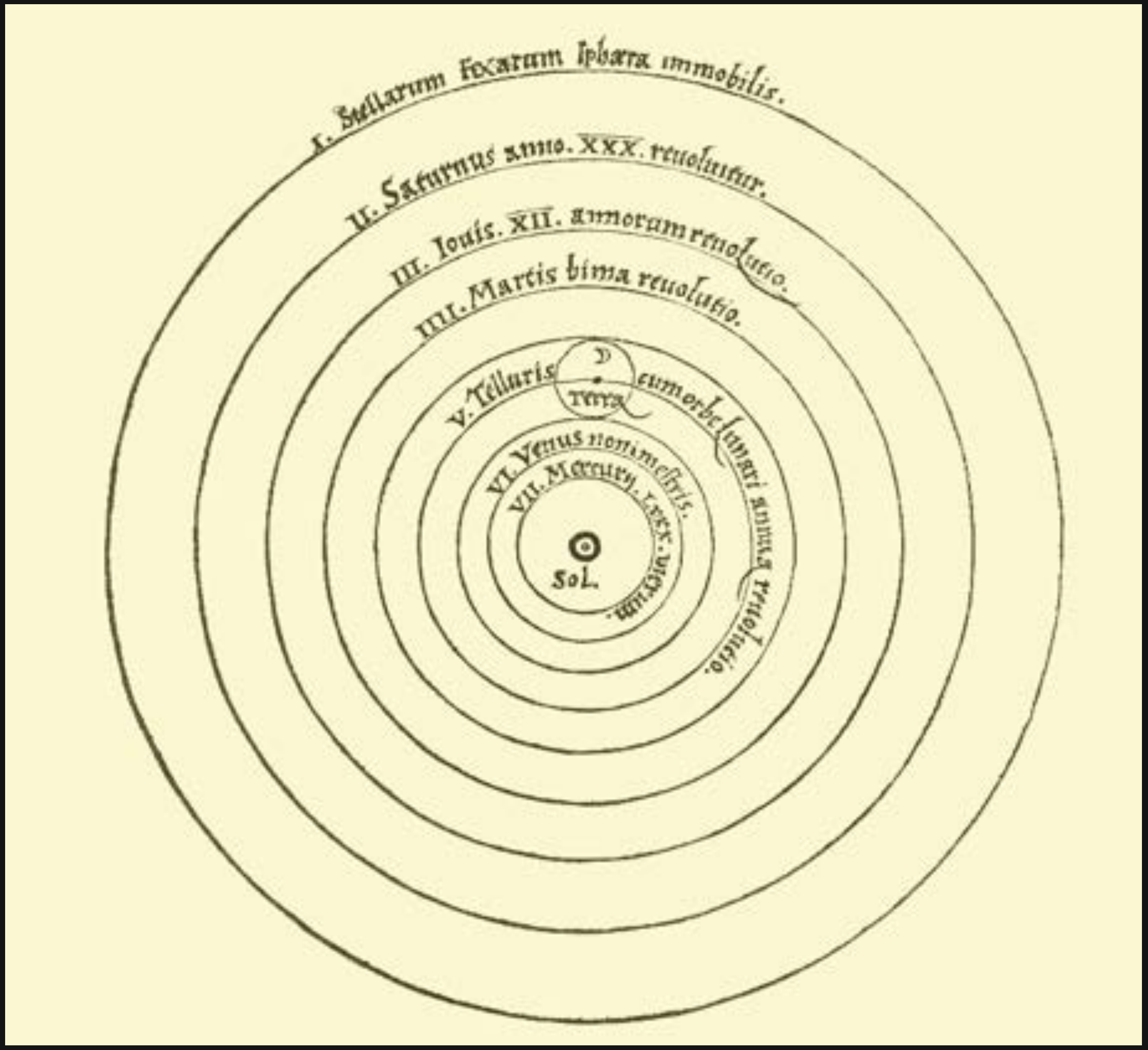
In modern notation:

$$\gamma(t) = R_0 \begin{pmatrix} \cos(\omega_0 t) \\ \sin(\omega_0 t) \end{pmatrix} + R_1 \begin{pmatrix} \cos(\omega_1 t) \\ \sin(\omega_1 t) \end{pmatrix} + R_2 \begin{pmatrix} \cos(\omega_2 t) \\ \sin(\omega_2 t) \end{pmatrix} + R_3 \begin{pmatrix} \cos(\omega_3 t) \\ \sin(\omega_3 t) \end{pmatrix} + R_4 \begin{pmatrix} \cos(\omega_4 t) \\ \sin(\omega_4 t) \end{pmatrix} + R_5 \begin{pmatrix} \cos(\omega_5 t) \\ \sin(\omega_5 t) \end{pmatrix} + \cdots$$



www.stevejtrettel.site/code/Ptolemy3/index.html

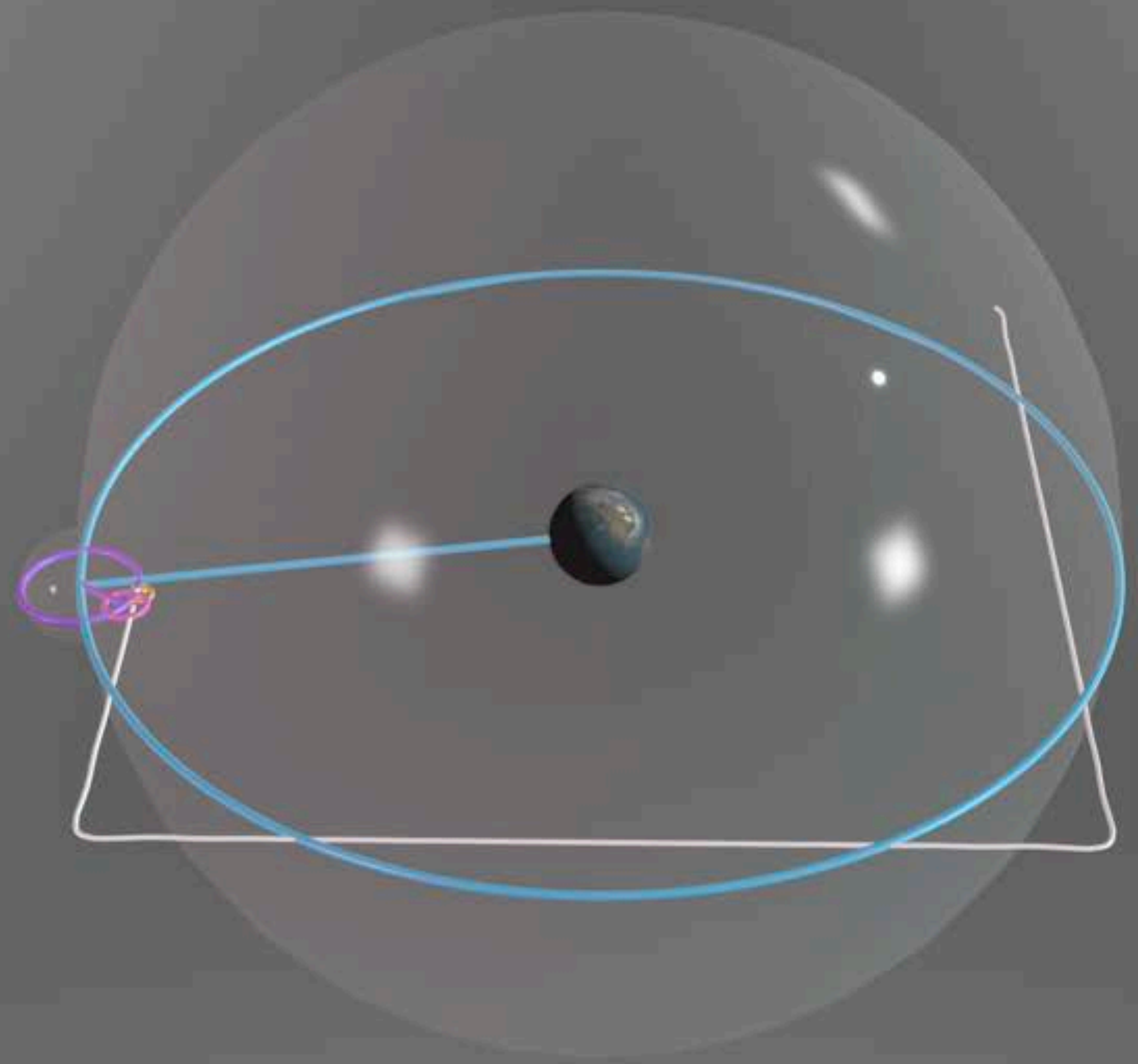
This theory was so accurate, that when Copernicus first (re)-introduced heliocentrism, the main objects where that it simply did not perform as well as the current predictions!



IS IBN AL SHATIR’S MODEL OF THE COSMOS…. FALSIFIABLE?

This becomes a very interesting mathematical question.

Can any arbitrary motion in the sky be modeled with sufficiently many epicycles?



www.stevejtrettel.site/code/SquareOrbit/index.html

We cannot answer this with math 115.

But we can phrase the question precisely:

Let $\gamma(t)$ be an arbitrary orbit in the plane. Is there a sequence of epicycles which converge to γ?

We cannot answer this with math 115.

But we can phrase the question precisely:

Let $\gamma(t) = (x(t), y(t))$ be an arbitrary orbit in the plane. Is there a sequence of functions

$$f_n(t) = \sum_{k=1}^{n} R_n \begin{pmatrix} \cos(nt) \\ \sin(nt) \end{pmatrix}$$

Which converge to γ?

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Such that $x_n(t) \to x(t)$ and $y_n(t) \to y(t)$ for each $t \in \mathbb{R}$?

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Answer: NO! Look at t=0: the y-coordinate is always equal to zero.

Let $\gamma(t) = (x(t), y(t))$ be an arbitrary orbit in the plane. Is there a sequence of functions

$$f_n(t) = \begin{pmatrix} x_n(t) \\ y_n(t) \end{pmatrix} = \sum_{k=1}^{n} R_n \begin{pmatrix} \cos(nt + \delta_k) \\ \sin(nt + \delta_k) \end{pmatrix}$$

Such that $x_n(t) \to x(t)$ and $y_n(t) \to y(t)$ for each $t \in \mathbb{R}$?

Let $x(t)$ be an arbitrary function. Is there a sequence of functions

$$x_n(t) = \sum_{k=1}^{n} a_k \cos(kt + \delta_k)$$

Such that $x_n(t) \to x(t)$?

Let $x(t)$ be an arbitrary function. Is there a sequence of functions

$$x_n(t) = \sum_{k=1}^{n} a_k \cos(kt + \delta_k)$$

No! This function has average value 0 for each n…

Let $x(t)$ be an arbitrary function. Is there a sequence of functions

$$x_n(t) = \sum_{k=1}^{n} a_k \cos(kt + \delta_k)$$

$$\int_0^{2\pi} \cos(kt + \delta_k)dt = 0 \implies \int_0^{2\pi} \sum_{k=1}^{n} a_k \cos(kt + \delta_k)dt = 0$$

Let $x(t)$ be an arbitrary function. Is there a sequence of functions

$$x_n(t) = \sum_{k=1}^{n} a_k \cos(kt + \delta_k)$$

Thus, such a sequence can never converge to a positive function x(t)

Let $x(t)$ be an arbitrary function. Is there a sequence of functions

$$x_n(t) = a_0 + \sum_{k=1}^{n} a_k \cos(kt + \delta_k)$$

Such that $x_n(t) \to x(t)$?

(Just bringing back the offset!)

Let $x(t)$ be an arbitrary function. Is there a sequence of functions

$$x_n(t) = \sum_{k=0}^{n} a_k \cos(kt + \delta_k)$$

Such that $x_n(t) \to x(t)$?

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$$x_n(t) = \sum_{k=0}^{n} a_k \cos(kt + \delta_k)$$

Such that $x_n(t) \to x(t)$?

No! Each of the x_n is periodic, with period dividing 2π.

Let $x(t)$ be an arbitrary 2π-periodic function. Is there a sequence of functions

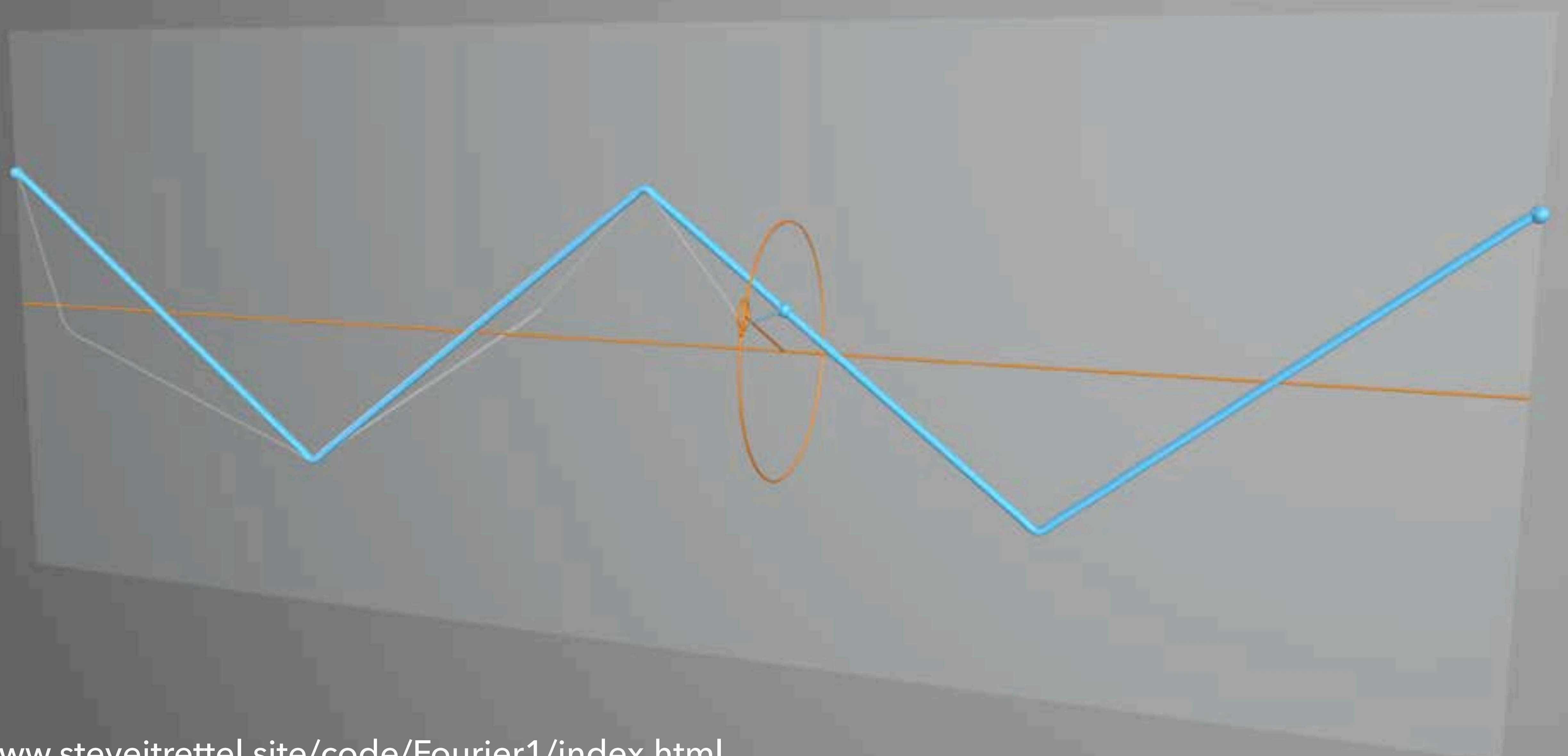
$$x_n(t) = \sum_{k=0}^{n} a_k \cos(kt + \delta_k)$$

Such that $x_n(t) \to x(t)$?

SOME EXAMPLES OF

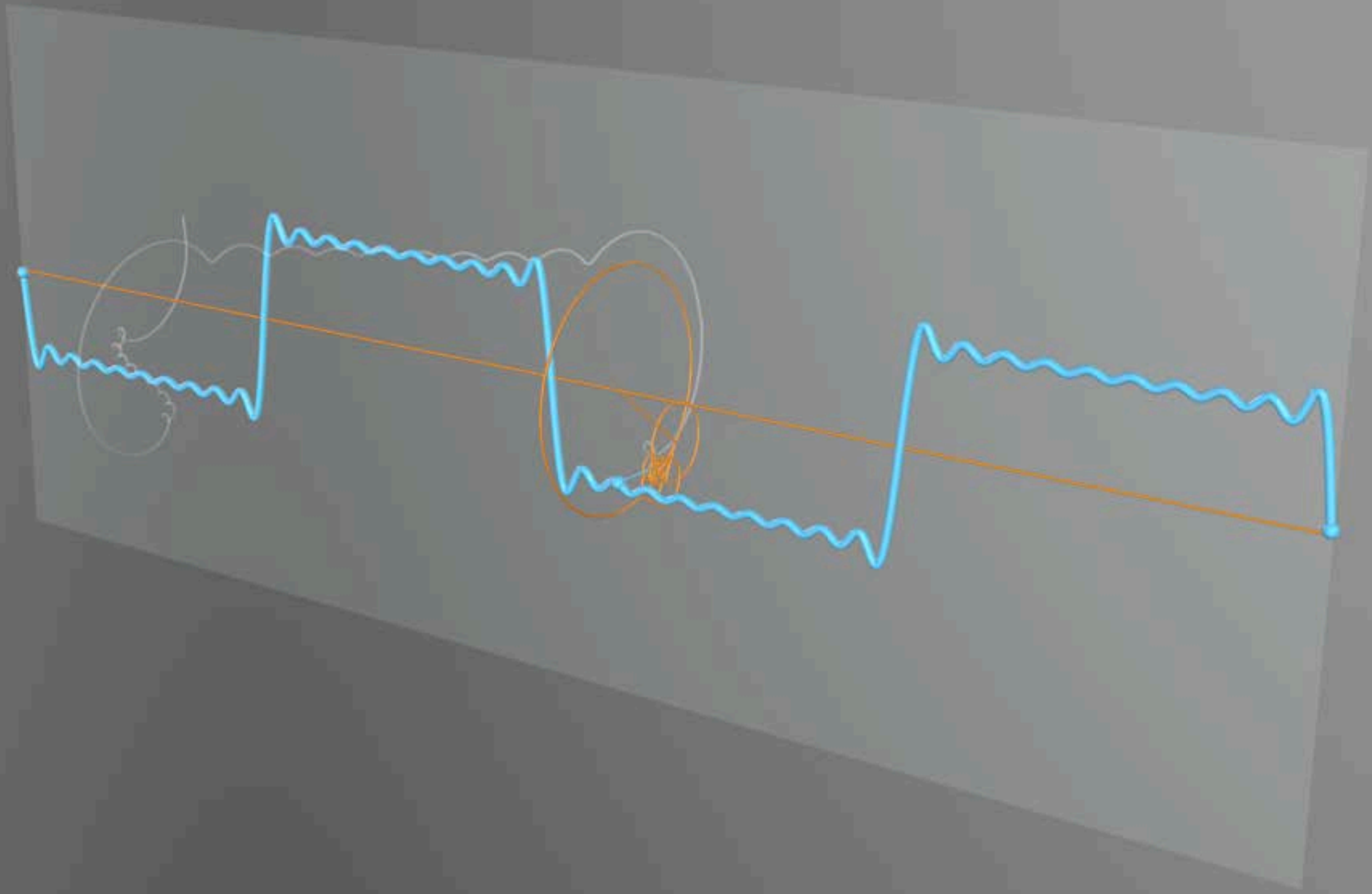
SINUSOIDAL SERIES

Animation of a sum of sinusoids



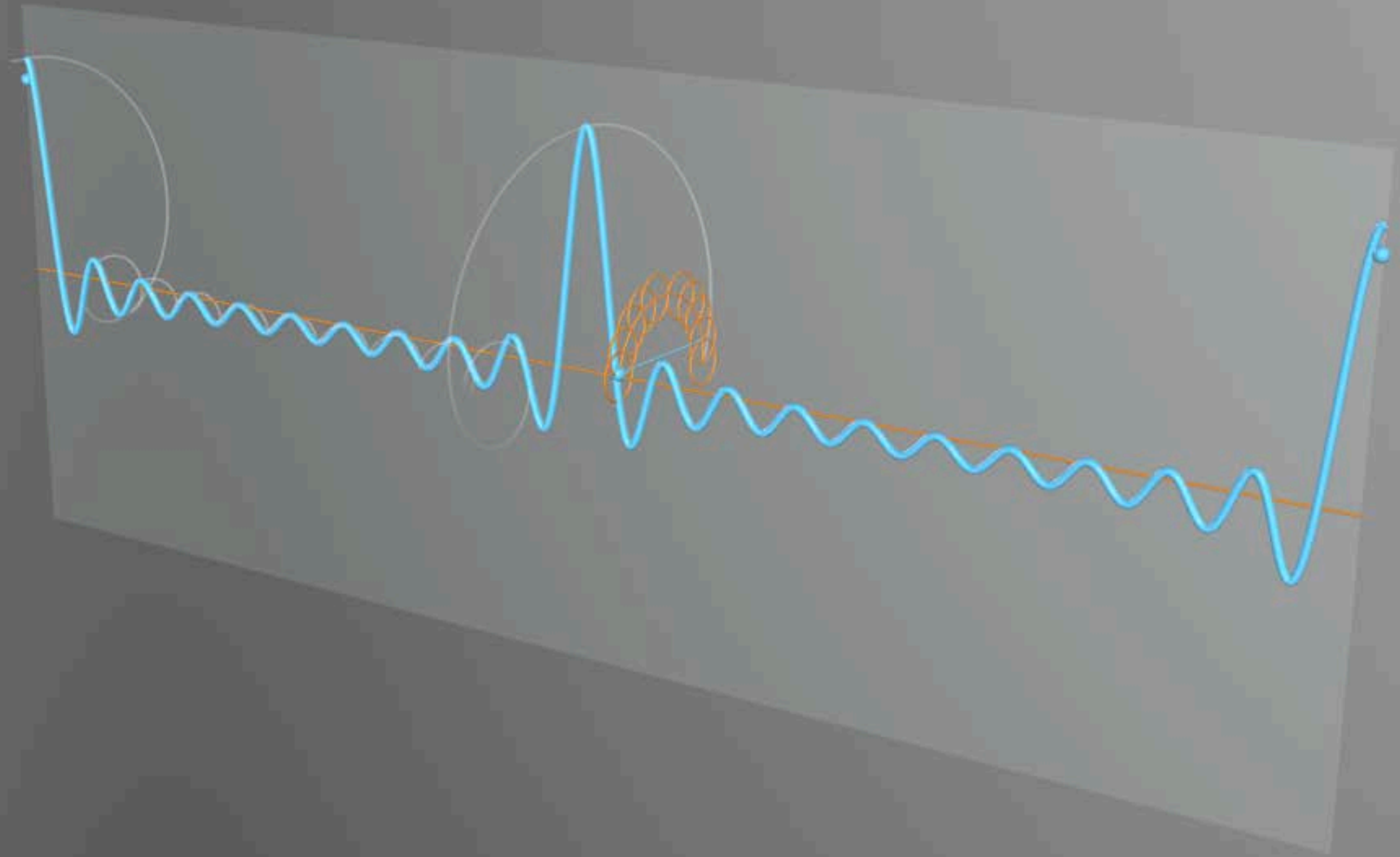
www.stevejtrettel.site/code/Fourier1/index.html

Do we need to assume $x(t)$ is continuous?



www.stevejtrettel.site/code/Fourier2/index.html

Do we need to assume $x(t)$ is bounded?



www.stevejtrettel.site/code/Fourier3/index.html

THE THEORY OF HEAT AND

JOSEPH FOURIER

Formally laid down the idea of a sum of sinusoids to approximate a function.

Came up in the context of trying to study heat.

$$f(t) = \lim_{n \to \infty} \sum_{k=0}^{n} a_k \cos(kt + \delta_k)$$

Formally laid down the idea of a sum of sinusoids to approximate a function.

Came up in the context of trying to study heat.

$$f(t) = \lim_{n \to \infty} \sum_{k=0}^{n} a_k \cos(kt) + b_k \sin(kt)$$

The basic idea:

If you can find a lot of easy solutions to a difficult problem, can you combine them to get a general solution?

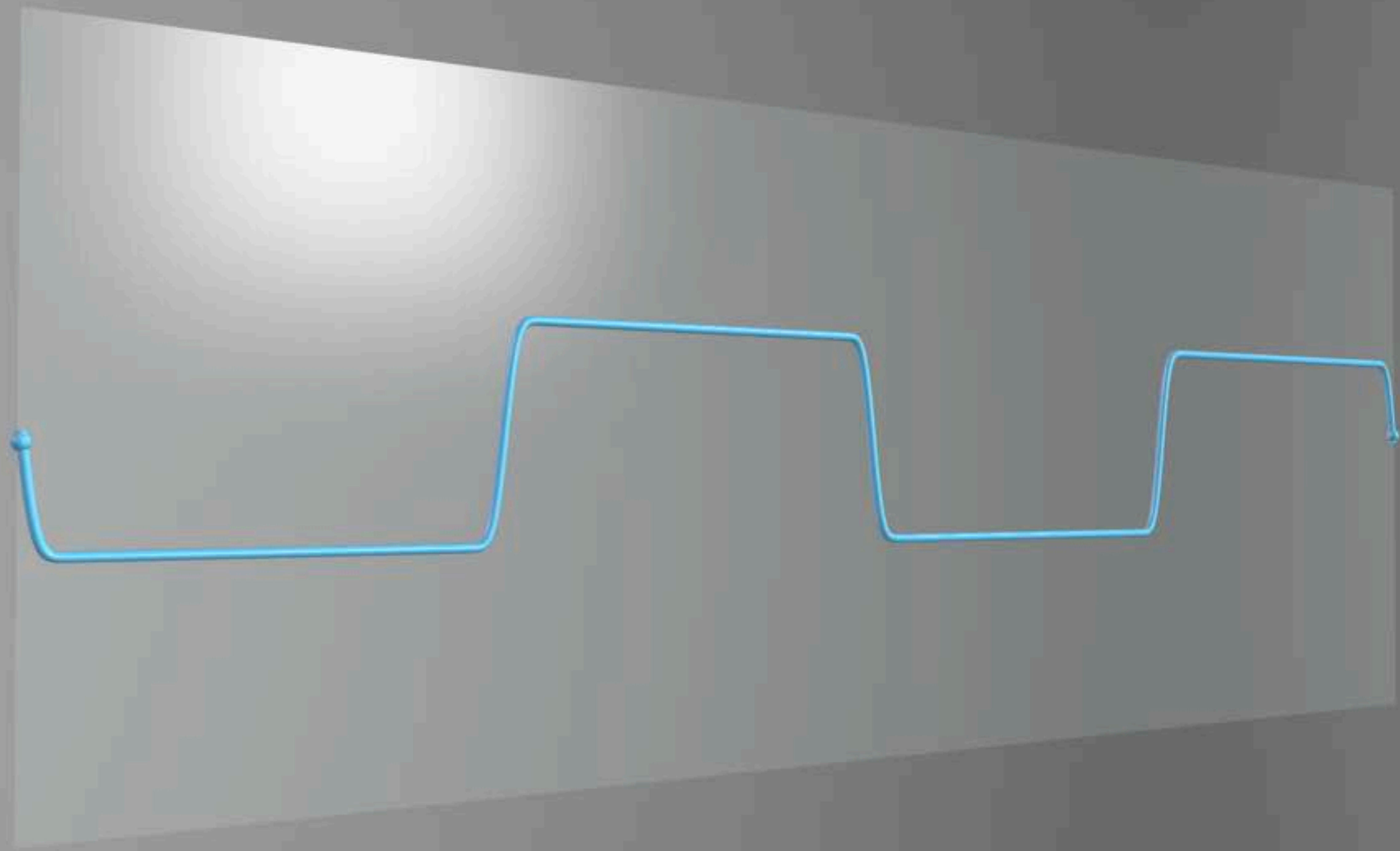
Sines and Cosines are easy to work with for problems involving second derivatives:

$$\frac{d^2}{dt^2}\sin(t) = -\sin(t) \qquad \frac{d^2}{dt^2}\cos(t) = -\cos(t)$$

The Heat Equation

Let $f(x, t)$ be the temperature of a material at position x, at time t. Then f solves

$$\frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2}$$



www.stevejtrettel.site/code/HeatEqn/index.html

$$\frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2}$$

Certain solutions are **easy to guess:**

$$f(x, t) = e^{-t} \sin(x)$$

$$f(x, t) = e^{-4t} \sin(2x)$$

$$f(x, t) = e^{-9t} \sin(3x)$$

$$\frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2}$$

Solutions are **easy to combine:**

If $f(x, t)$ is a solution and $g(x, t)$ is a solution then $af(x, t) + bg(x, t)$ is a solution.

$$\frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2}$$

Solutions are **easy to combine:**

If $f(x, t)$ is a solution and $g(x, t)$ is a solution then $af(x, t) + bg(x, t)$ is a solution.

$$\sum_{k=0}^{n} a_k e^{-n^2 t} \cos(nt) + b_n e^{-n^2 t} \sin(nt)$$

Fourier's Main Question:

How many solutions are of the form I have discovered?

$$\sum_{k=0}^{n} a_k e^{-n^2 t}\cos(nt) + b_n e^{-n^2 t}\sin(nt)$$

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How many solutions are of the form I have discovered?

$$\sum_{k=0}^{n} a_k e^{-n^2 t} \cos(nt) + b_n e^{-n^2 t} \sin(nt)$$

Like before, all are periodic. But how many of the periodic solutions are of this form?

Fourier's Main Question:

How many solutions are of the form I have discovered?

$$\sum_{k=0}^{n} a_k e^{-n^2 t} \cos(nt) + b_n e^{-n^2 t} \sin(nt)$$

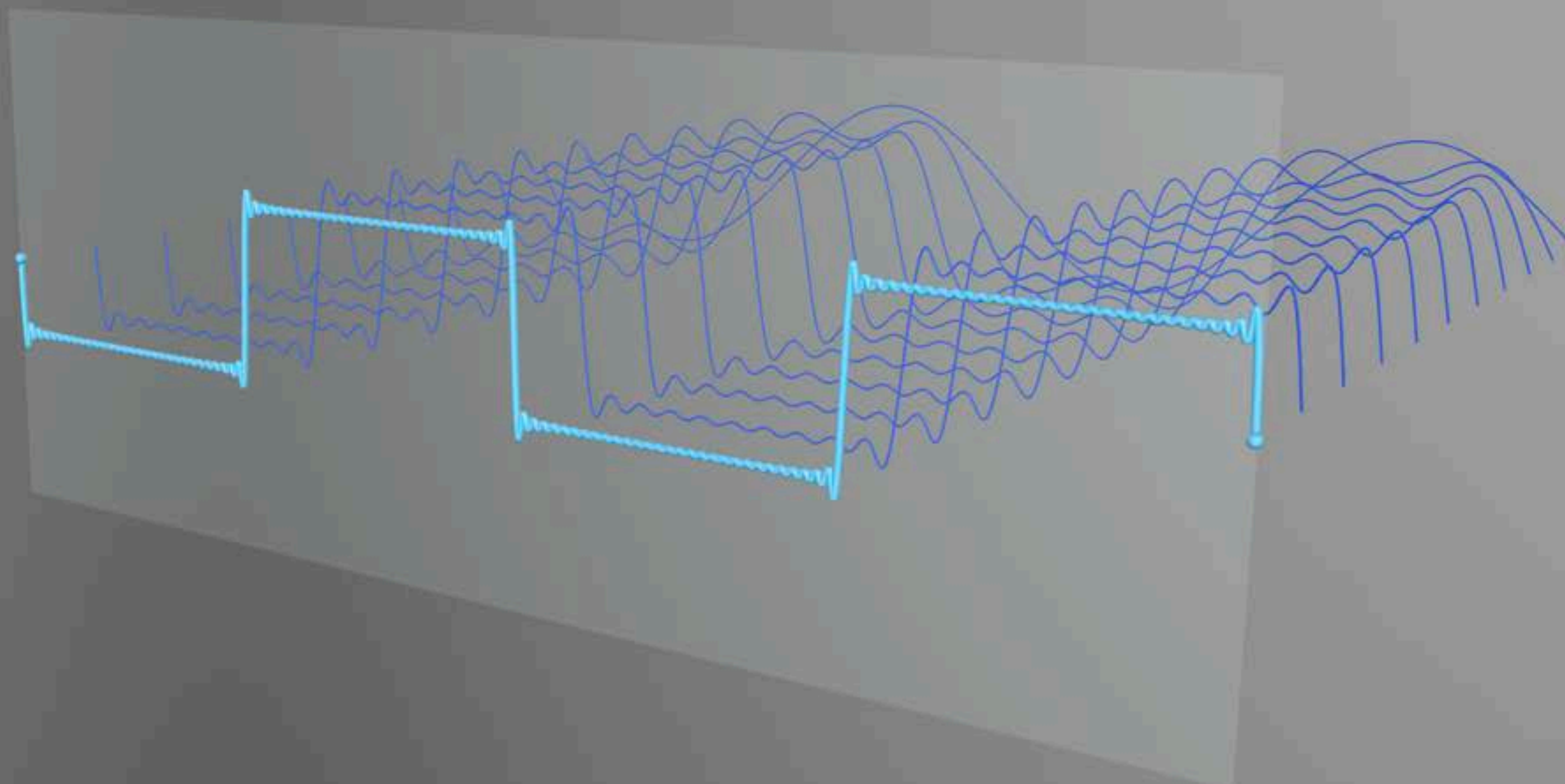
If I measure the temperature at time =0 to be $f(x)$, can I find the right solution in this form?

Plug in t=0:

If our solution is a sum of sinusoids, then

$$f(x,0) = \sum_{k=0}^{n} a_k \cos(nt) + b_n \sin(nt)$$

The initial condition is a sum of sinusoids.



www.stevejtrettel.site/code/Fourier4/index.html

Thus:

The class of functions we can solve the heat equation

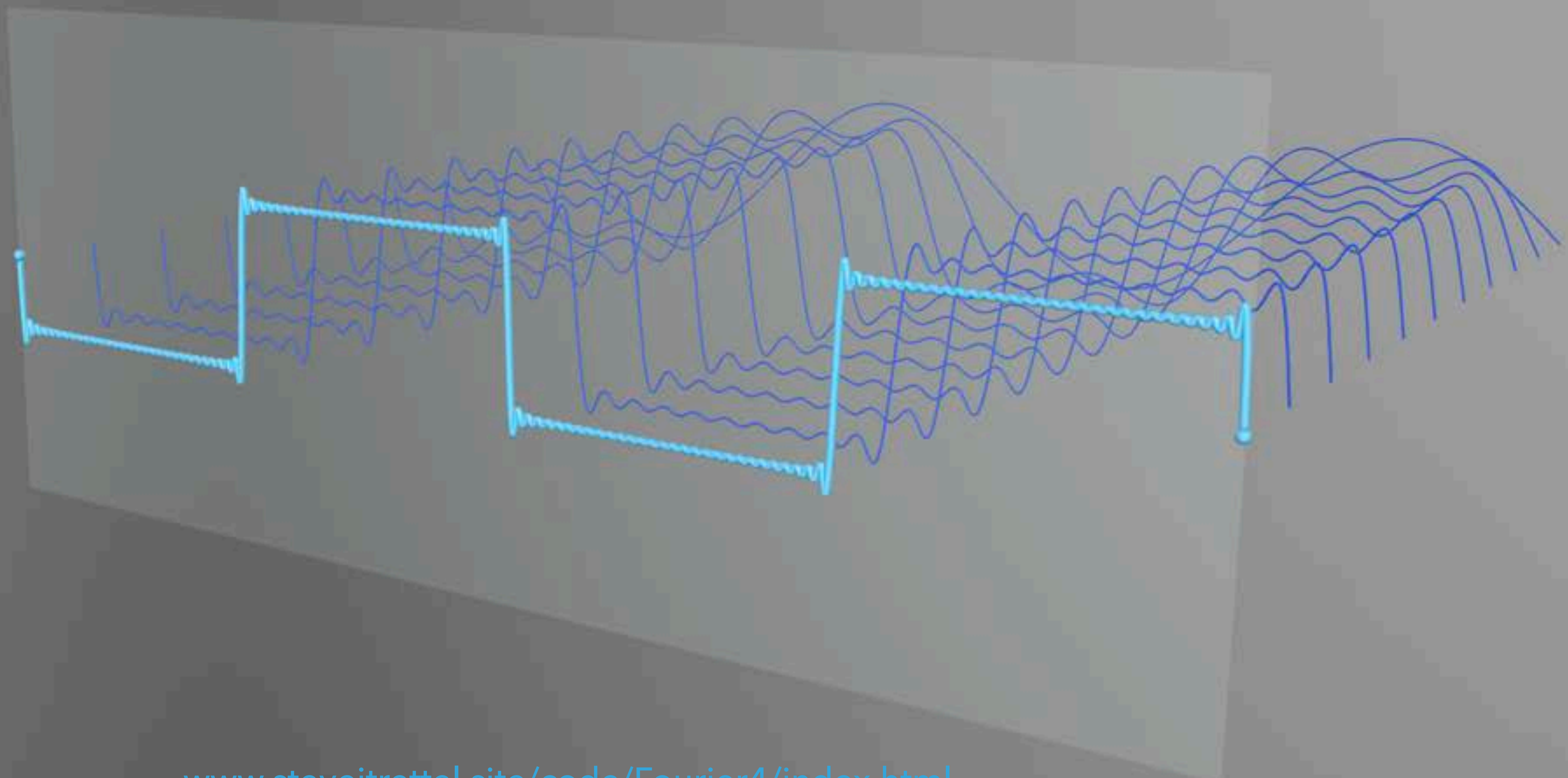
$$\sum_{k=0}^{n} a_k \cos(nt) + b_n \sin(nt)$$

Equals the class of functions for which are possible epicycle orbits!

Exactly which functions these are, and how the theory of convergence plays out for them, will give us a first view into where analysis can go beyond 115.

THE DISCOVERY OF A

MYSTERIOUS FORMULA



www.stevejtrettel.site/code/Fourier4/index.html

To solve the heat equation for initial condition f(x), Fourier **assumed** that f could be written as a sum of sinusoids, and derived a formula:

$$f(x) = \sum_{k=0}^{n} a_k \cos(nx) + b_n \sin(nx)$$

To solve the heat equation for initial condition f(x), Fourier **assumed** that f could be written as a sum of sinusoids, and derived a formula:

$$f(x) = \sum_{k=0}^{\infty} a_k \cos(kx) + b_k \sin(kx)$$

$$a_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\cos(kx)dx \qquad b_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\sin(kx)dx$$

That is: **if f(x) is a limit of a sinusoidal series**, then the coefficients of that series are given by:

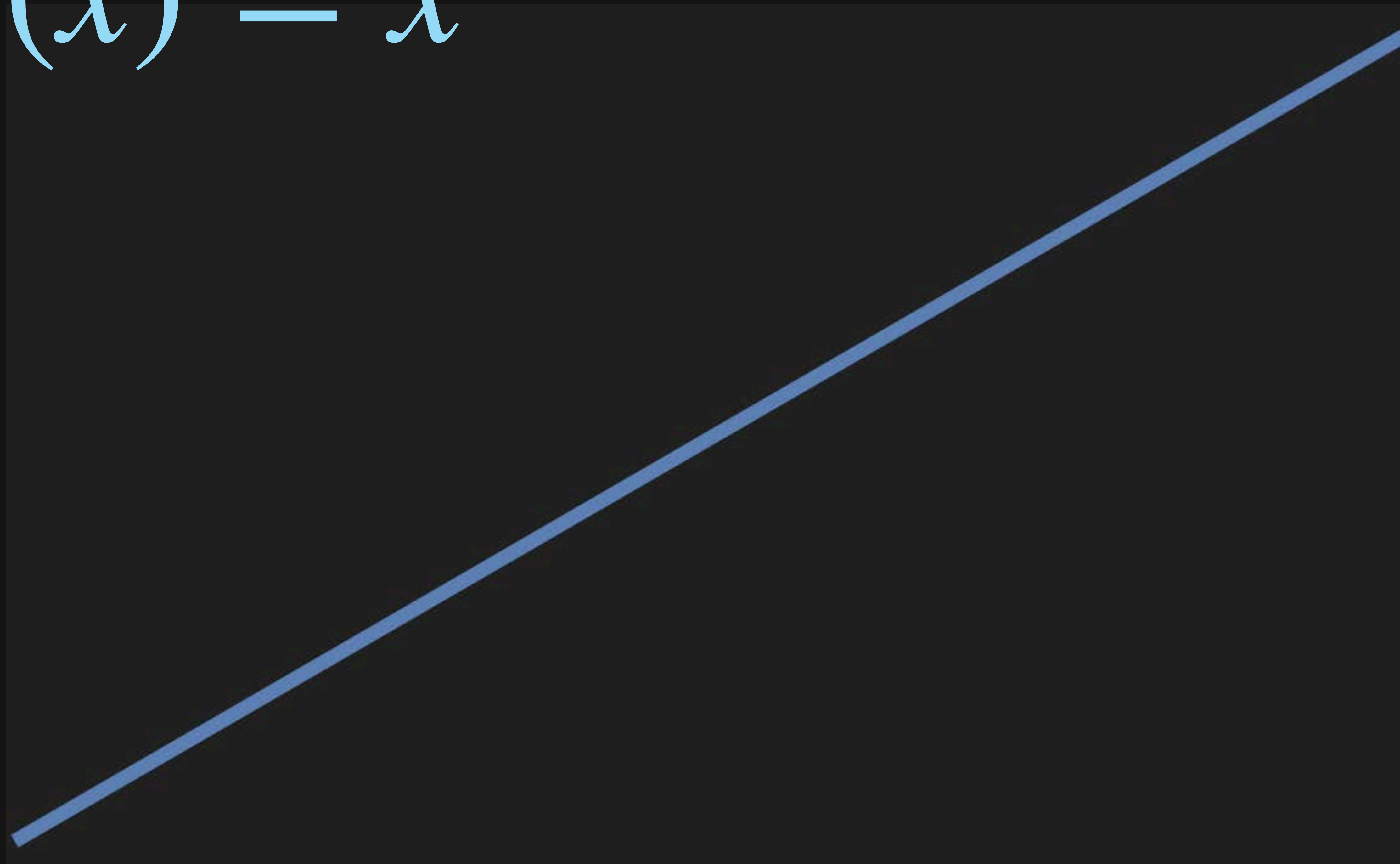
$$f(x) = \sum_{k=0}^{n} a_k \cos(nx) + b_n \sin(nx)$$

$$a_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx$$

$$b_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx$$

Experiment time: take some functions and plug them into this formula, see what you get:

$$f(x) = x$$



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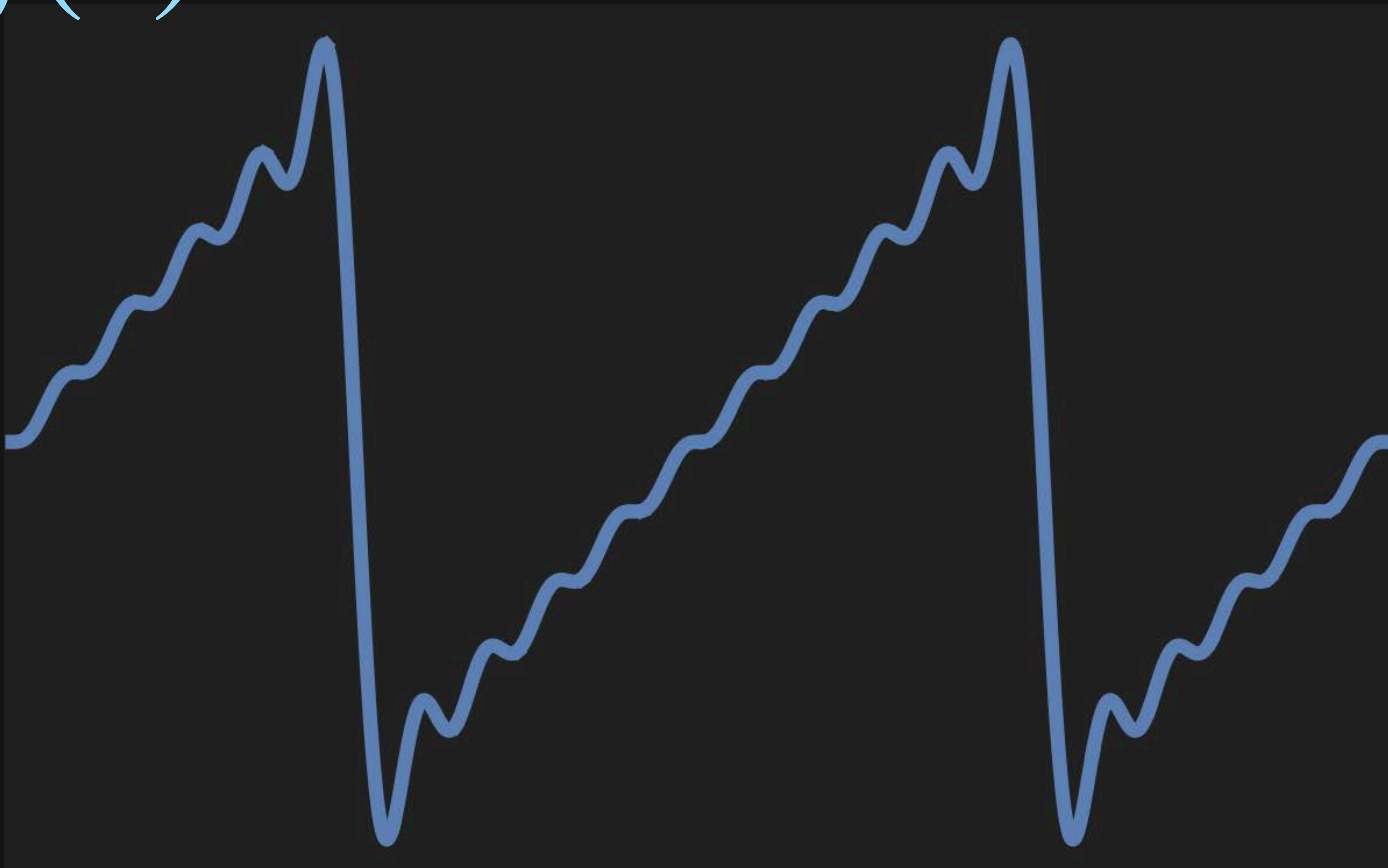
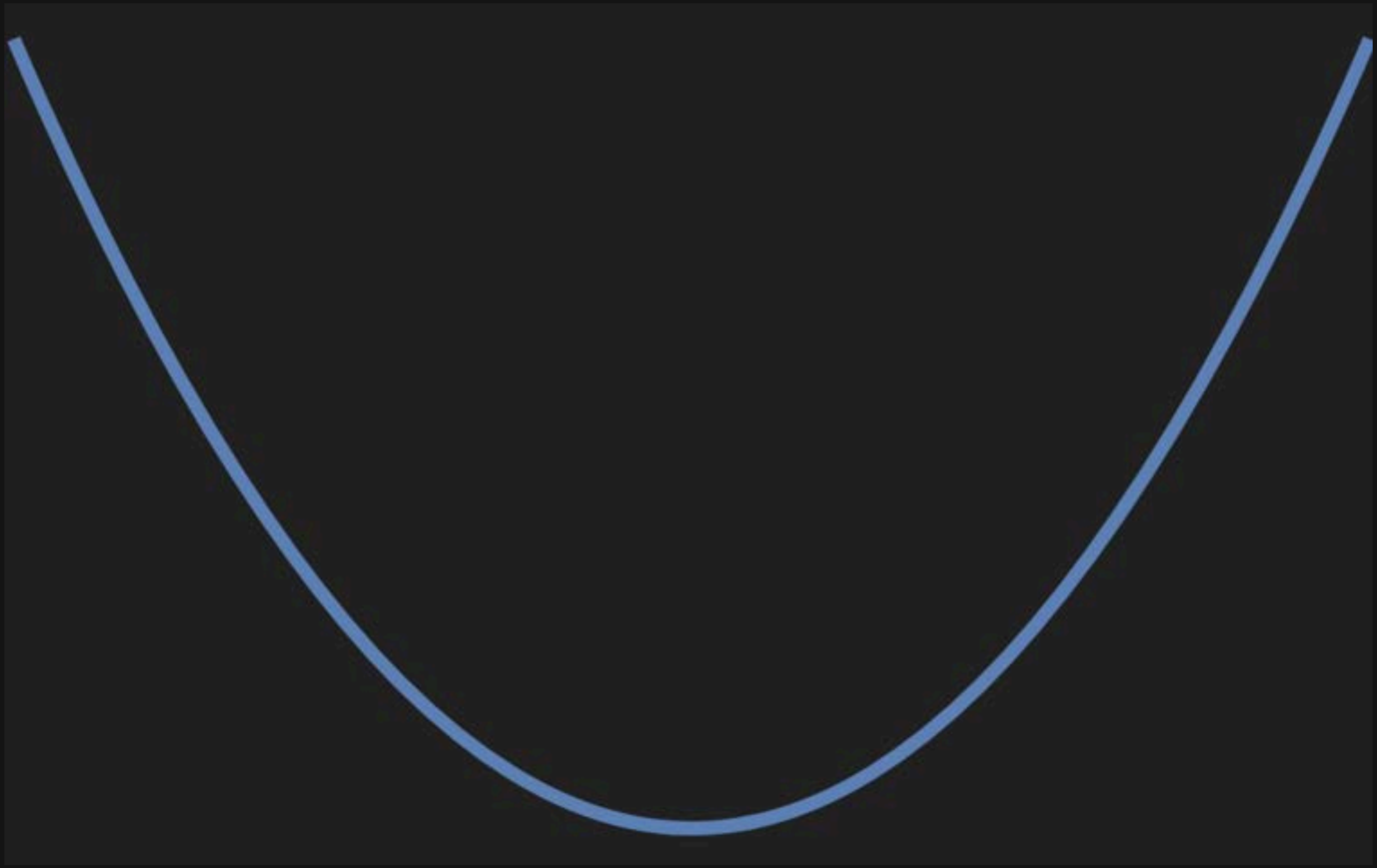


$$f(x) = x^2$$



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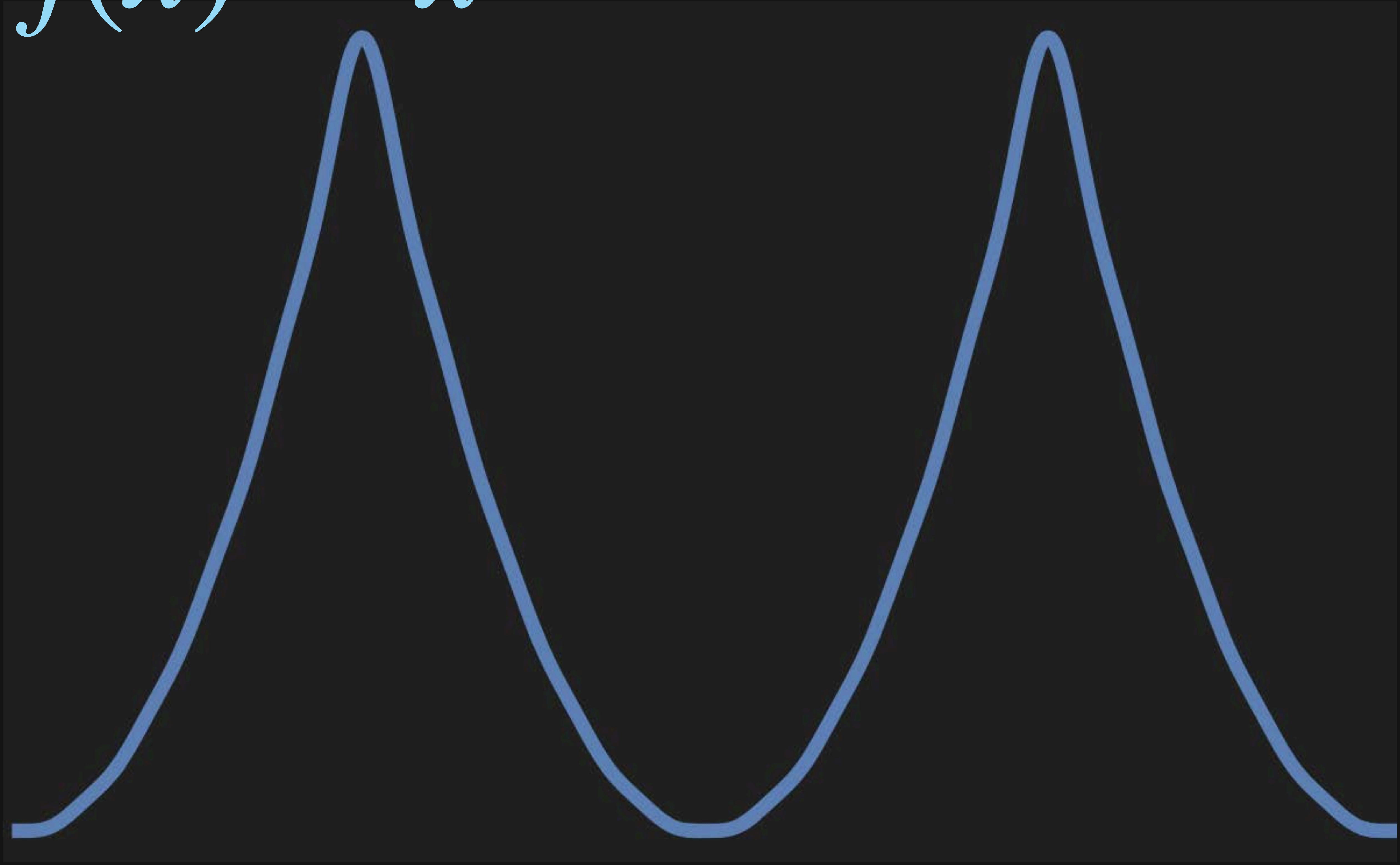
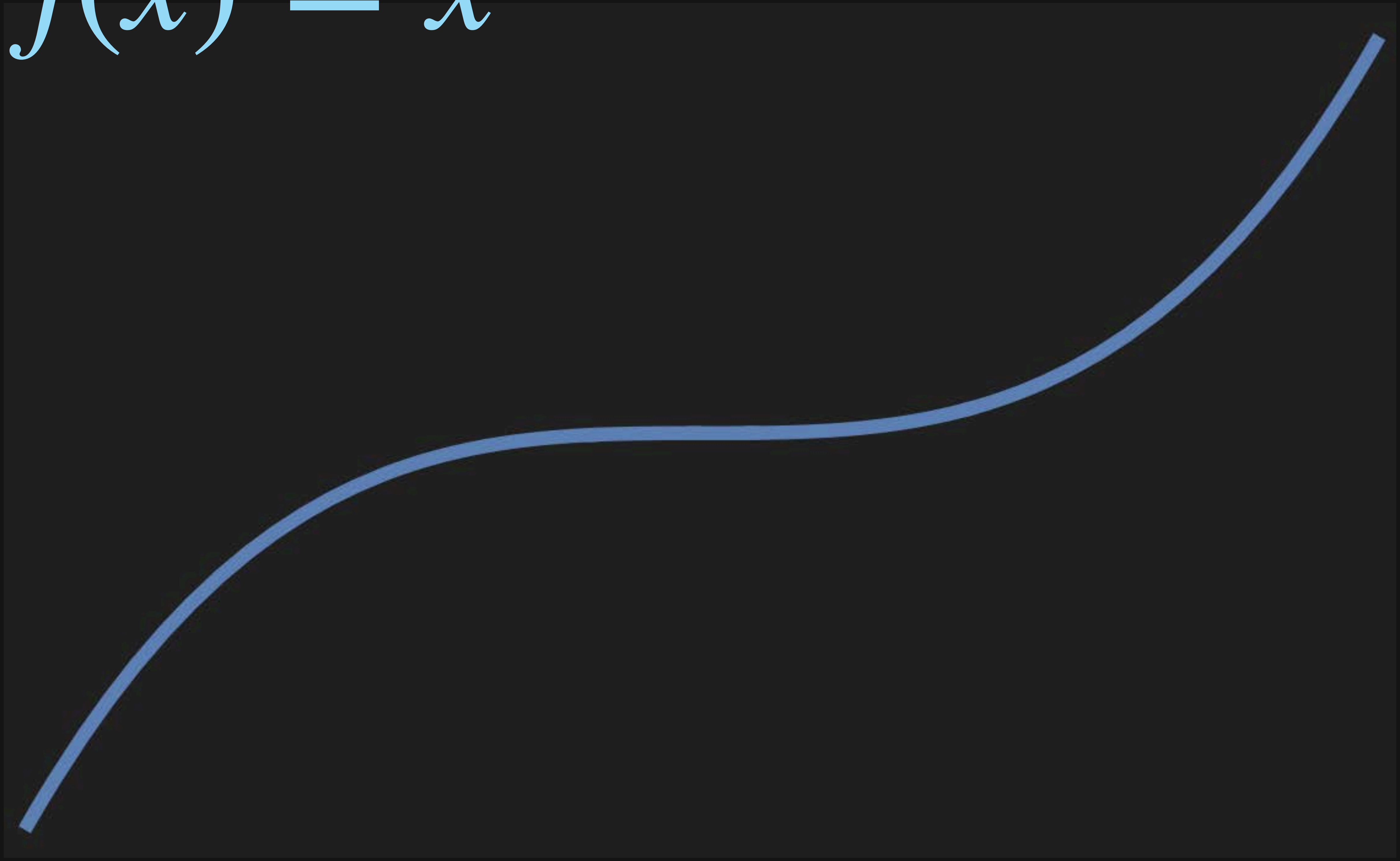


$$f(x) = x^3$$



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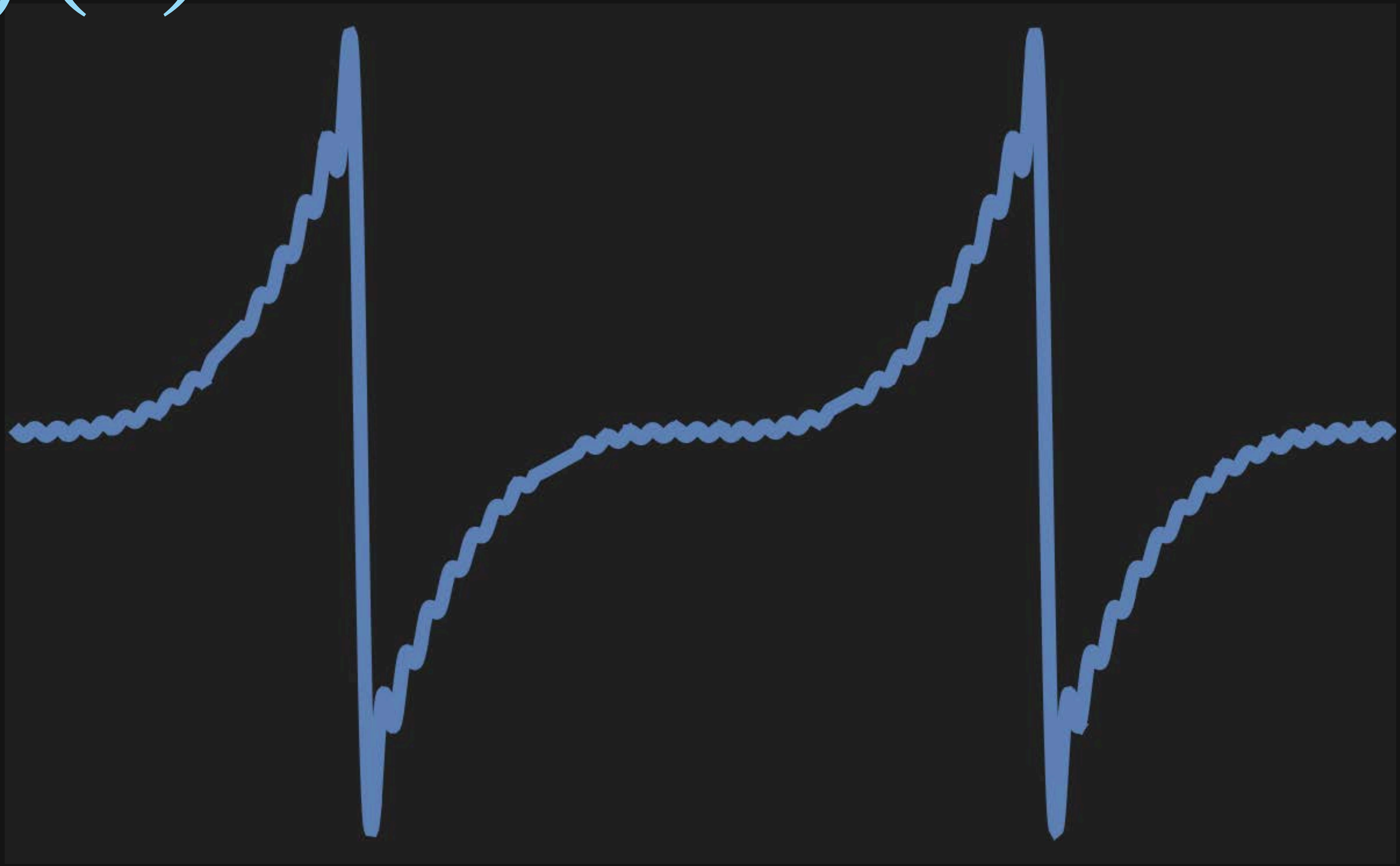


Results:

The formula takes f(x), and creates a *periodic extension* of its values on $(-\pi, \pi)$

Seems to work fine even when this extension is discontinuous!

If its continuous, the partial sums look smoother and less jagged.

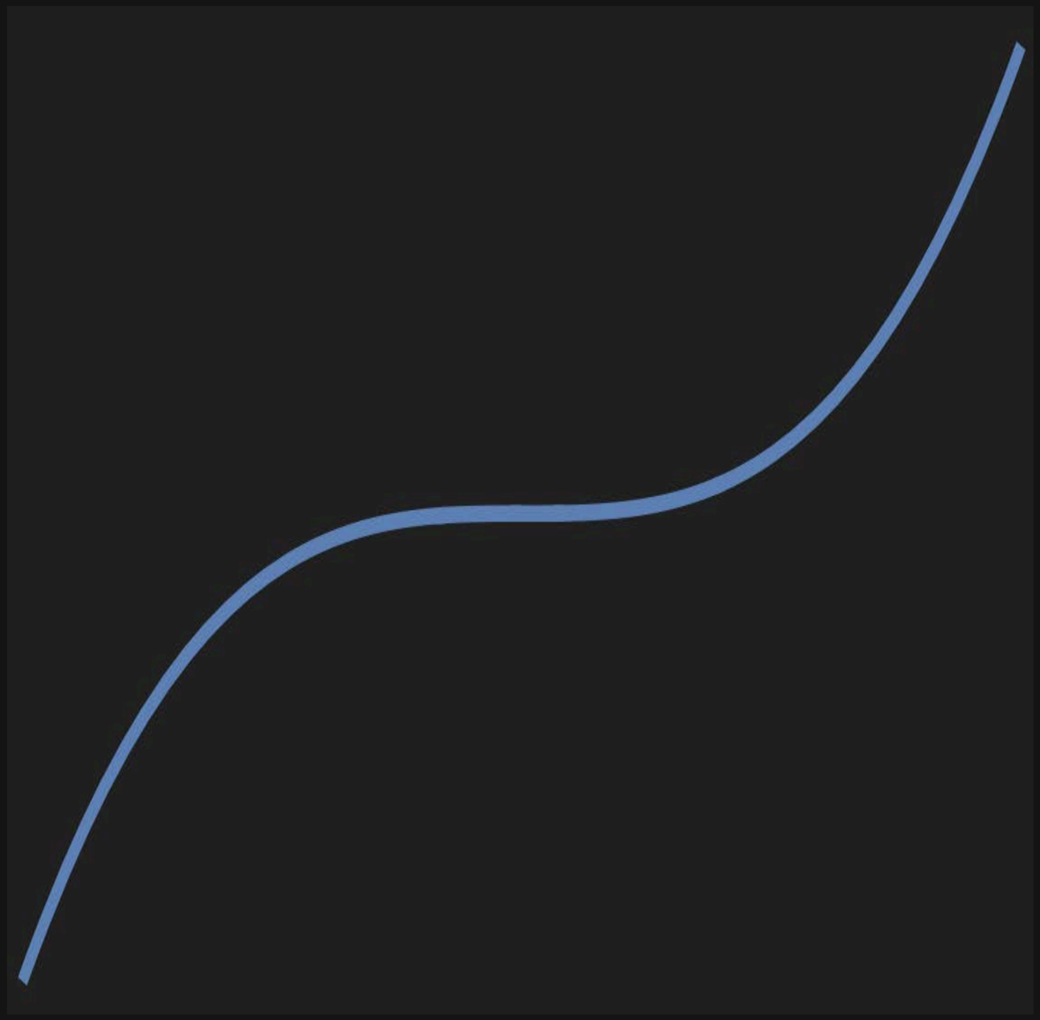
Setting things up more formally:

Given a function f(x), define the *periodic extension* $\bar{f}(x)$ *as*

$$\bar{f}(x) = f(x \mod 2\pi)$$

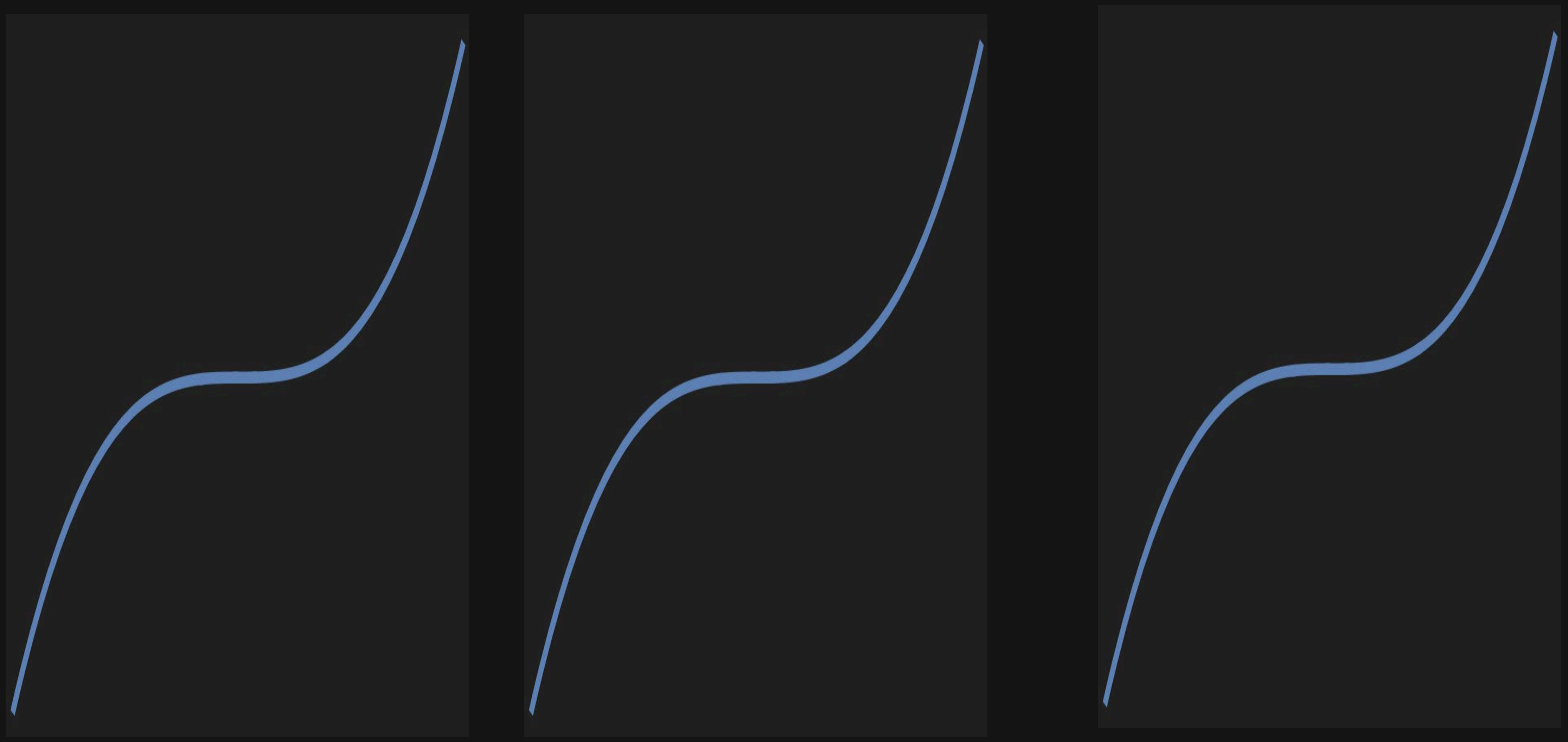
With $x \mod 2\pi \in (-\pi, \pi]$

Setting things up more formally:



$$f(x) = x^3$$

Setting things up more formally:



$$\bar{f}(x) = f(x \mod 2\pi)$$

Setting things up more formally:

Given a function f(x), define the *fourier expansion* of f(x) to be

$$\tilde{f}(x) = \sum_{k=0}^{\infty} a_k \cos(nx) + b_n \sin(nx)$$

The Main Question

For which f(x) does

$$\bar{f}(x) = \tilde{f}(x)?$$

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$$\bar{f}(x) = \tilde{f}(x)?$$

Step 1: Look at the convergence of $f_n \to \tilde{f}$

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This cannot always be uniform convergence:

Each partial sum is continuous (its a finite sum of continuous functions). But we have seen discontinuous limits.

The Pointwise Question:

For which f(x) does

$$\bar{f}(x) = \tilde{f}(x)?$$

A first result:

If f(x) is has a jump discontinuity at x=a and $\tilde{f}$ converges at x=a, then

$$\tilde{f}(a) = \frac{1}{2}\left(\lim_{x\to a^+} f(x) + \lim_{x\to a^-} f(x) \right)$$

A first result:

If f(x) is has a jump discontinuity at x=a and $\tilde{f}$ converges at x=a, then

$$\tilde{f}(a) = \frac{1}{2}\left(\lim_{x \to a^+} f(x) + \lim_{x \to a^-} f(x) \right)$$

Theorem from class

If f is integrable on [a,c] and on [c,b] then it is integrable on [a,b] and

$$\int_a^c f dx + \int_c^b f dx = \int_a^b f dx$$

We used this to prove that piecewise functions are integrable, and their value does not depend on the value at c.

But this leads to a kind of strange situation:

$$f(x) = \begin{cases} -1 & x < 0 \\ 0 & x = 0 \\ 1 & x > 0 \end{cases}$$

$$g(x) = \begin{cases} -1 & x < 0 \\ 1732 & x = 0 \\ 1 & x > 0 \end{cases}$$

But this leads to a kind of strange situation:

The fourier coefficients ak, bk are defined by *integrals: so they are the same for f and g!*

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Thus $\tilde{f}(x) = \tilde{g}(x)$ even though $f \neq g$.

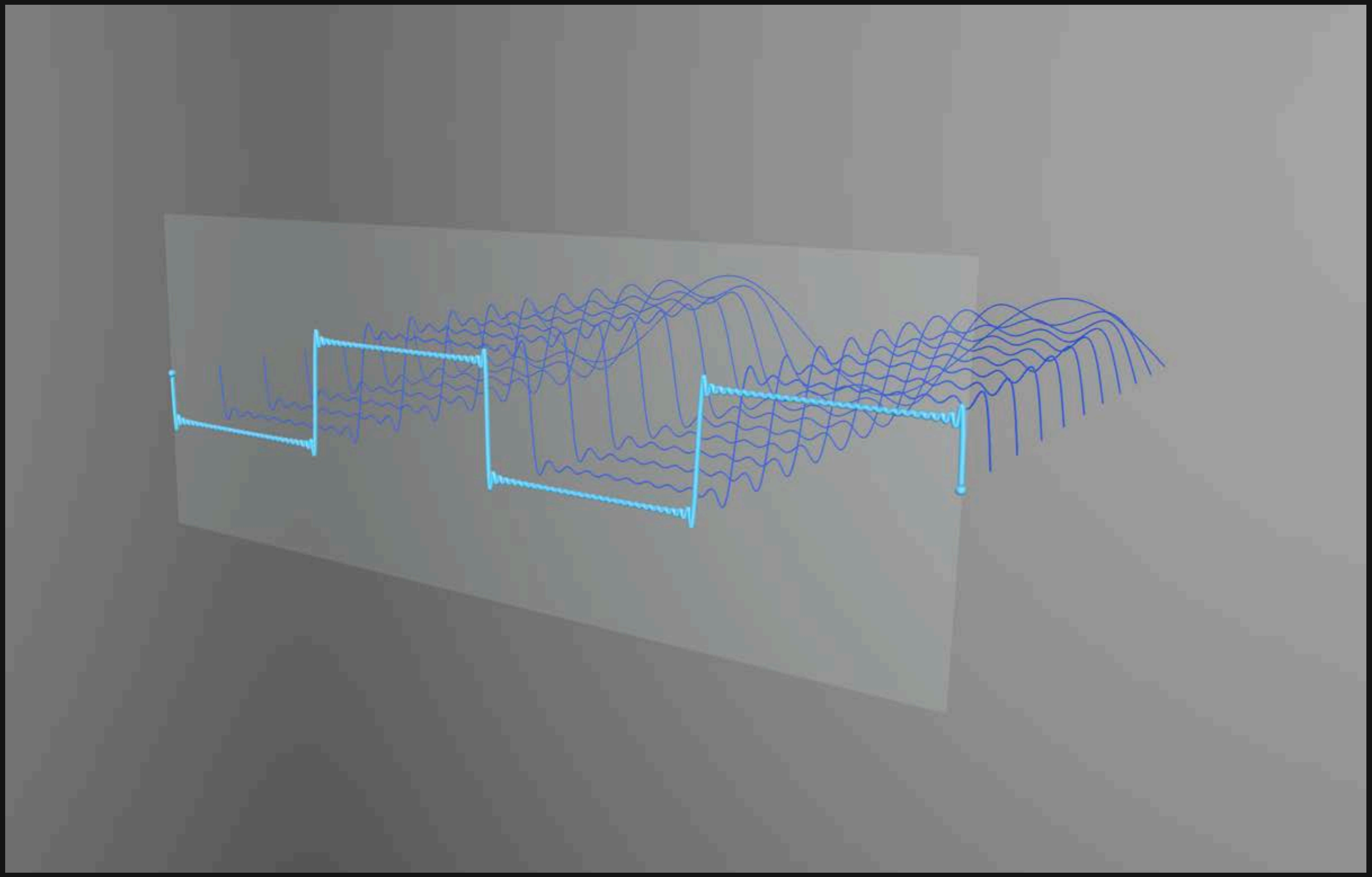
But this leads to a kind of strange situation:

The fourier coefficients ak, bk are defined by *integrals: so they are the same for f and g!*

Thus $\tilde{f}(x) = \tilde{g}(x)$ even though $f \neq g$.

So these series don't converge uniformly OR pointwise…

Still its “obvious” that in some sense, the series does to the right sort of thing! It just misses the value at one point



How do we formalize this?

BEYOND 115 PART 1:

ABSTRACTING THE IDEA OF CONVERGENCE

Recall the notion of convergence for sequences of real numbers:

$$\forall \epsilon \, \exists N \, \forall n > N \, |s_n - L| < \epsilon$$

This relies on the absolute value being a means of measuring the distance between two numbers.

Can we think about convergence for functions as measuring some sort of **distance between functions**?

Uniform convergence:

$$\forall \epsilon \, \exists N \, \forall n > N \, \forall x \in S \, |f_n(x) - f(x)| < \epsilon$$

Can we think of this as a "distance"?

Definition: Let f, g be two functions defined on S. Then the *sup-distance* between them is:

$$d_{\sup}(f, g) = \sup\{\, |f(x) - g(x)| : x \in X\}$$

Uniform convergence:

$$\forall \epsilon \, \exists N \, \forall n > N \, \forall x \in S \, |f_n(x) - f(x)| < \epsilon$$

This just says $d_{\sup}(f_n, f) < \epsilon$

Uniform Convergence Rephrased:

A sequence f_n converges uniformly to f on a set S if

$$\forall \epsilon \, \exists N \, \forall n > N \quad d_{\sup}(f_n, f) < \epsilon$$

Idea: finding different notions of 'distance' between functions can lead to different notions of convergence.

BEYOND 115 PART 2:

FUNCTION SPACES

The Big Leap in Abstraction:

Just like we did not study particular numbers, but rather studied the *set of reals,* we should not study individual functions, but rather *function spaces:*

The Big Leap in Abstraction:

Just like we did not study particular numbers, but rather studied the *set of reals*, we should not study individual functions, but rather *function spaces:*

Example Function Spaces:

$$C^0(\mathbb{R}) = \{\text{continuous functions on } \mathbb{R}\}$$

$$C^1(\mathbb{R}) = \{\text{differentiable functions on } \mathbb{R}\}$$

$$L^\infty(\mathbb{R}) = \{\text{bounded functions on } \mathbb{R}\}$$

Analysis on a Function Space

Given a function space, we can basically start analysis over again, but at this “next level”

Define some notion of distance to take the place of the absolute value.

Use all the definitions from analysis, but replace numbers with functions and absolute values with this distance.

See what theorems are true!

Analysis on

$$L^\infty(\mathbb{R}) = \{\text{bounded functions on } \mathbb{R}\}$$

The correct notion of distance here is the sup-distance! (Called the L-infinty norm)

$$\|f - g\|_\infty = \sup\{\, |f(x) - g(x)| : x \in X\}$$

The theorems about convergence in L^∞ are just the theorems we proved about uniform convergence.

Other notions of distance:

To understand fourier series, we want a notion of convergence which ignores the value at single points:

An integral-based distance:

Say that two functions are close to one another, if the *area between their graphs* is small!

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$$d(f, g) = \int |f(x) - g(x)|\, dx$$

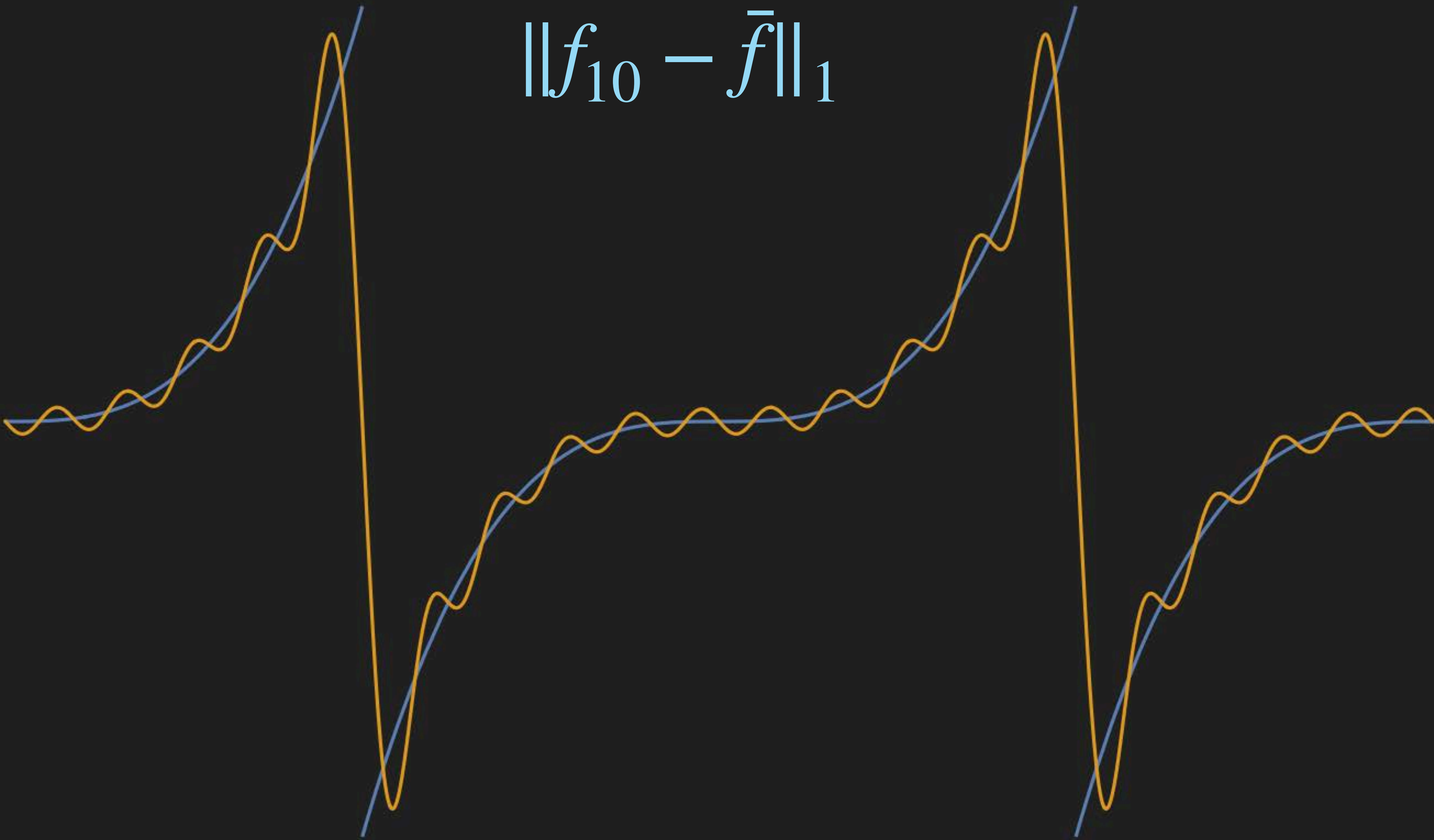
An integral-based distance:

Say that two functions are close to one another, if the *area between their graphs* is small!

$$\|f - g\|_1 = \int |f(x) - g(x)|\, dx$$

The space of integrable functions, together with this notion of distance, is called $L^1(\mathbb{R})$

$$\|f_{10} - \bar{f}\|_1$$



An integral-based distance:

Say that two functions are close to one another, if the *area between their graphs* is small!

$$\|f - g\|_2 = \sqrt{\int (f(x) - g(x))^2 dx}$$

The space of integrable functions, together with this notion of distance, is called $L^2(\mathbb{R})$

The Space $L^2([0,2\pi])$

The functions: all functions f: $[0,2\pi] \to \mathbb{R}$ which are square integrable: that is,

$$\int_0^{2\pi} f^2 dx < \infty$$

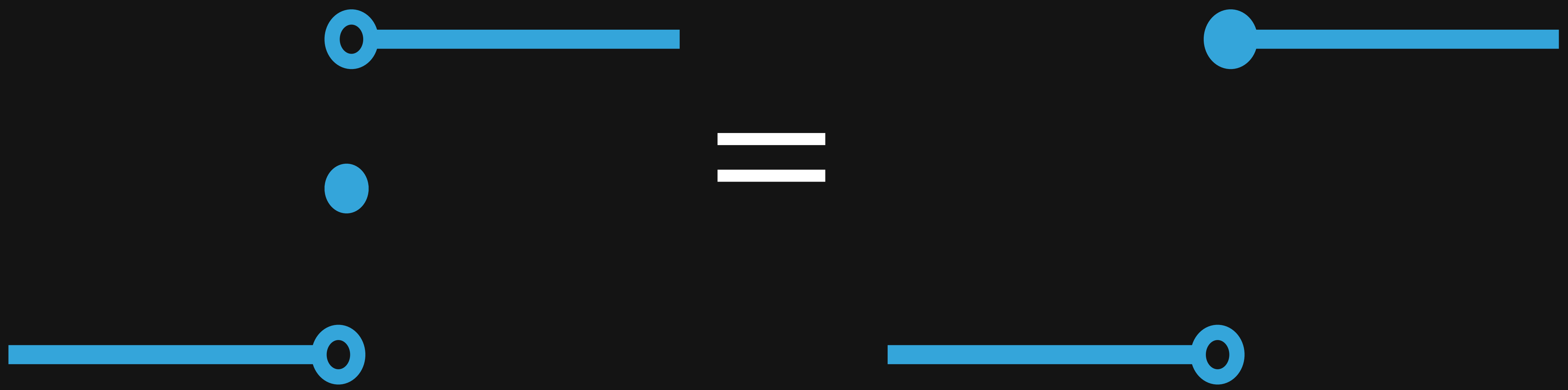
Distance: the L2 distance

$$\|f - g\|_2 = \sqrt{\int (f(x) - g(x))^2 dx}$$

The reason we like

The Space $L^2([0,2\pi])$

If two functions are equal at all but finitely many points - their L2 distance apart is equal to zero!



The reason we like

The Space $L^2([0,2\pi])$

If we work inside of L2, then the fact that these functions

$$f(x) = \begin{cases} -1 & x < 0 \\ 0 & x = 0 \\ 1 & x > 0 \end{cases} \qquad g(x) = \begin{cases} -1 & x < 0 \\ 1732 & x = 0 \\ 1 & x > 0 \end{cases}$$

Have the same fourier series is not surprising: because they are the same* L2 function!

*Theres a ton of details here....

The Space $L^2([0,2\pi])$

And the fundamental theorem for Fourier Analysis.

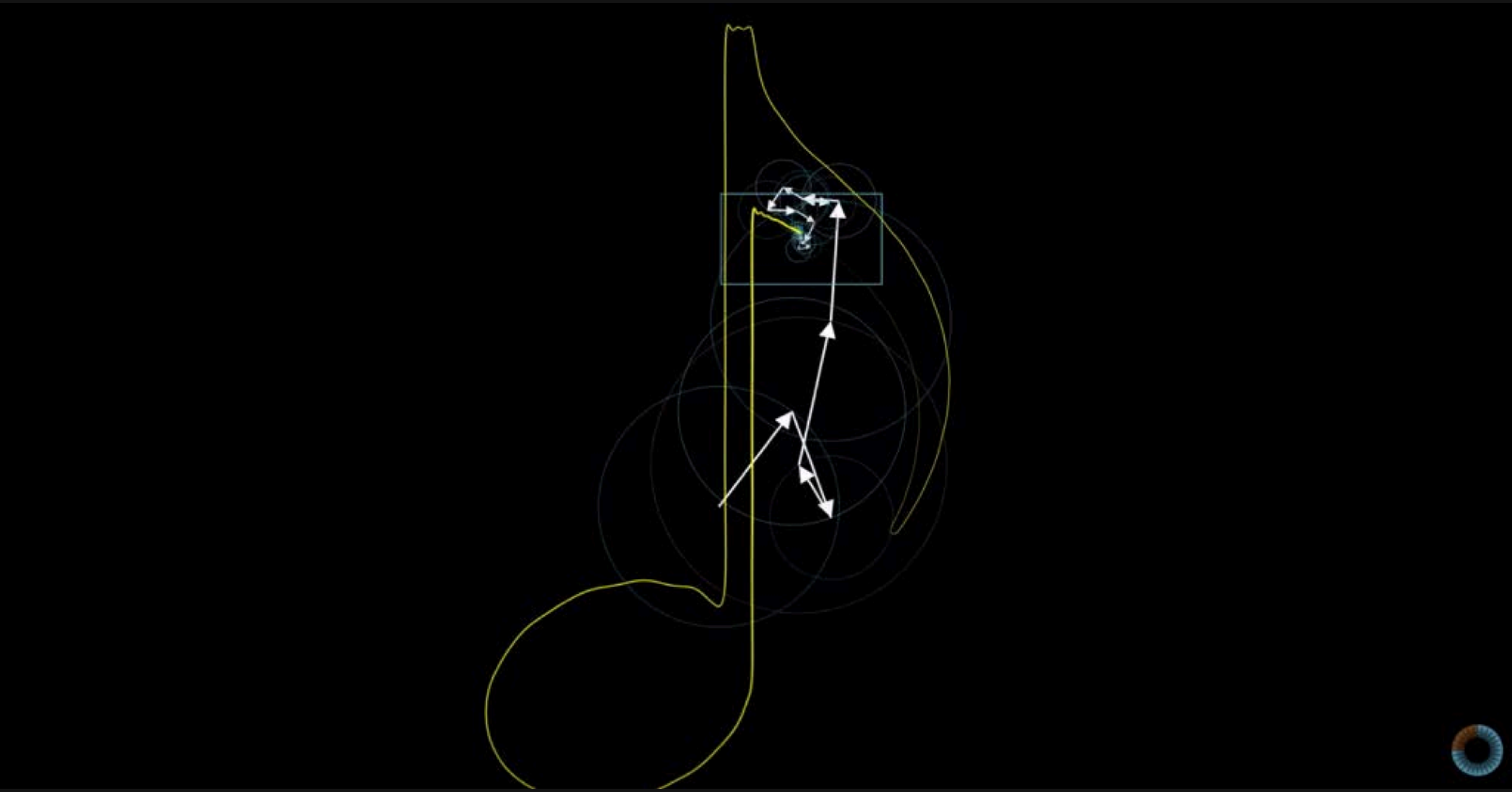
For every function in $L^2([0,2\pi])$ the Fourier series converges and $\bar{f} = \tilde{f}$, where this equality means these two functions are distance zero apart in the L2 norm.

We've done it!

Finally made all the parts of our original question precise enough to prove it.

Every square-integrable function can be made by epicycles.

The heat equation has a solution for every square-integrable initial condition.



https://youtu.be/-qgreAUpPwM

BEYOND 115 PART 3:

TO FUNCTION SPACES AND BEYOND!

Working with L^2

This space is very helpful for working with convergence when you don't care about the particular values at any given point - as we've seen!

But this space has some problems we need to fix:

How do we take a limit of L2 functions?

How do we take a derivative of L2 functions?

Limits in L^2

Remember - we are doing “Analysis level 2” now. What did we do when we encountered this problem before?

Not all sequences of rationals had a limit. So - we defined the reals to be exactly “rationals and limits of sequences of rationals” to fix this.

Limits in L^2

Remember - we are doing "Analysis level 2" now. What did we do when we encountered this problem before?

Not all sequences of rationals had a limit. So - we defined the reals to be exactly "rationals and limits of sequences of rationals" to fix this.

Similarly, we can define new things to serve as the limits of L2 functions.

Limits in L^2

We can define new things to serve as the limits of L2 functions.

Just like the new numbers (irrationals) were no longer rational, the new objects here are no longer functions!

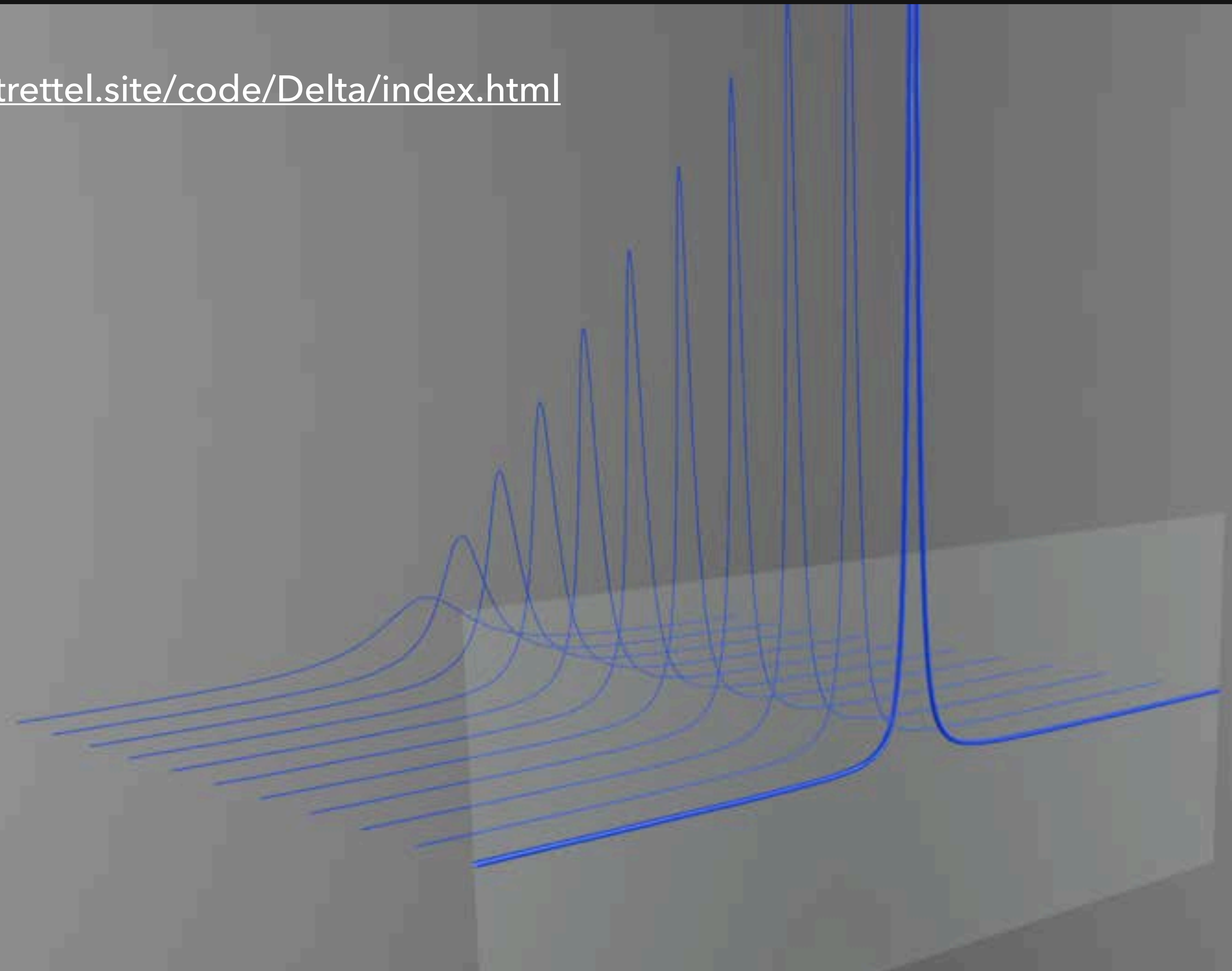
They are something called *distributions:*

"Definition" A distribution is the limit of a sequence of L2 functions.

Limits in L^2

The most famous example is the Dirac δ distribution:

www.stevejtrettel.site/code/Delta/index.html



Derivatives in L^2

There are plenty of differentiable functions in L2, but also non-differentiable ones.

How can we extend the idea of a "derivative" to apply even to a non-differentiable function?

Derivatives in L^2

There are plenty of differentiable functions in L2, but also non-differentiable ones.

How can we extend the idea of a “derivative” to apply even to a non-differentiable function?

Think back to “Analysis Level 1” - did we encounter any analogous situations?

Derivatives in L^2

When we only knew how to define the exponential on rationals, how did we extend it to the irrationals?

Derivatives in L^2

When we only knew how to define the exponential on rationals, how did we extend it to the irrationals?

$$e^a = \lim_{n\to\infty} e^{a_n}$$

Derivatives in L^2

When we only knew how to define the exponential on rationals, how did we extend it to the irrationals?

$$e^a = \lim_{n \to \infty} e^{a_n}$$

Can we define the derivative like this?

Derivatives in L^2

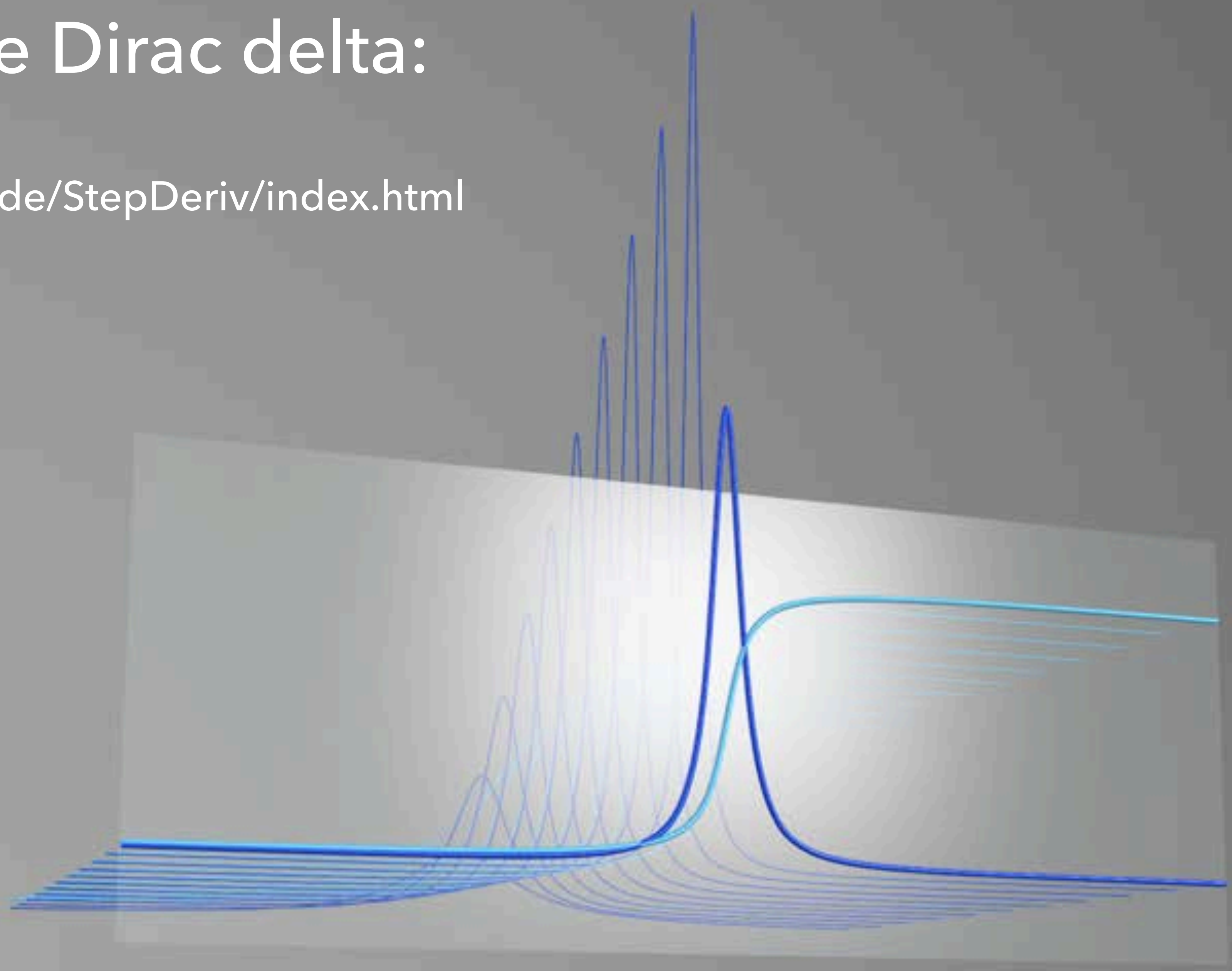
Theorem: Differentiable functions are dense in L2!

Definition: the *distributional derivative* of a non-differentiable function f is the *limit of derivatives of a sequence converging to f.*

Derivatives in L^2

The most common example: the derivative of a step function is the Dirac delta:

www.stevejtrettel.site/code/StepDeriv/index.html



BEYOND 115 PART 4:

FUNCTIONAL ANALYSIS

The coolest fact in analysis*

Calculus is infinite dimensional linear algebra!

*of course, this is just my opinion. But its true!!

The coolest fact in analysis*

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Vectors are things you can add and multiply

$$av + bw$$

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The coolest fact in analysis*

Calculus is infinite dimensional linear algebra!

Functions are things you can add and multiply

$$af(x) + bg(x)$$

***of course, this is just my opinion. But its true!!**

The coolest fact in analysis*

Calculus is infinite dimensional linear algebra!

Functions are vectors!

$$af(x) + bg(x)$$

***of course, this is just my opinion. But its true!!**

The coolest fact in analysis*

Calculus is infinite dimensional linear algebra!

A vector space is……a space of vectors

A function space is a vector space, where all the vectors are functions!

***of course, this is just my opinion. But its true!!**

The coolest fact in analysis*

Calculus is infinite dimensional linear algebra!

A linear transformation on a vector space V is a map

$$A : V \to V$$

Such that

$$A(cv + w) = cAv + Aw$$

***of course, this is just my opinion. But its true!!**

The coolest fact in analysis*

Calculus is infinite dimensional linear algebra!

A linear operator on a function space L is a map

$$F\colon L^2 \to L^2$$

Such that

$$F(cf+g) = cF(f) + F(g)$$

***of course, this is just my opinion. But its true!!**

The coolest fact in analysis*

Calculus is infinite dimensional linear algebra!

The derivative is a linear operator!

$$\frac{d}{dx} : L^2 \to L^2$$

$$(cf + g)' = cf' + g'$$

***of course, this is just my opinion. But its true!!**

The coolest fact in analysis*

Calculus is infinite dimensional linear algebra!

So is the integral

$$\int : L^2 \to L^2$$

$$\int (cf + g)dx = c\int f dx + \int g dx$$

***of course, this is just my opinion. But its true!!**

The coolest fact in analysis*

Calculus is infinite dimensional linear algebra!

A basis for a vector space is a set $\{v_1, \dots\}$ such that every vector is a linear combination

$$v = \sum_i a_i v_i$$

***of course, this is just my opinion. But its true!!**

The coolest fact in analysis*

Calculus is infinite dimensional linear algebra!

Our theorem earlier said that every function in L2 can be written as a Fourier Series.

$$f = \sum_i a_i \cos(nx) + b_i \sin(nx)$$

***of course, this is just my opinion. But its true!!**

The coolest fact in analysis*

Calculus is infinite dimensional linear algebra!

Our theorem earlier said that every function in L2 can be written as a Fourier Series.

This means the trigonometric functions are a basis for L2!

***of course, this is just my opinion. But its true!!**

The coolest fact in analysis*

Calculus is infinite dimensional linear algebra!

This basis is particularly well - suited to working with the second derivative:

$$\frac{d}{dx}\sin(nx) = -n^2\sin(x)$$

***of course, this is just my opinion. But its true!!**

The coolest fact in analysis*

Calculus is infinite dimensional linear algebra!

A vector v is an eigenvector of a linear transformation A if

$$Av = \lambda v$$

***of course, this is just my opinion. But its true!!**

The coolest fact in analysis*

Calculus is infinite dimensional linear algebra!

A function f is an eigenfunction of a linear operator D if

$$D(f) = \lambda f$$

***of course, this is just my opinion. But its true!!**

The coolest fact in analysis*

Calculus is infinite dimensional linear algebra!

Trigonometric functions are the eigenfunctions of the second derivative operator.

Fourier series are writing a function in the basis of trigonometric functions.

***of course, this is just my opinion. But its true!!**

The coolest fact in analysis*

Calculus is infinite dimensional linear algebra!

How do we write a vector in a basis?

$$v = \sum_i a_i v_i$$

$$a_i = \frac{v \cdot v_i}{\|v_i\|^2}$$

***of course, this is just my opinion. But its true!!**

The coolest fact in analysis*

Calculus is infinite dimensional linear algebra!

How do we write a function in the trigonometric basis?

$$\tilde{f}(x) = \sum_{k=0}^{\infty} a_k \cos(nx)$$

$$a_i = \frac{f \cdot \cos(nx)}{\| \cos(nx) \|^2}$$

***of course, this is just my opinion. But its true!!**

The coolest fact in analysis*

Calculus is infinite dimensional linear algebra!

How do we write a function in the trigonometric basis?

$$\tilde{f}(x) = \sum_{k=0}^{\infty} a_k \cos(nx)$$

$$a_i = \frac{\int_{-\pi}^{\pi} f(x)\cos(nx)}{\|\cos(nx)\|_2^2}$$

***of course, this is just my opinion. But its true!!**

The coolest fact in analysis*

Calculus is infinite dimensional linear algebra!

How do we write a function in the trigonometric basis?

$$\tilde{f}(x) = \sum_{k=0}^{\infty} a_k \cos(nx)$$

$$a_i = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx$$

***of course, this is just my opinion. But its true!!**

BEYOND 115 PART 5:

FUNCTIONAL ANALYSIS IN HIGHER DIMENSIONS

Epicycles were the first hint that the trigonometric functions formed a basis for function spaces!

Fourier then used this fact (without knowing the bigger picture) to solve the heat equation in an ingenious way:

> Since the equation involves the second derivative, it becomes trivial to solve if you write functions in a basis of eigenfunctions for the second derivative!

http://www.stevejtrettel.site/code/WaveEqn/index.html

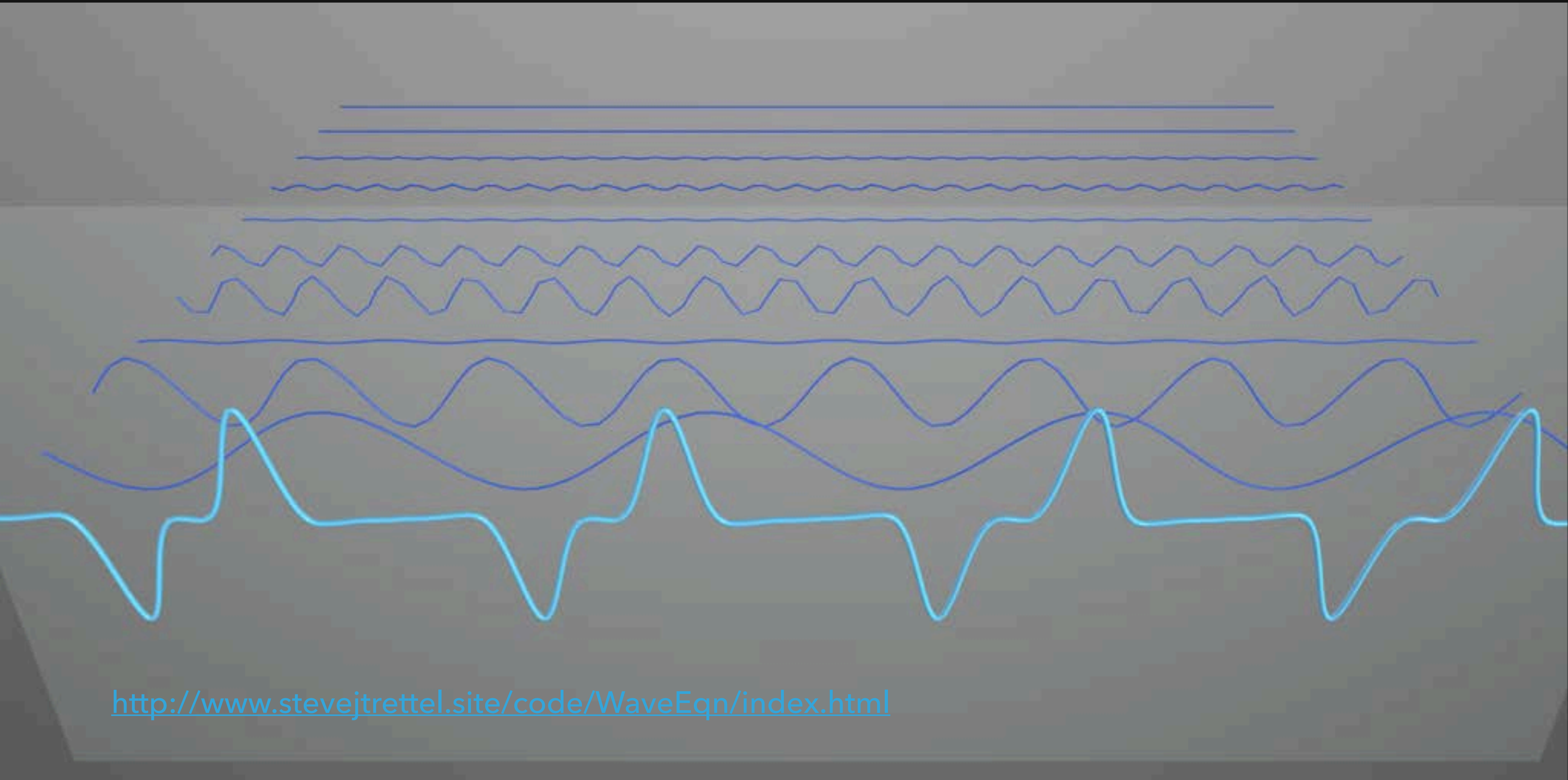
Epicycles were the first hint that the trigonometric functions formed a basis for function spaces!

This process works much more generally!

Given a differential equation $f^{(n)} = Df$ on a function space L, can solve it by finding a basis of eigenfunctions for D in L!

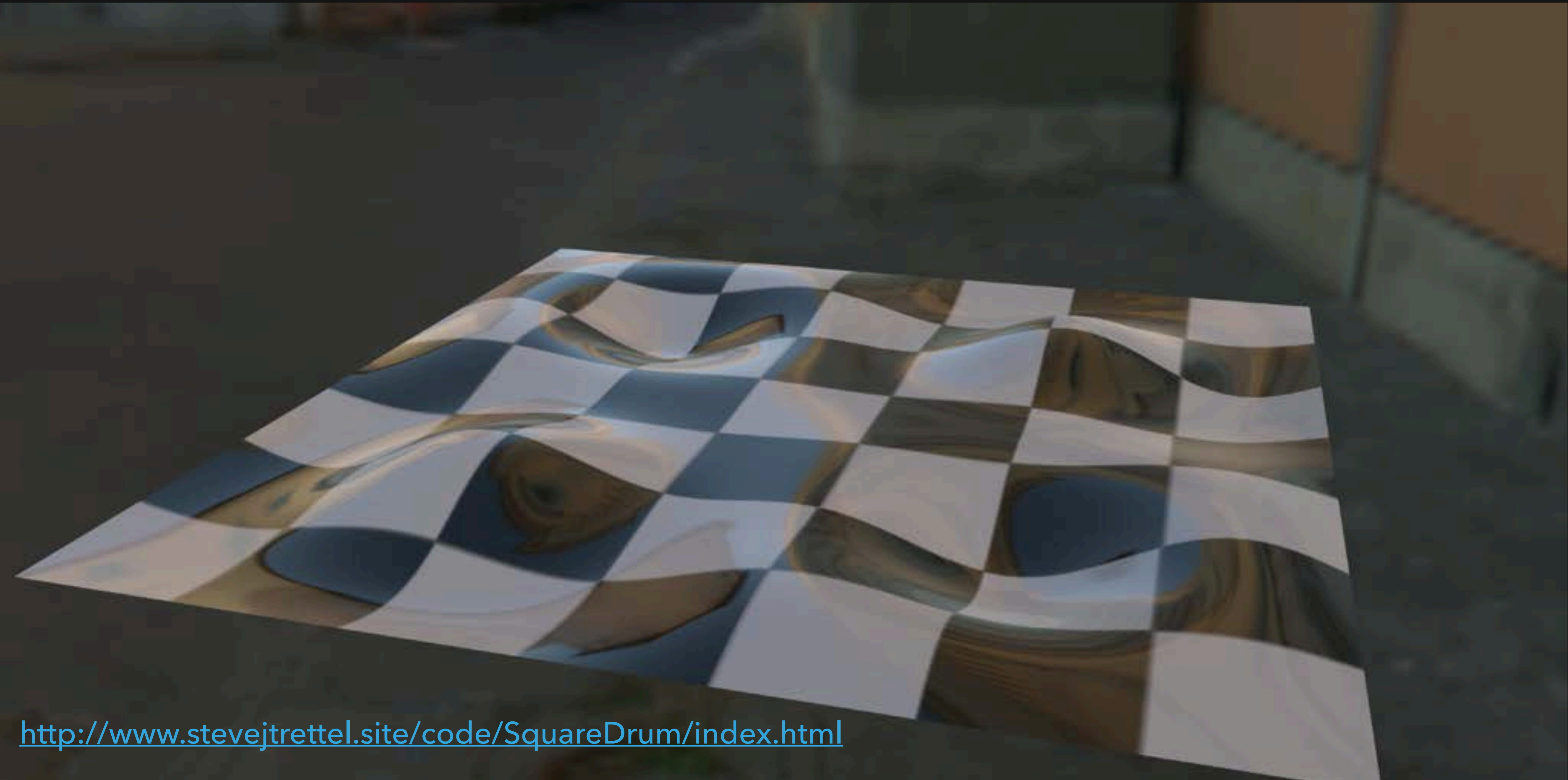
http://www.stevejtrettel.site/code/WaveEqn/index.html

The Wave Equation and $L^2([a,b])$



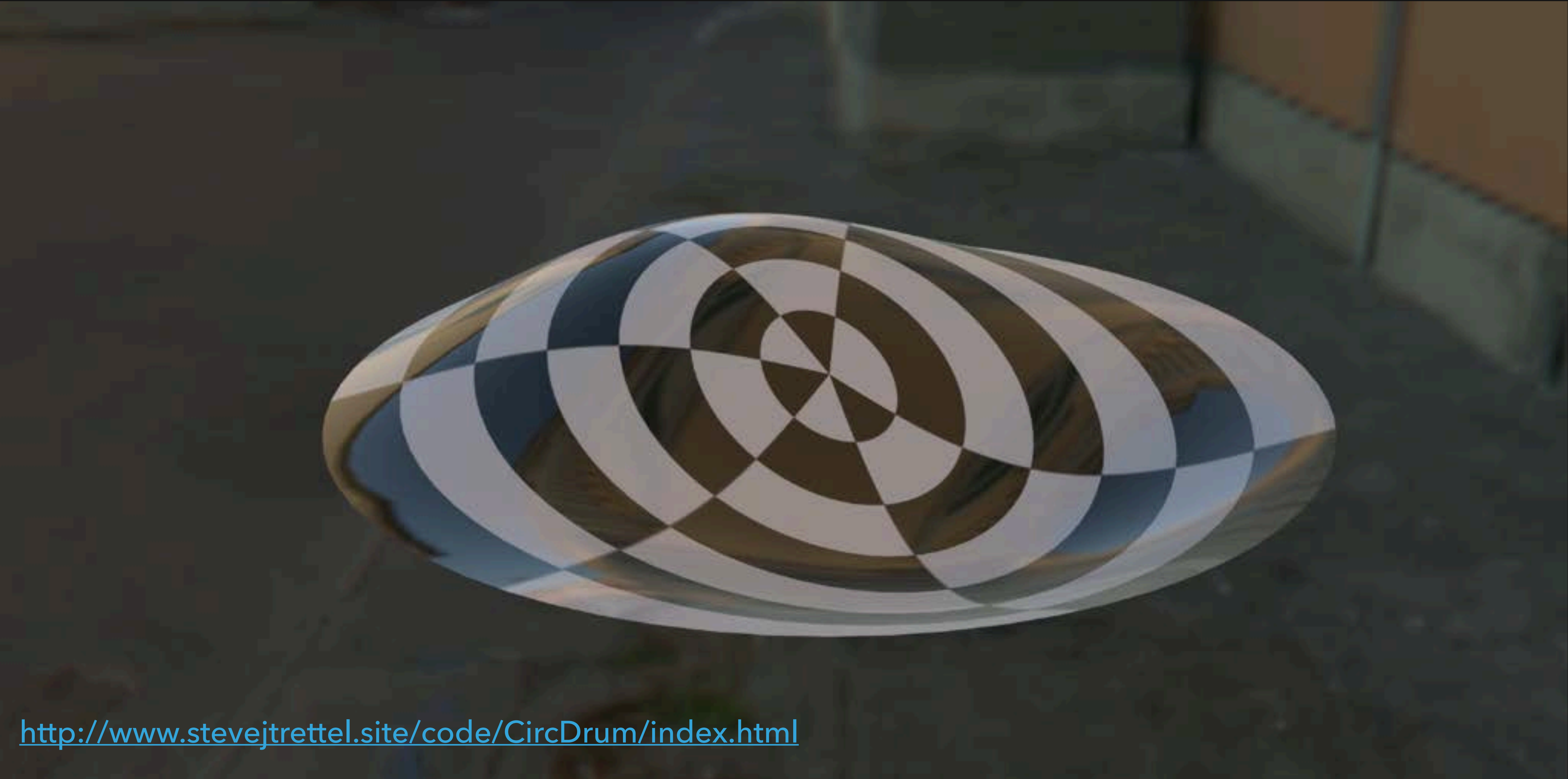
http://www.stevejtrettel.site/code/WaveEqn/index.html

The Wave Equation and L^2(square)



http://www.stevejtrettel.site/code/SquareDrum/index.html

The Wave Equation and L^2(disk)



http://www.stevejtrettel.site/code/CircDrum/index.html

Quantum Mechanics

$$L^2(\mathbb{R}^3)$$