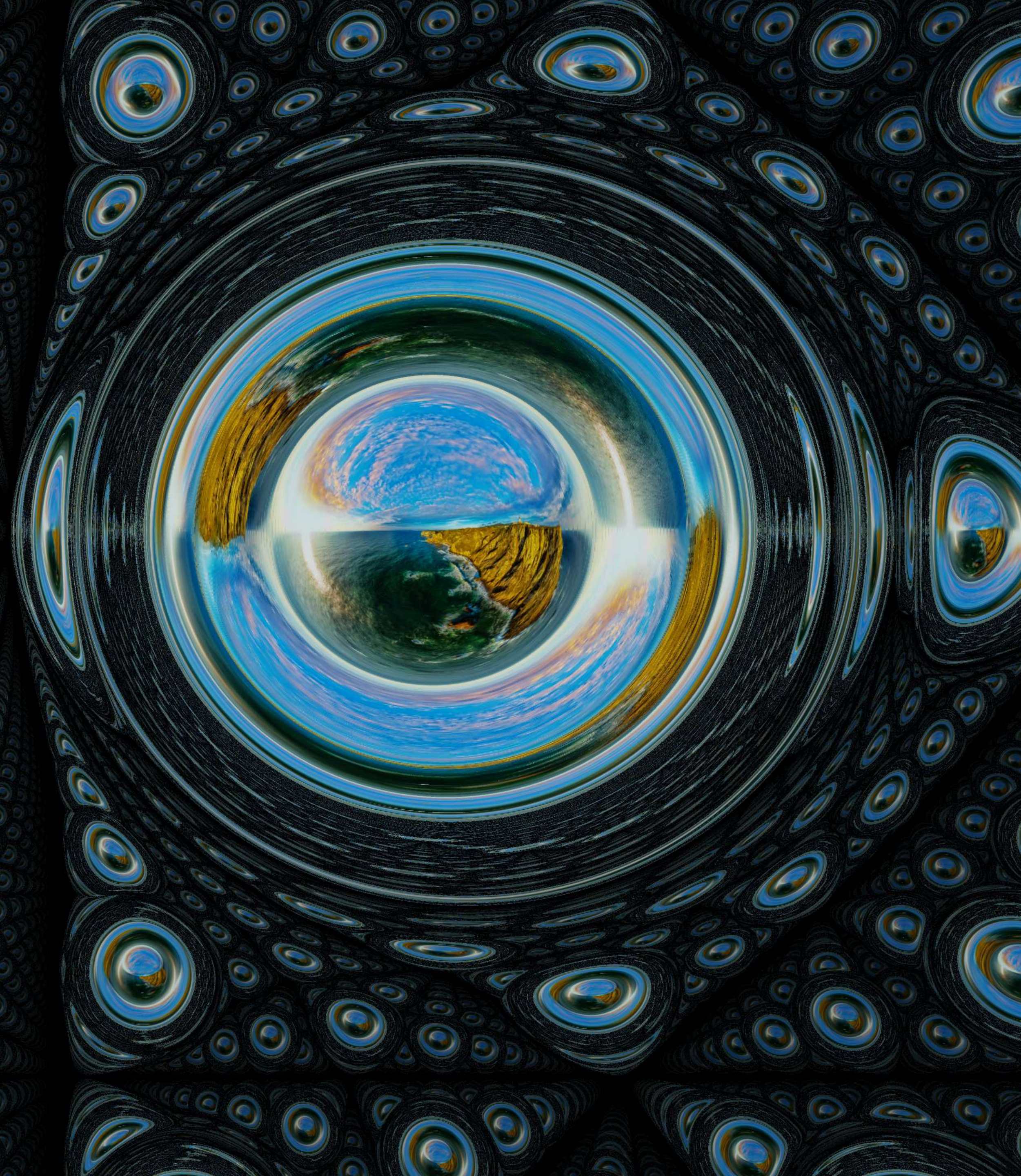


What do 3-Manifolds Look Like?

Steve Trettel

*Joint work with Remi Coulon, Sabetta
Matsumoto, and Henry Segerman*



I Raytracing in Manifolds

a Topology & Vision

b Geometry and Vision

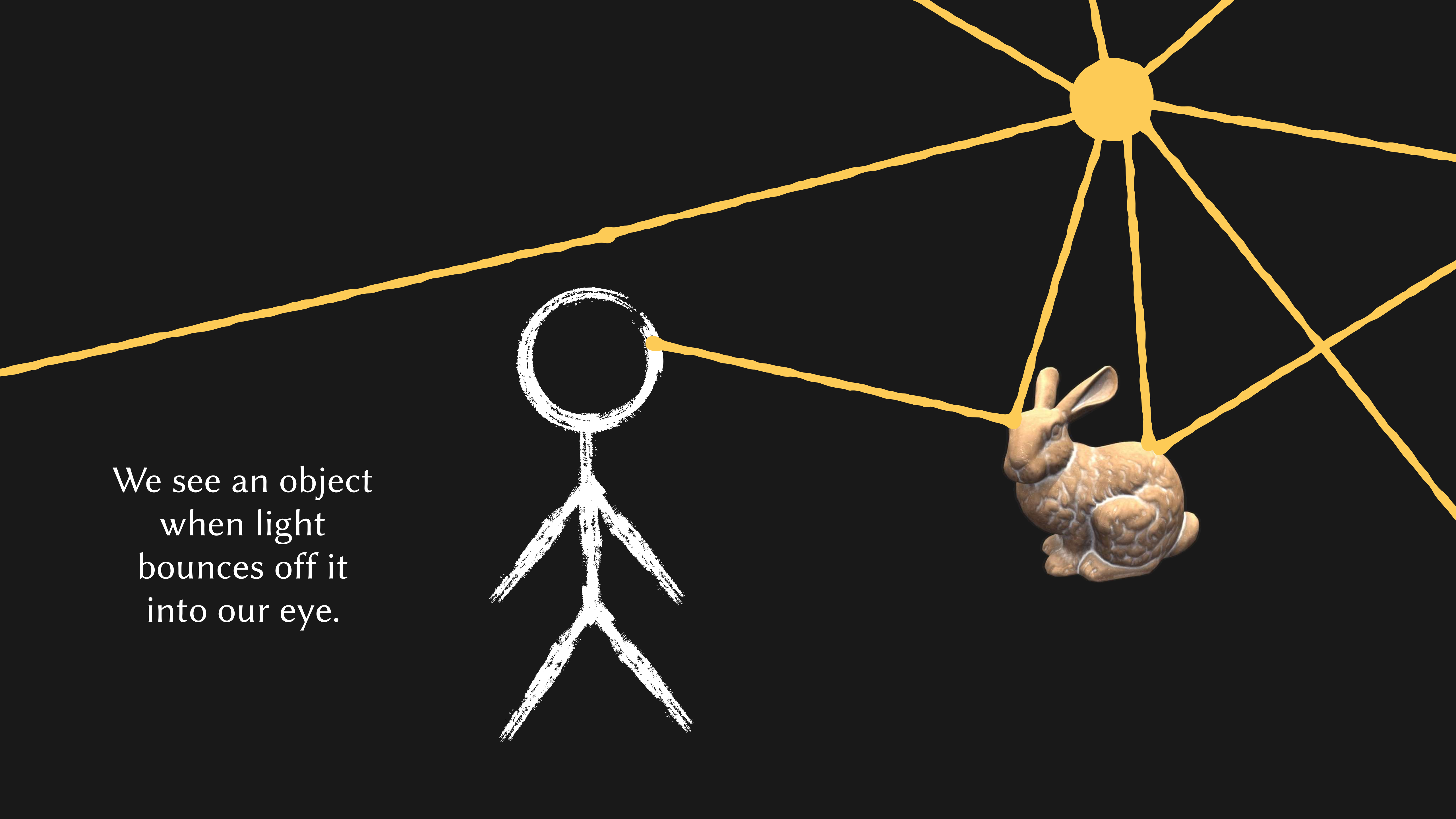
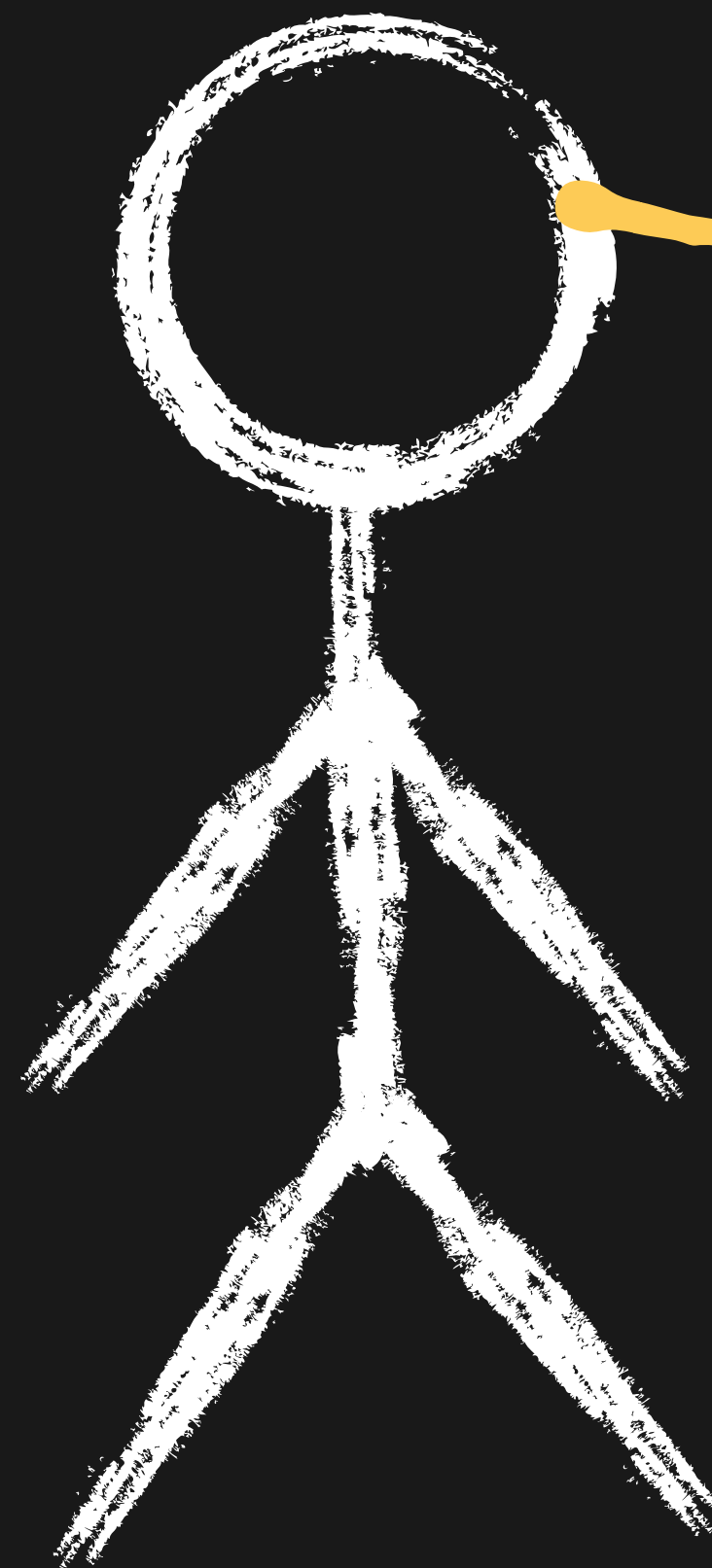
II Visualizing General 3 Manifolds

a Geometric Decomposition

b Putting the Pieces Together

I Raytracing in Manifolds

We see an object
when light
bounces off it
into our eye.



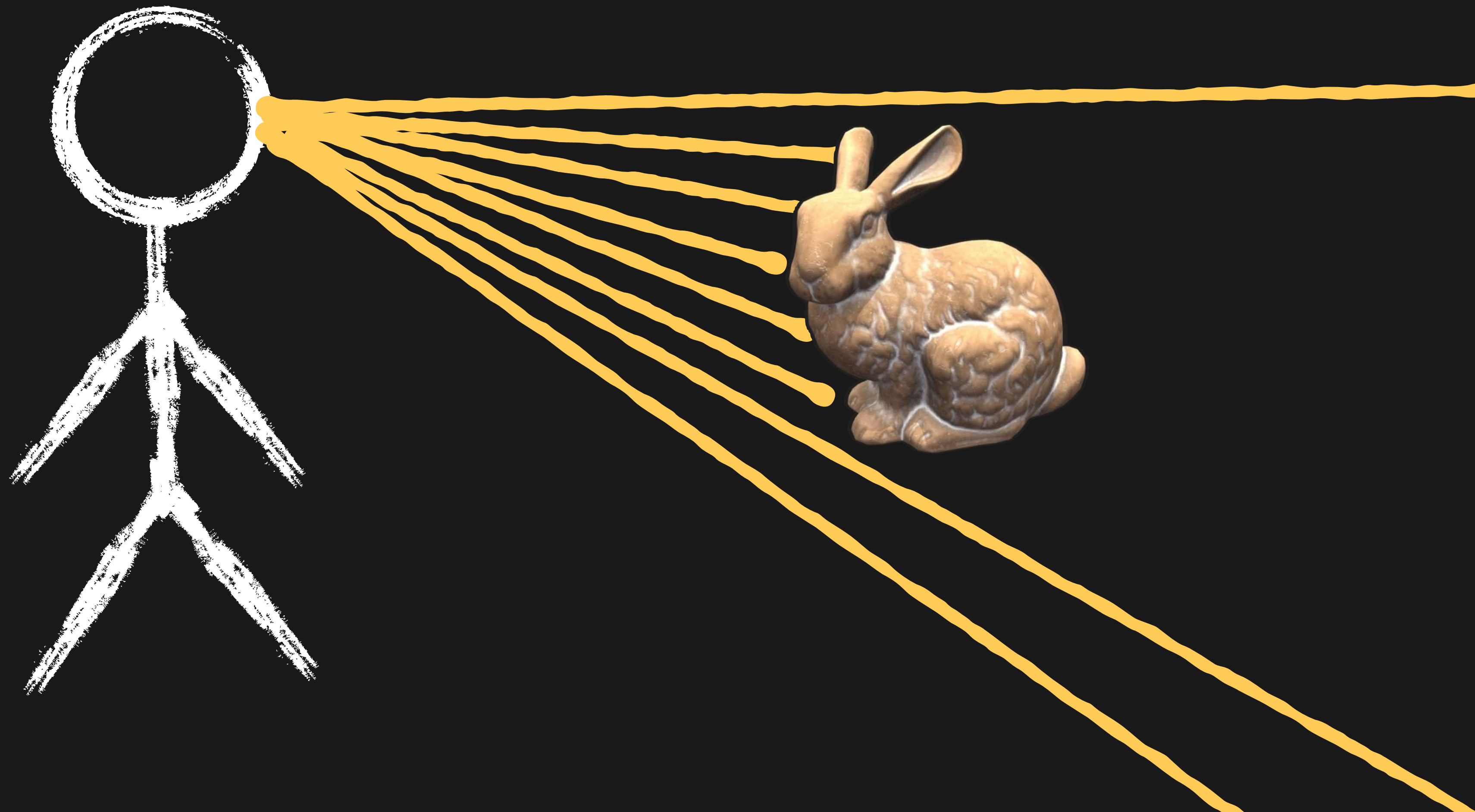
$$\text{viz} \left(\text{🐰} \right) = \mathbb{P} \exp_{\text{👁}}^{-1} \left(\text{🐰} \right) \in \mathbb{P} T_{\text{👁}} \text{Space}$$

This is the inverse
of the exponential
map in Riemannian
geometry

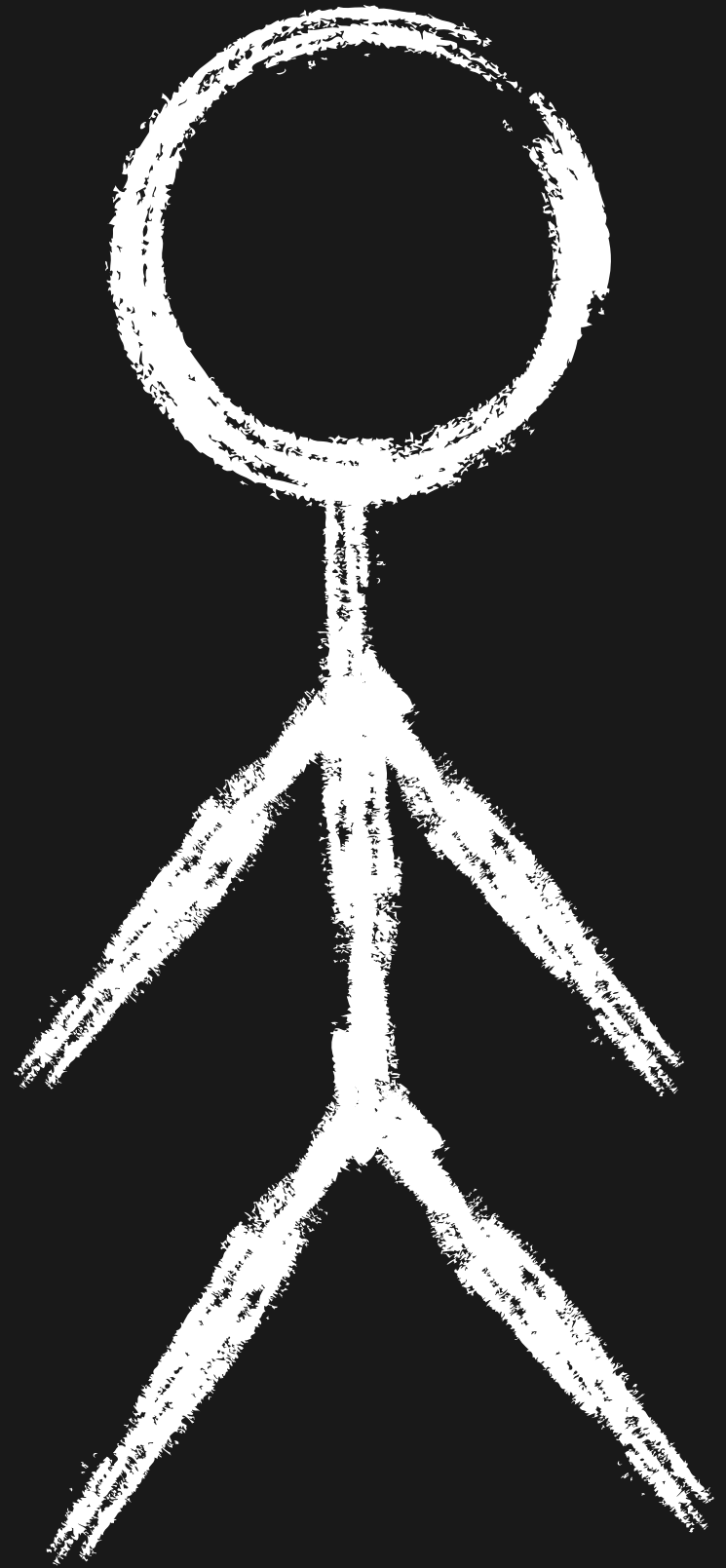


On the computer...reverse this!

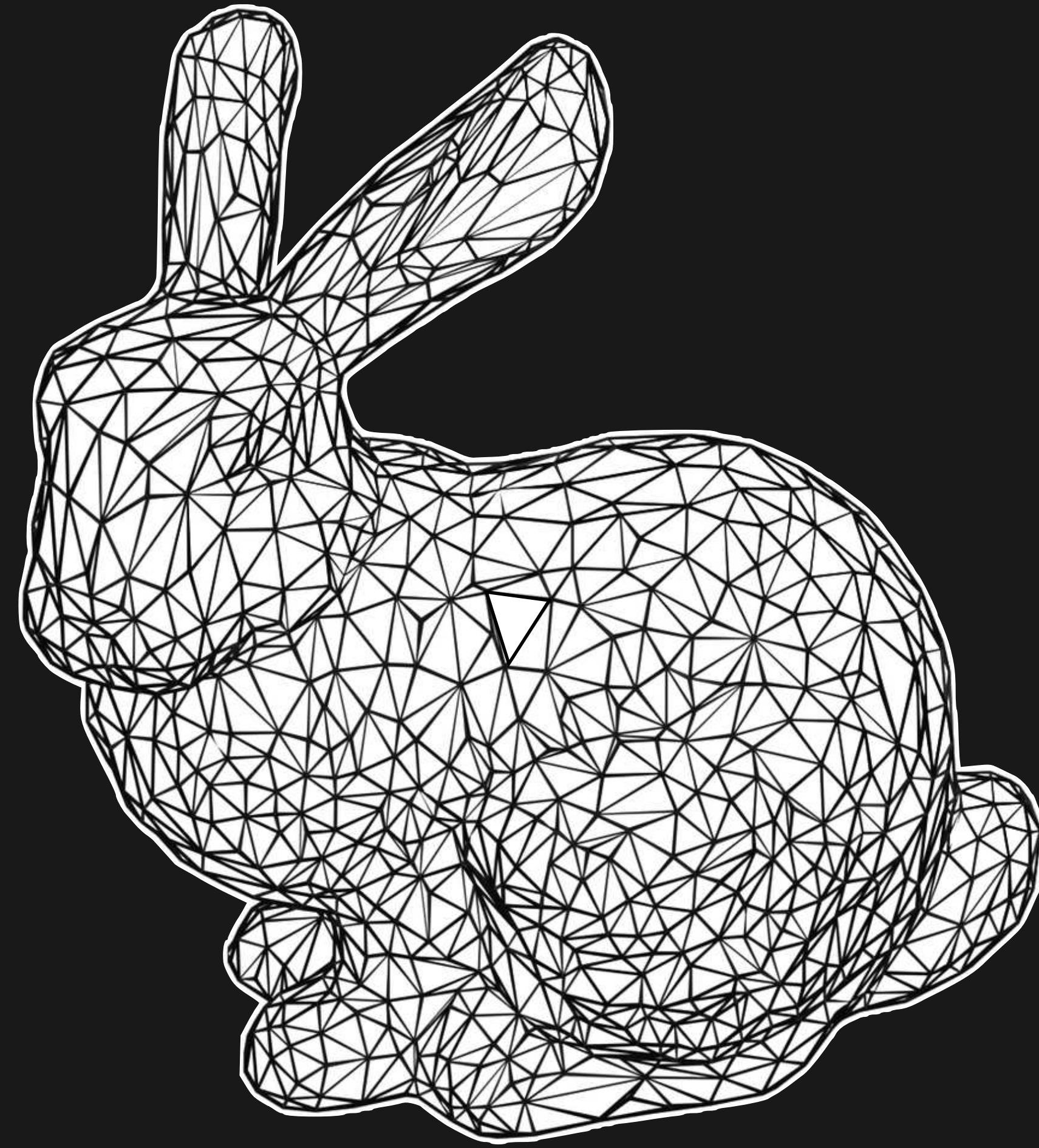
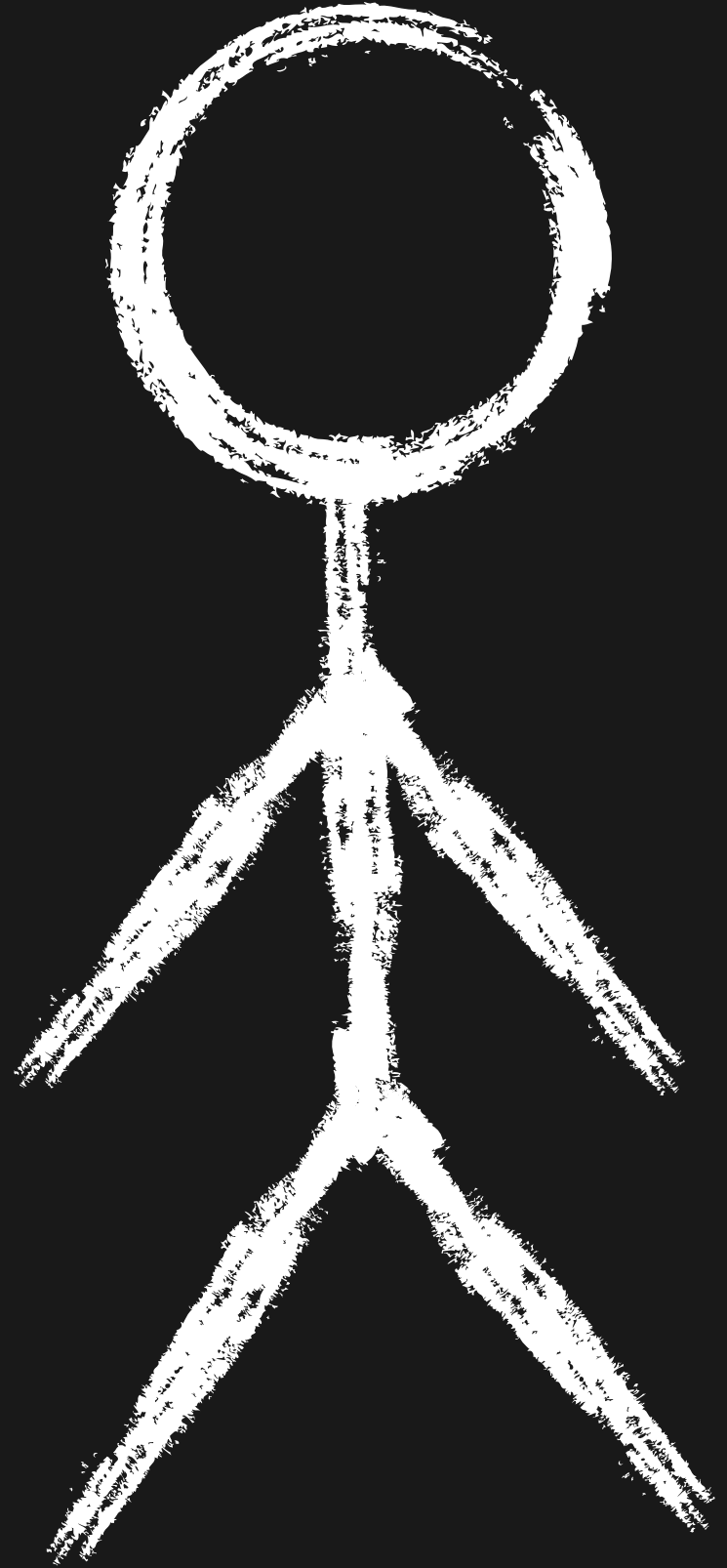
Calculate for each initial direction in the tangent space at your eye a geodesic out into the world, and see what it hits



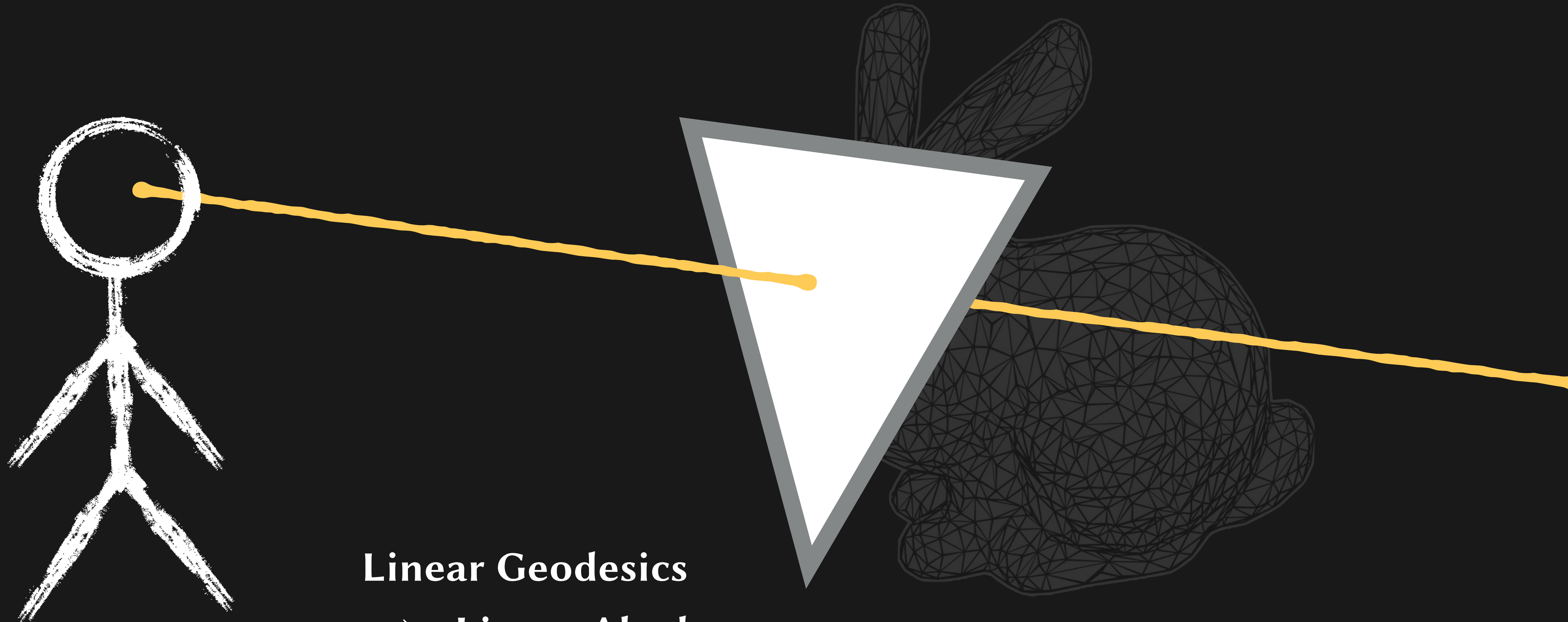
Computing Intersections



Computing Intersections

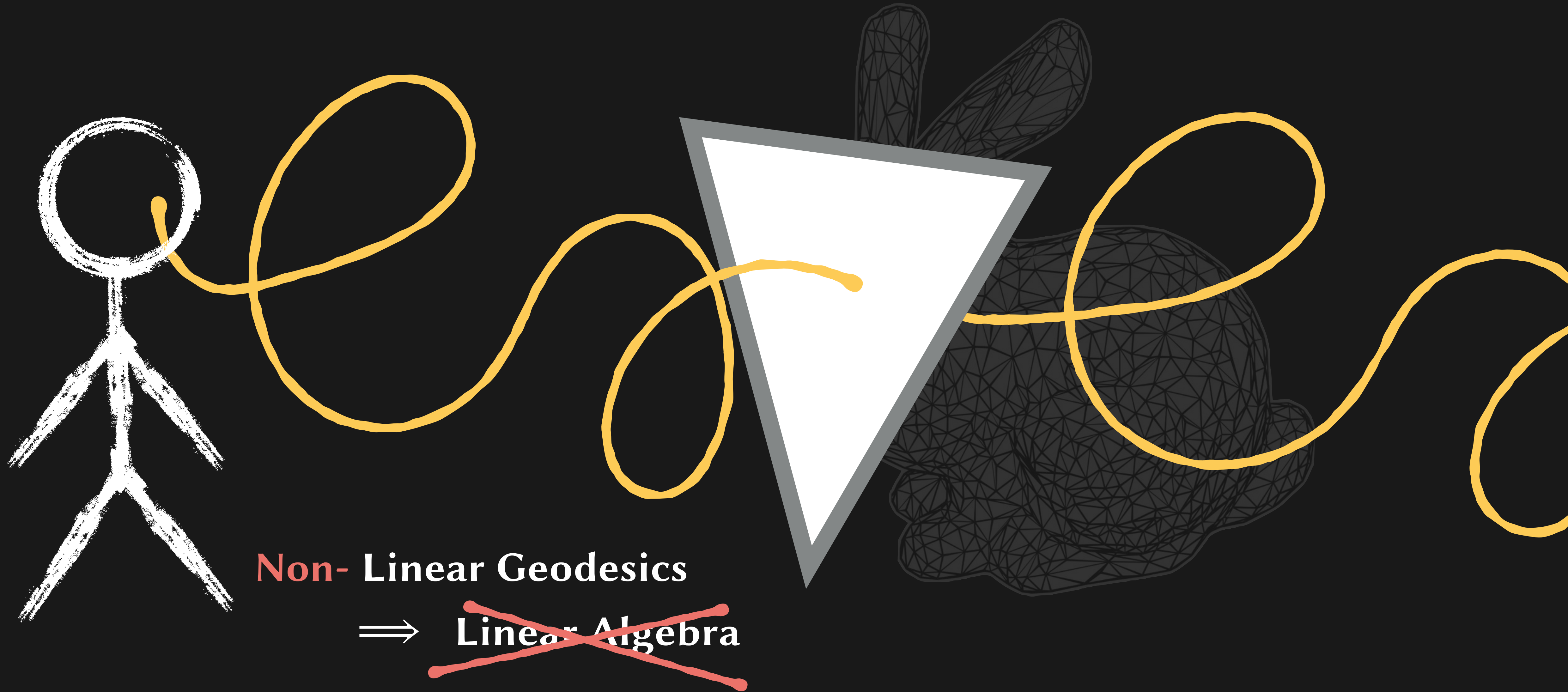


Computing Intersections



Linear Geodesics
 $\Rightarrow$ Linear Algebra

Computing Intersections



$$dx^2 + dy^2 + (dz - xdy)^2$$



The Raymarching Algorithm

Compute the distance to object

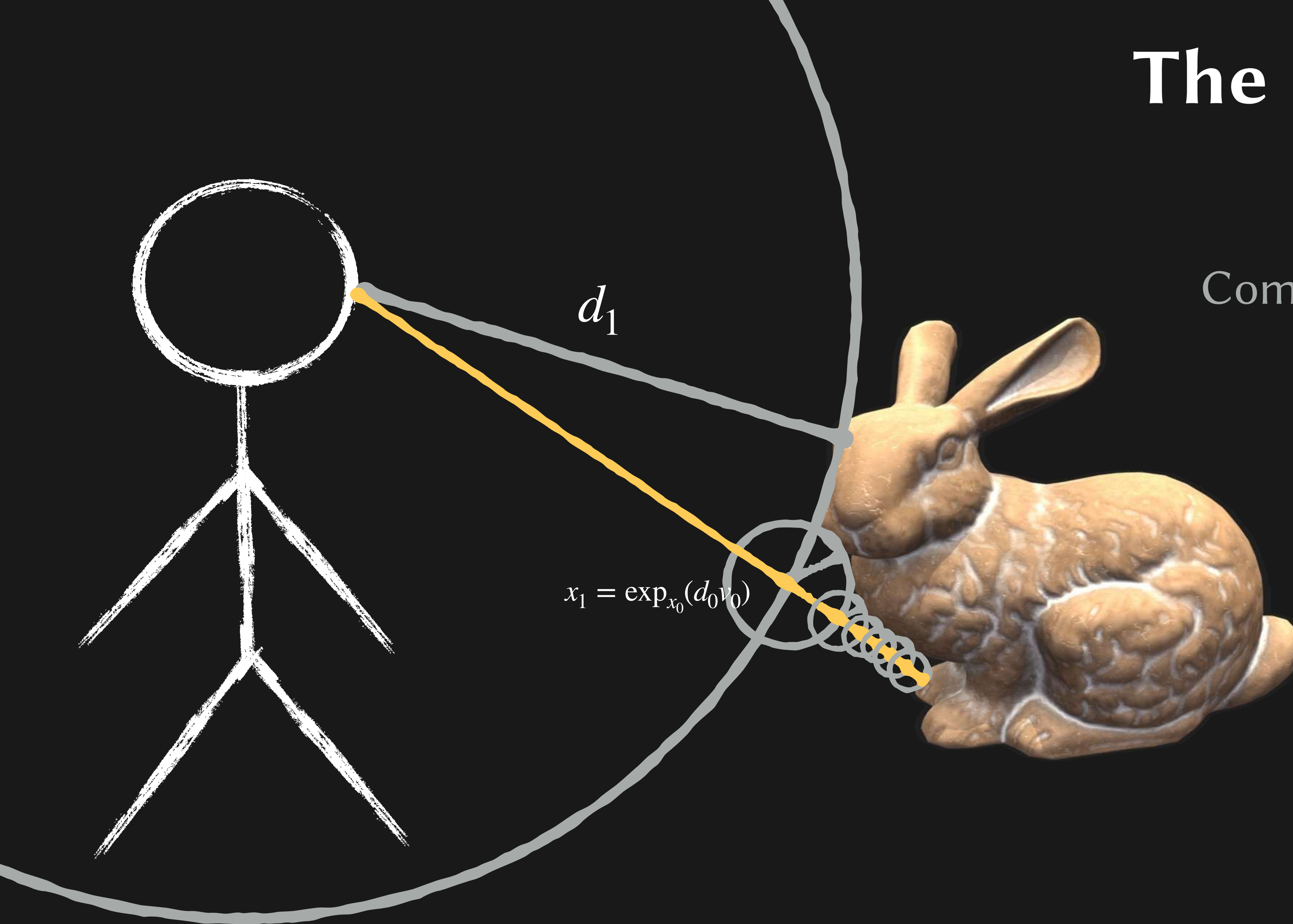
$$d_n = \text{dist}(x_n, \text{Obj})$$

March this distance
along geodesic

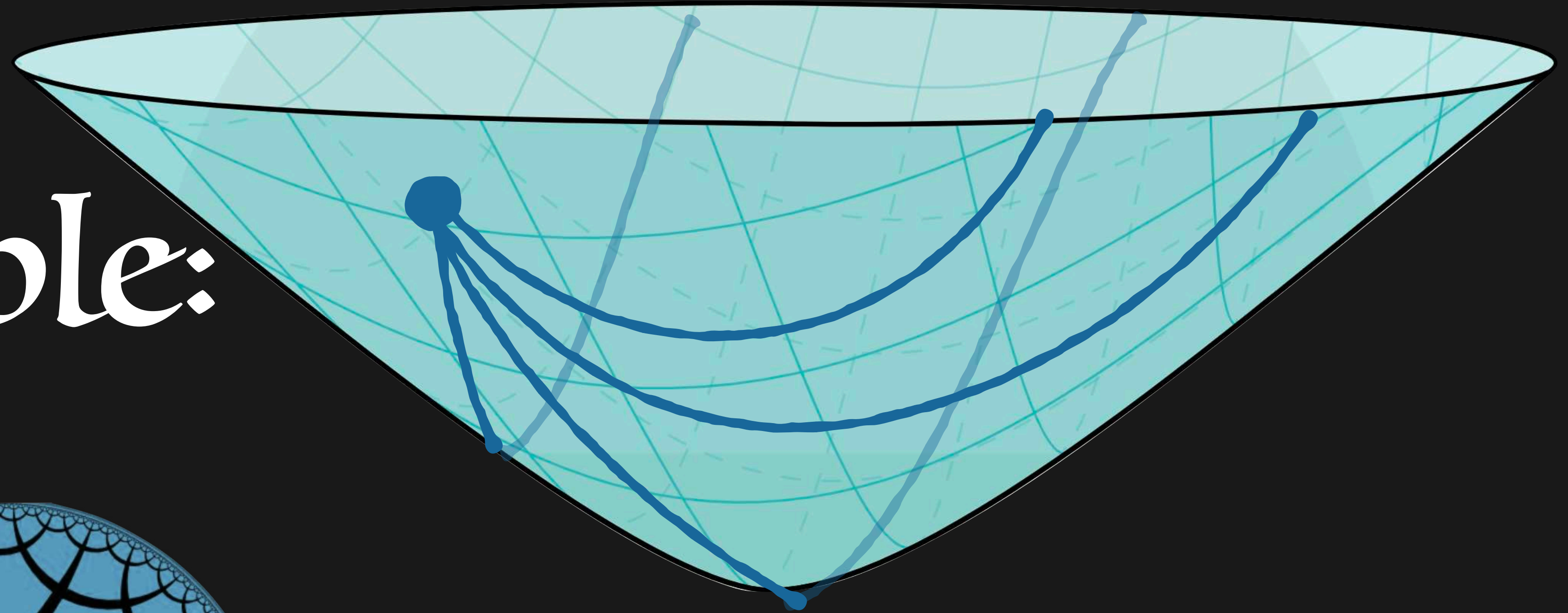
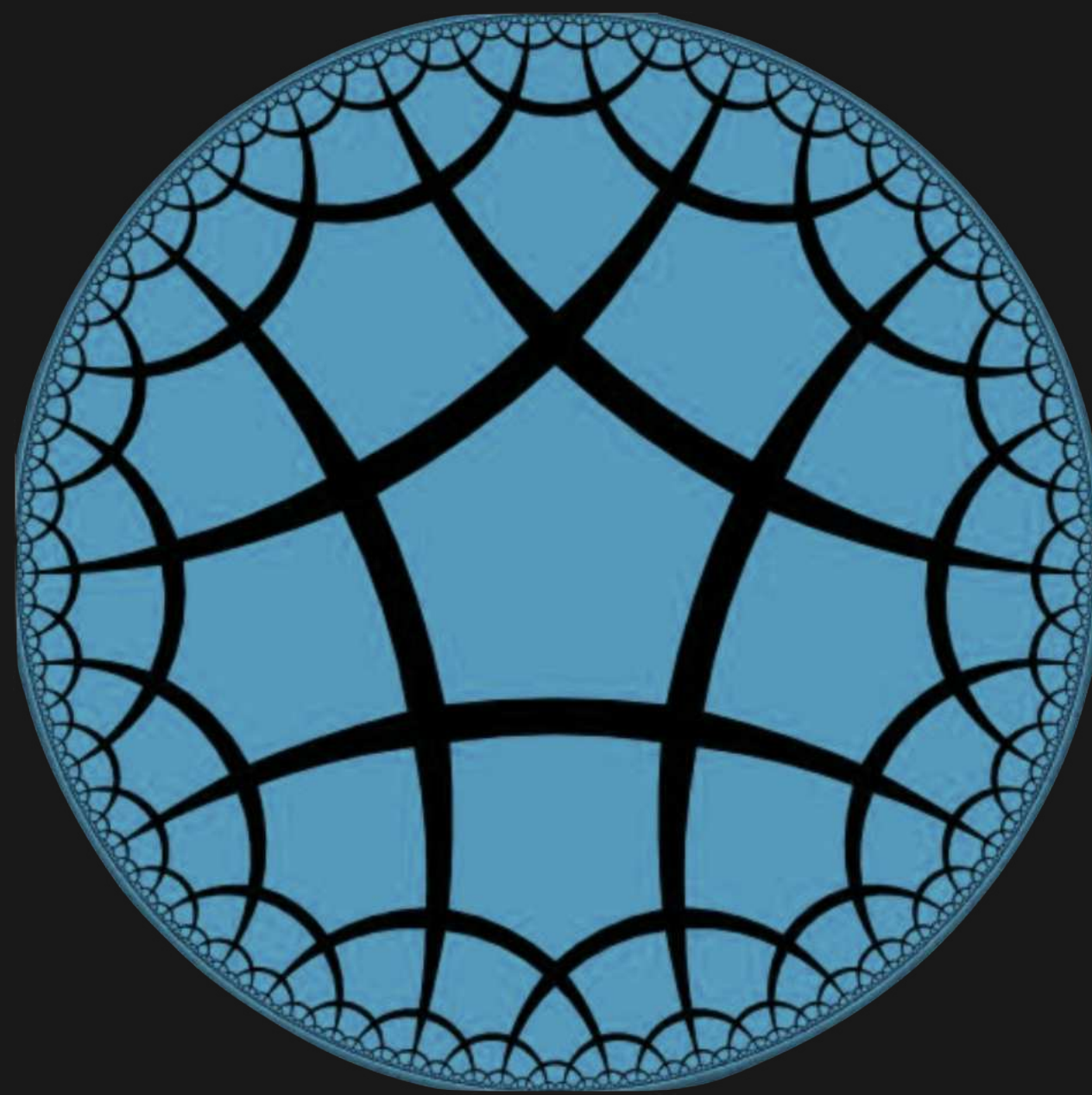
$$x_{n+1} = \exp_{x_n}(d_n v_n)$$

$$v_{n+1} = \left. \frac{d}{dt} \right|_{t=d_n} \exp_{x_n}(tv_n)$$

Repeat until $d_N \approx 0$



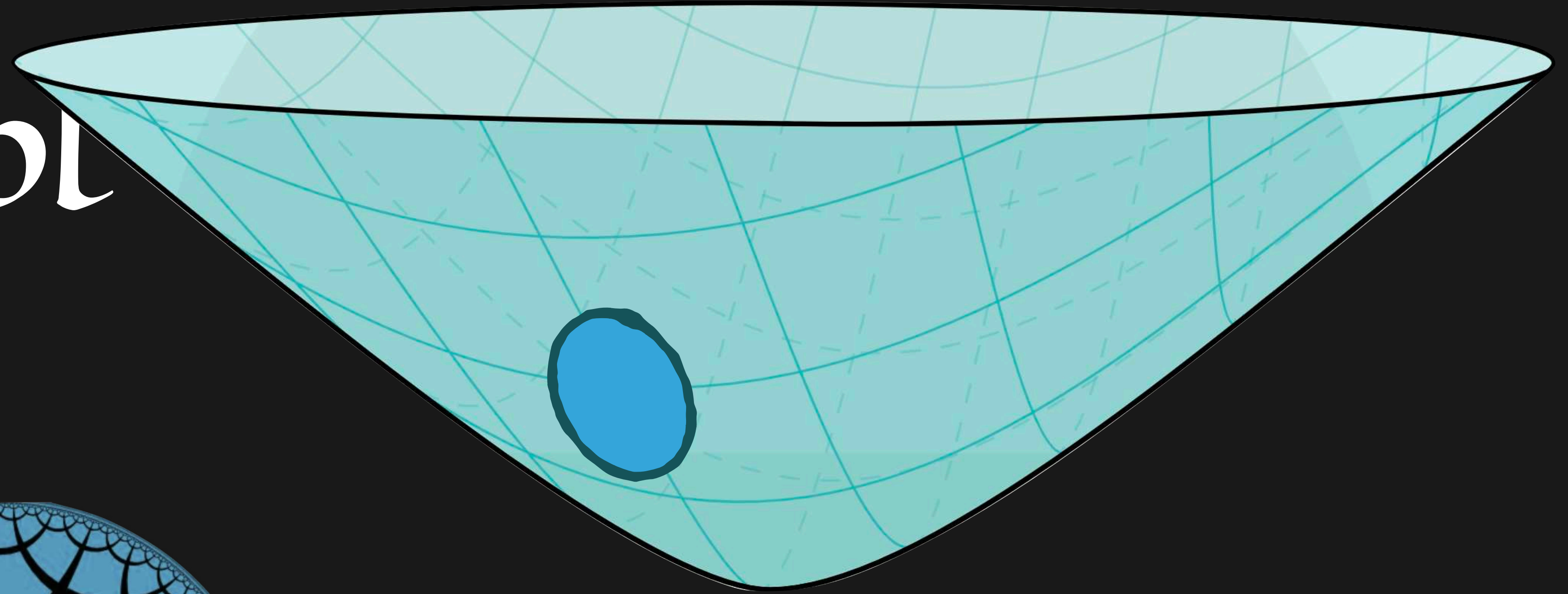
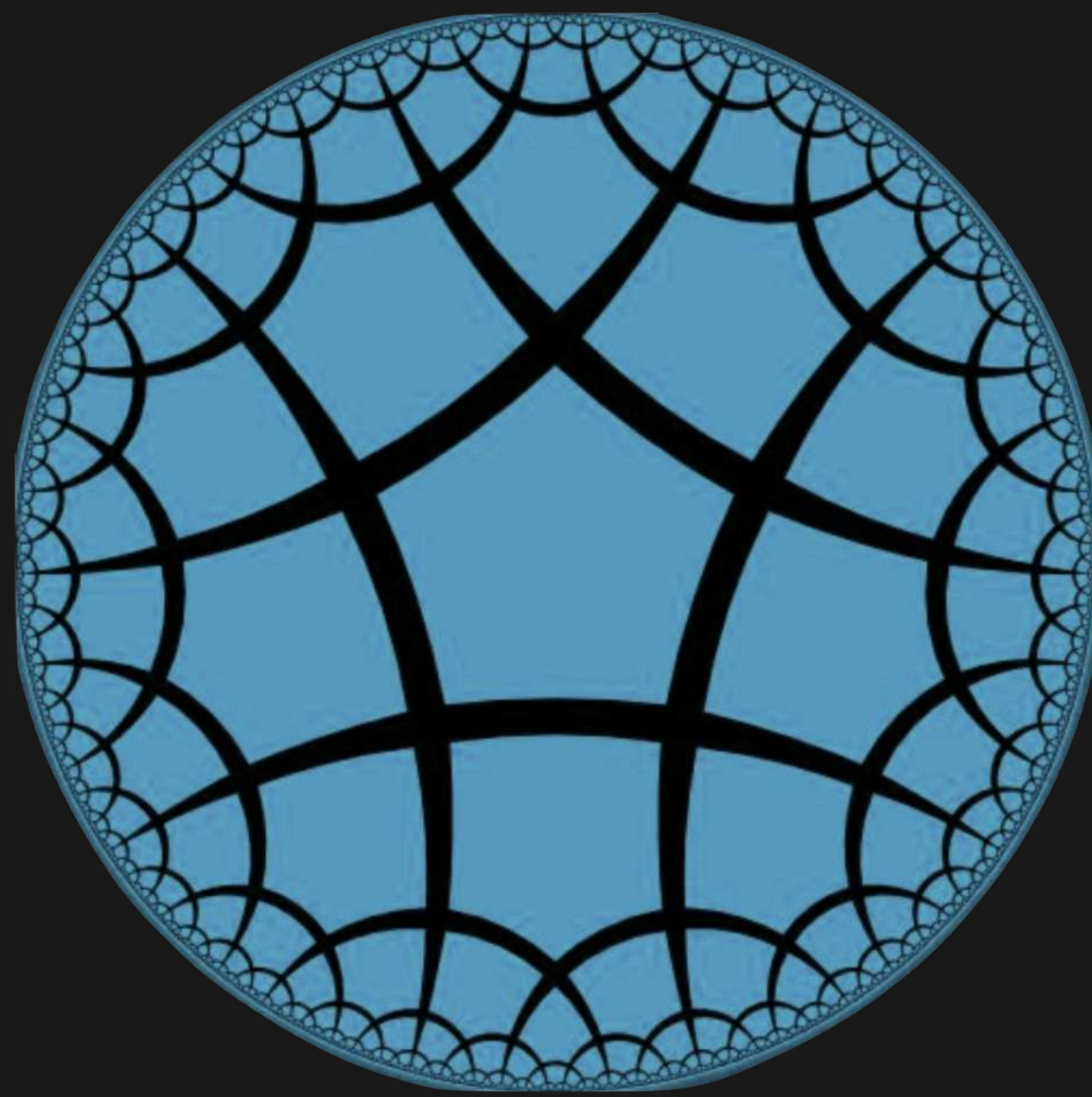
Example:



Hyperbolic Space

Geodesics are easy to compute:
intersection with a linear space.

Example:

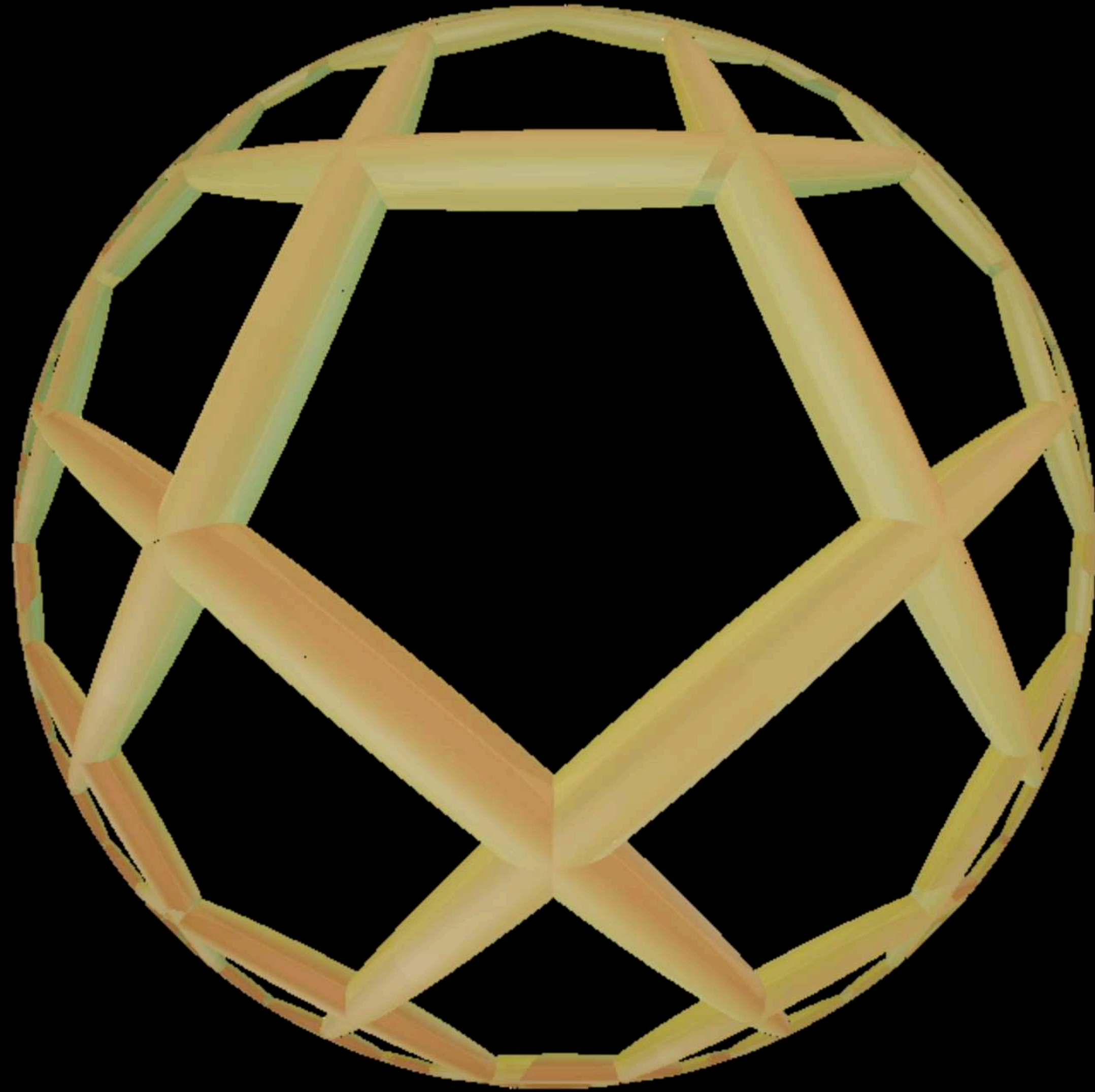


Hyperbolic Space

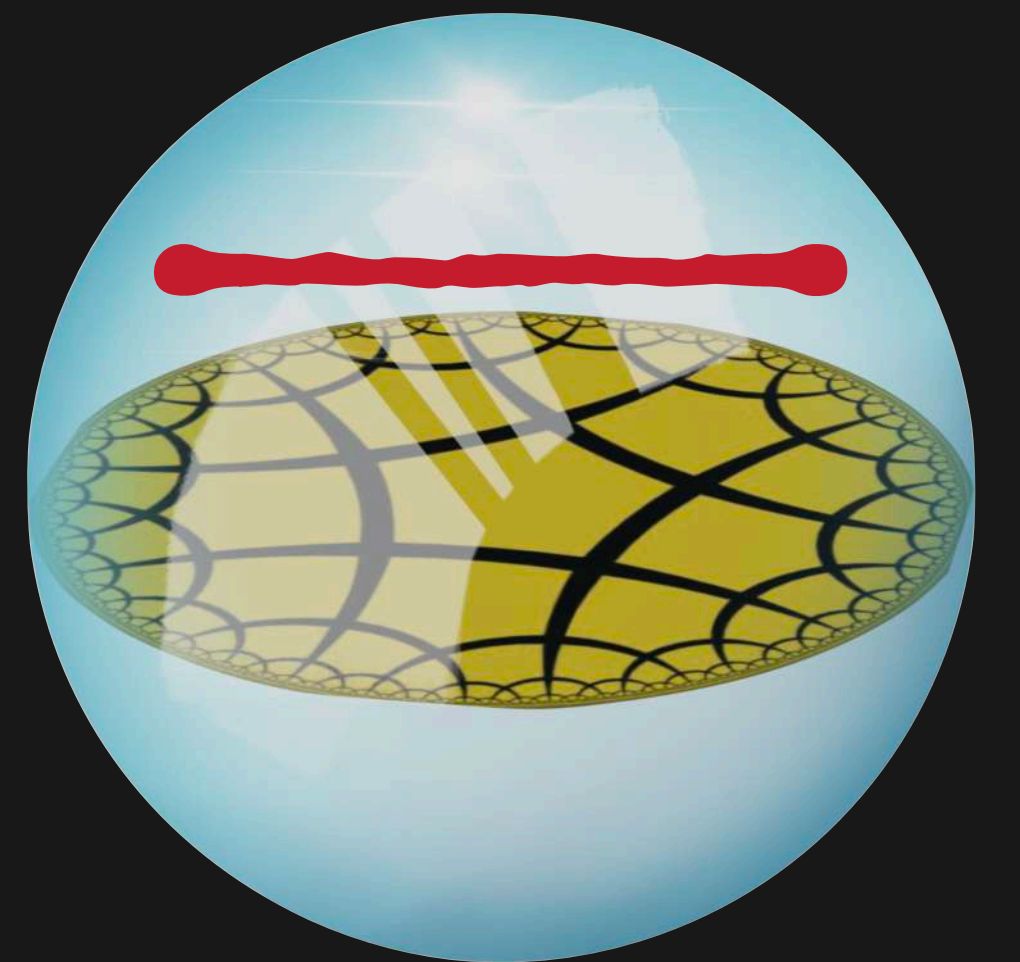
Geodesics are easy to compute:
intersection with a linear space.

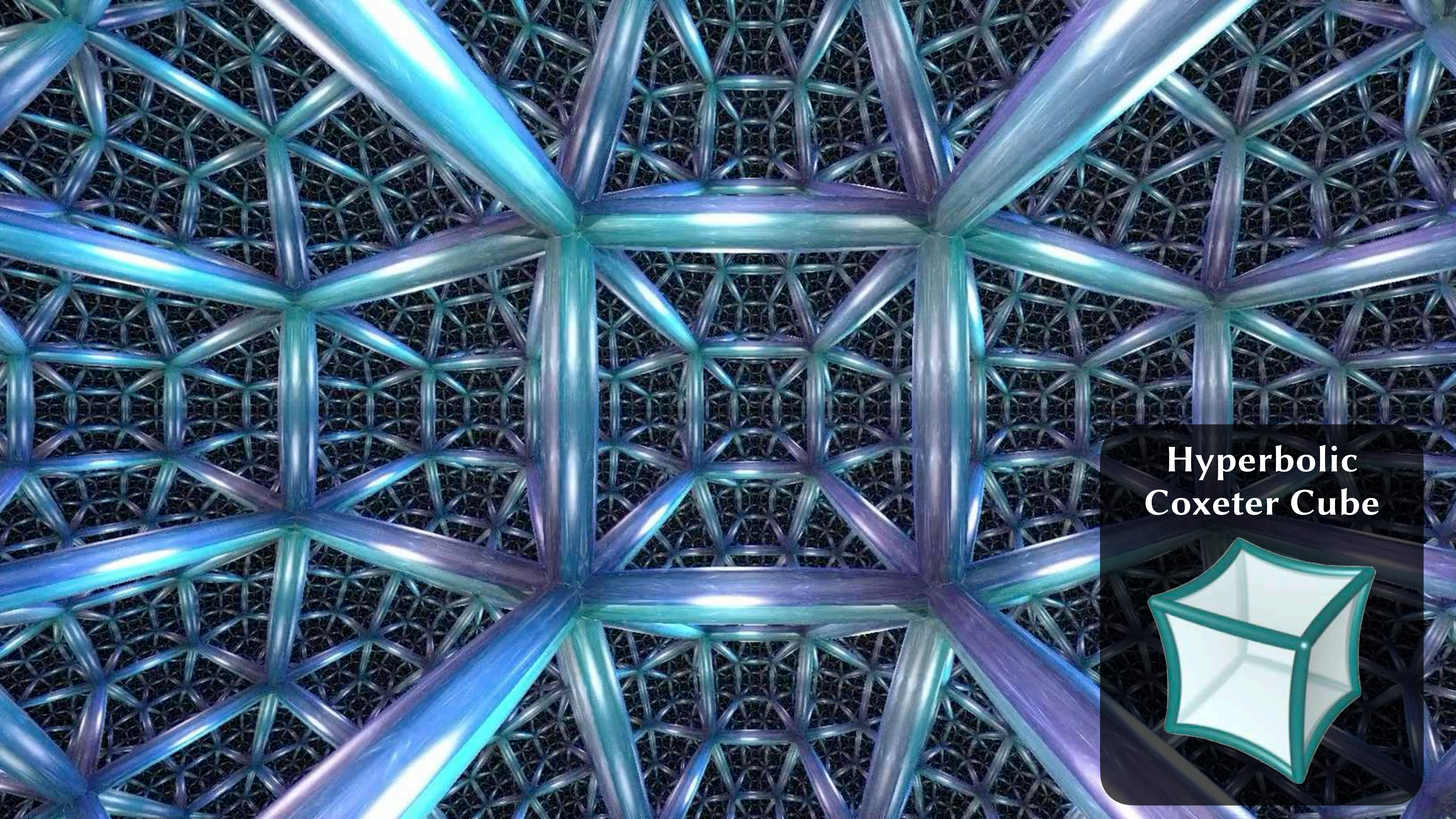
Distance is easy to compute: Minkowski
dot product.

Hyperbolic Space



RA Pentagon
Tiling





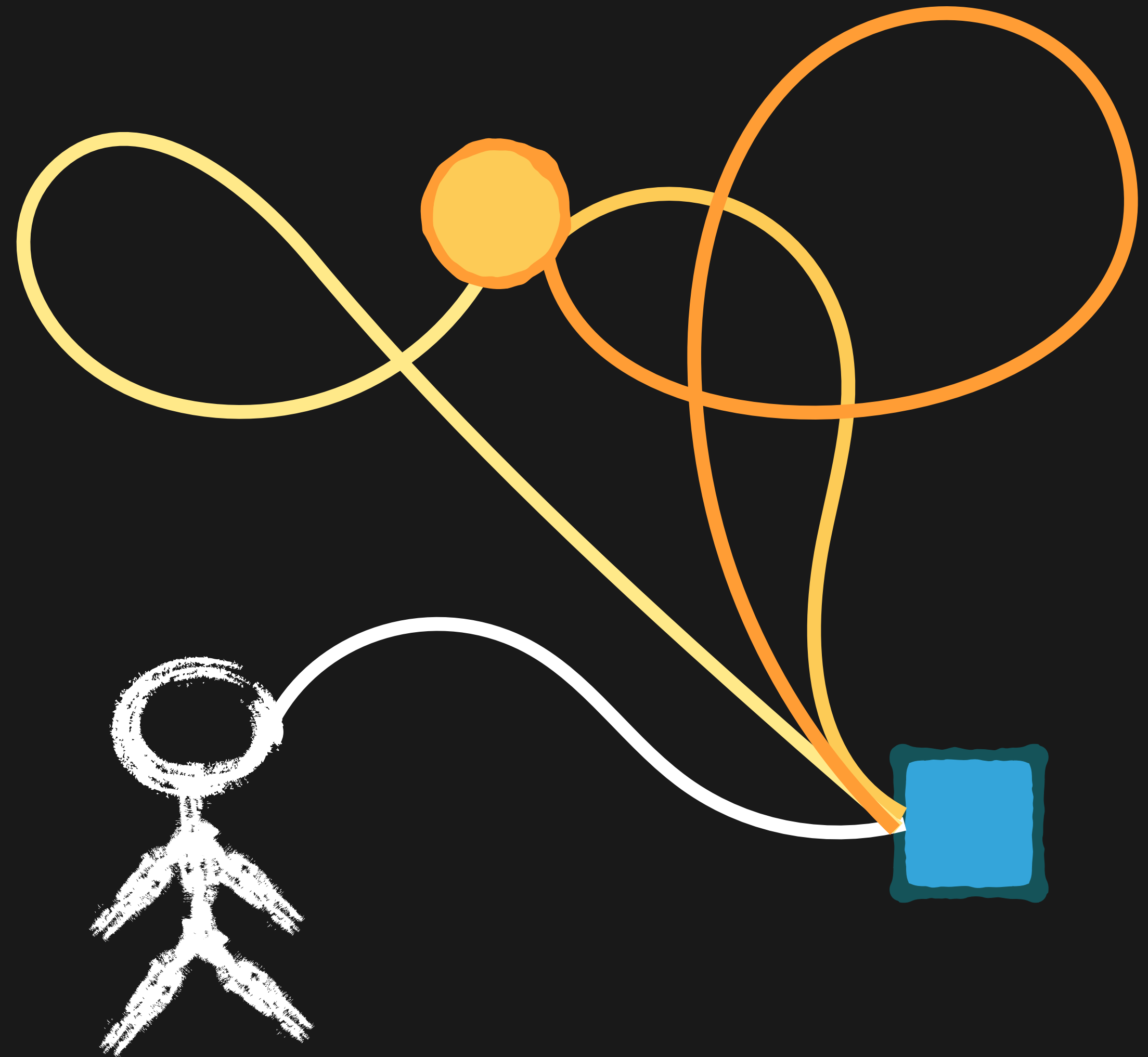
**Hyperbolic
Coxeter Cube**



Of course, there's details:

To compute lighting, need to
find geodesics given endpoints

In positive curvature, there's
non-uniqueness to contend with

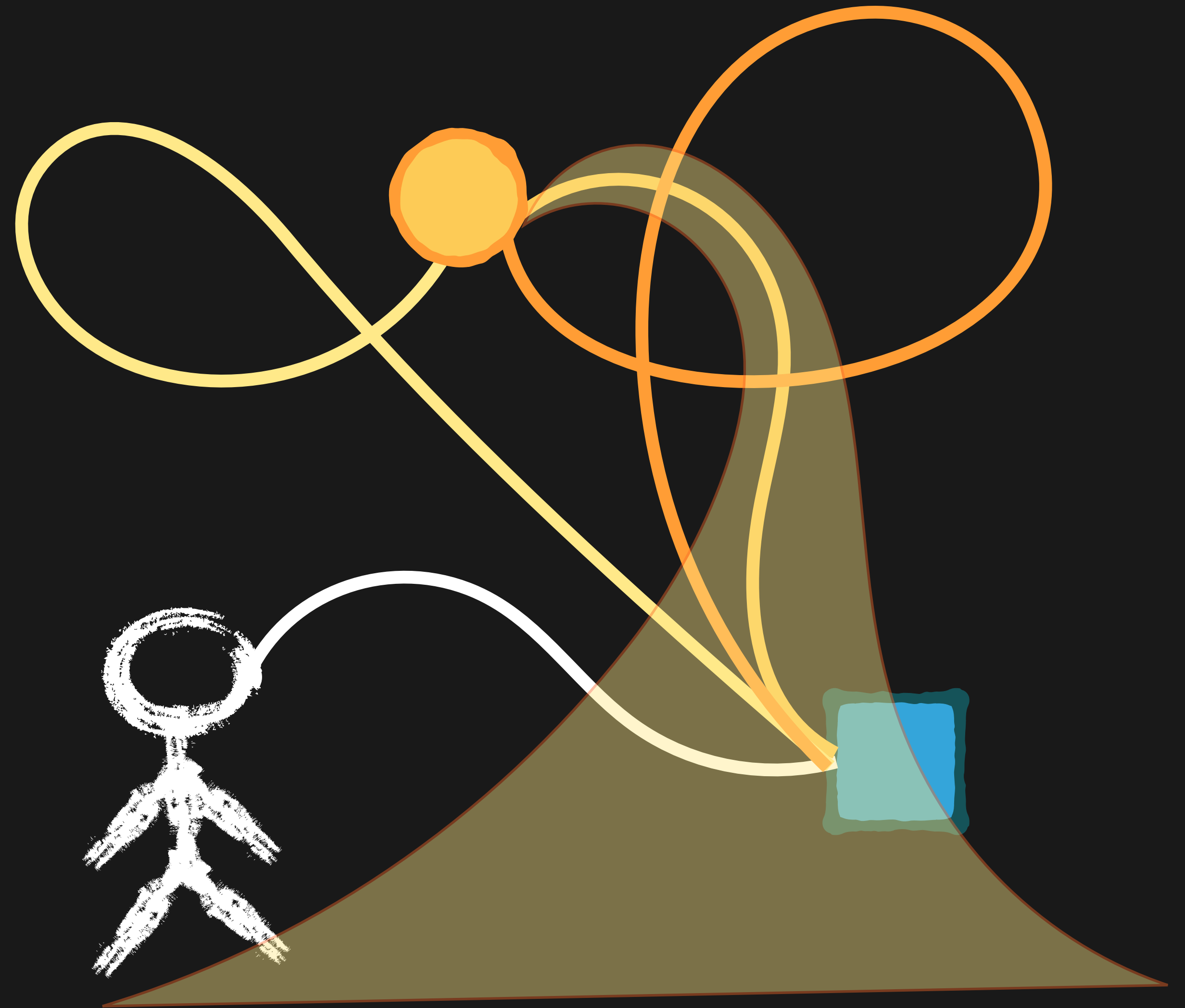


Of course, there's details:

To compute lighting, need to
find geodesics given endpoints

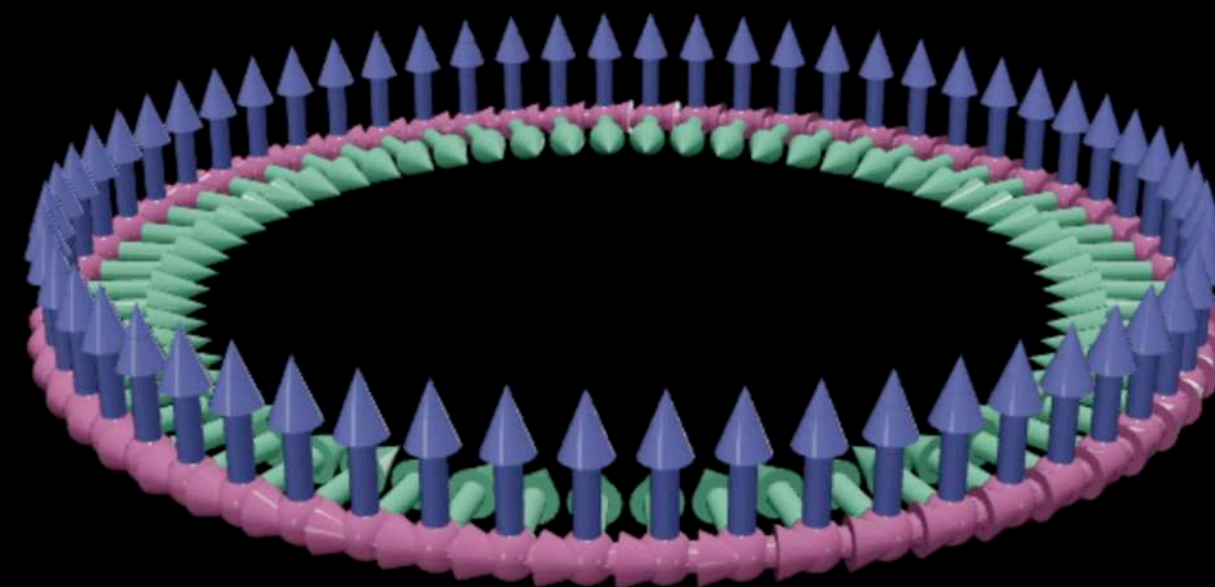
In positive curvature, there's
non-uniqueness to contend with

Intensity is computed via
Jacobi fields



Of course, there's details:

To move around, need to take
input from computer / headset

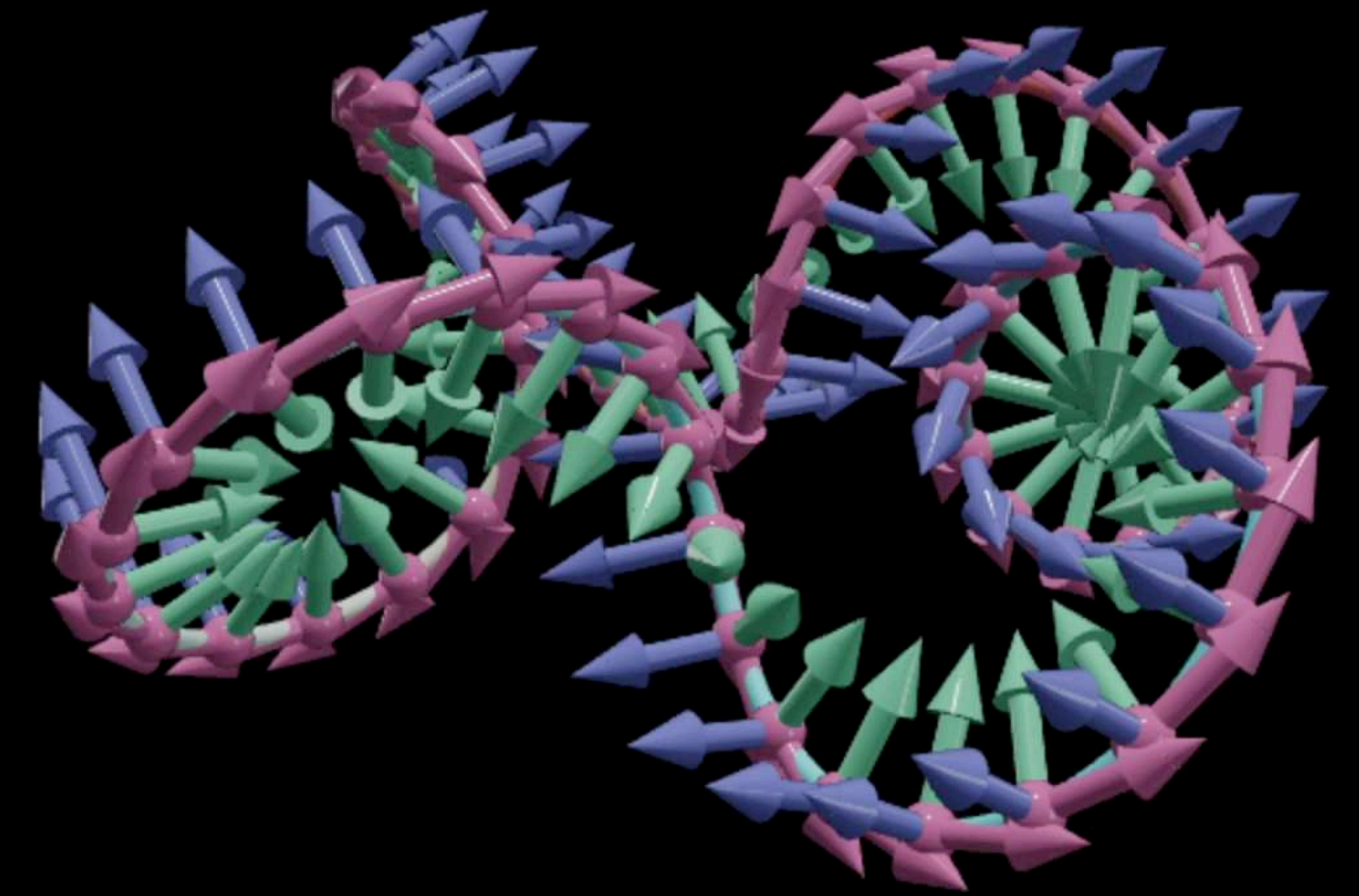


$$I \rightarrow \mathbb{R}^3$$
$$I \rightarrow \mathrm{SO}(3)$$

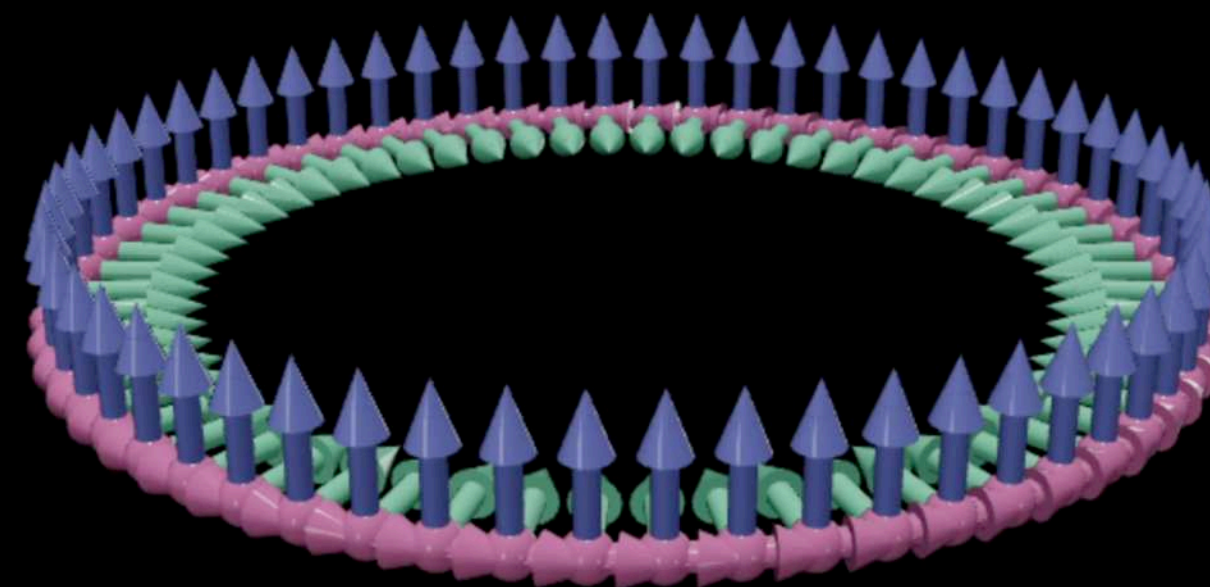
Of course, there's details:

To move around, need to take
input from computer / headset

Then lift this to the
frame bundle of your
manifold



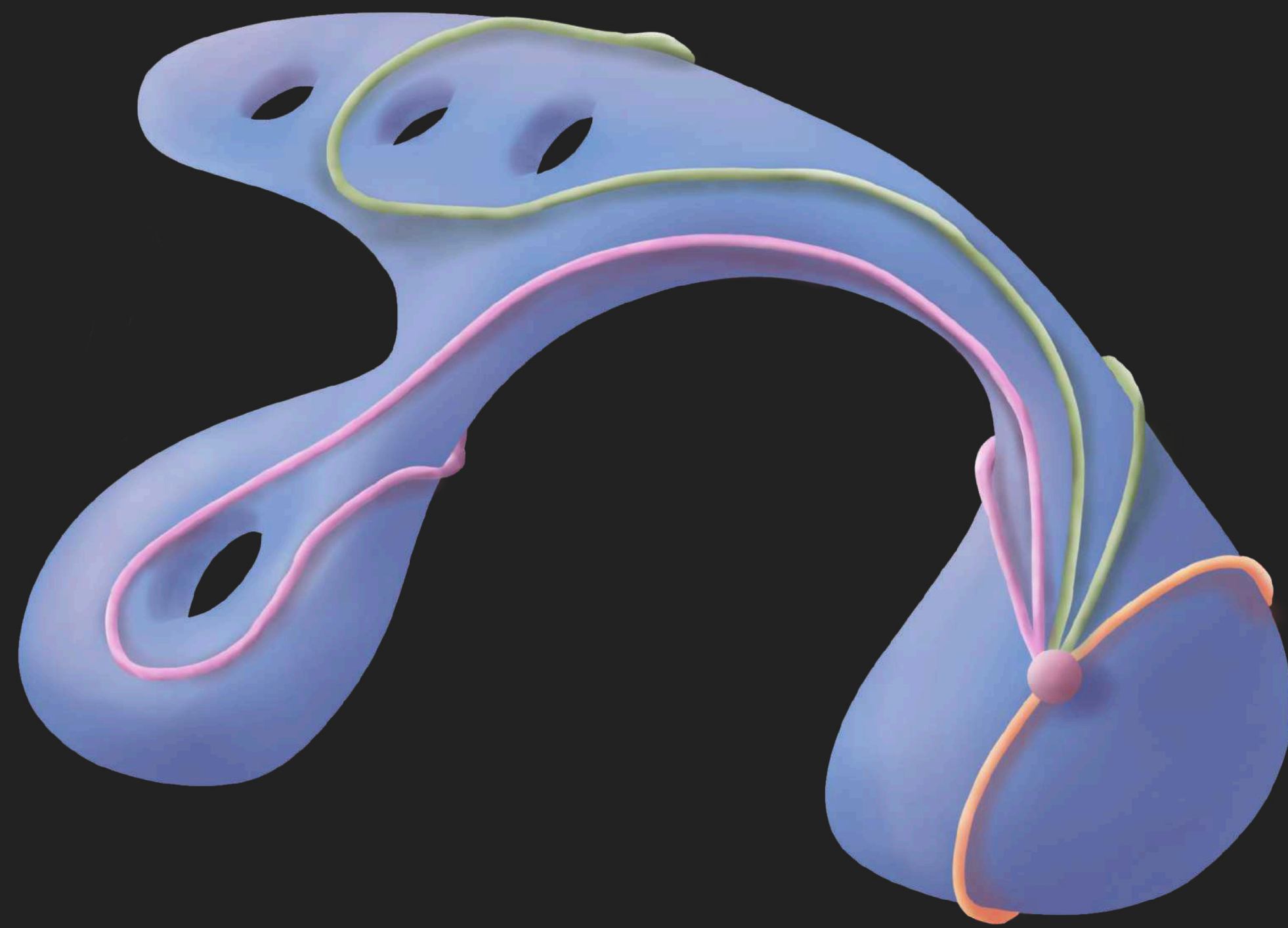
$$I \rightarrow \mathcal{F}(M, g)$$



$$I \rightarrow \mathbb{R}^3$$

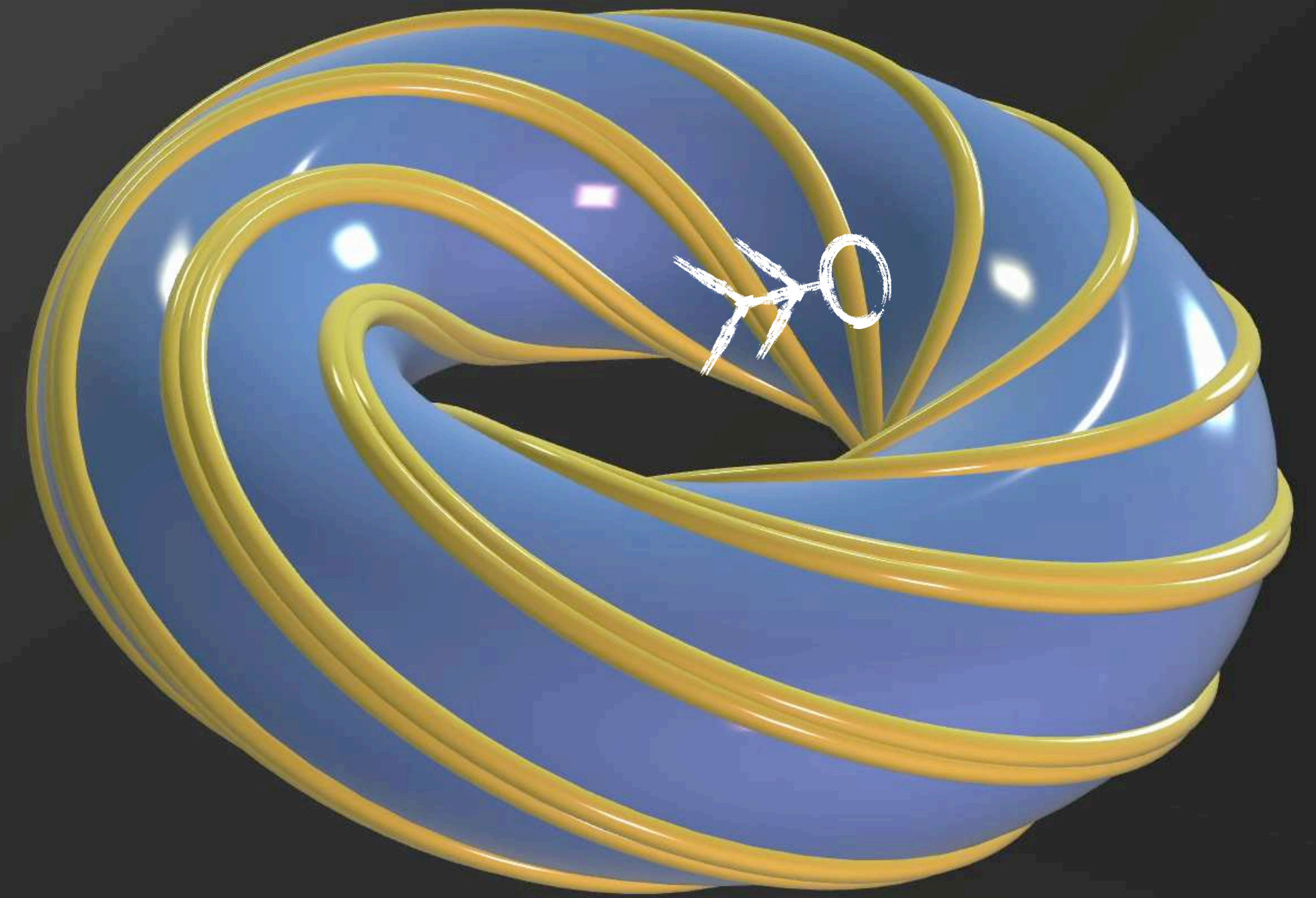
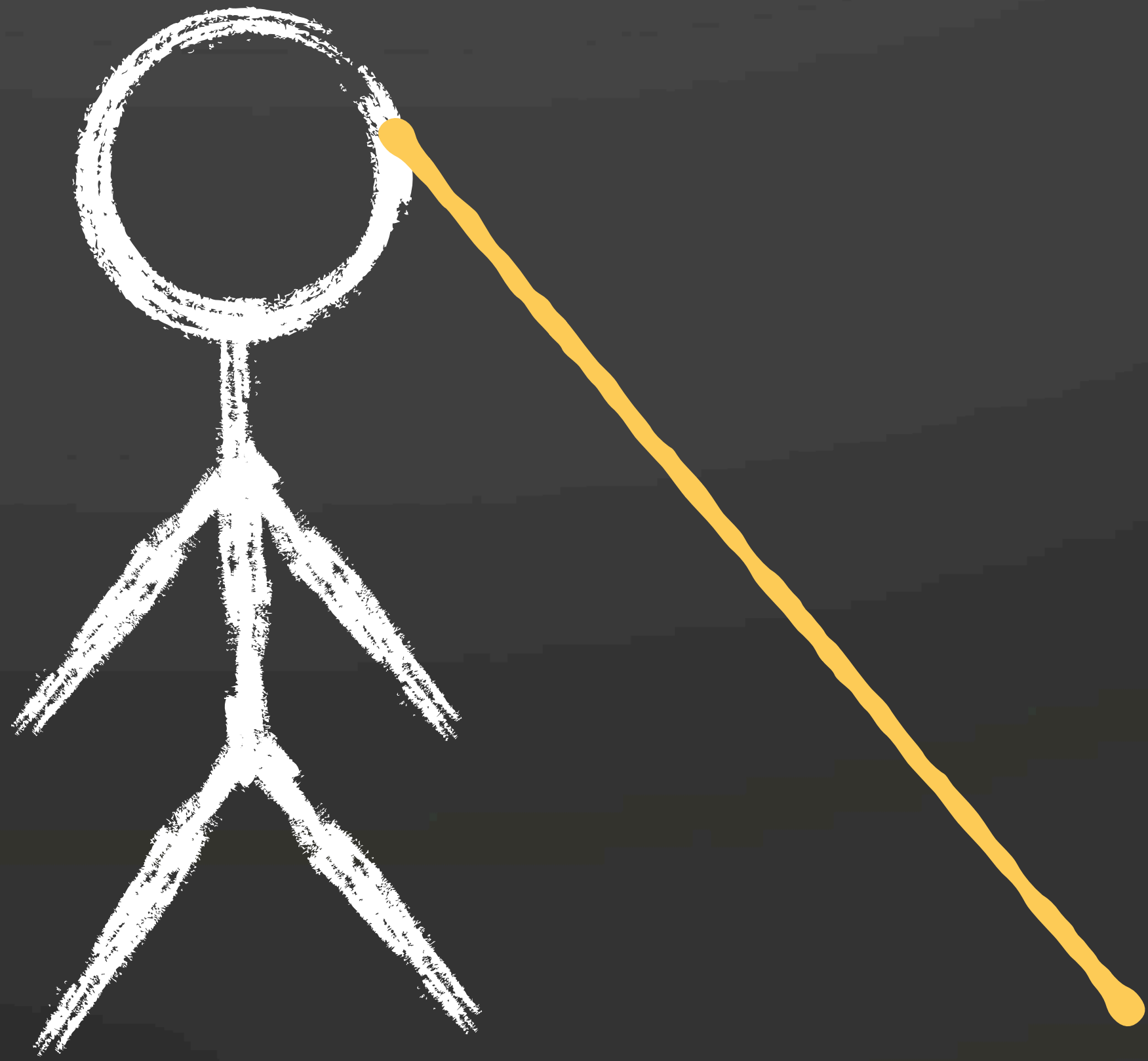
$$I \rightarrow \mathrm{SO}(3)$$

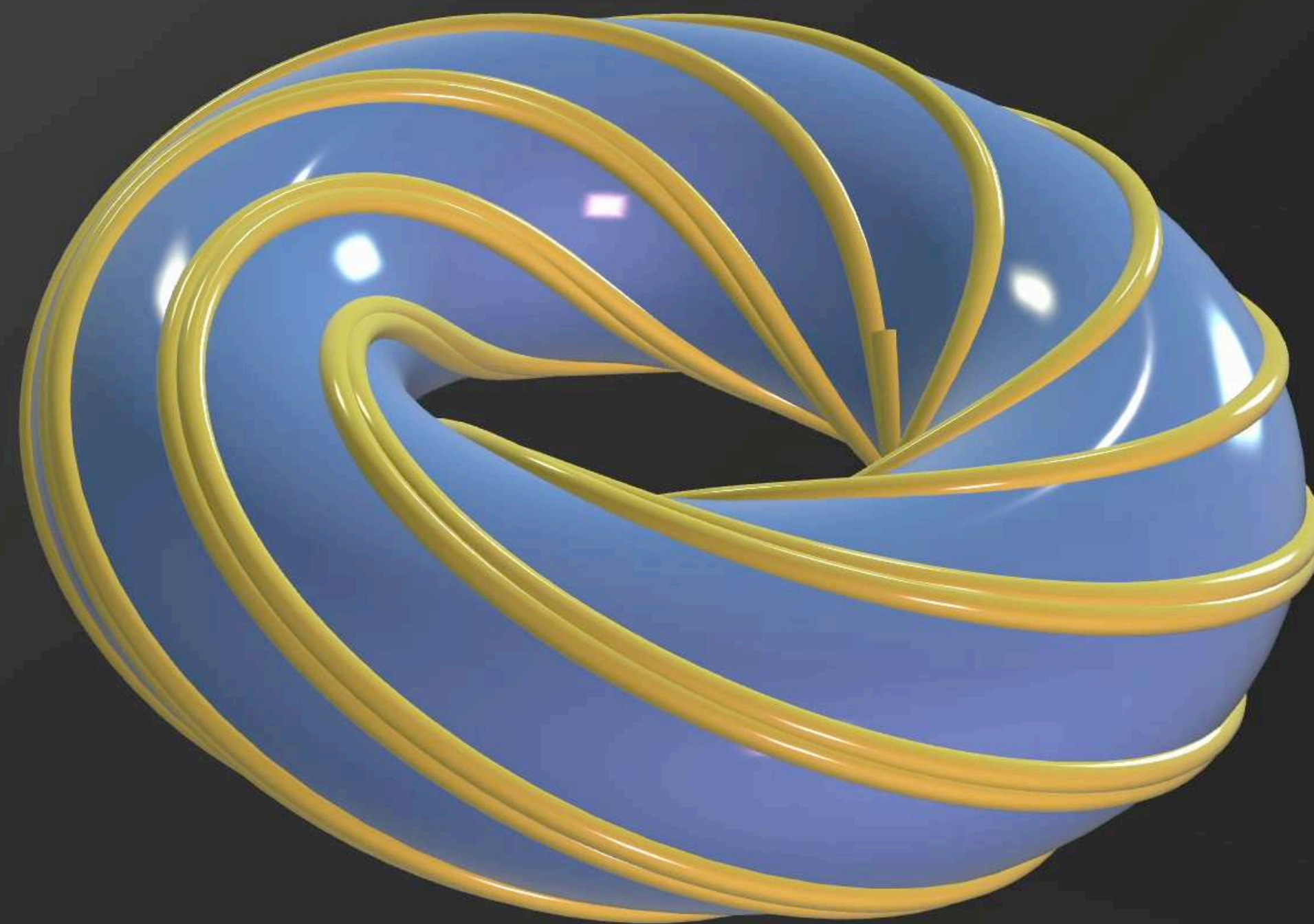
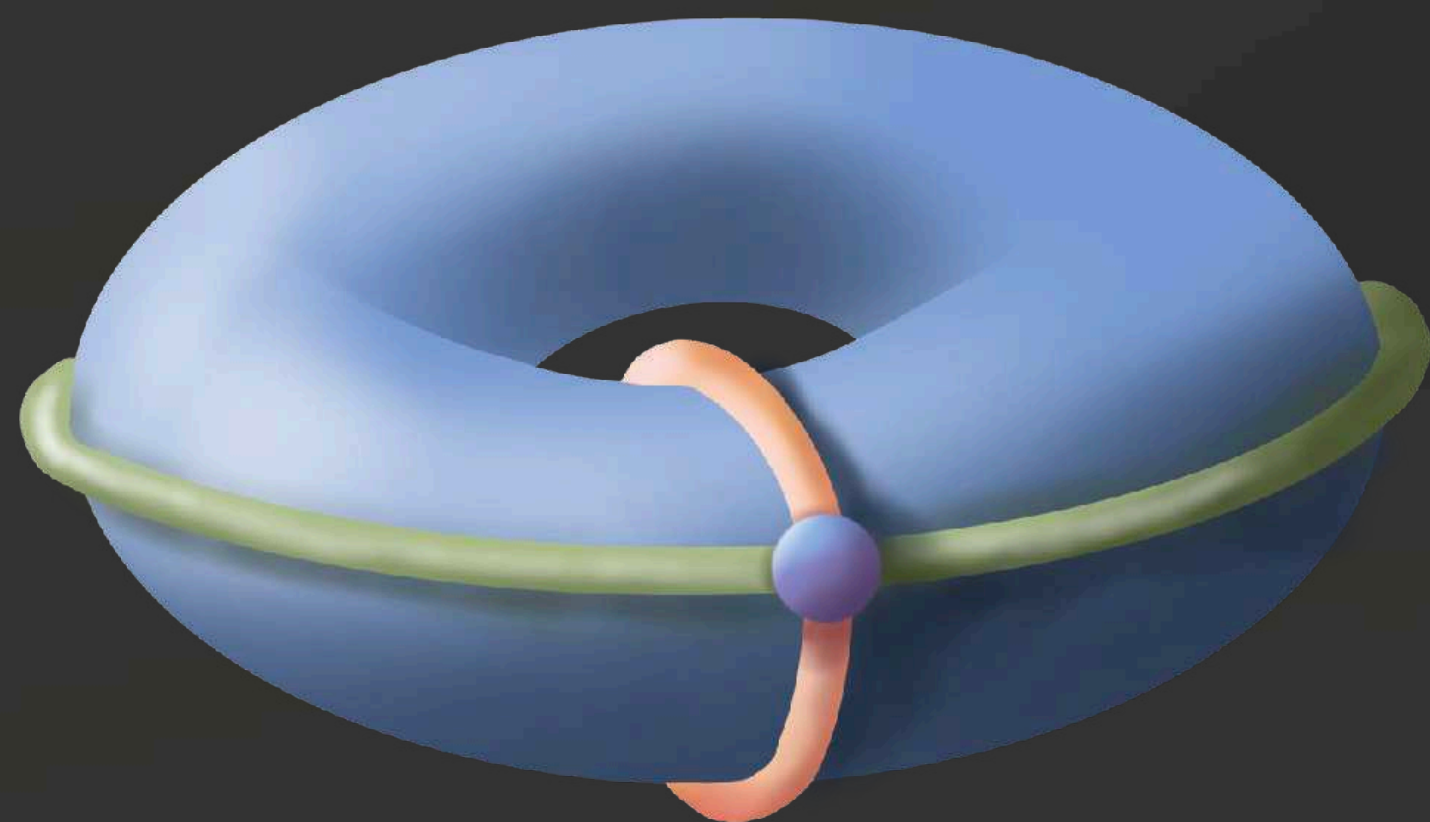
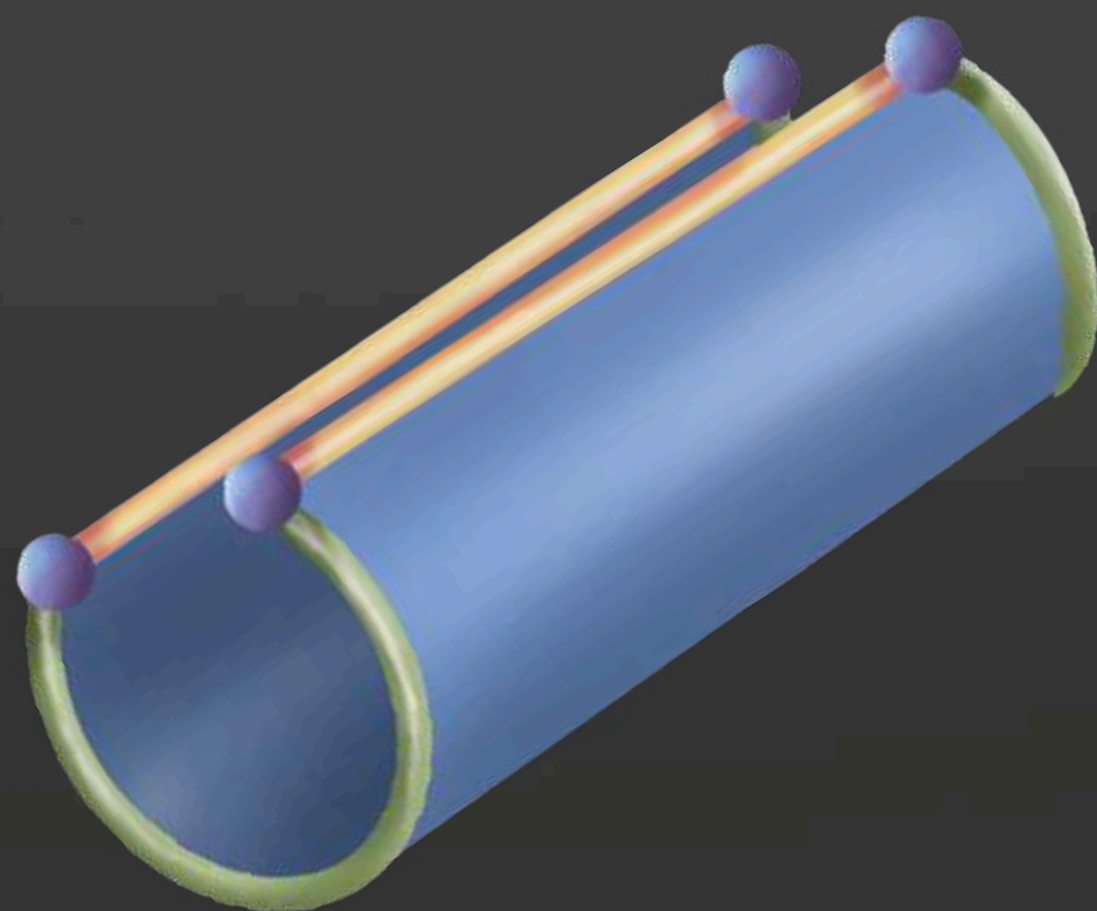
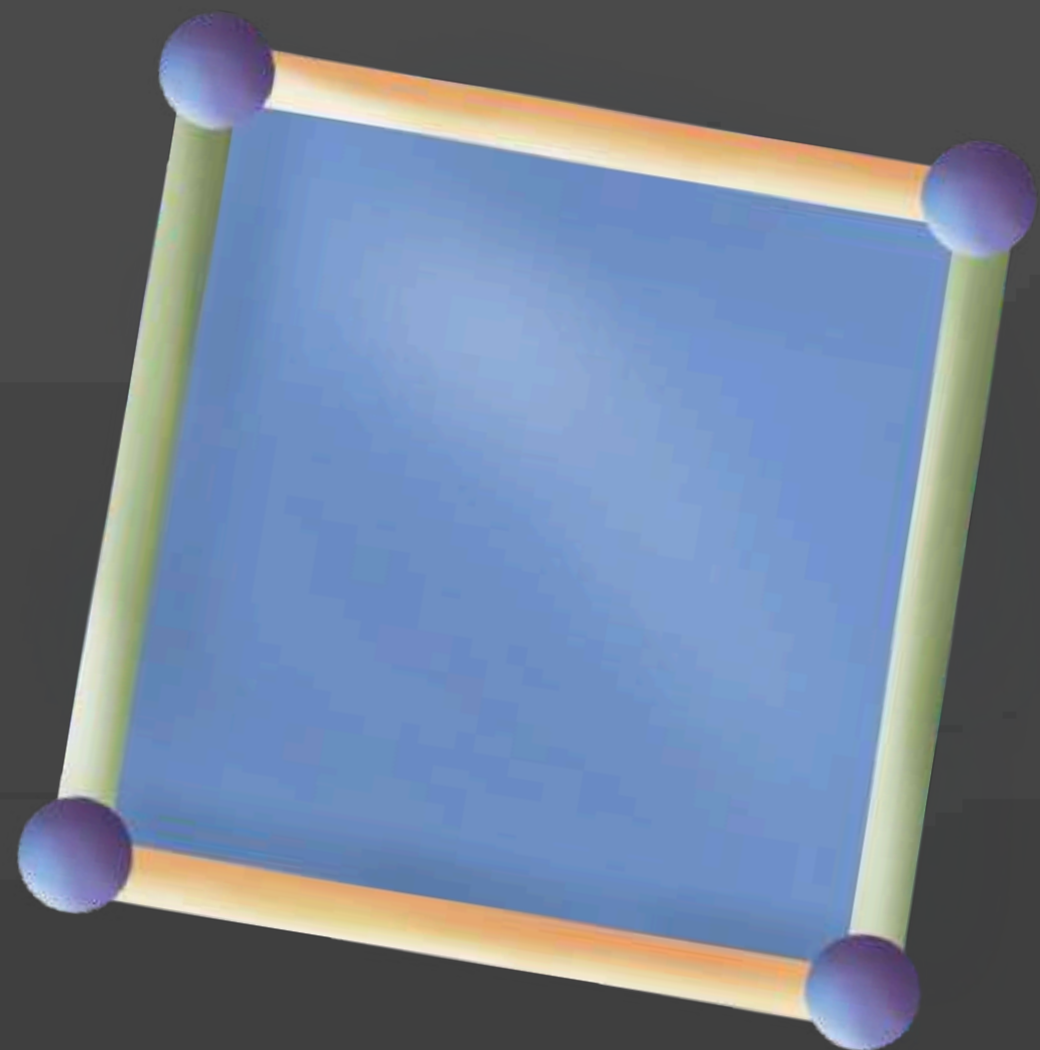
Ia Topology & Vision



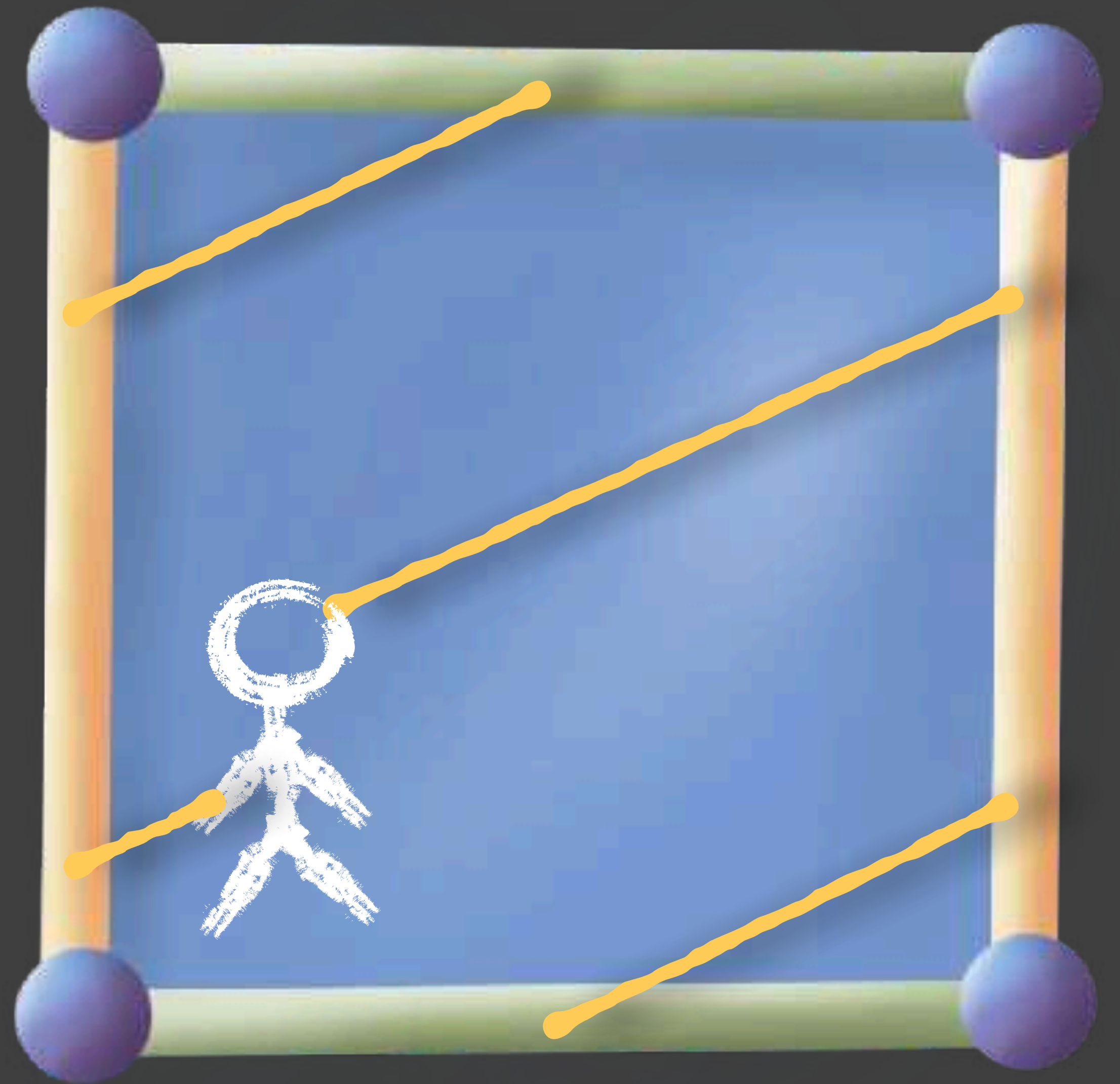
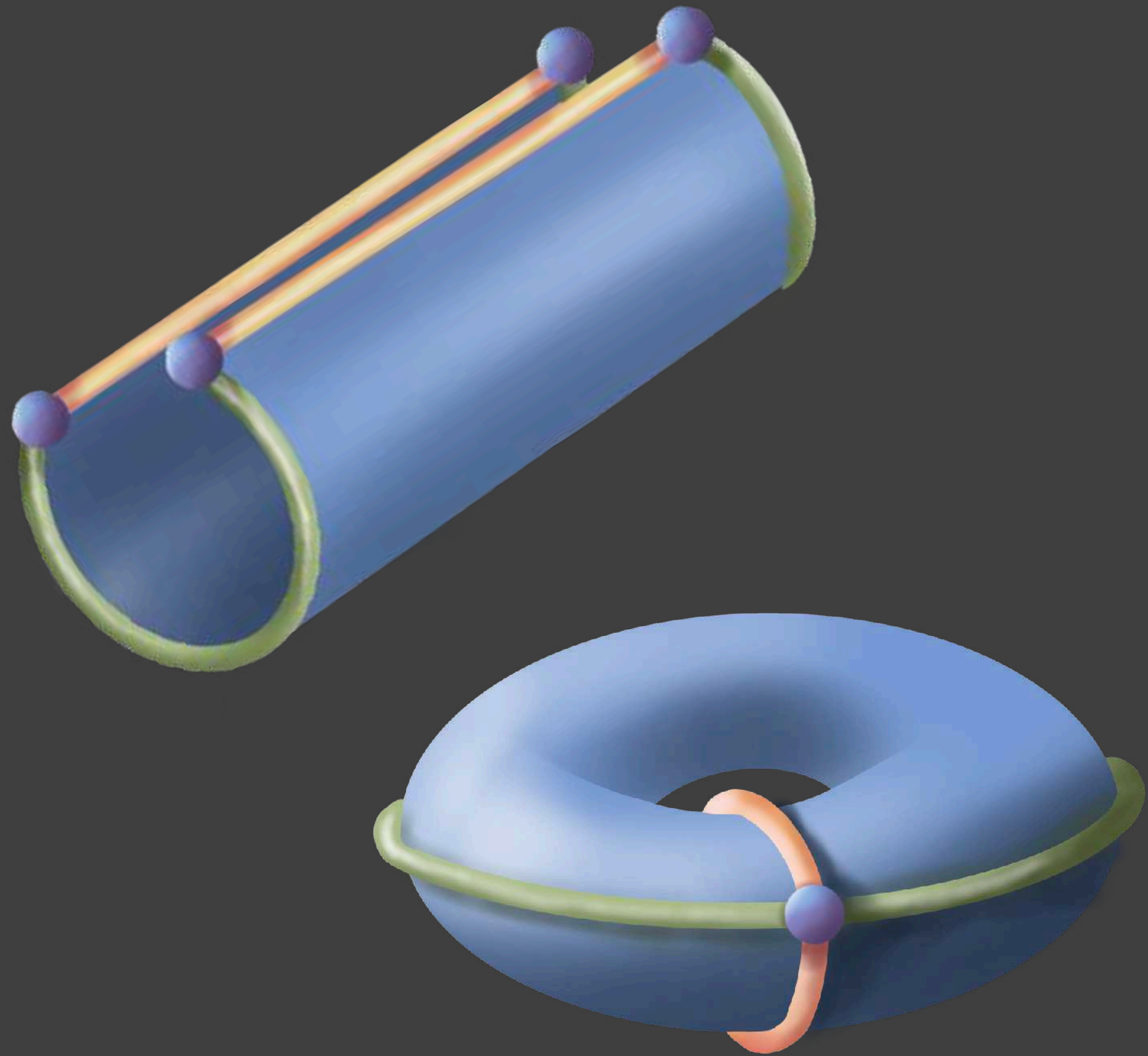
*Raytracing in
multiply
connected
manifolds*

Light on a closed manifold spirals around
and around.....



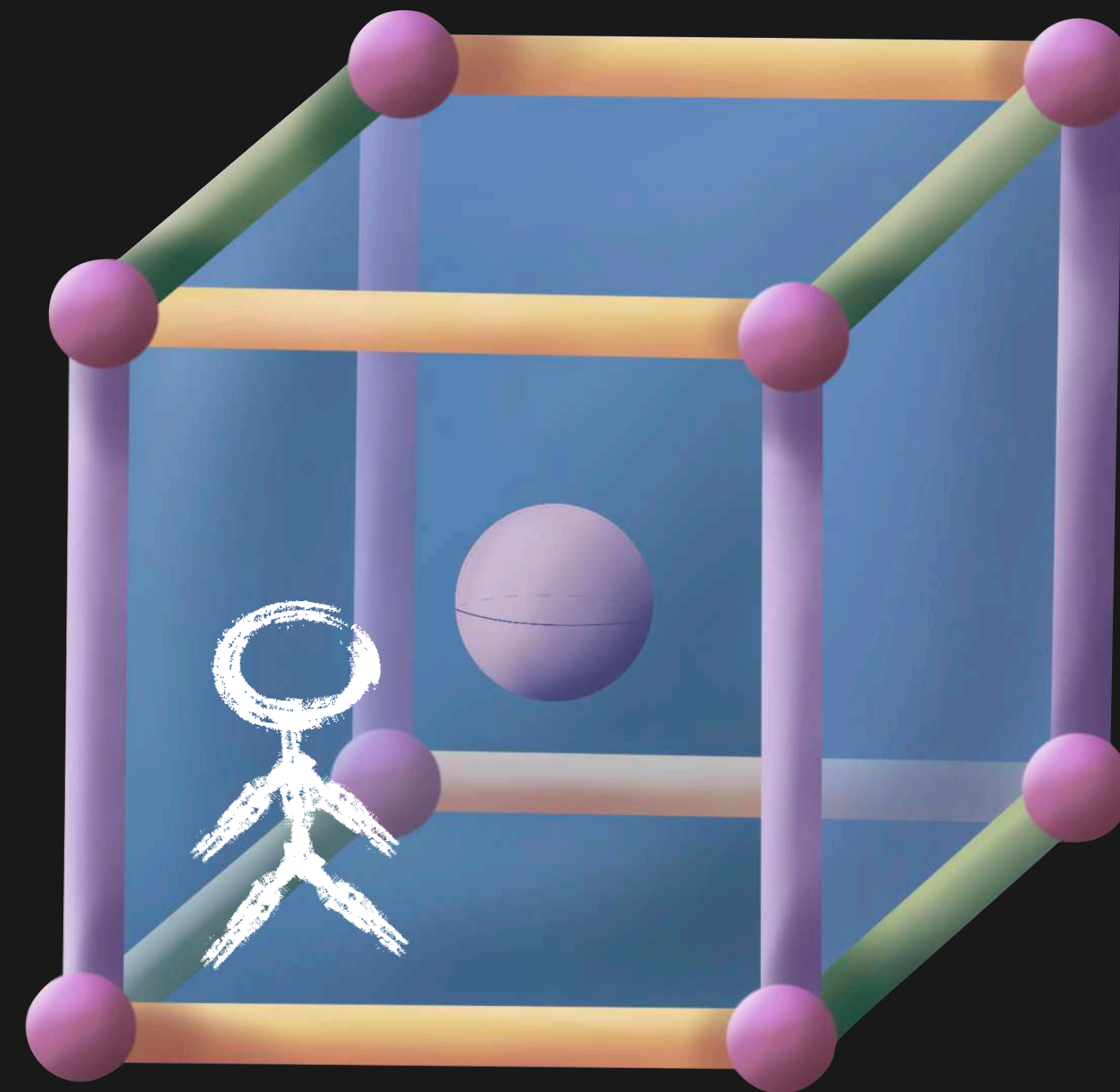
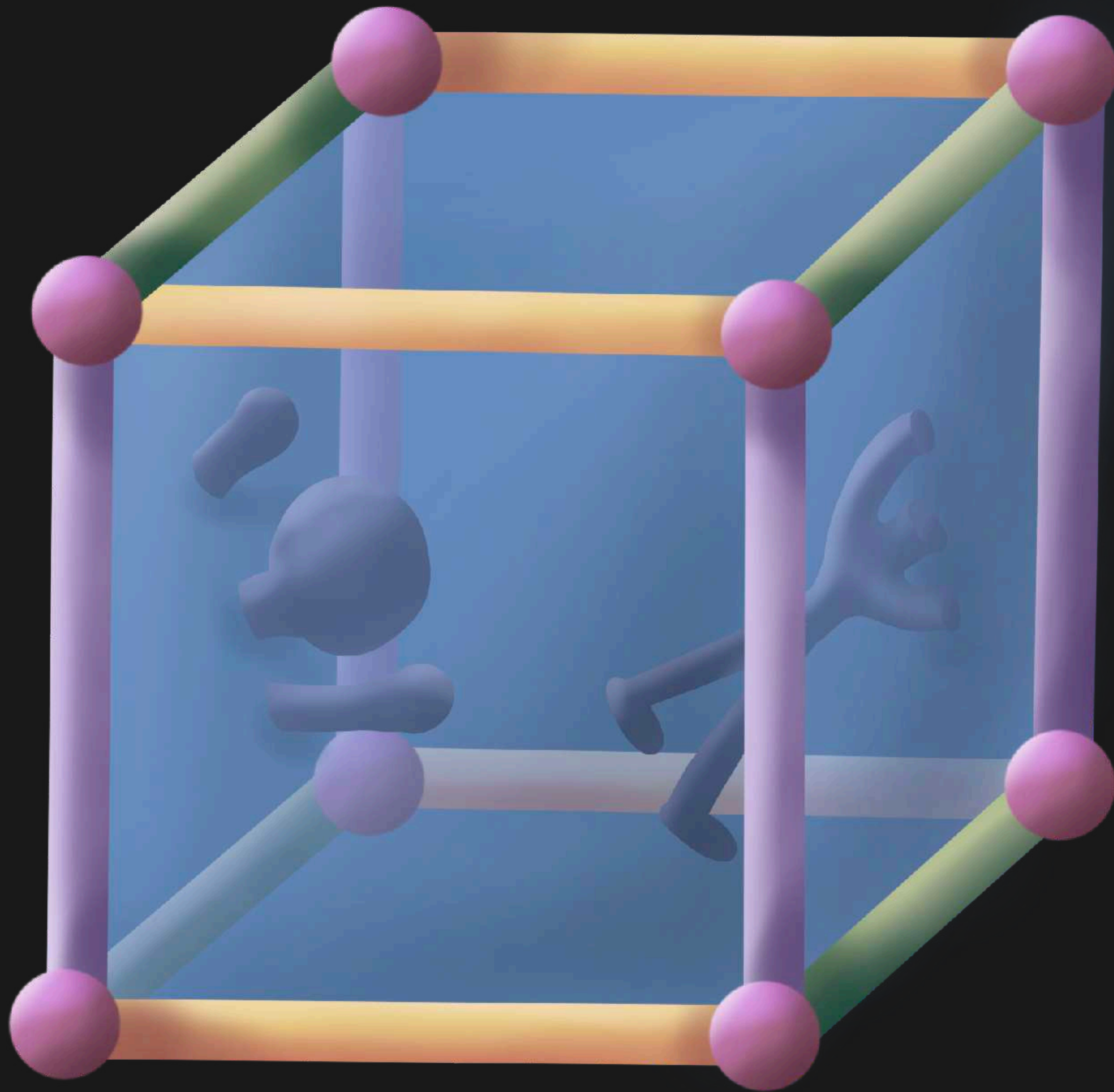


Covering spaces & raytracing

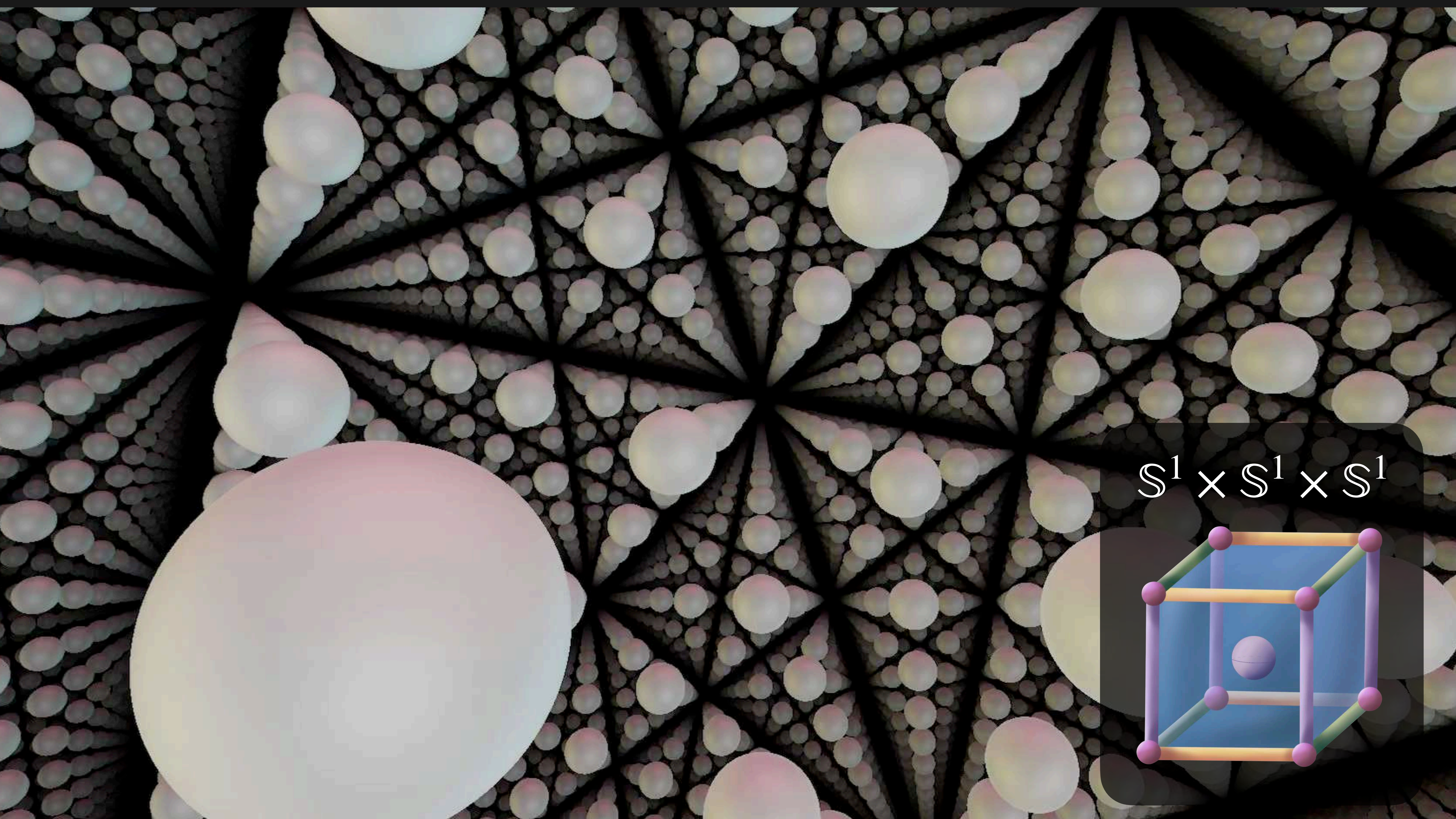


The 3-Torus

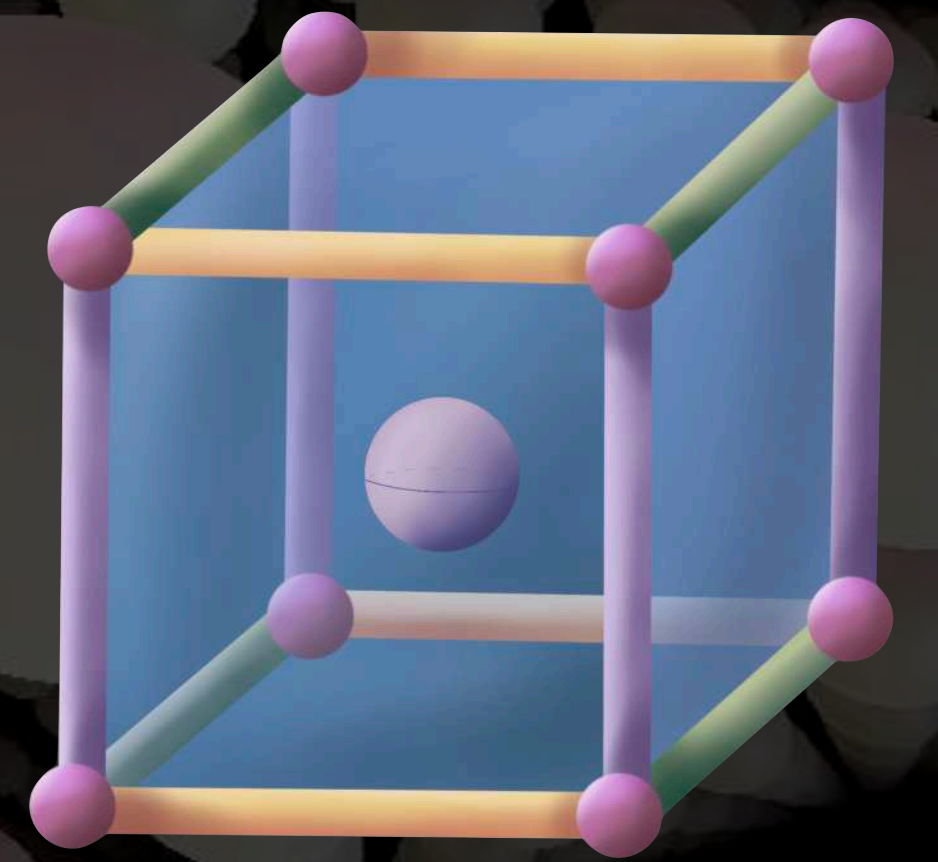
Fundamental domain is a cube / parallelepiped with opposite sides identified via translation.

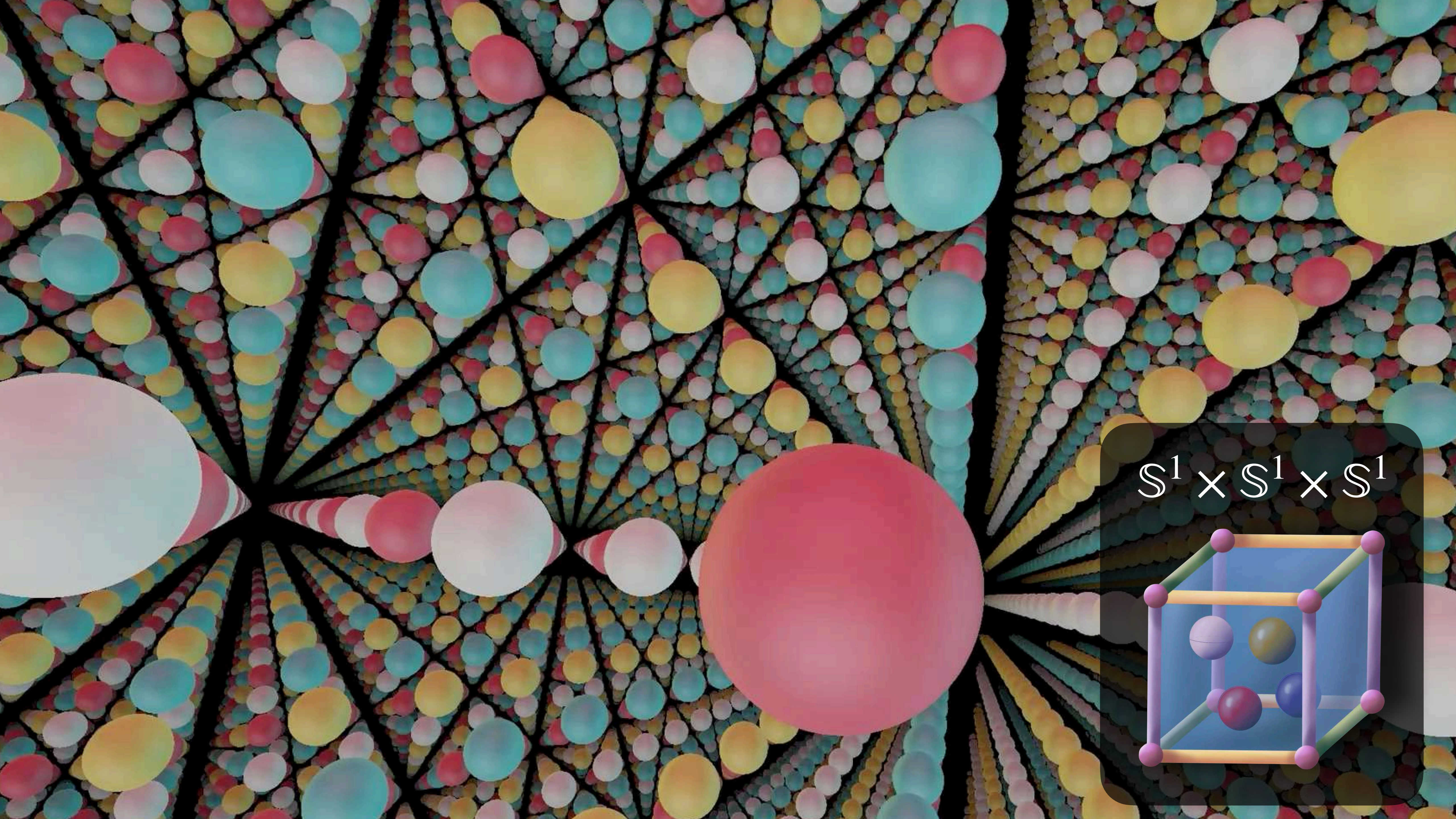


What do you expect to see, if the 3-torus is empty except for a single white sphere?

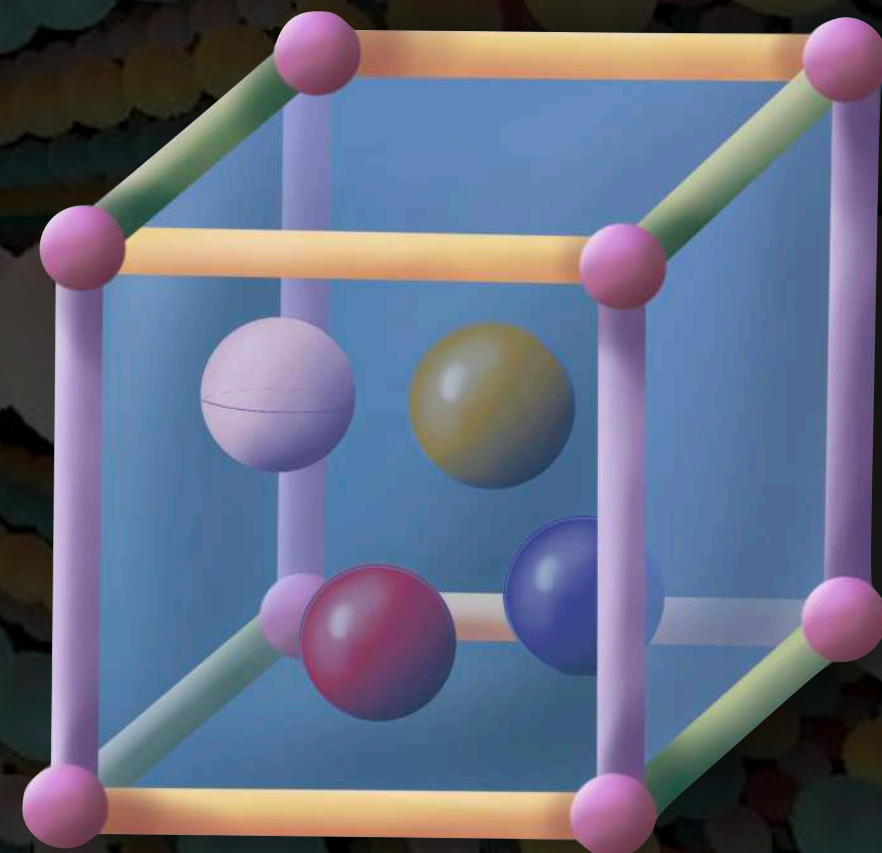


$$\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$$

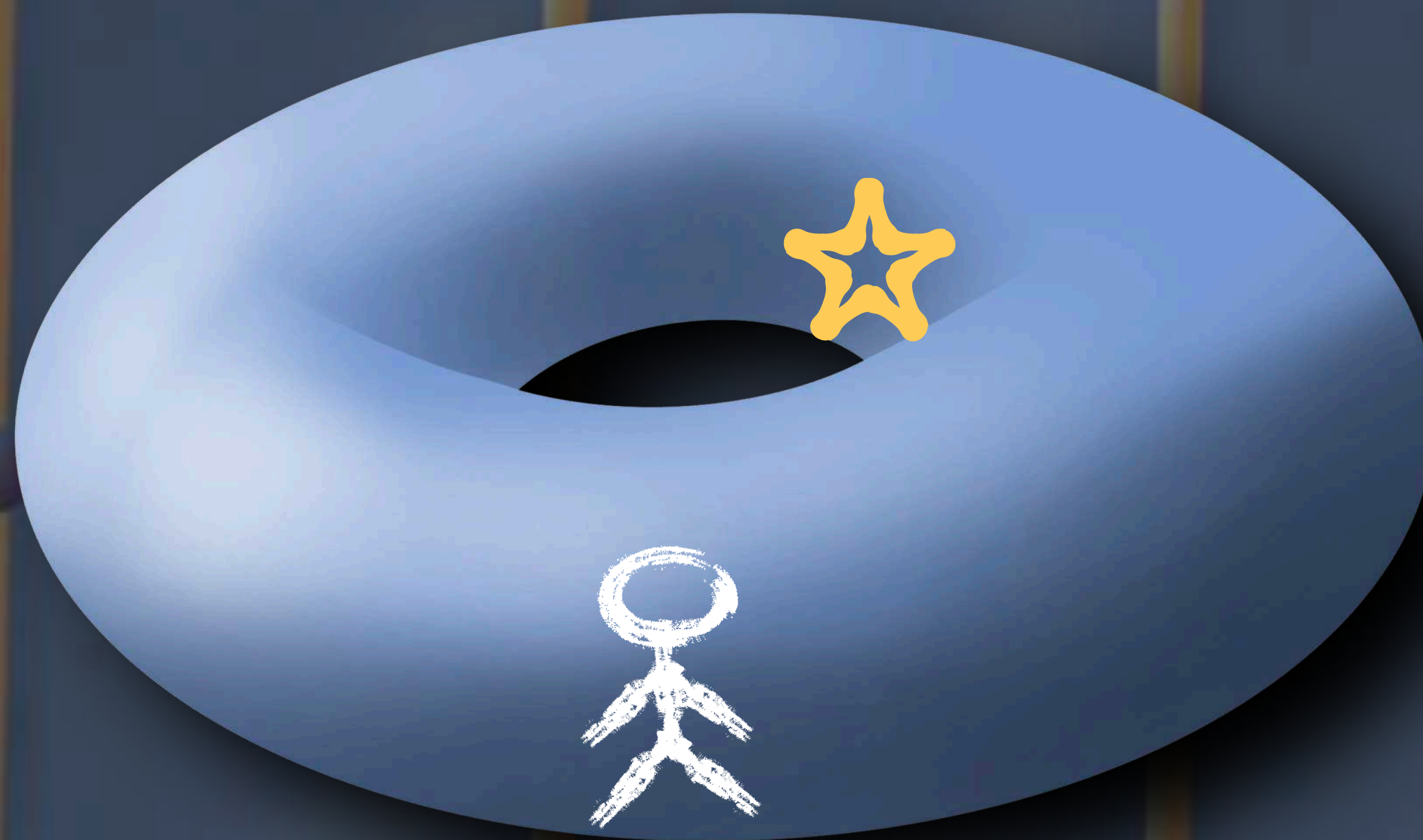


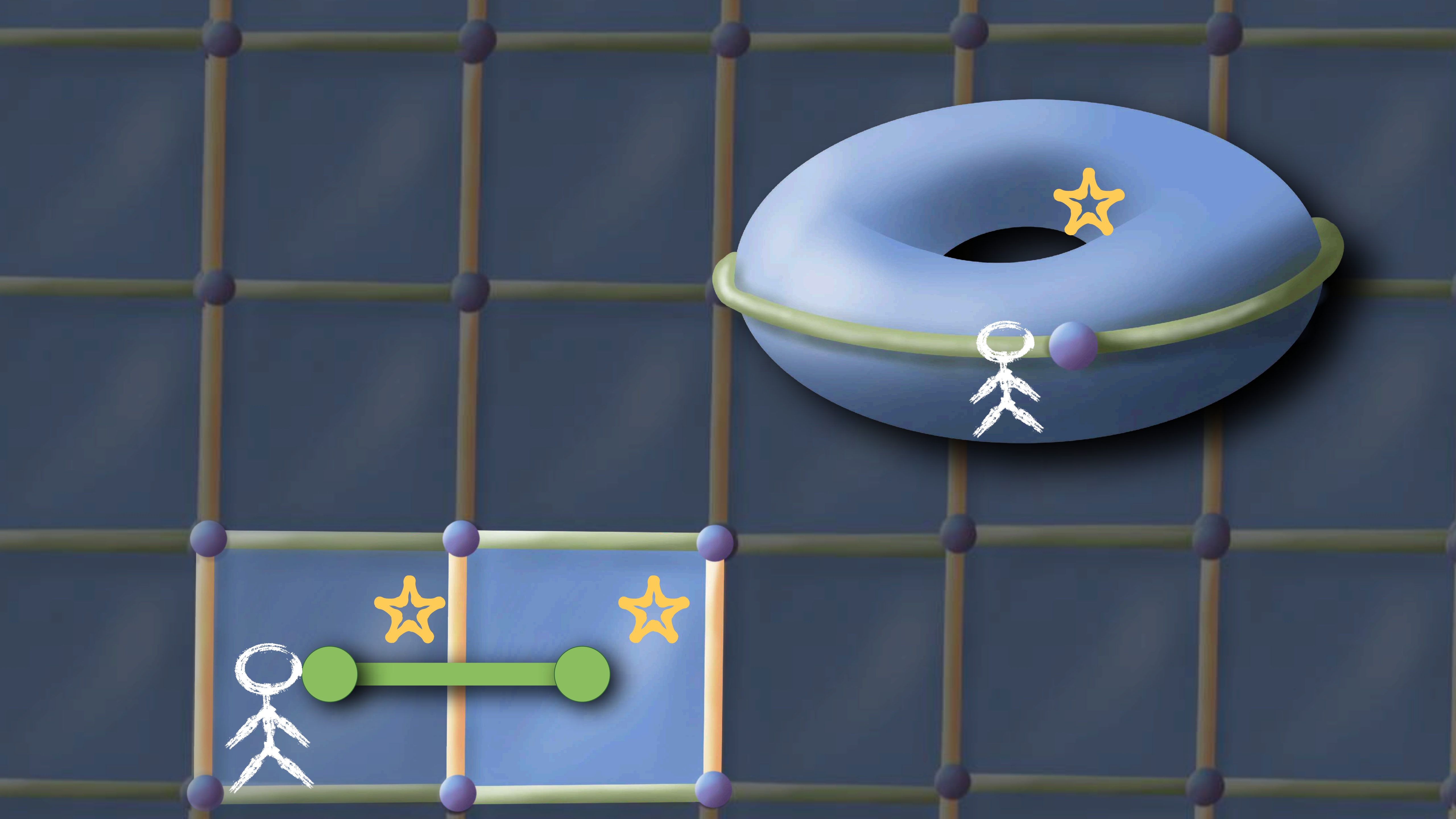


$$\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$$



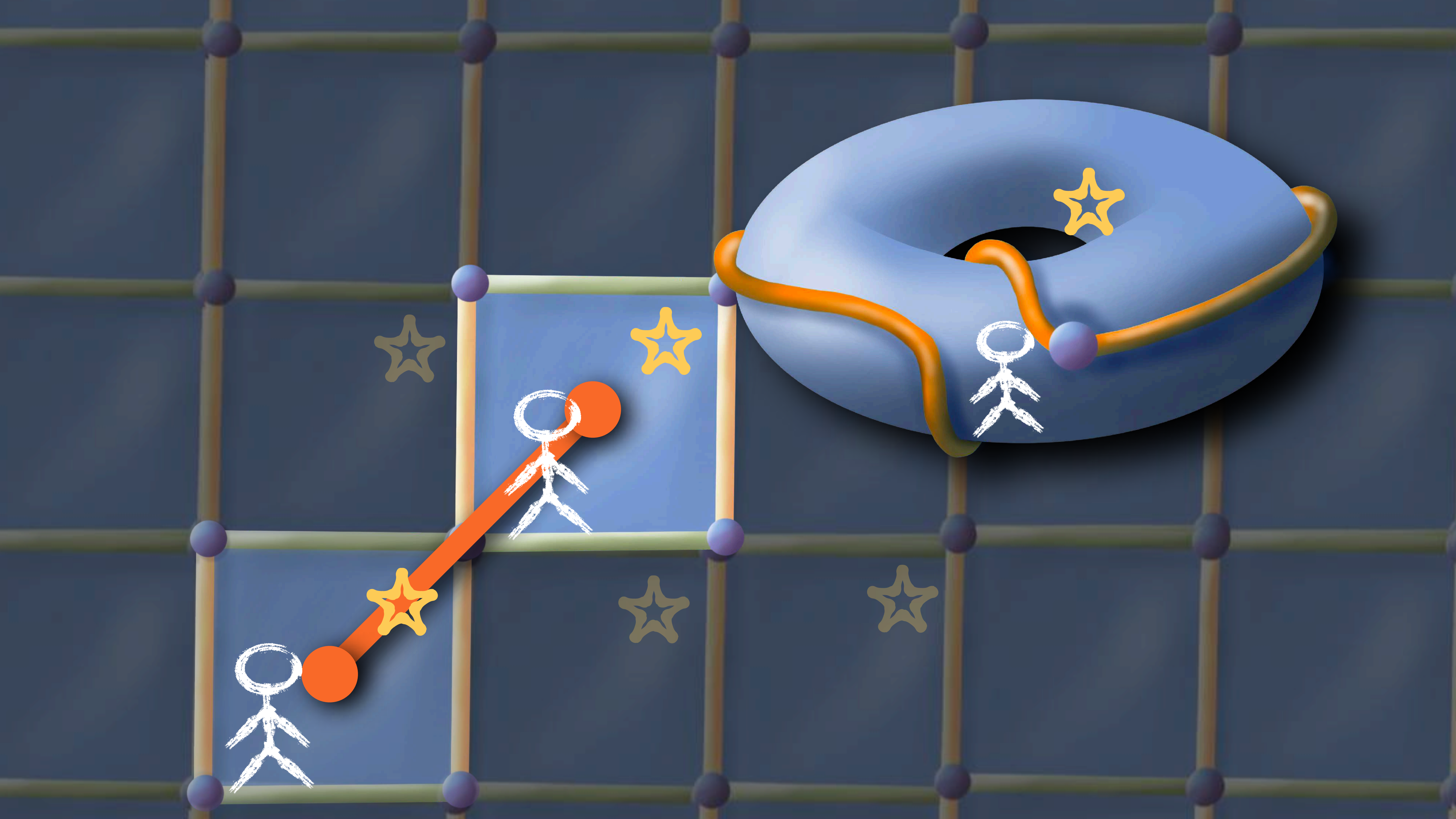
How do we
interpret what
we are seeing?



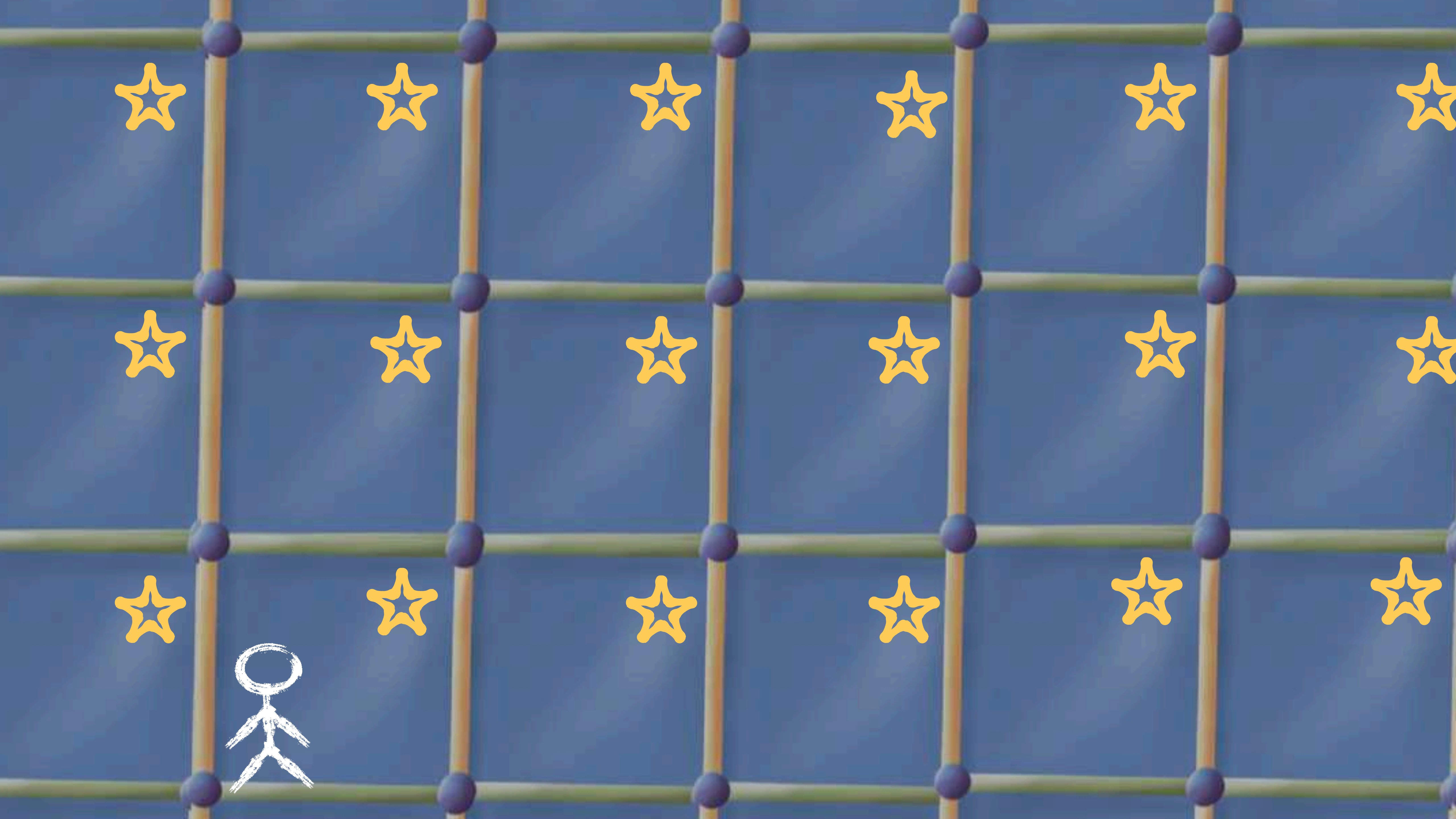




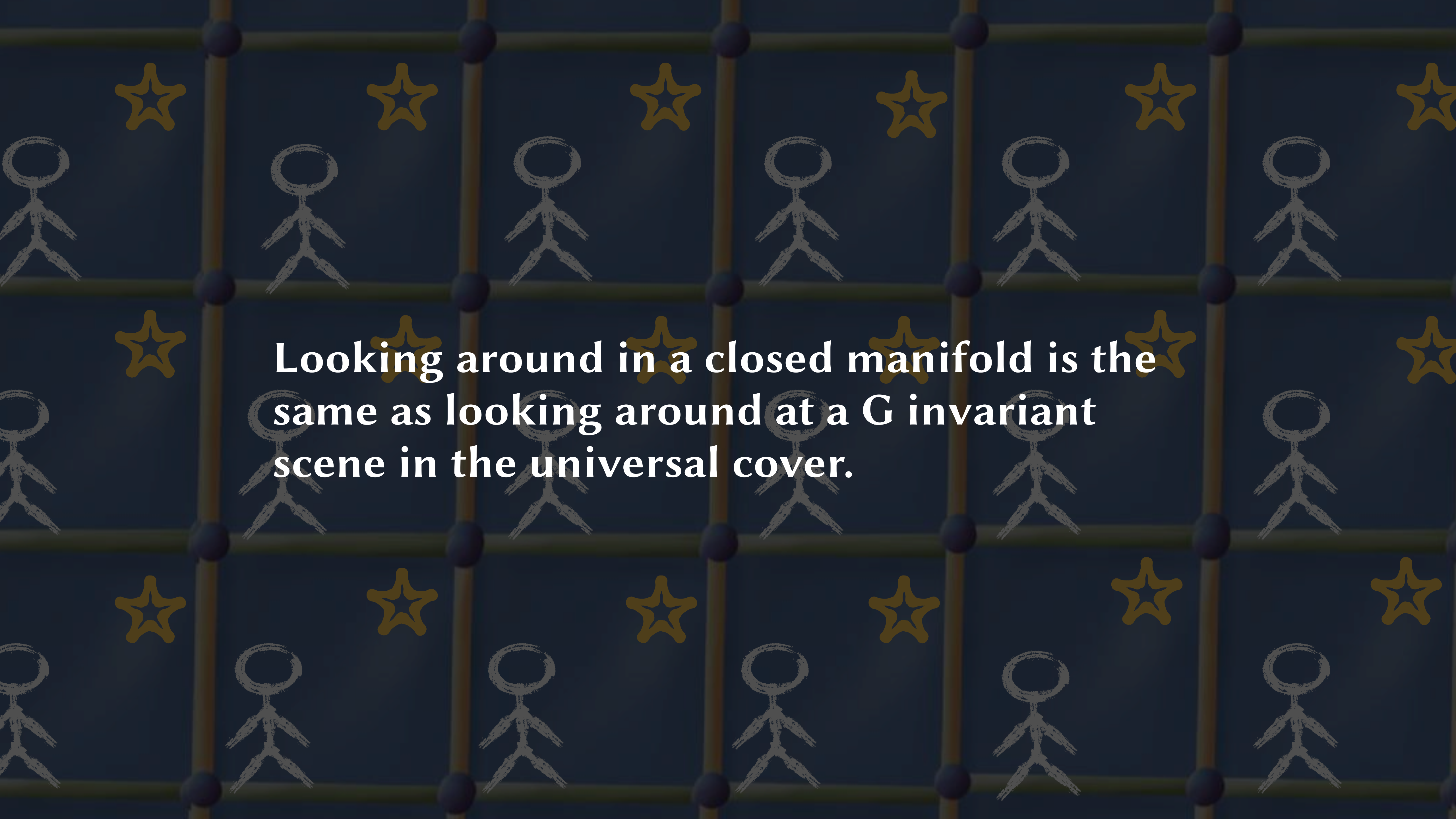






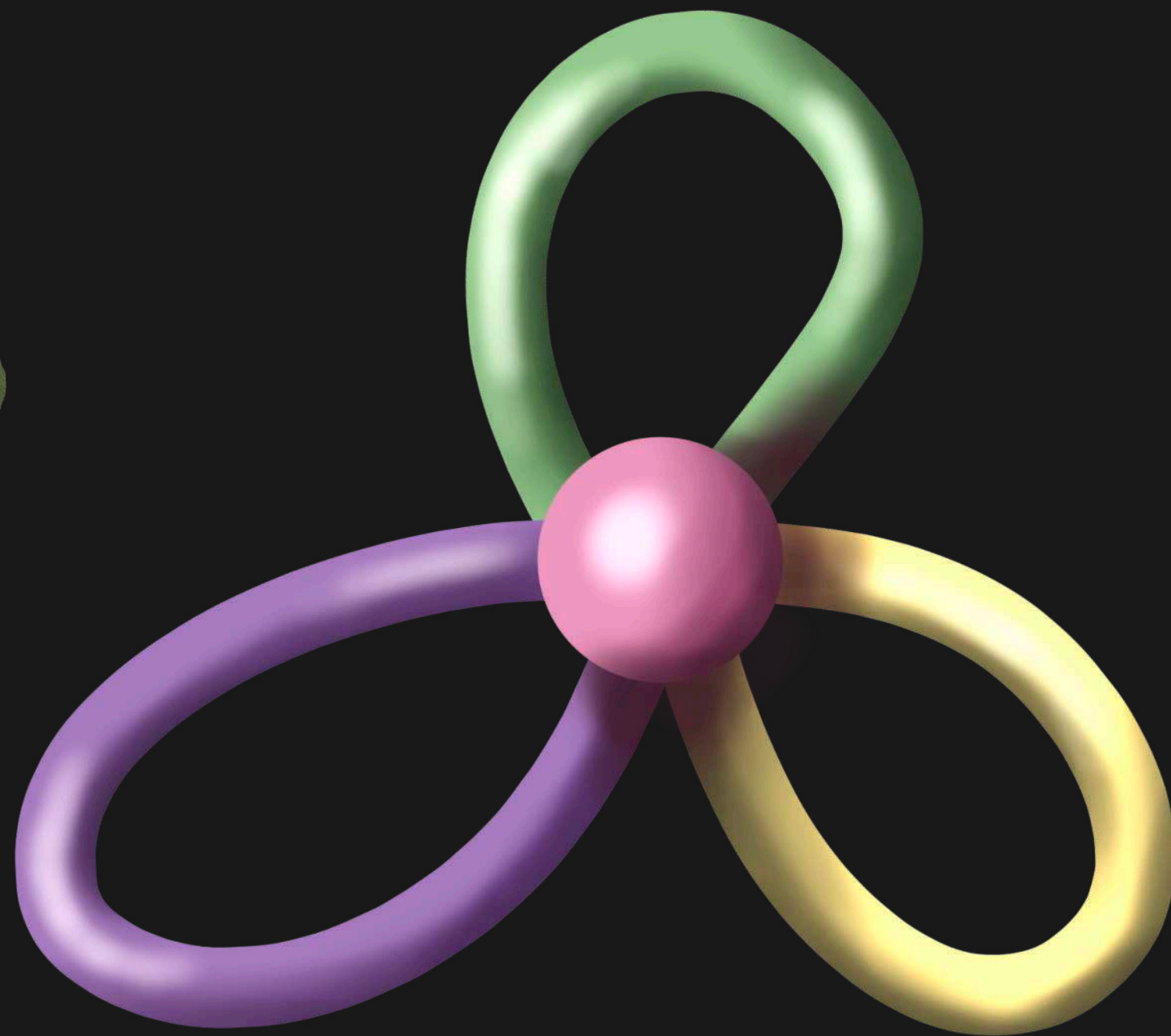
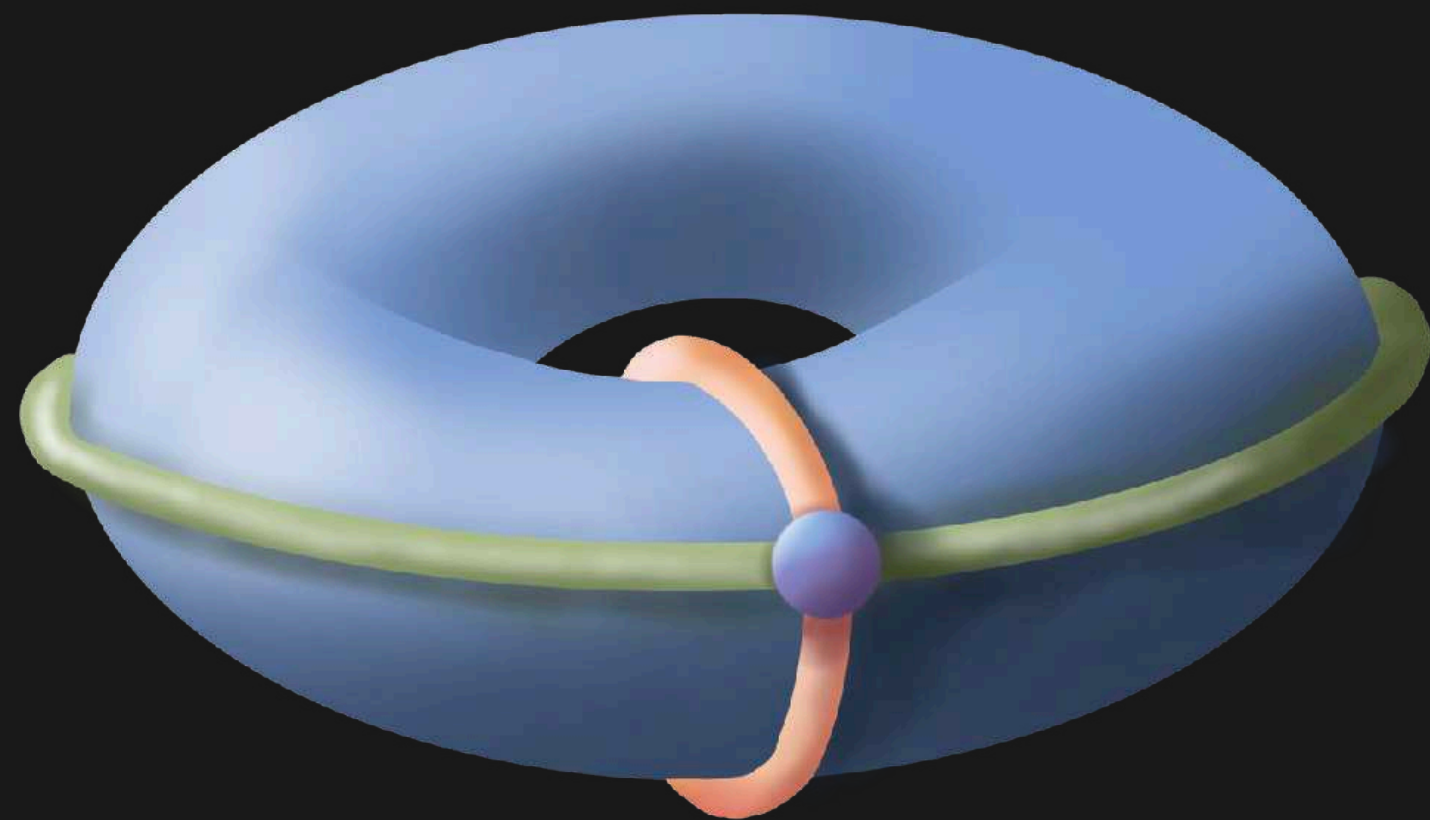




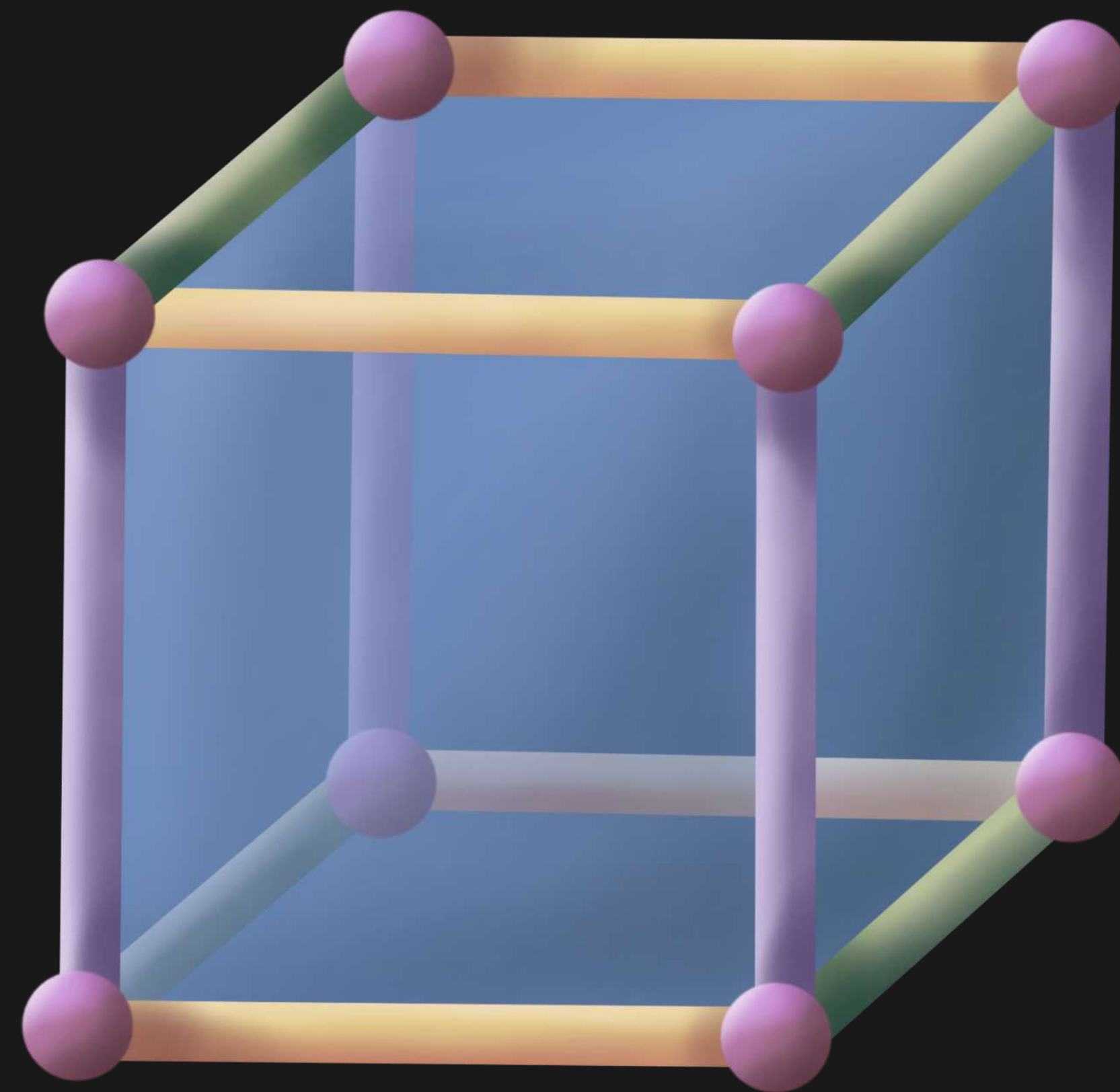
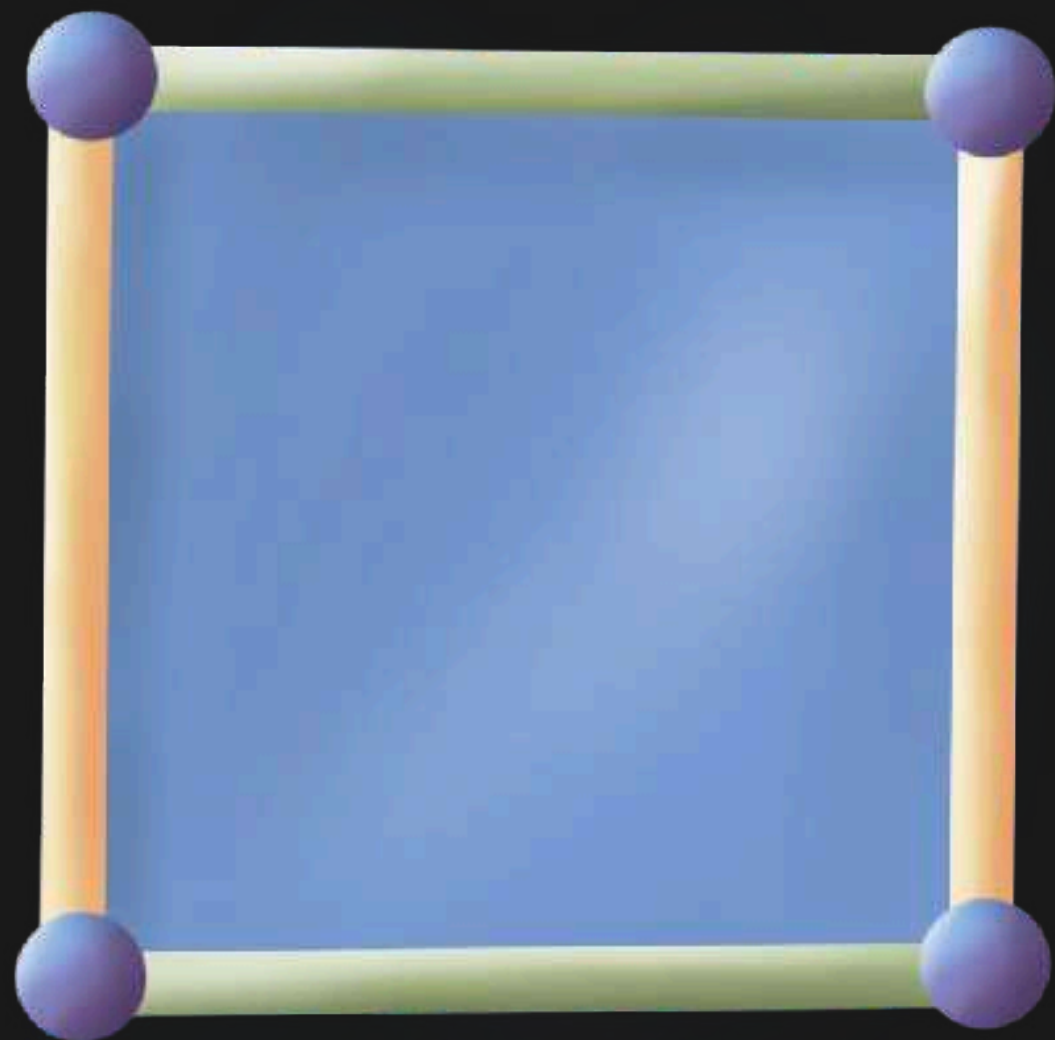
The background features a repeating pattern of stylized stick figures and stars. The stick figures are white with a circular head and three limbs, arranged in a grid. The stars are yellow with a five-pointed shape, also in a grid. The background is dark blue with a grid of thin yellow lines and small dark blue dots at the intersections.

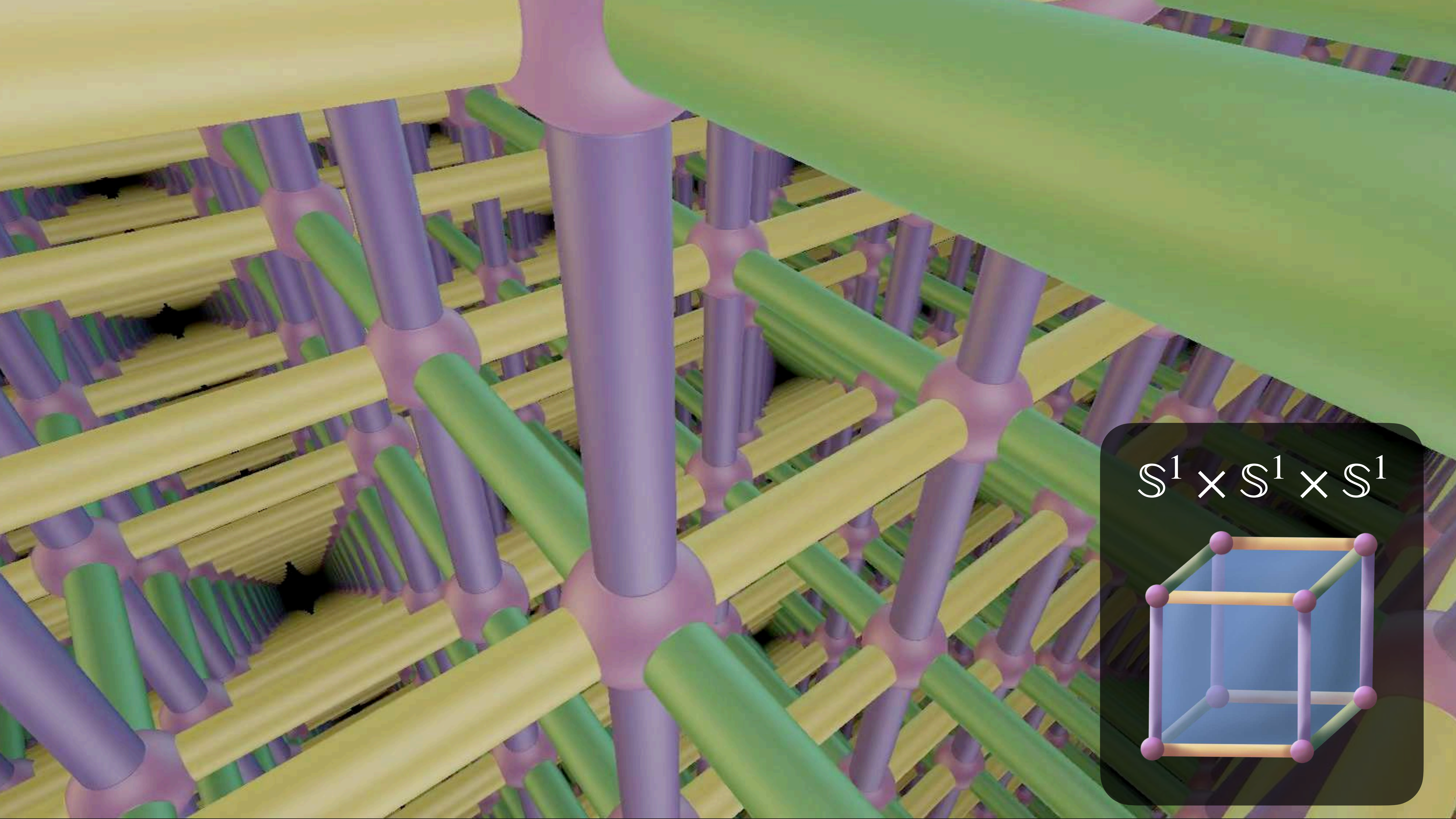
Looking around in a closed manifold is the
same as looking around at a G invariant
scene in the universal cover.

The 3-Torus

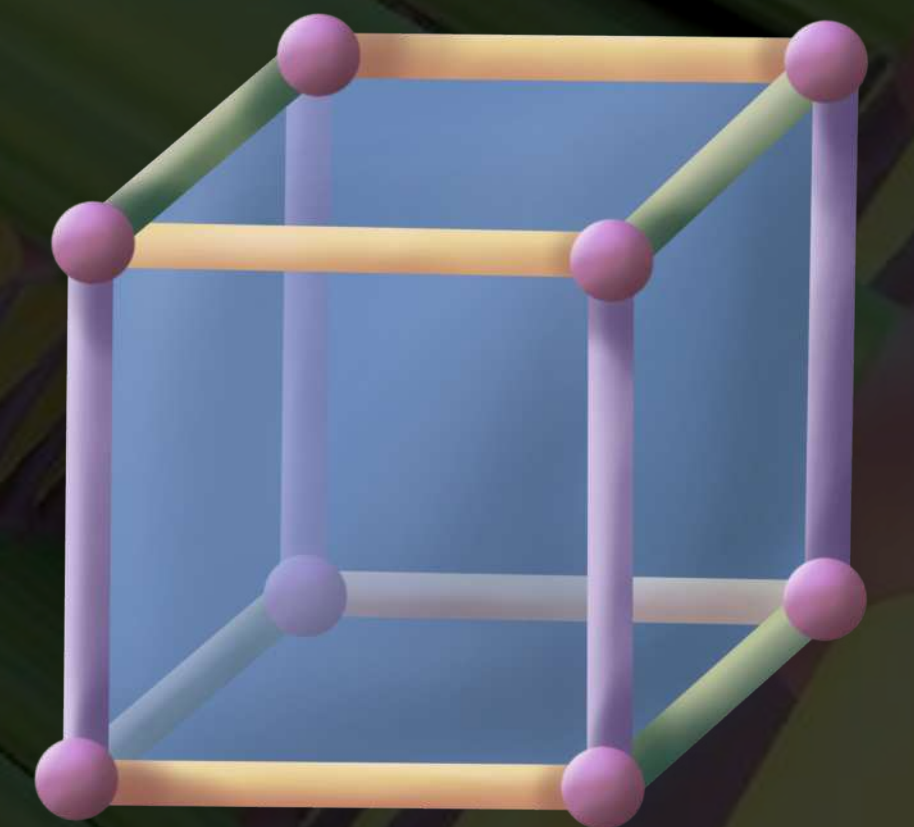


Generators of π_1





$$\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$$



Seifert-Weber Dodecahedral Space

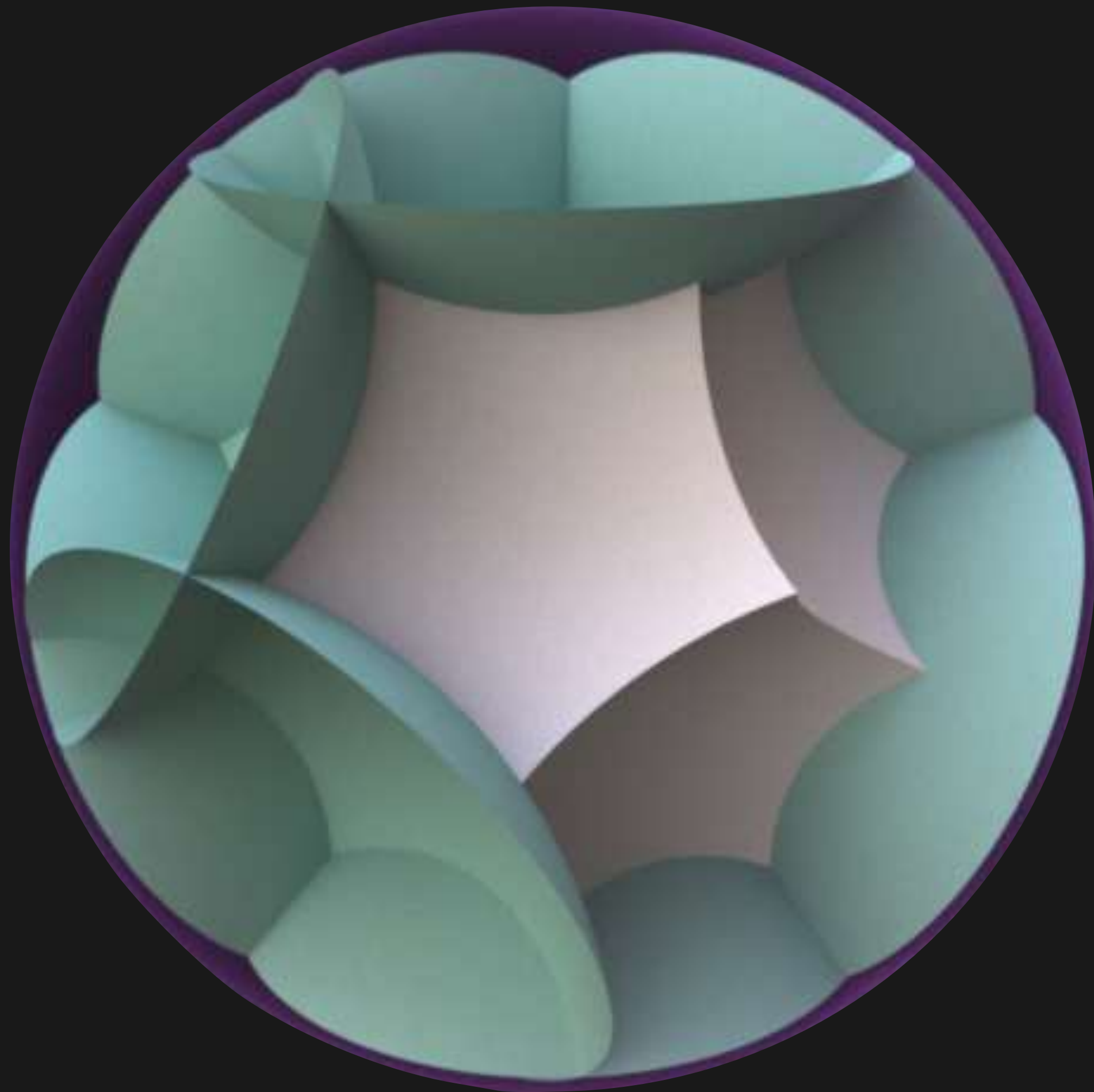
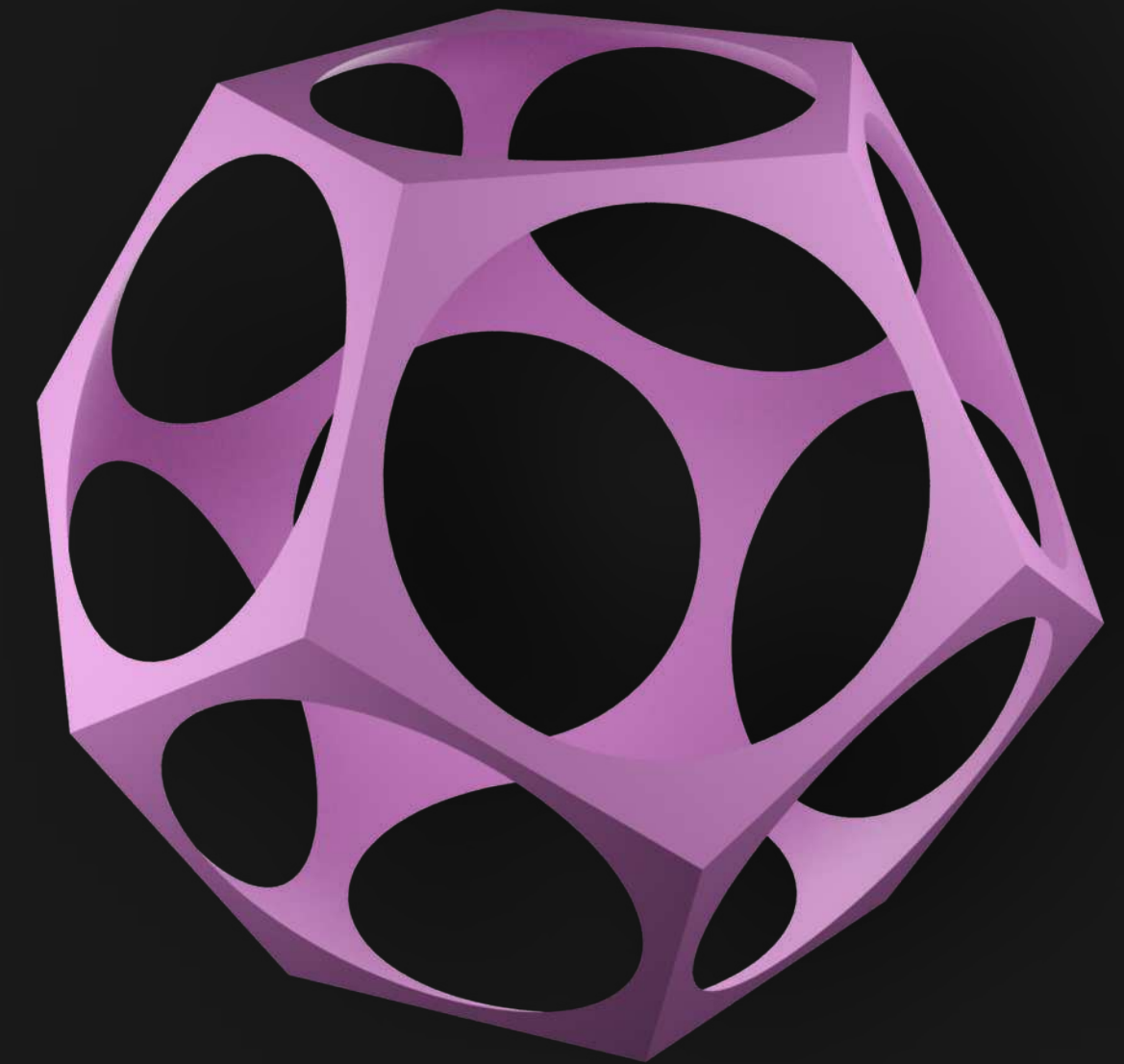


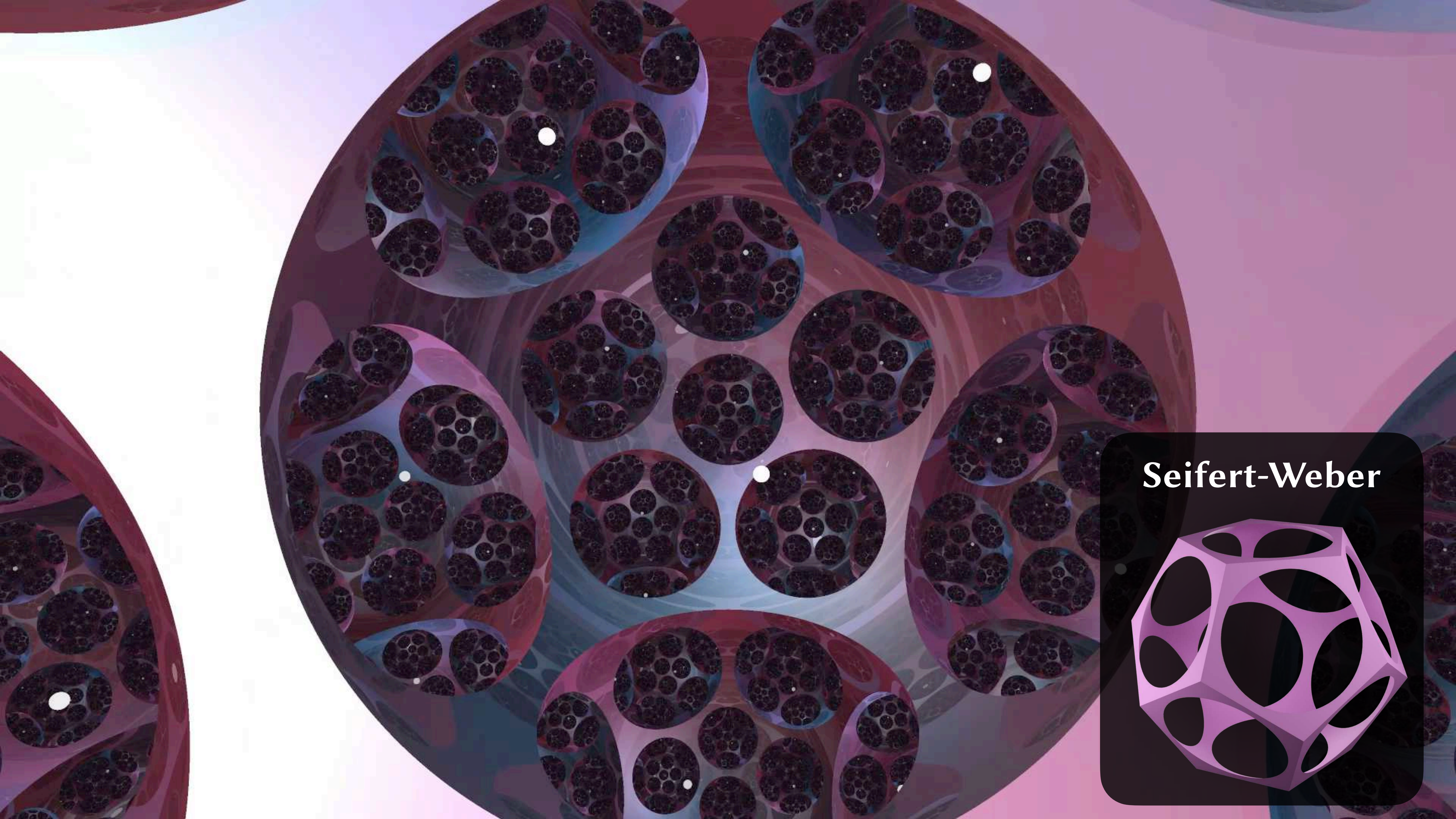
Image: Arnaud Cheritat

Fundamental domain is a hyperbolic dodecahedron, with opposite sides paired by a $3/10$ twist.

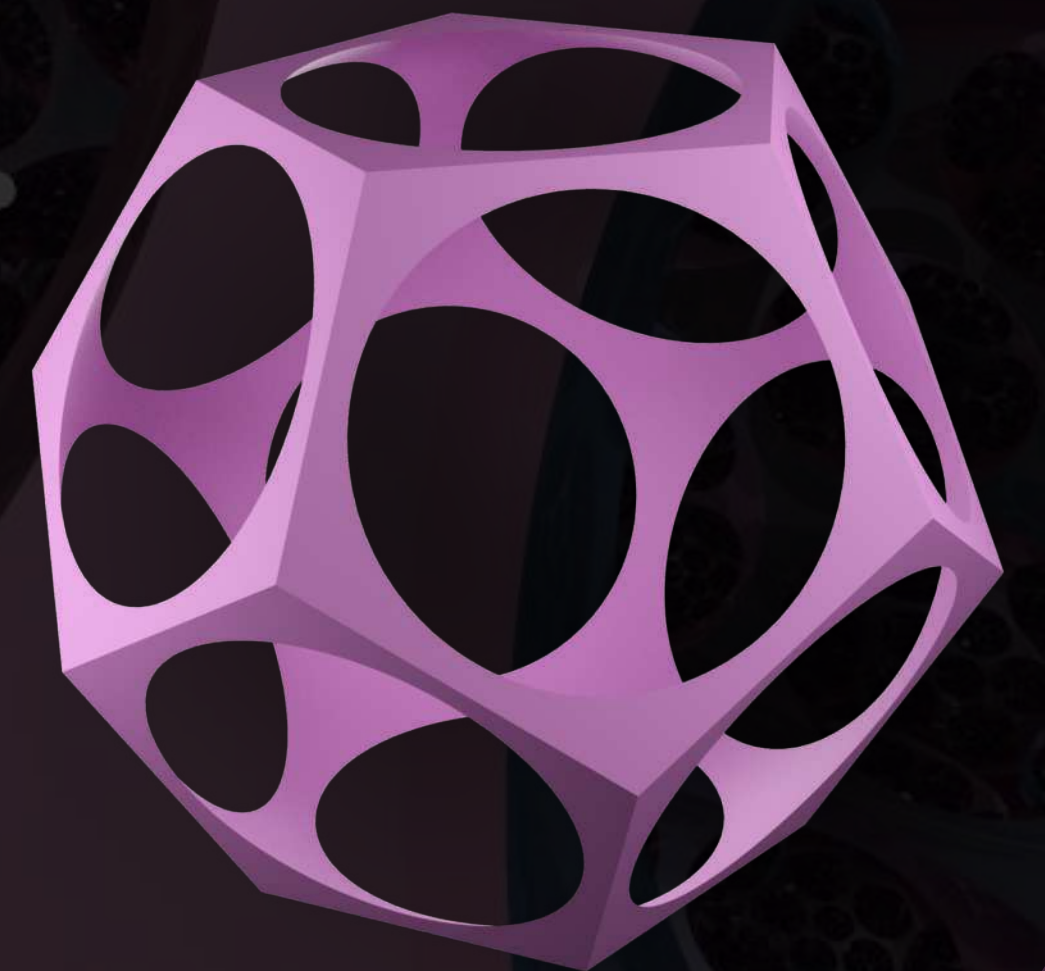


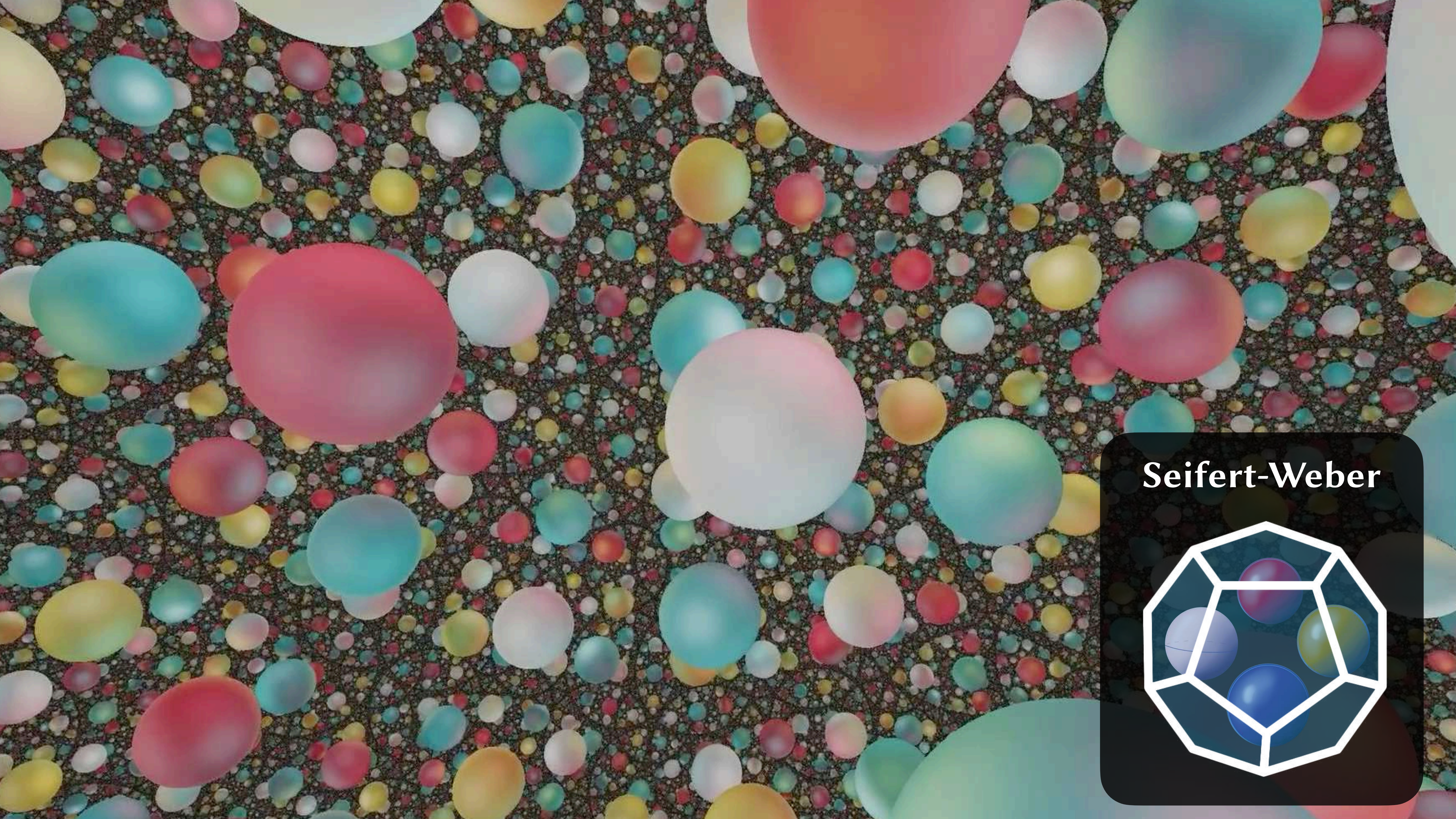
To raytrace:

Domain walls built by deleting a hyperbolic sphere from its center



Seifert-Weber

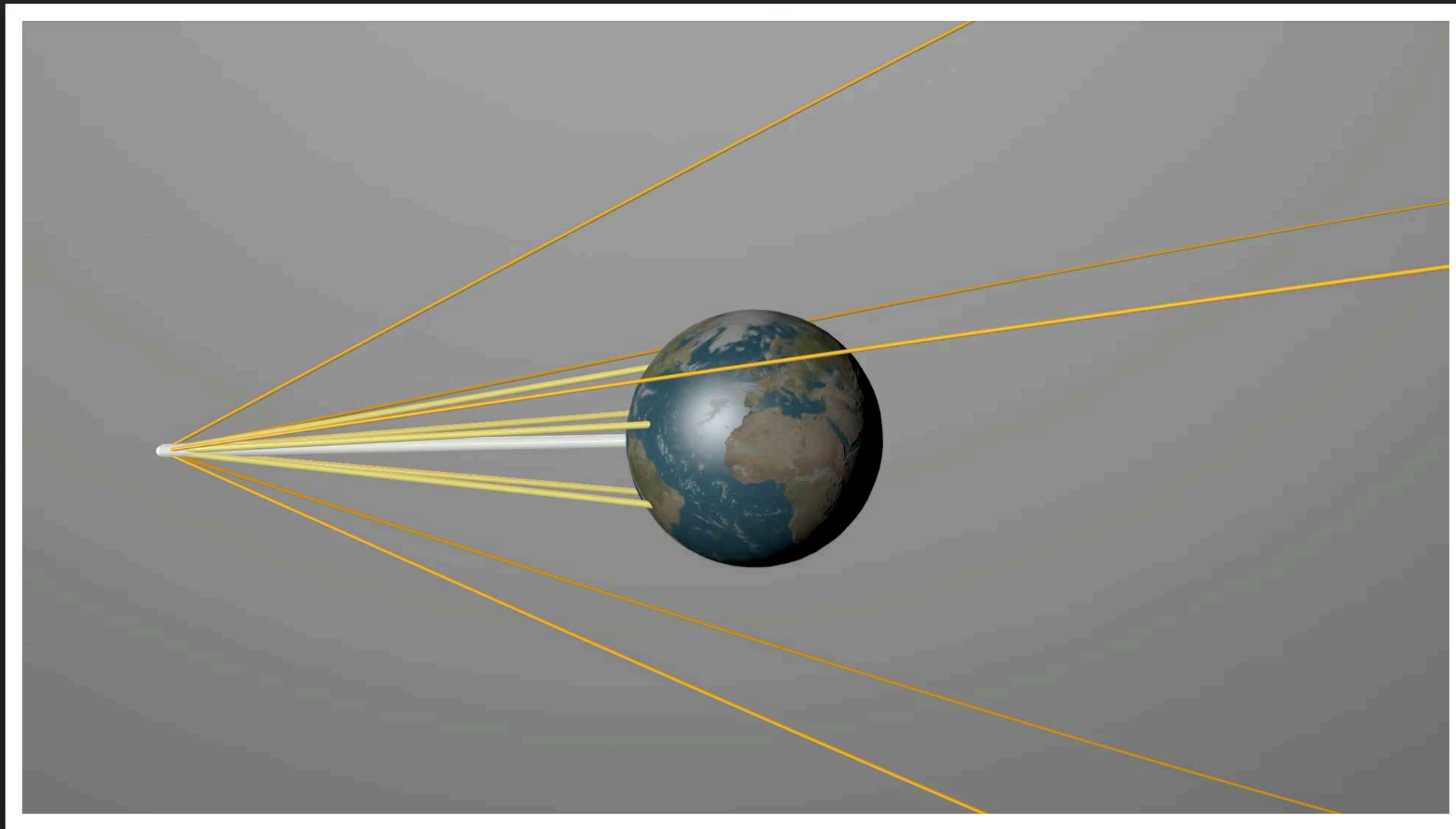




Seifert-Weber



IIb Geometry & Vision



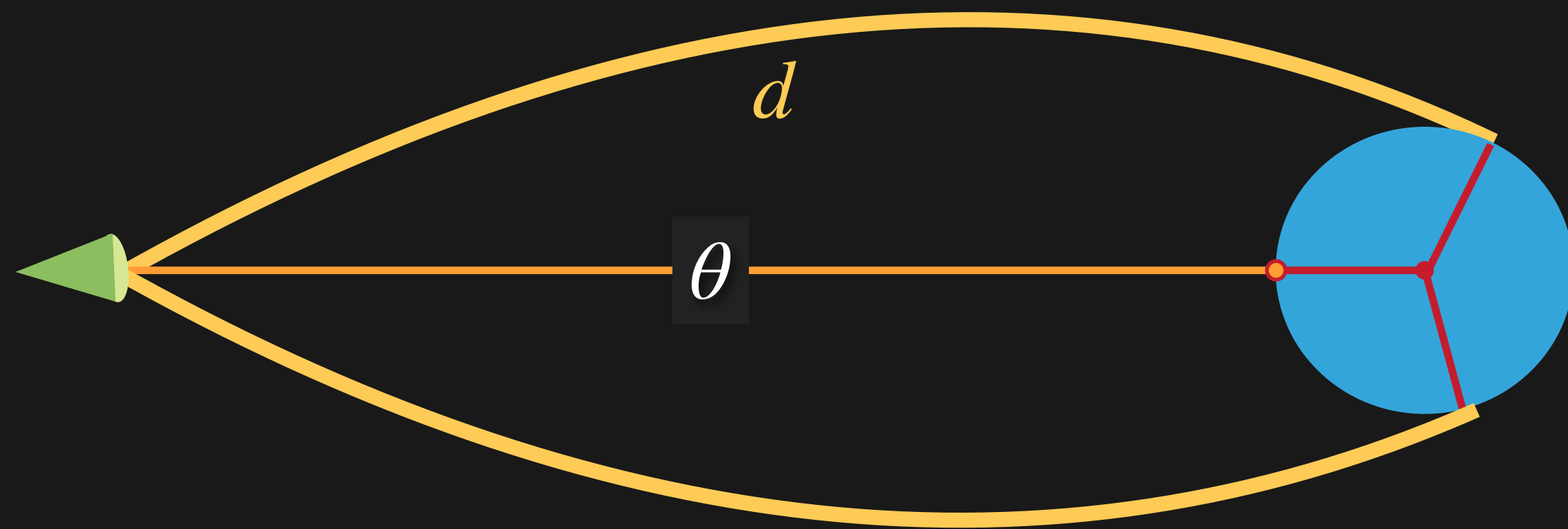
Geodesics and
Curvature

$$J'' + \mathfrak{R}J = 0$$

Positive Curvature

Geodesics reconverge

Distant objects appear large

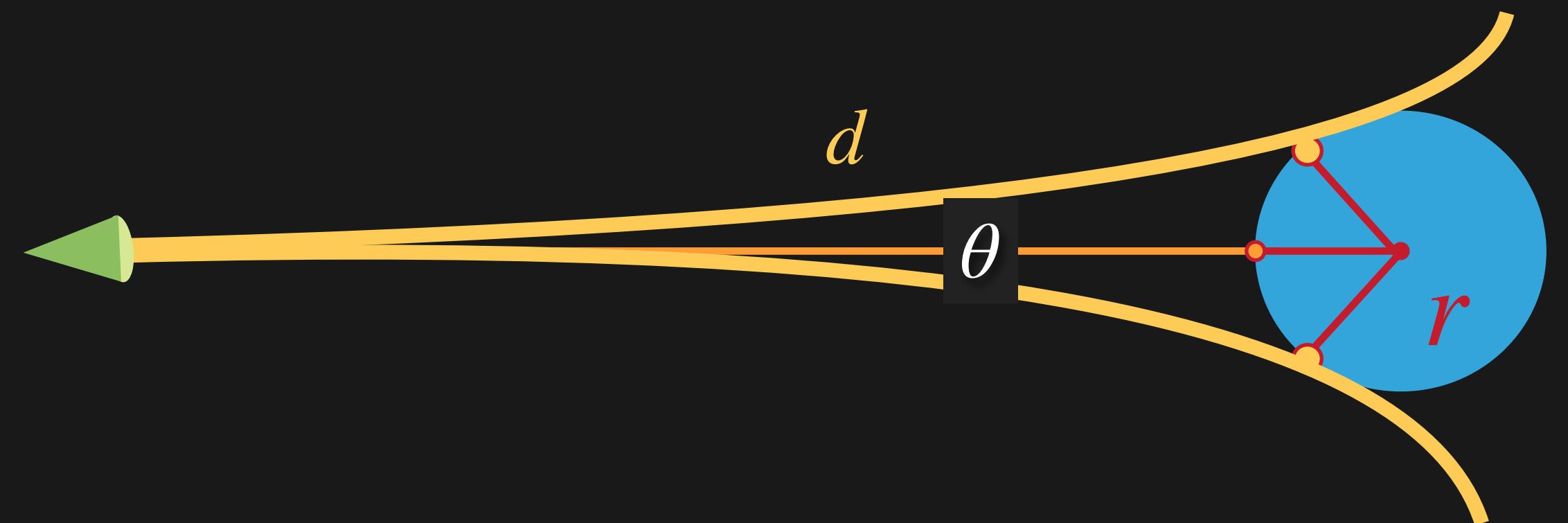


$$\theta \approx 2 \tan^{-1} \left(\frac{\tan r}{\sin d} \right) \cong \frac{2r}{\sin(d)}$$

Negative Curvature

Geodesics diverge

Distant objects appear small



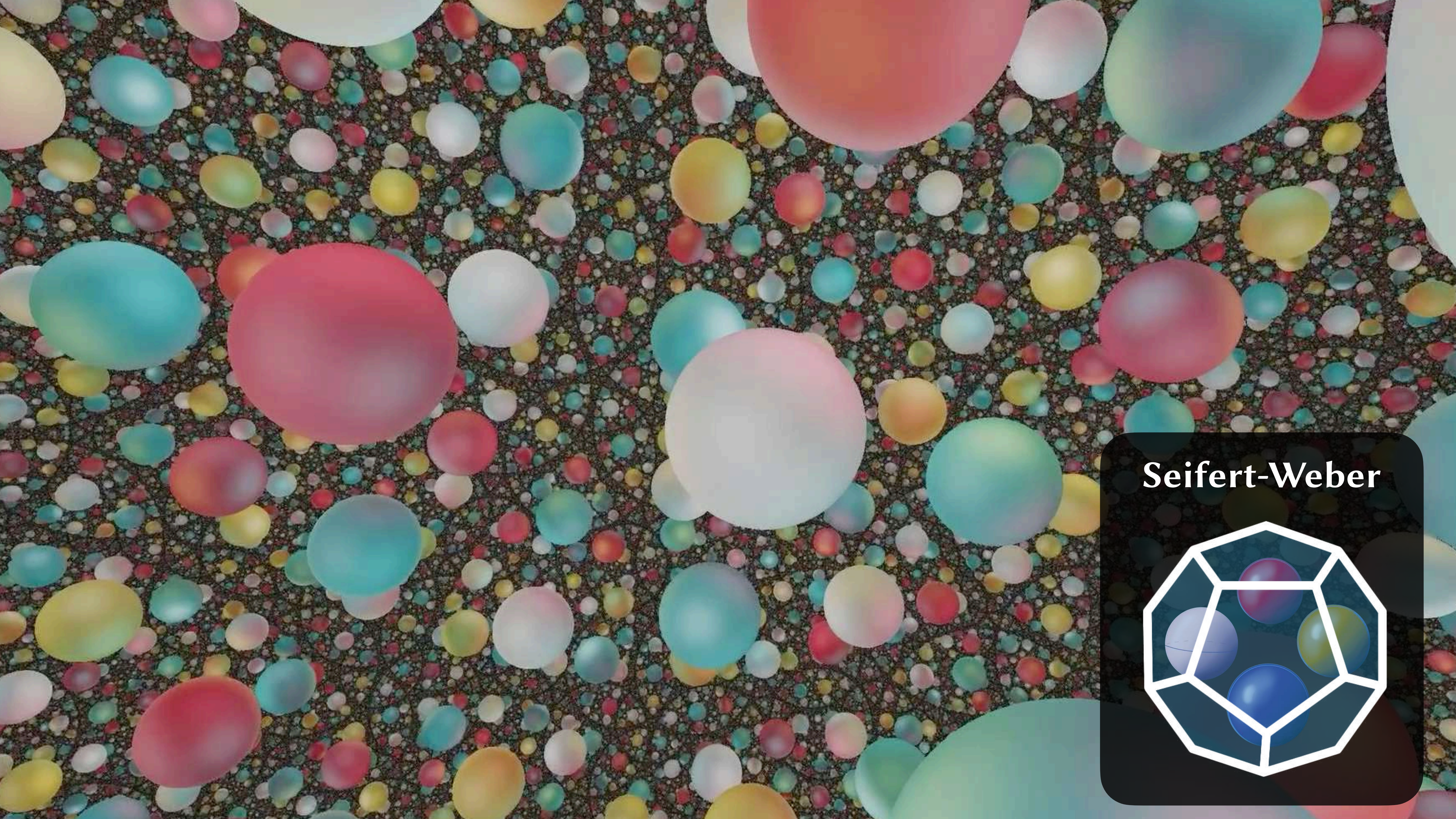
$$\theta \approx 2 \tan^{-1} \left(\frac{\tanh r}{\sinh d} \right) \cong \frac{2r}{e^d}$$

Positive Curvature



Negative Curvature





Seifert-Weber

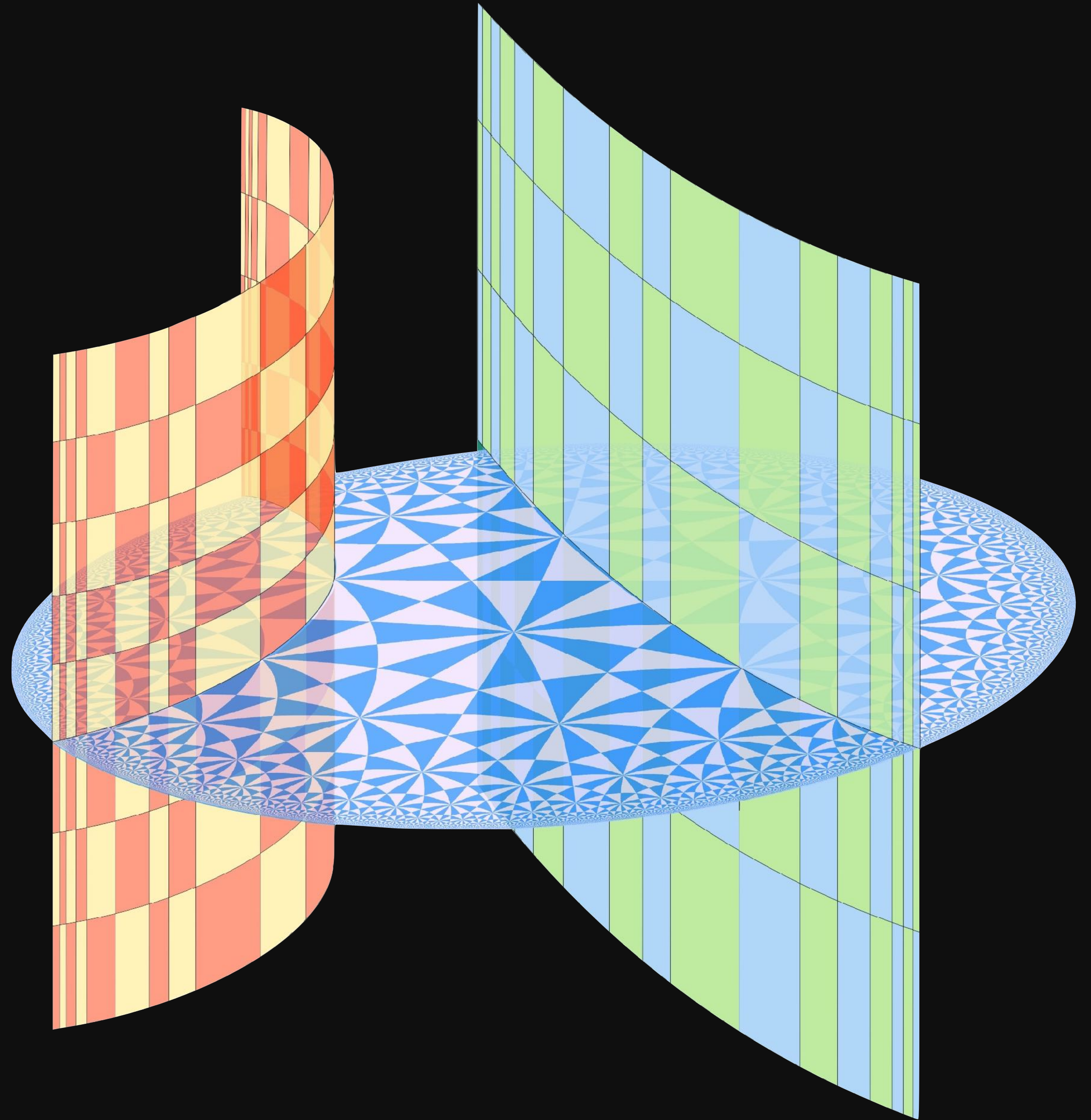


Poincare Homology Sphere



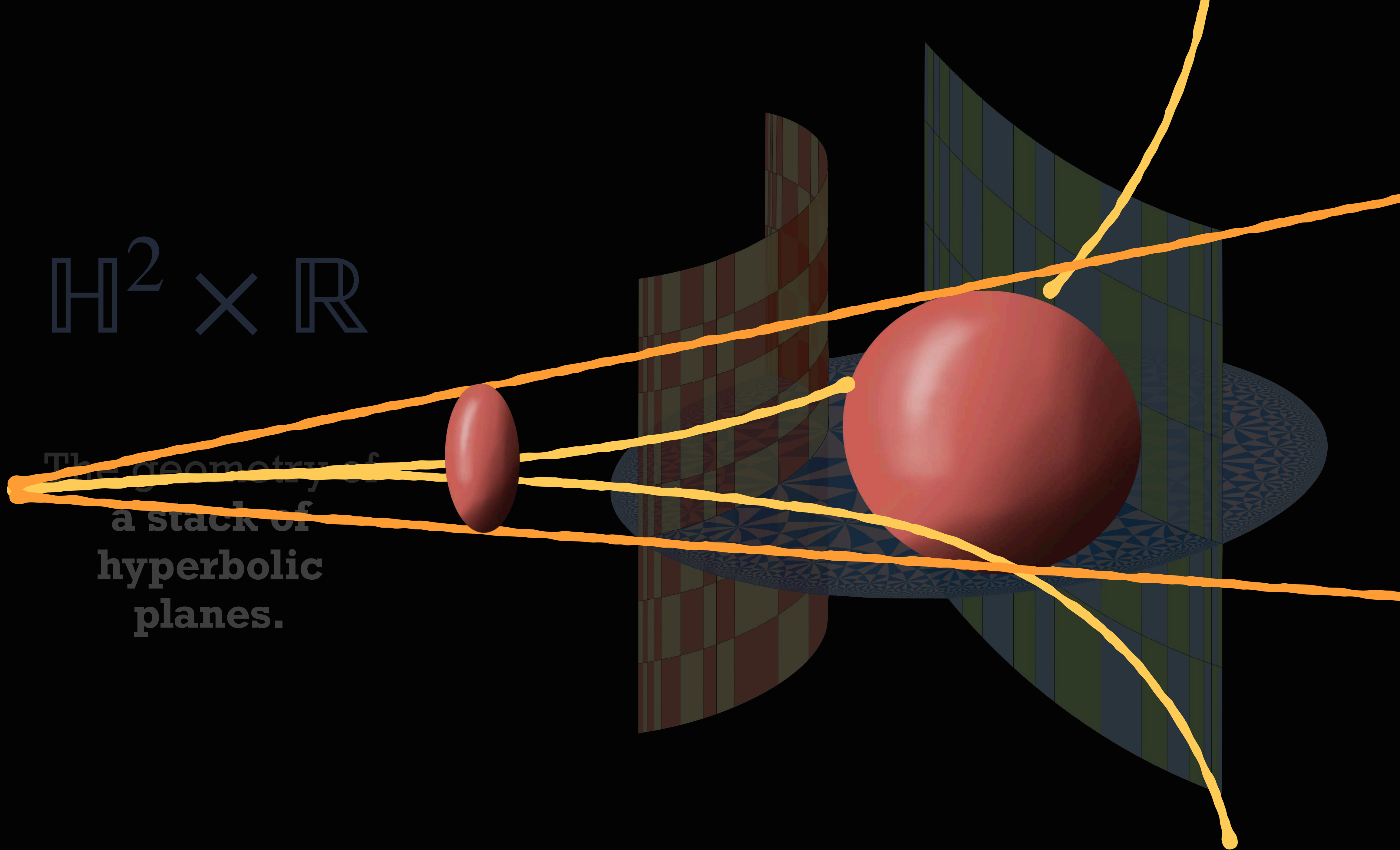
$$\mathbb{H}^2 \times \mathbb{R}$$

The geometry of
**a stack of
hyperbolic
planes.**

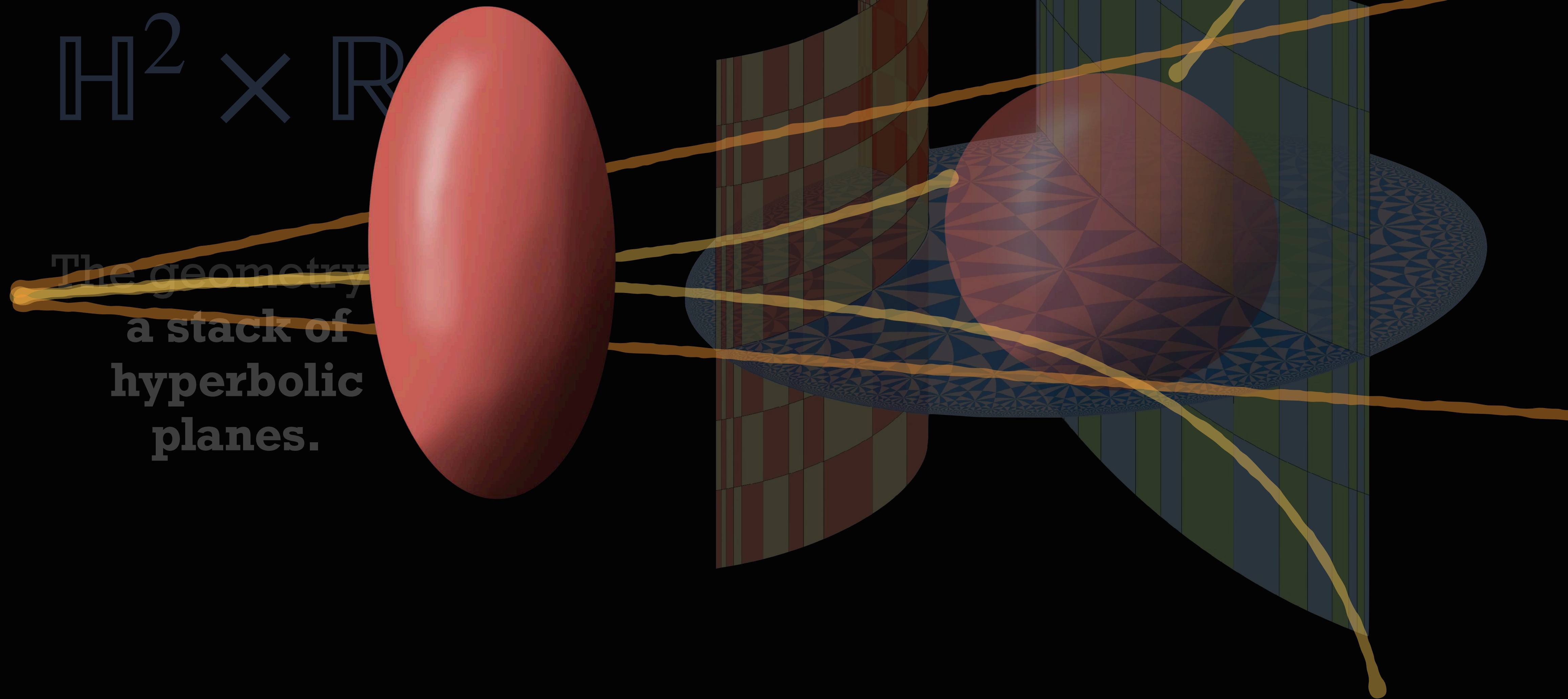


$$\mathbb{H}^2 \times \mathbb{R}$$

The geometry of
a stack of
hyperbolic
planes.



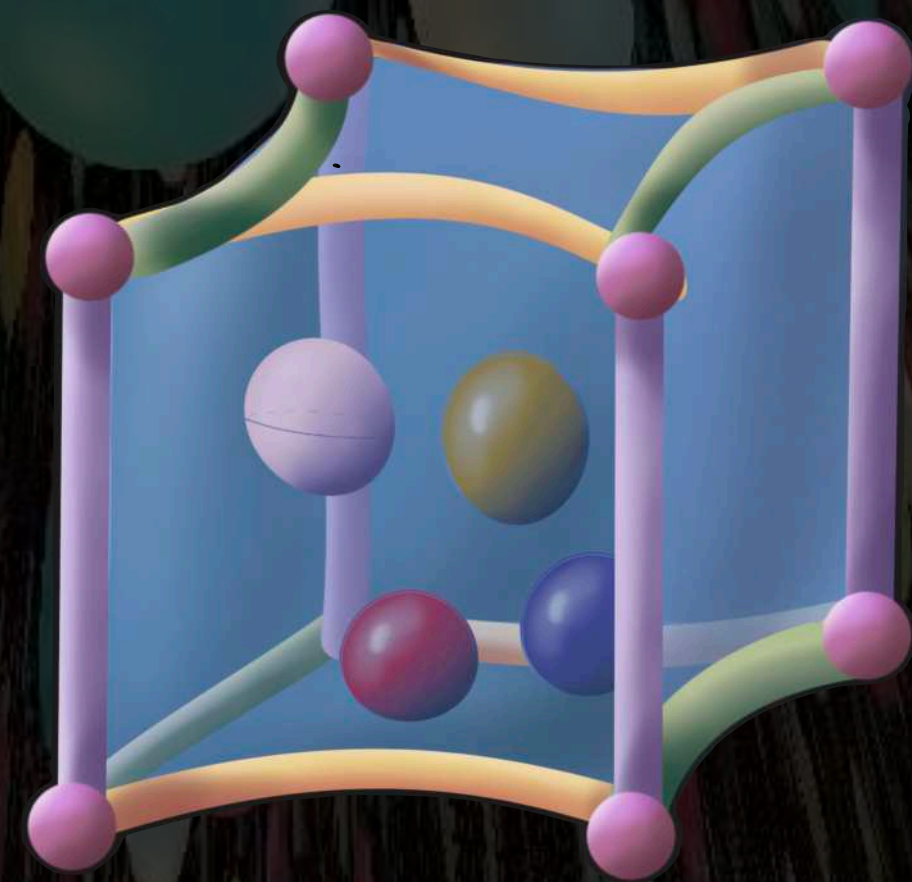
Different rates of geodesic divergence can
affect the aspect ratio of objects in vision



$H^2 \times \mathbb{R}$

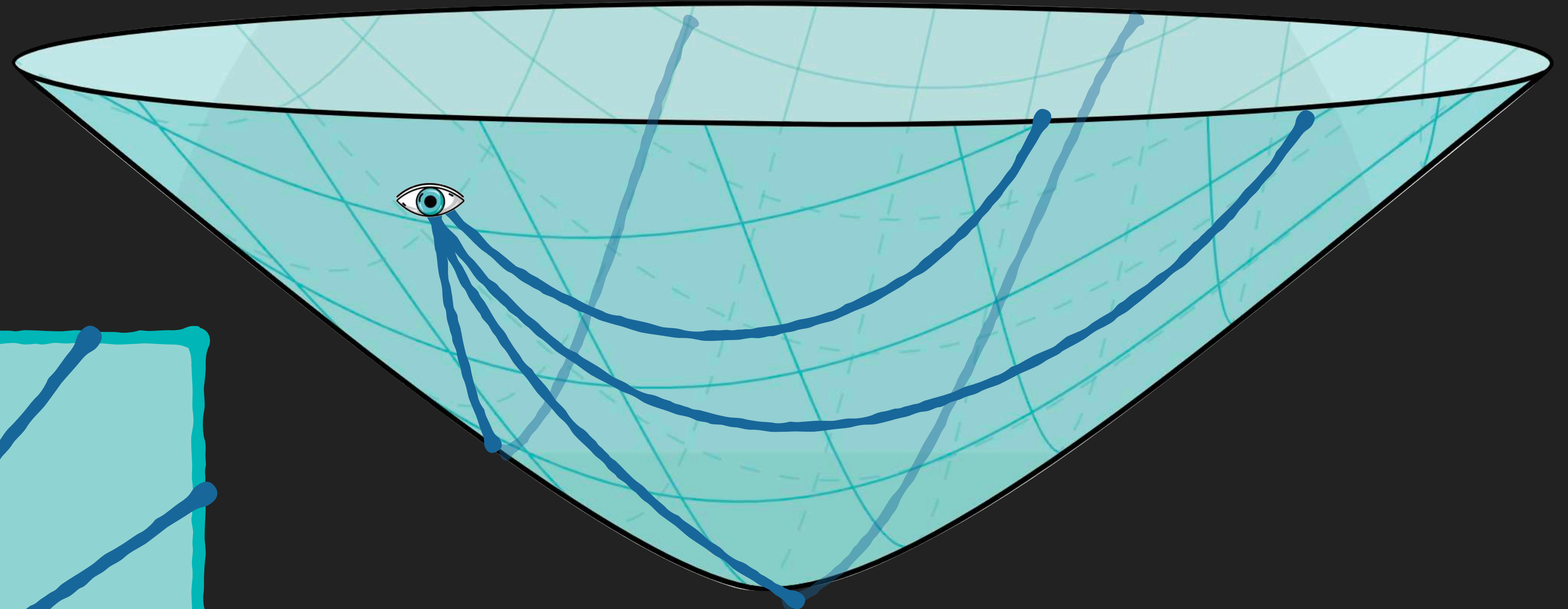
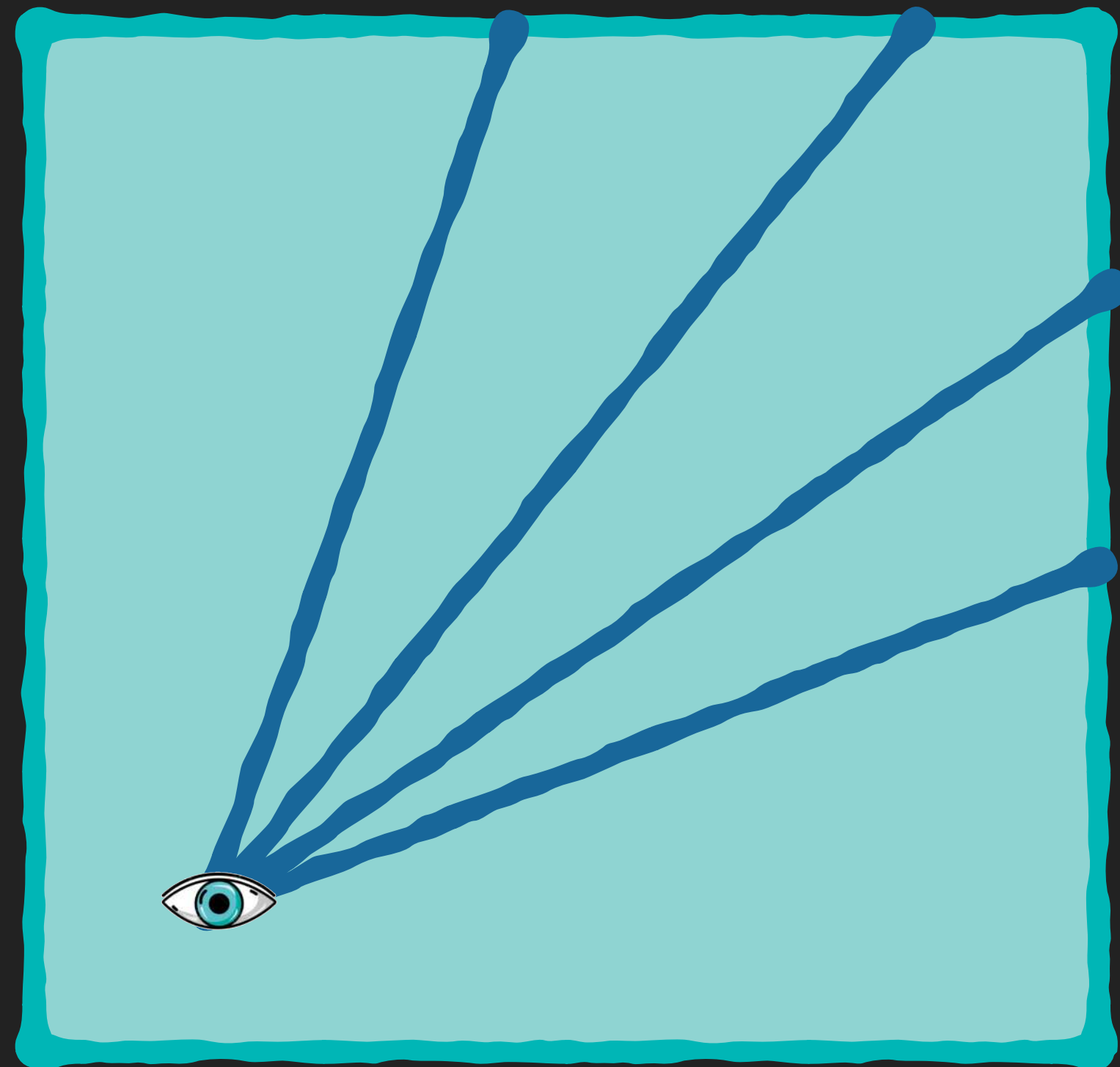
The geometry
of a stack of
hyperbolic
planes.

Vision in $\mathbb{H}^2 \times \mathbb{R}$



So we may not be able to trust the size or aspect ratio of things in our vision, but how does the general 'topology' of what we see relate to reality?

exp 



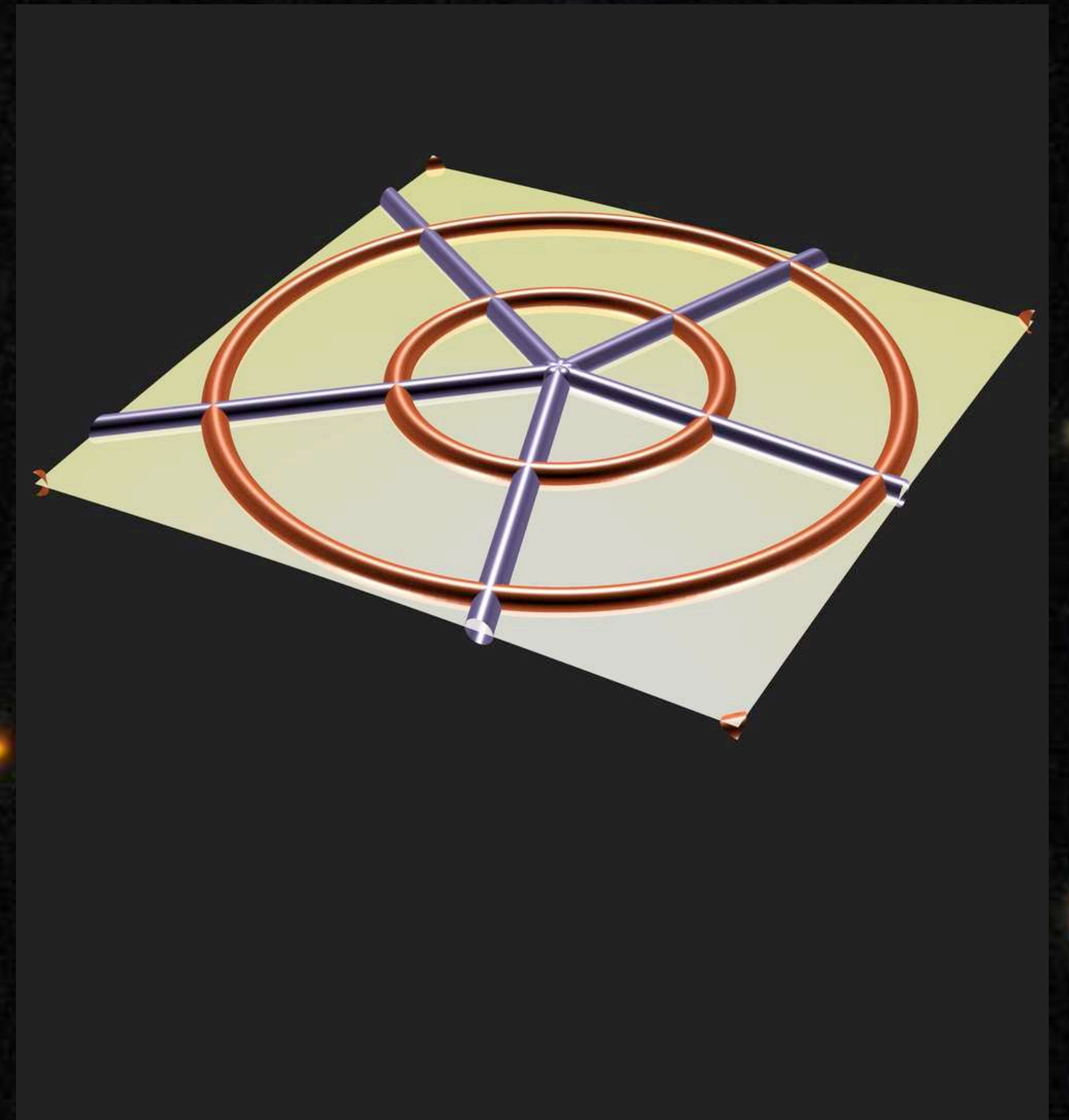
Cartan-Hadamard

In nonpositive curvature, our vision is diffeomorphic to reality

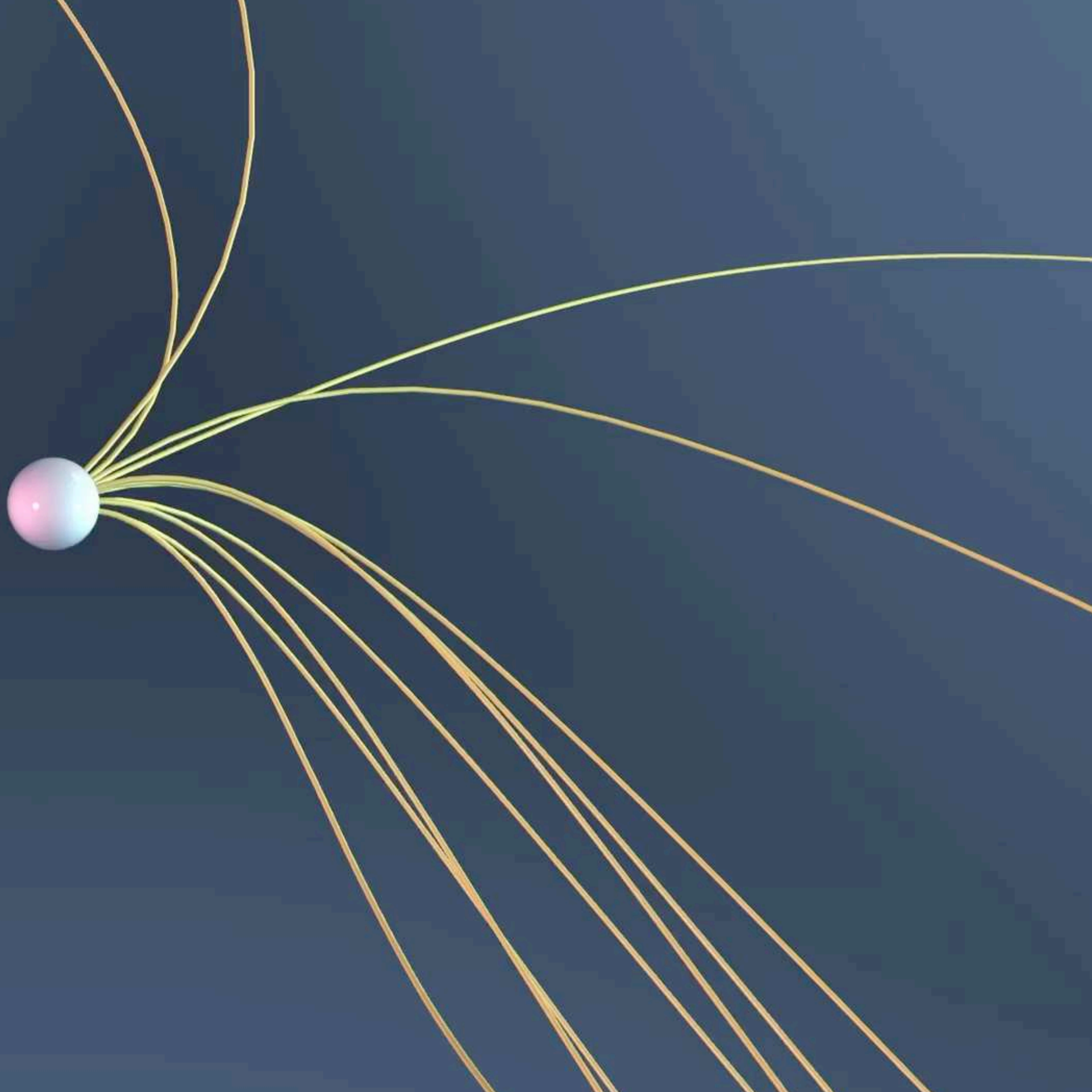
Exponential map in positive
curvature need not be injective

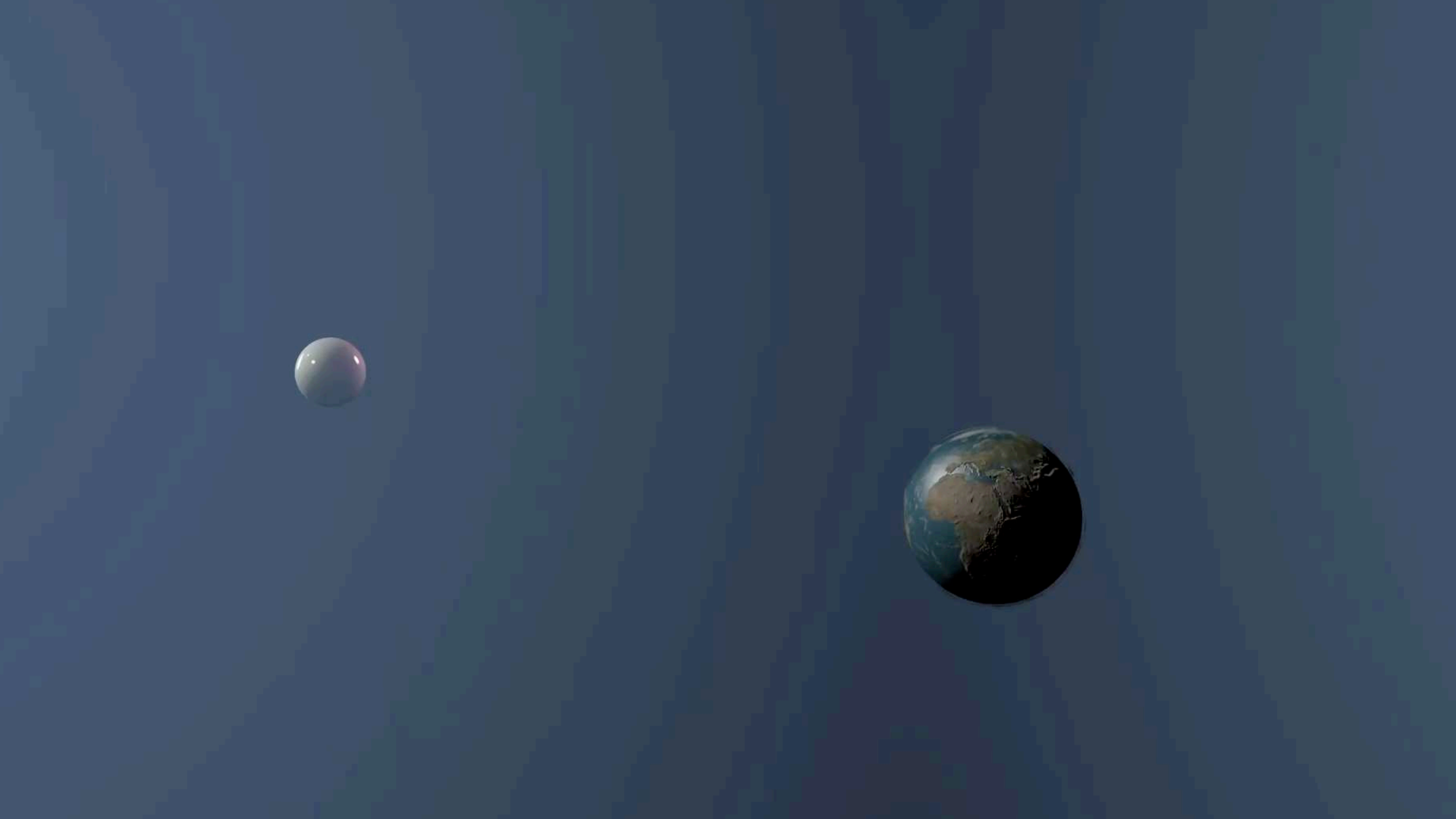
$$\exp : \mathbb{R}^2 \rightarrow \mathbb{S}^2$$

This can cause 'Lensing' Mirages

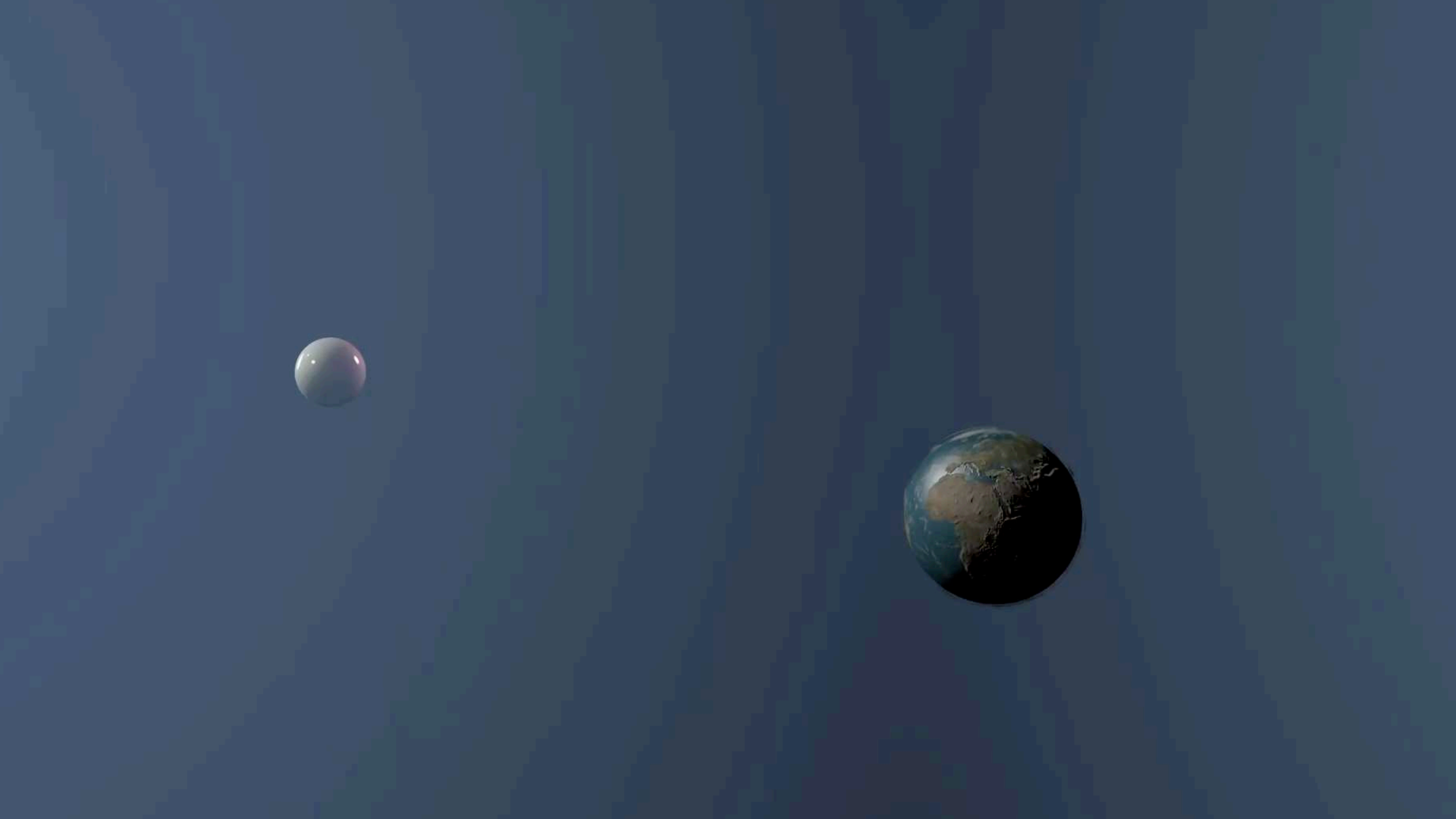


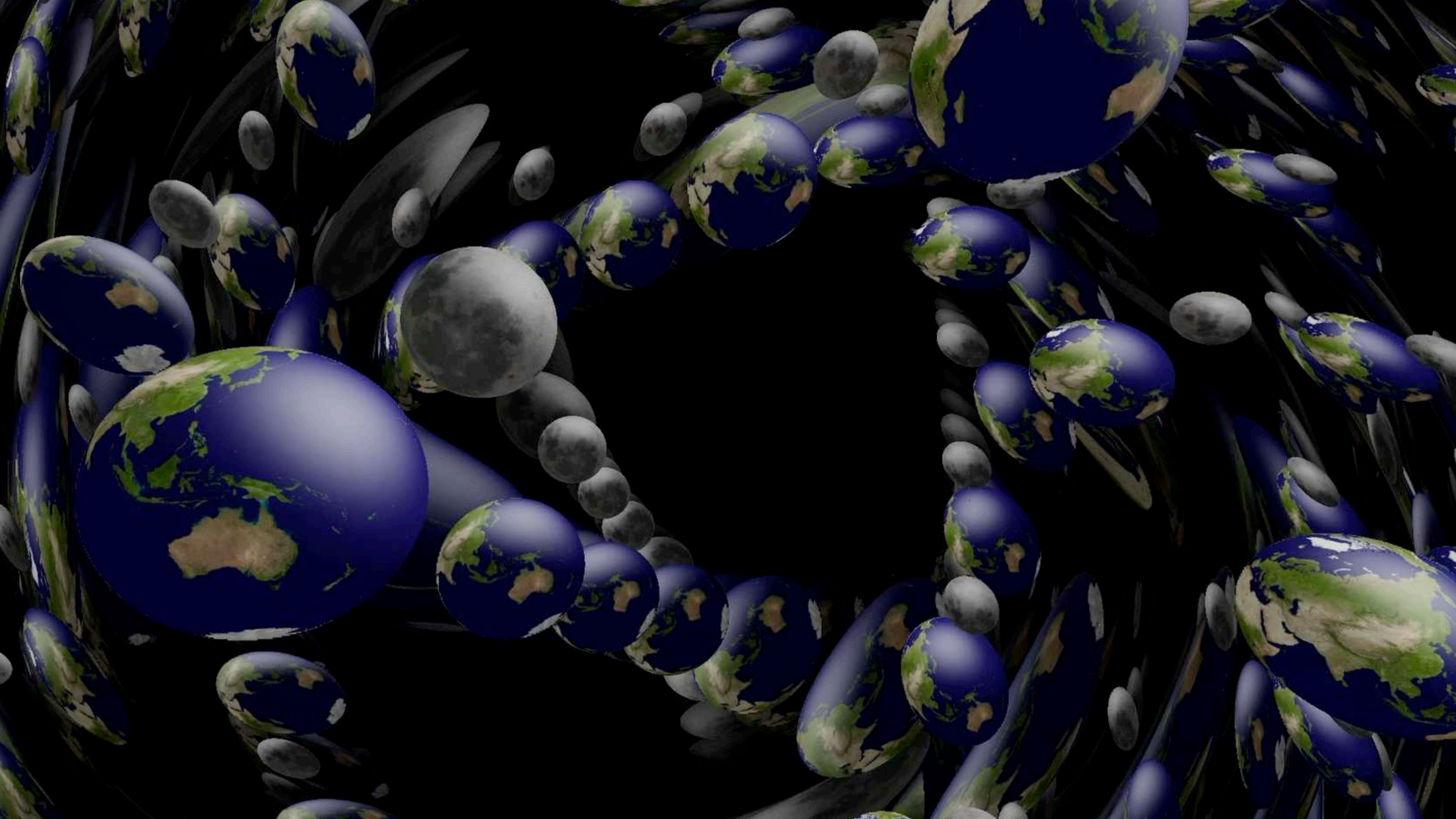
Nil Geometry



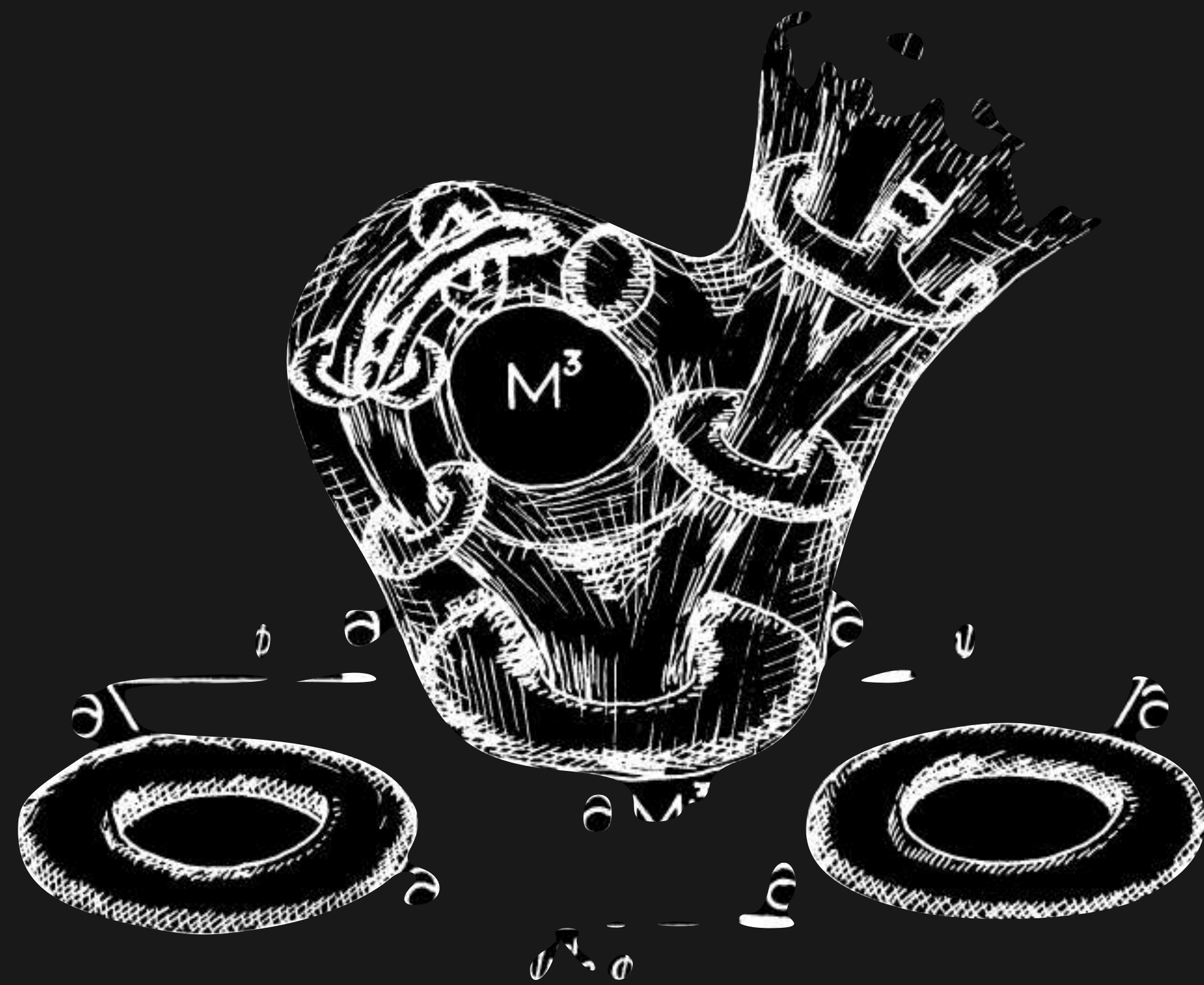








II Seeing 3-Manifolds



So far we've only seen some
very special 3-manifolds.

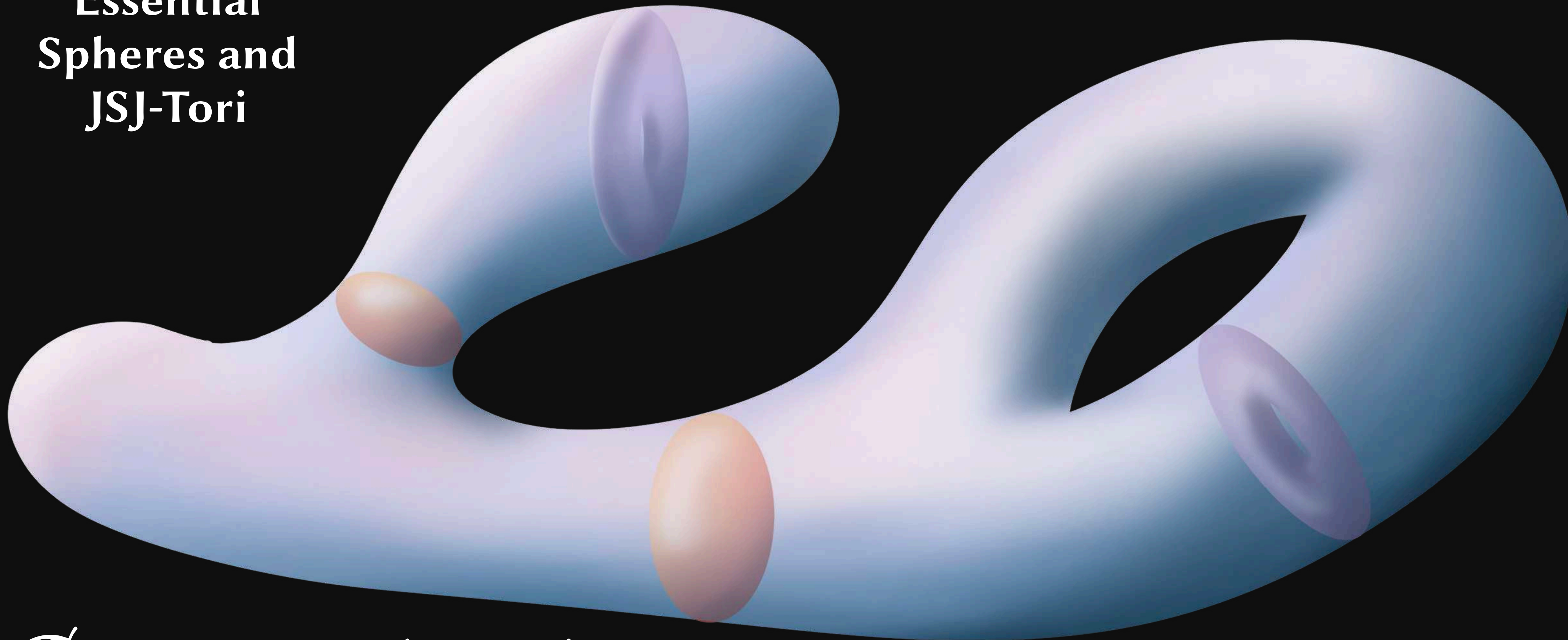
(Quotients of homogeneous spaces)

Is there a hope of a
general picture?



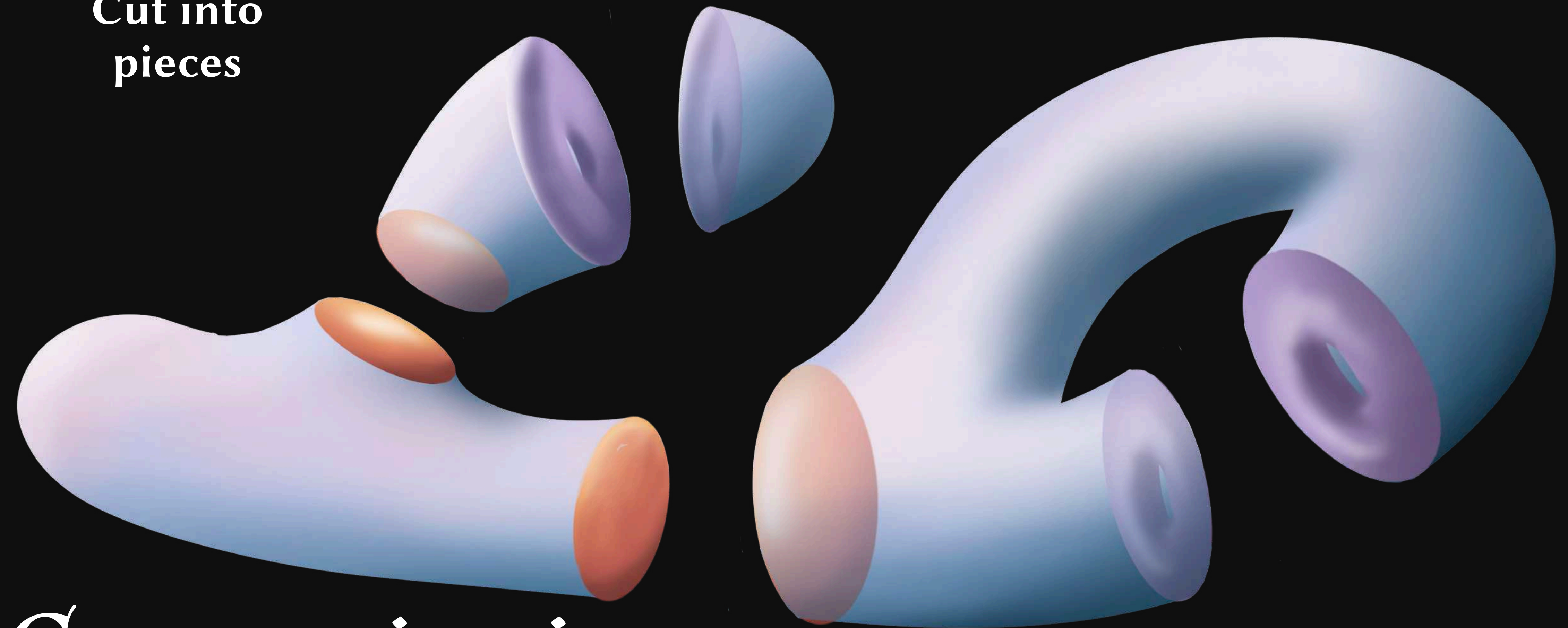
Geometrization
on

Essential
Spheres and
JSJ-Tori



Geometrization
on

Cut into
pieces



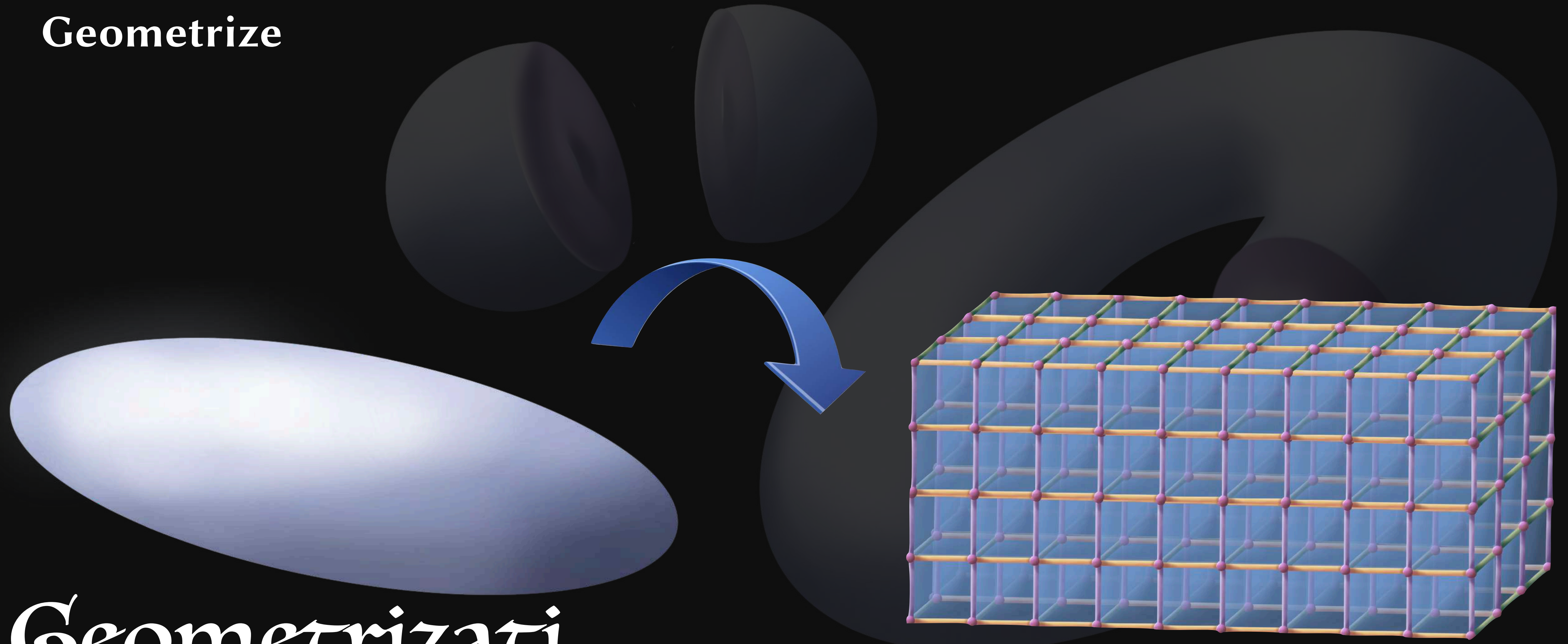
Geometrization

Fill 3-Balls



Geometrization
on

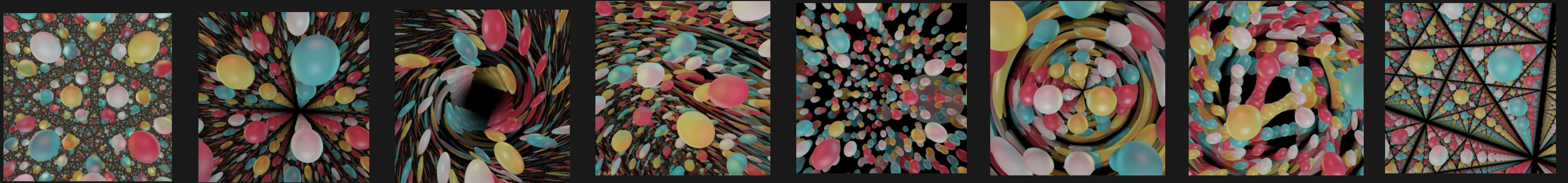
Geometrize



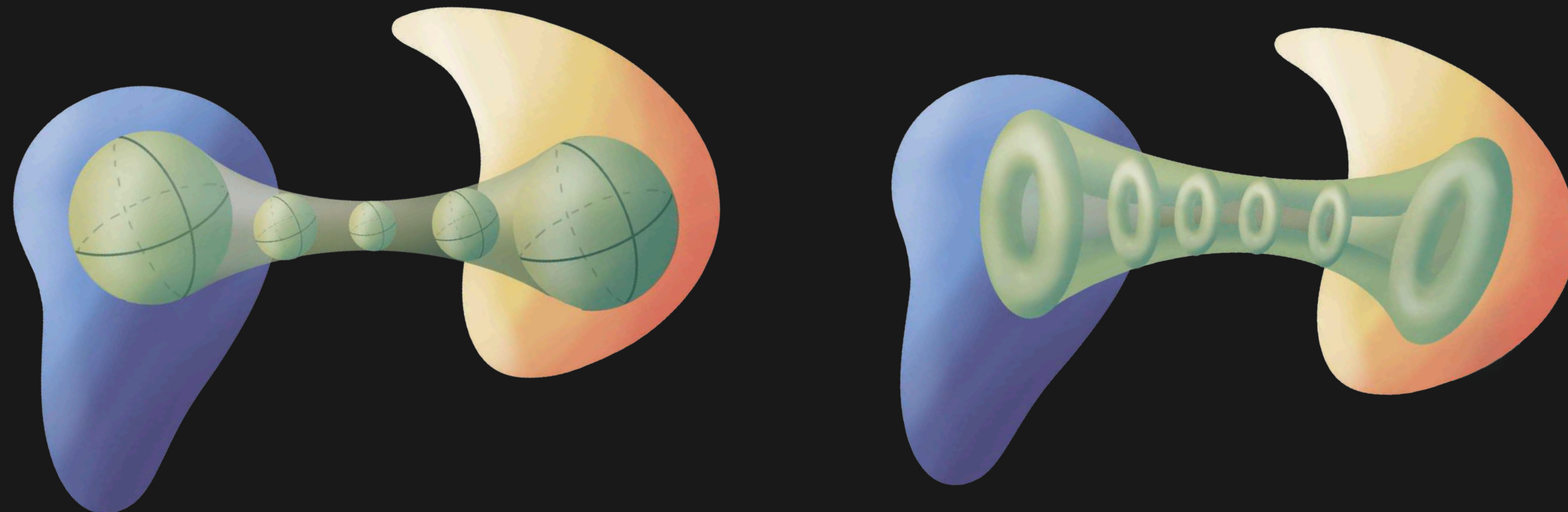
Geometrization

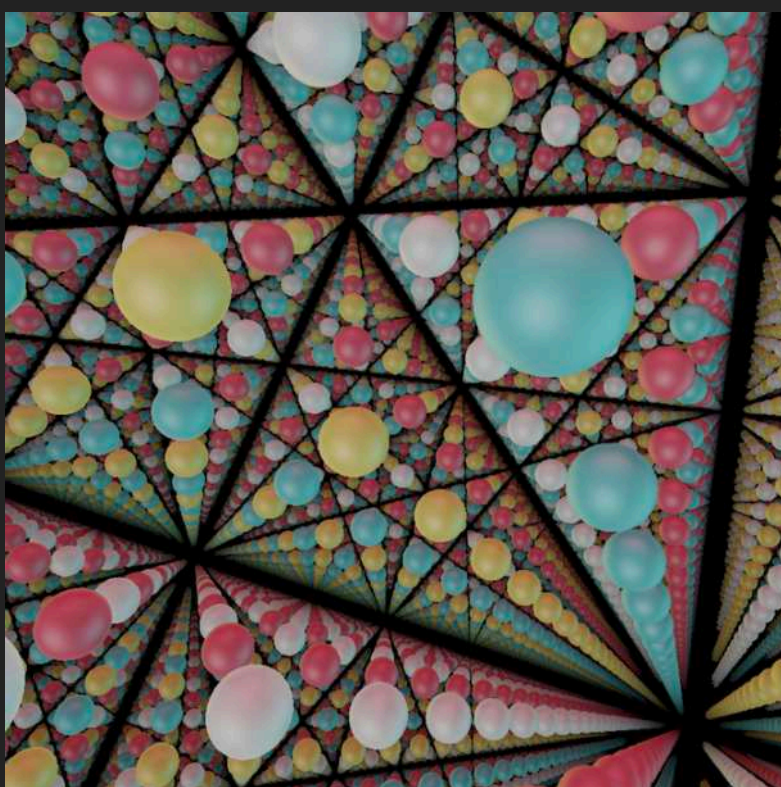
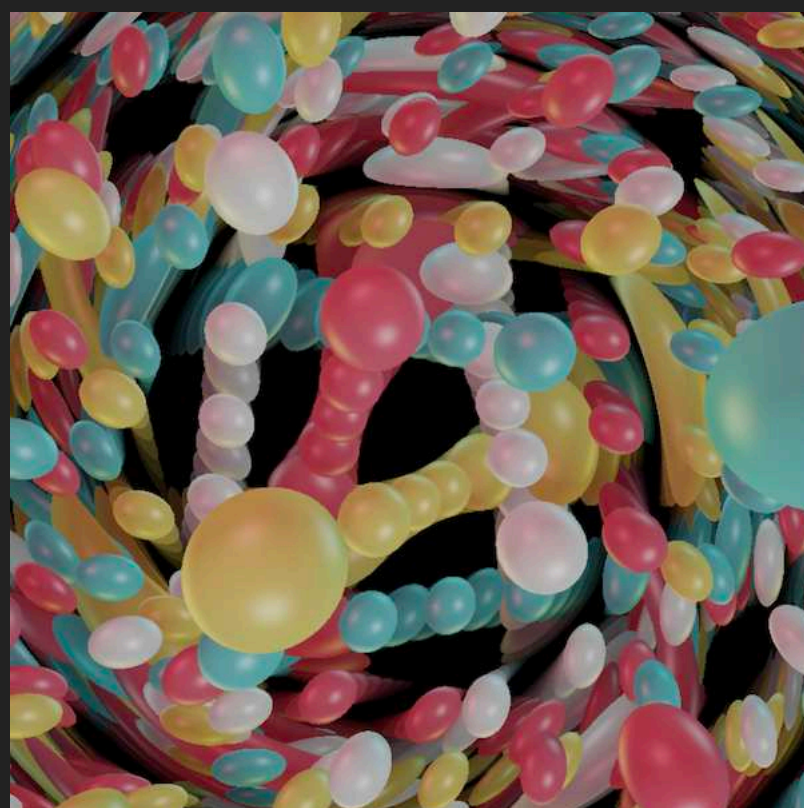
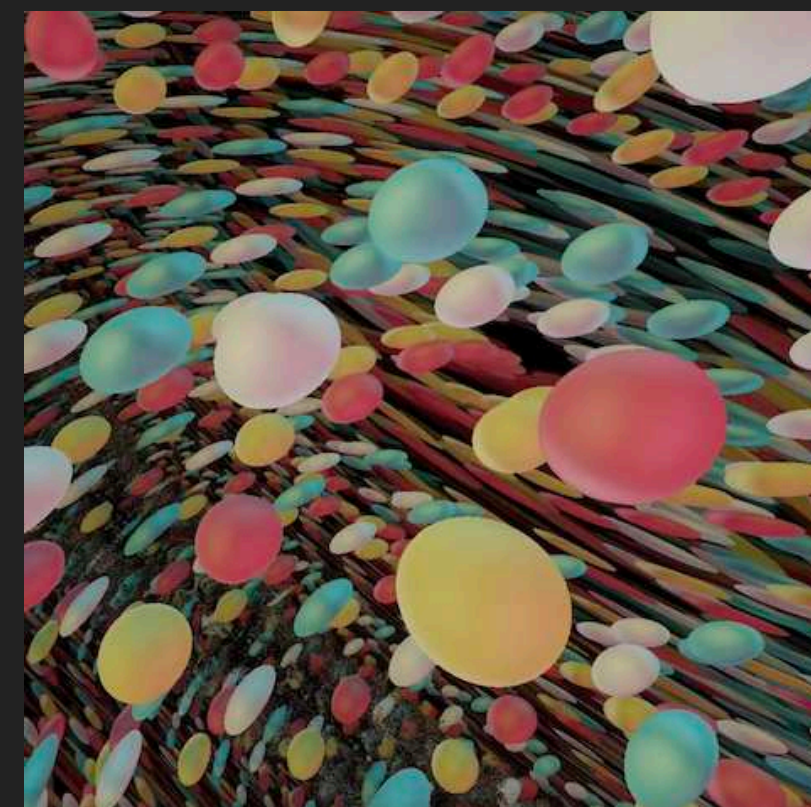
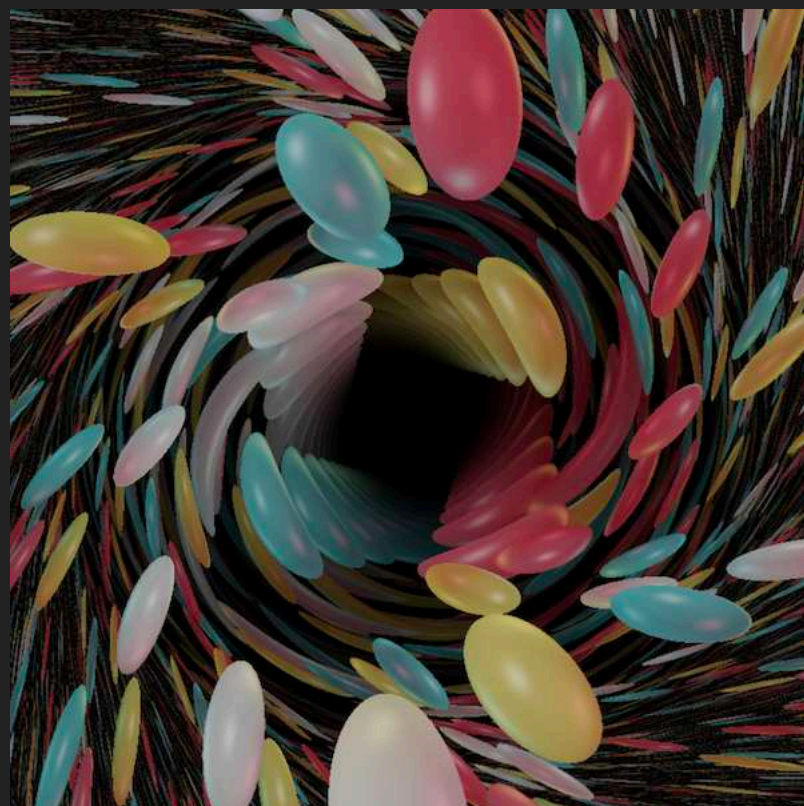
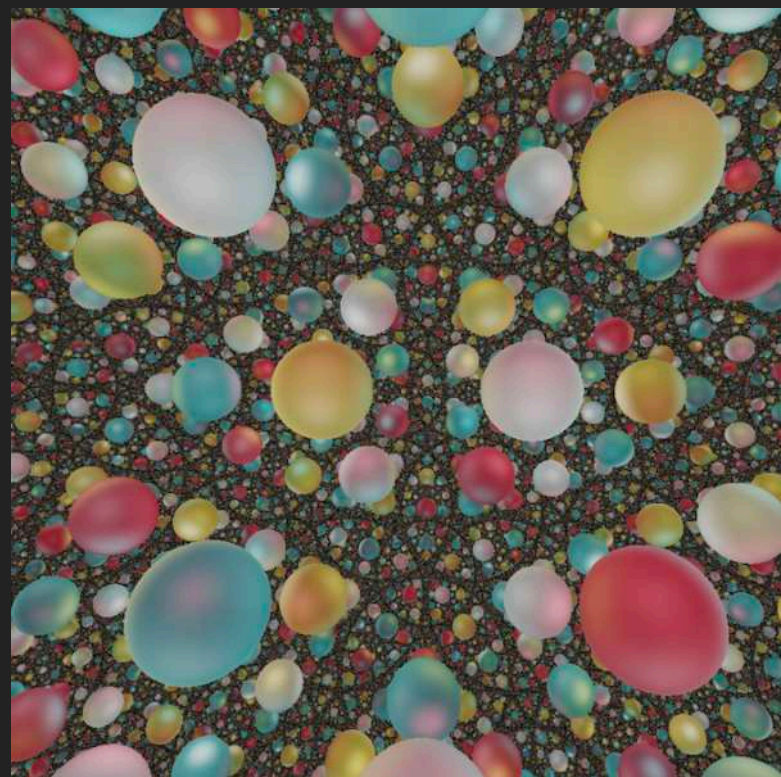
3 Manifolds are like Legos:

Eight general types of bricks



Two types of connectors



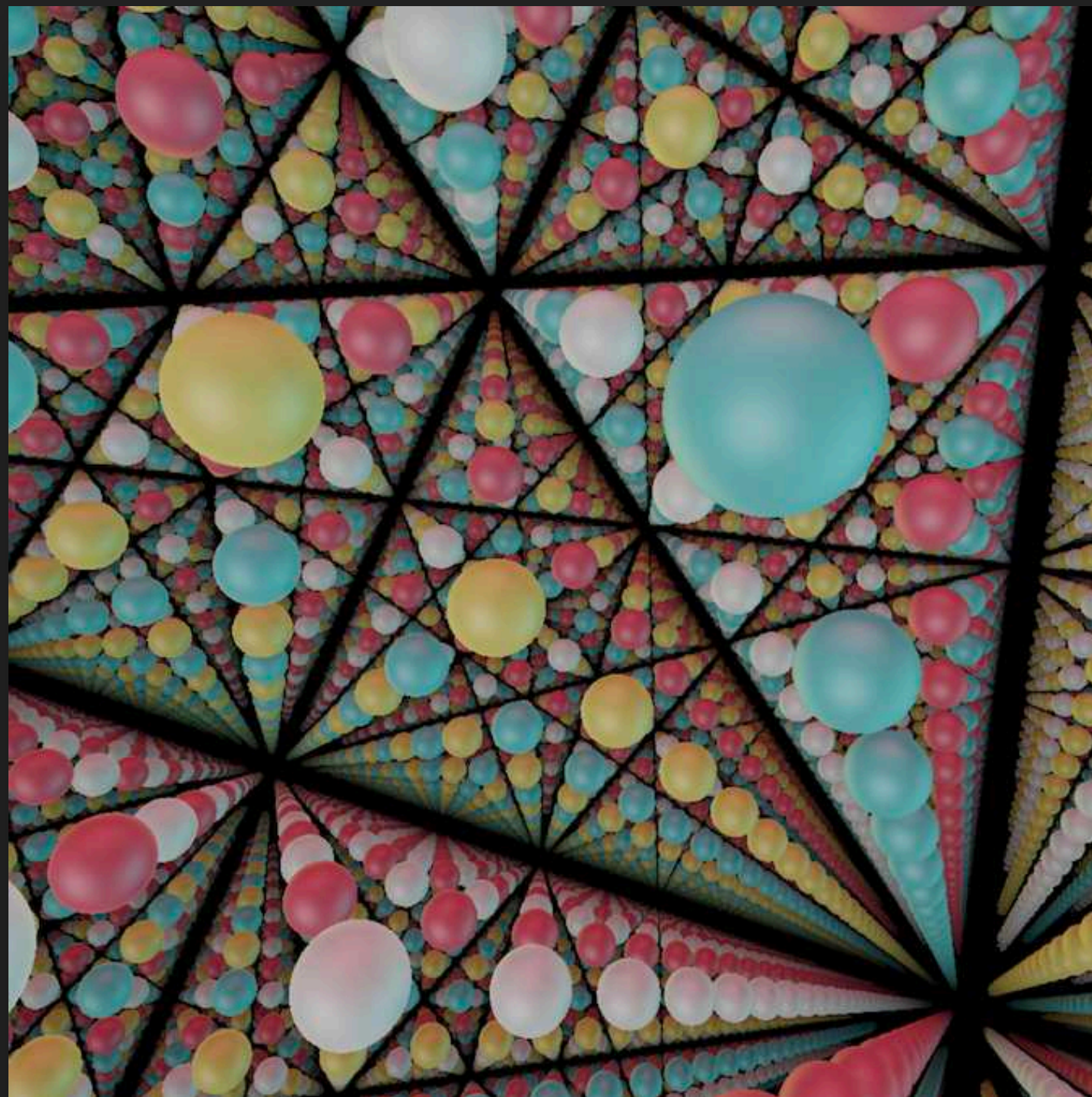


Lego Bricks of
Geometrization

IIa

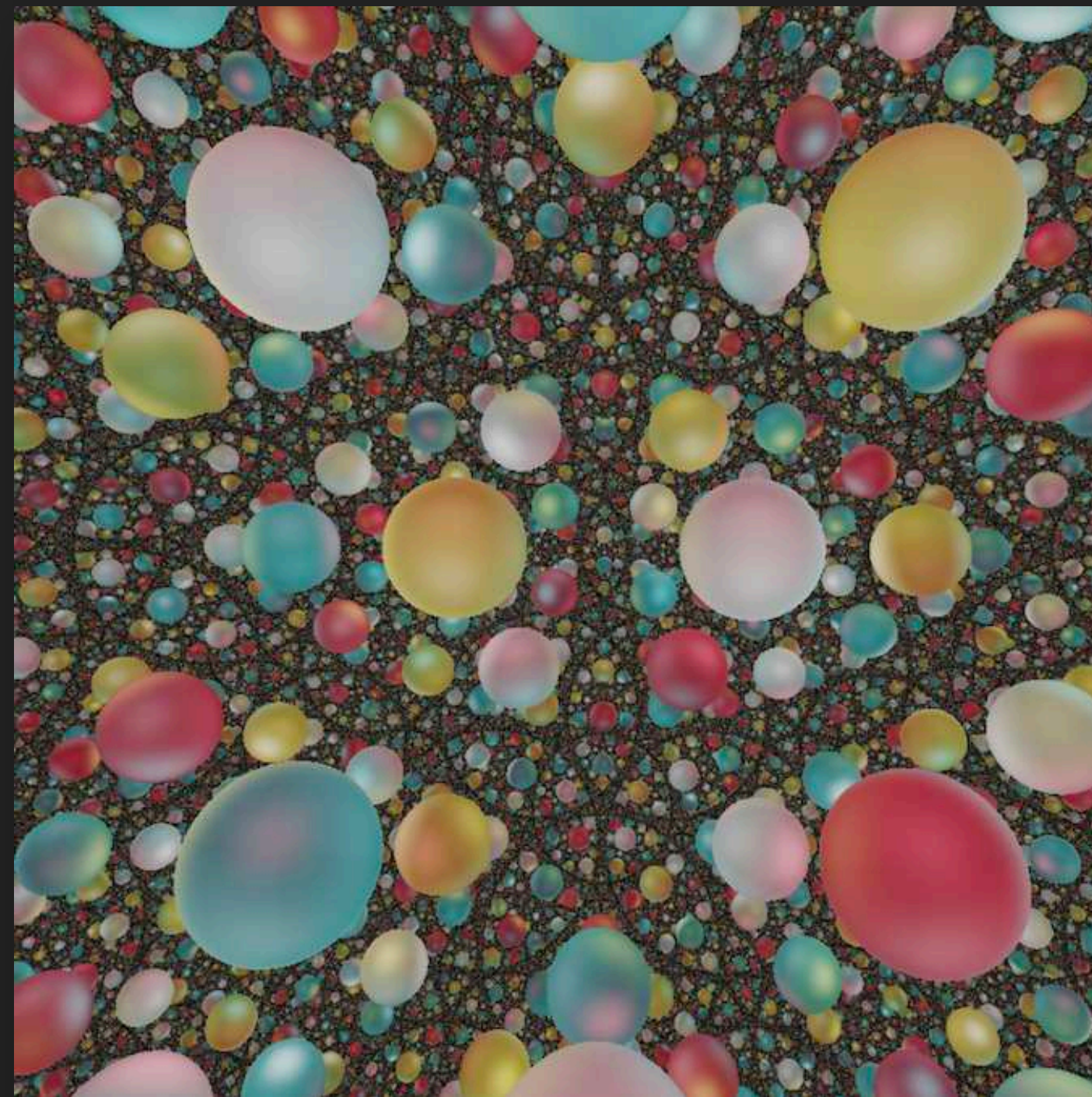
The
Thurston
Geometries

Constant Curvature



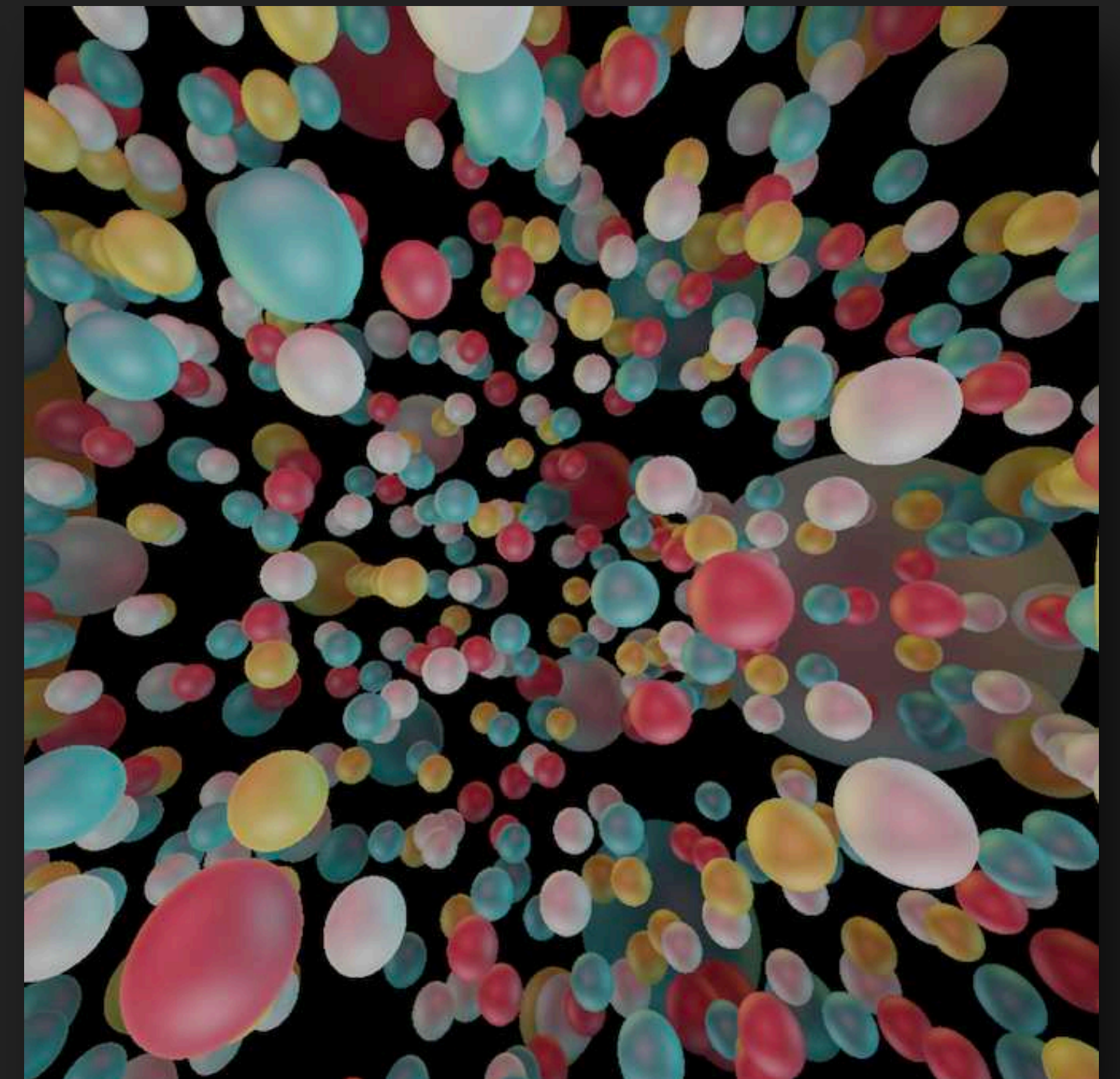
Euclidean

Ten closed manifolds, all finitely covered by 3-Torus.



Hyperbolic

Tons of closed manifolds!



Spherical

Manifolds determined by finite subgroups of $SO(3)$

Product Geometries



$$\mathbb{H}^2 \times \mathbb{E}$$

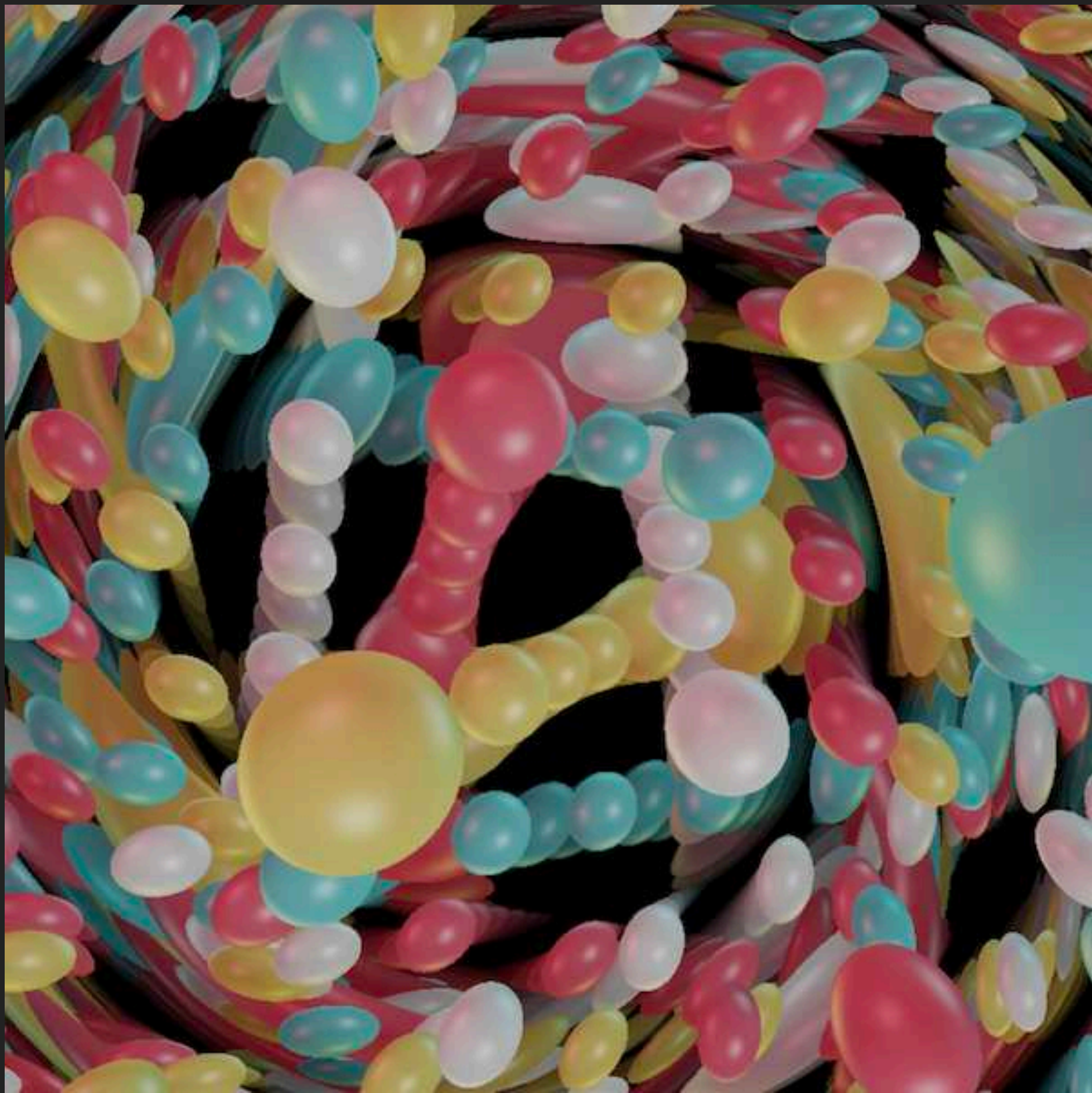
Finite quotients of
Hyperbolic Surface $\times \mathbb{S}^1$



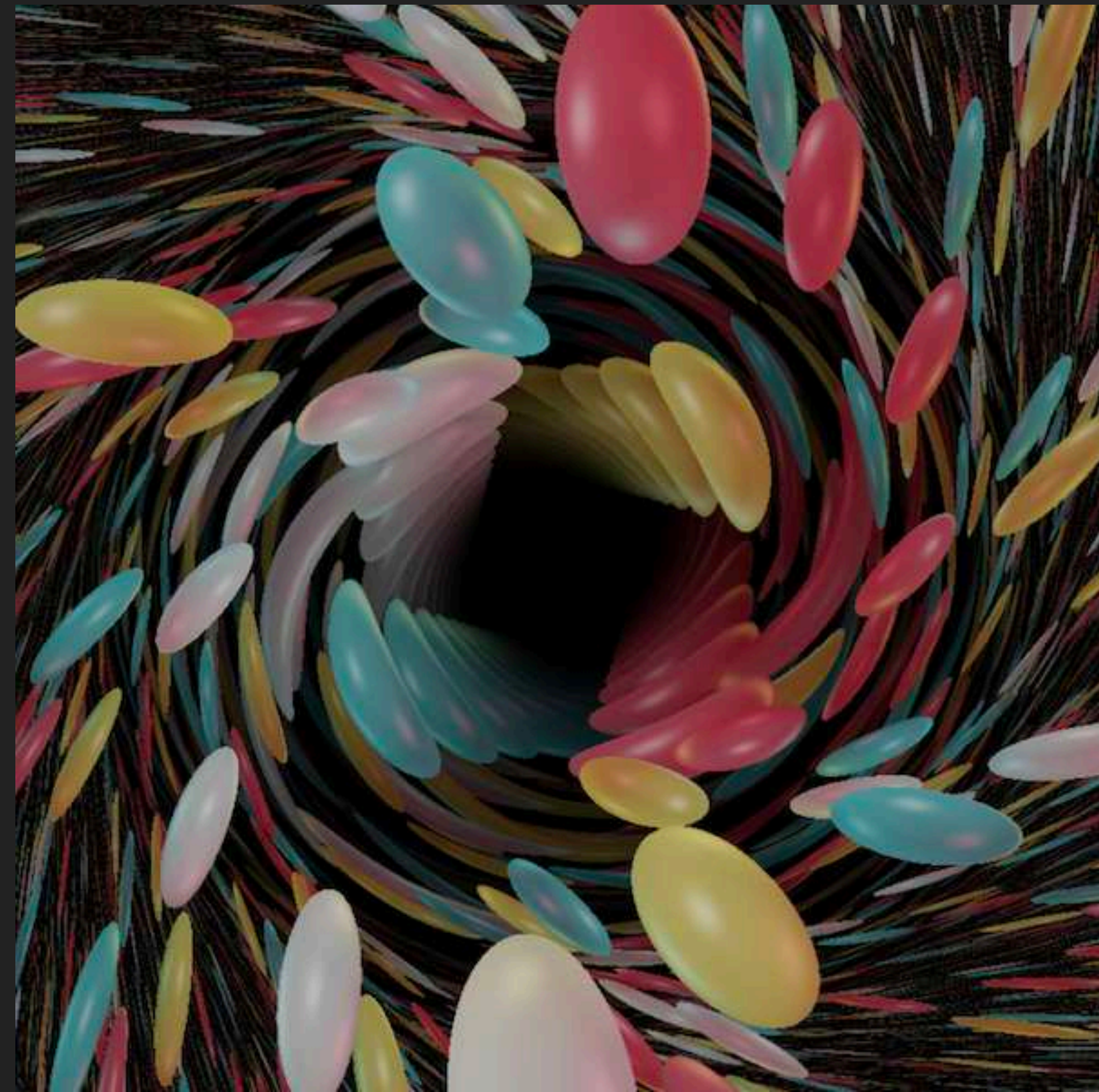
$$\mathbb{S}^2 \times \mathbb{E}$$

Only **FOUR** closed manifolds
with this geometry!

"Twisted" Products



Nil



SL2

Circle bundles over Euclidean/Hyperbolic orbifolds of nonzero Euler class



nil

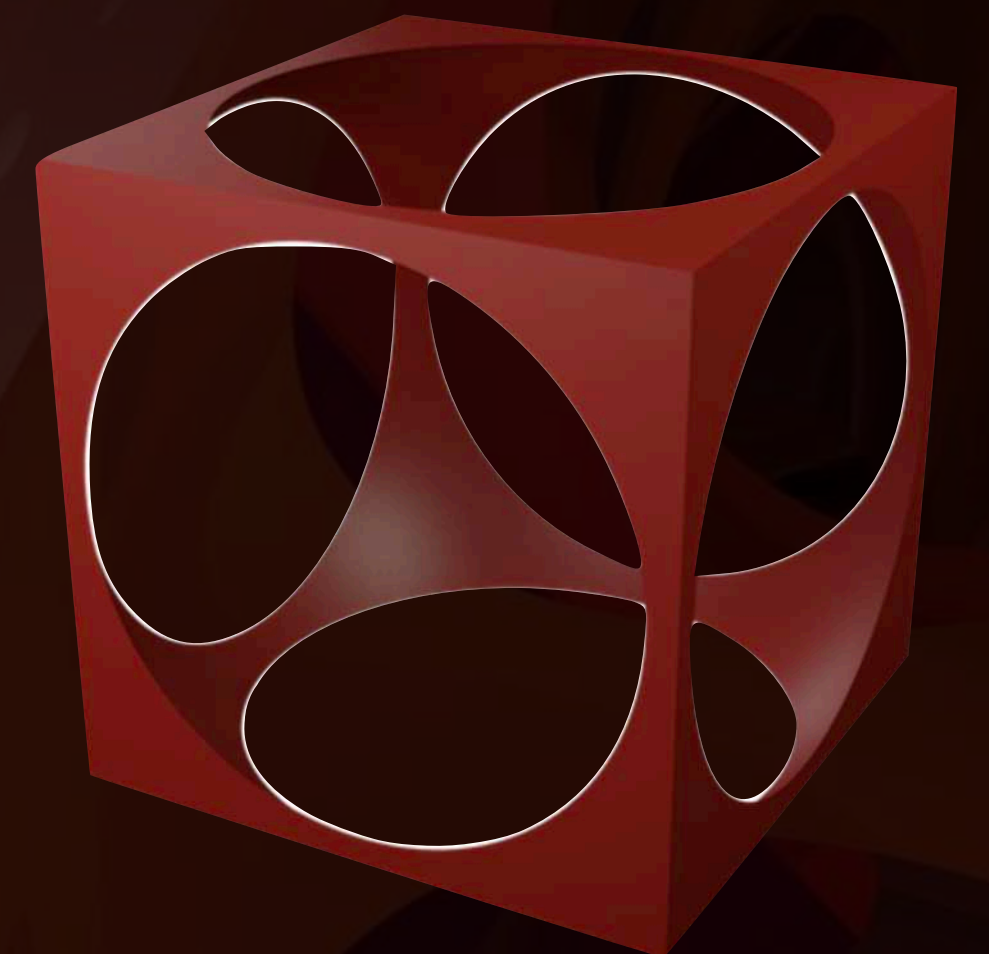
$$dx^2 + dy^2 + (dz - xdy)^2$$

Dehn Twist Mapping Torus



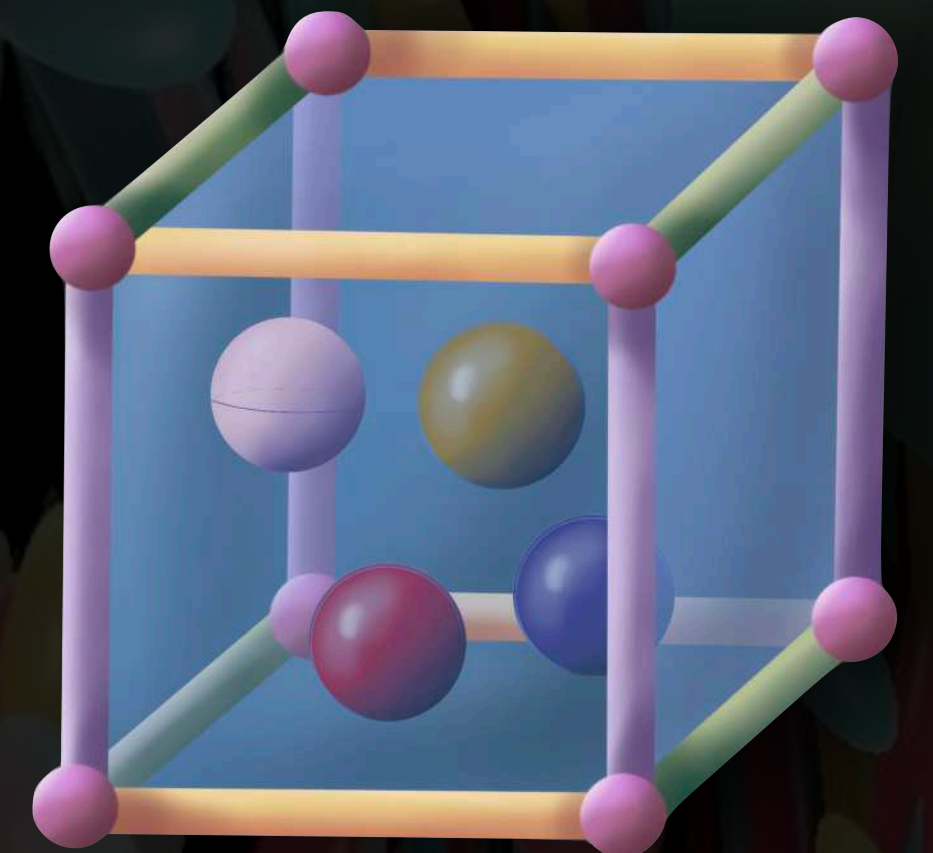
$$f: \pi_1 T^2 \rightarrow \pi_1 T^2 \quad (x, y) \mapsto (x + y, y)$$

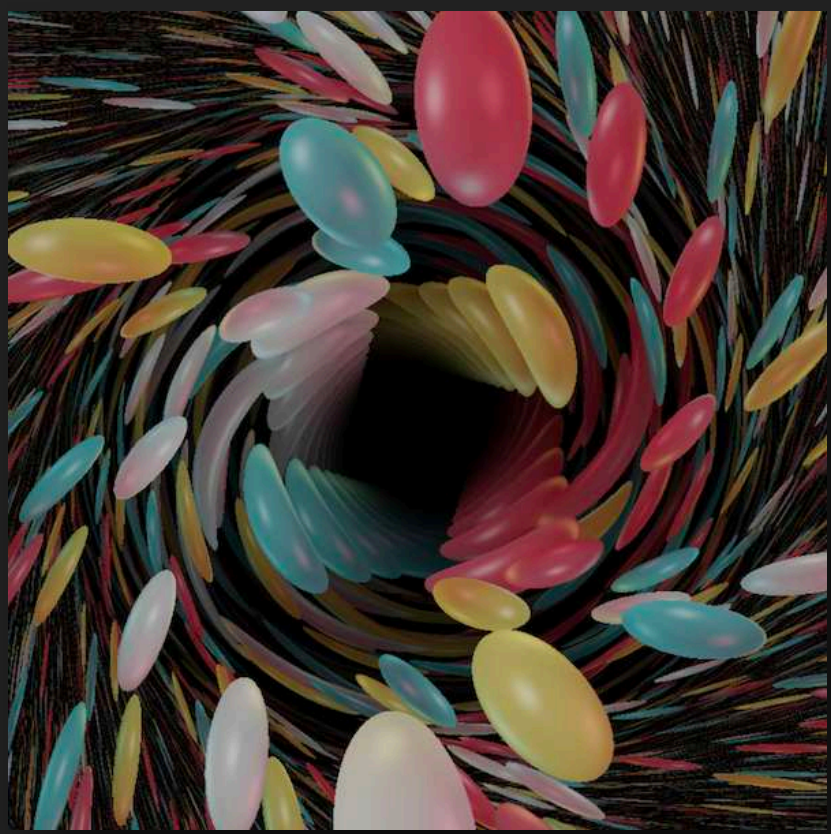
Dehn Twist Torus Bundle





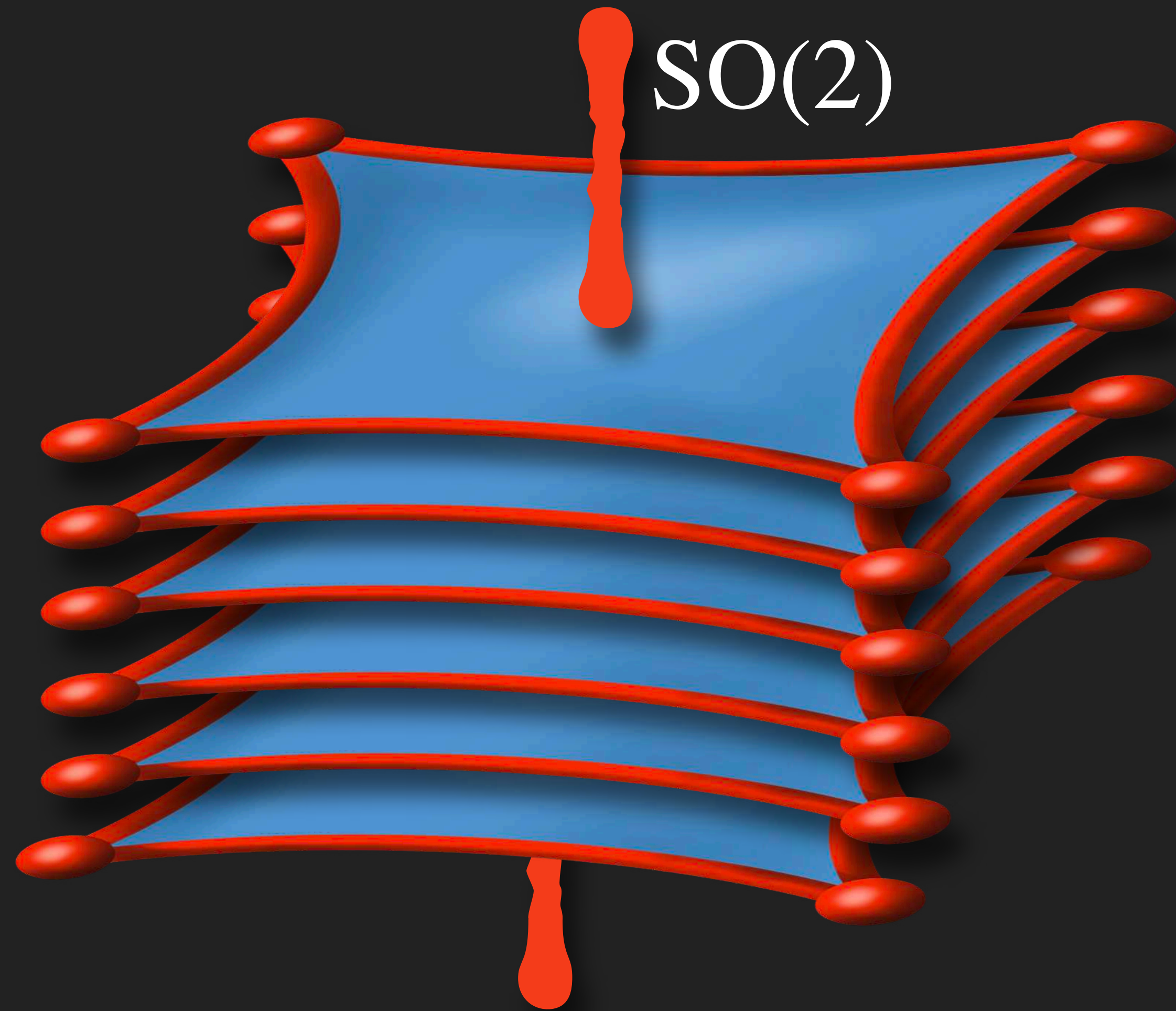
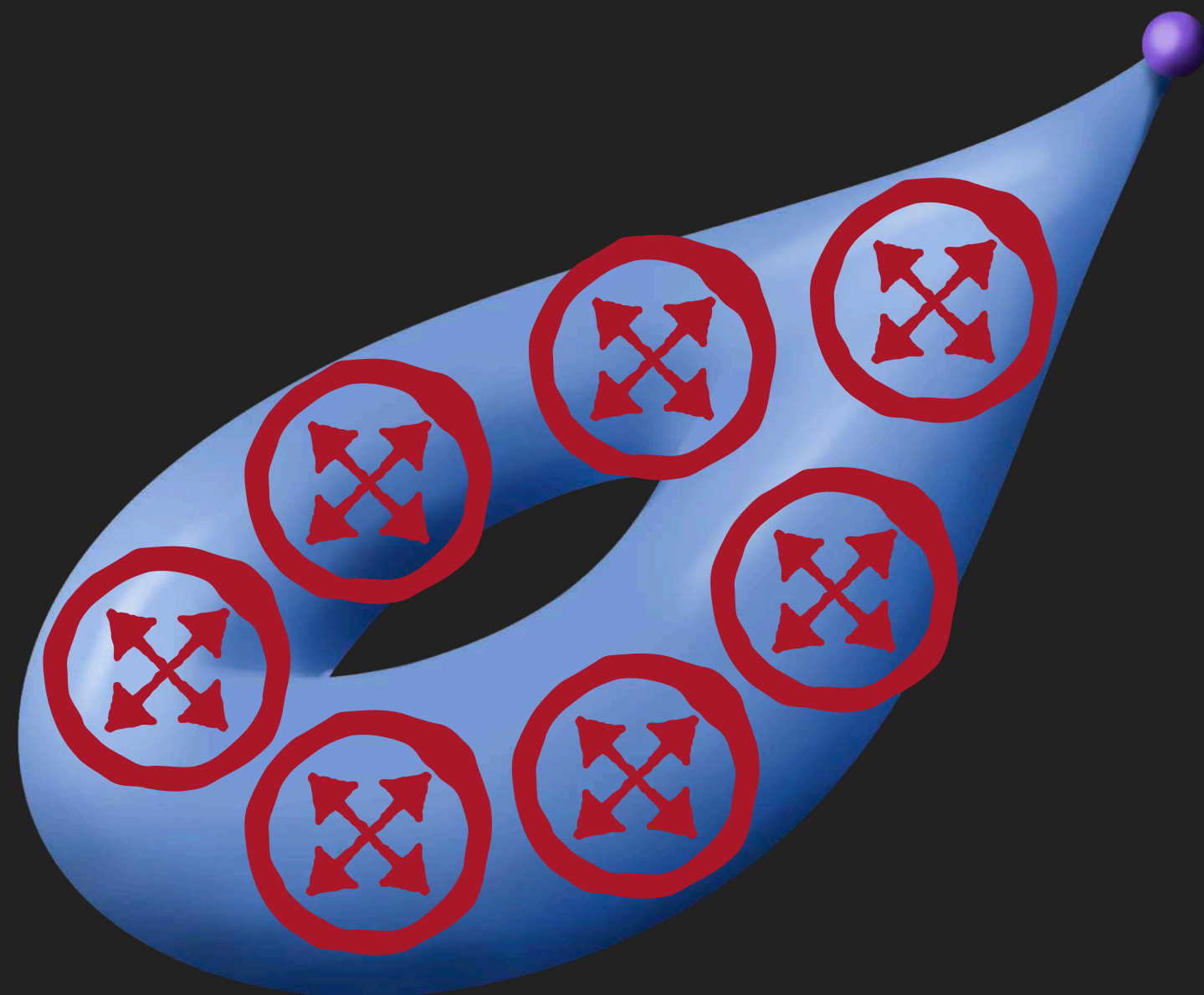
Dehn Twist Torus Bundle

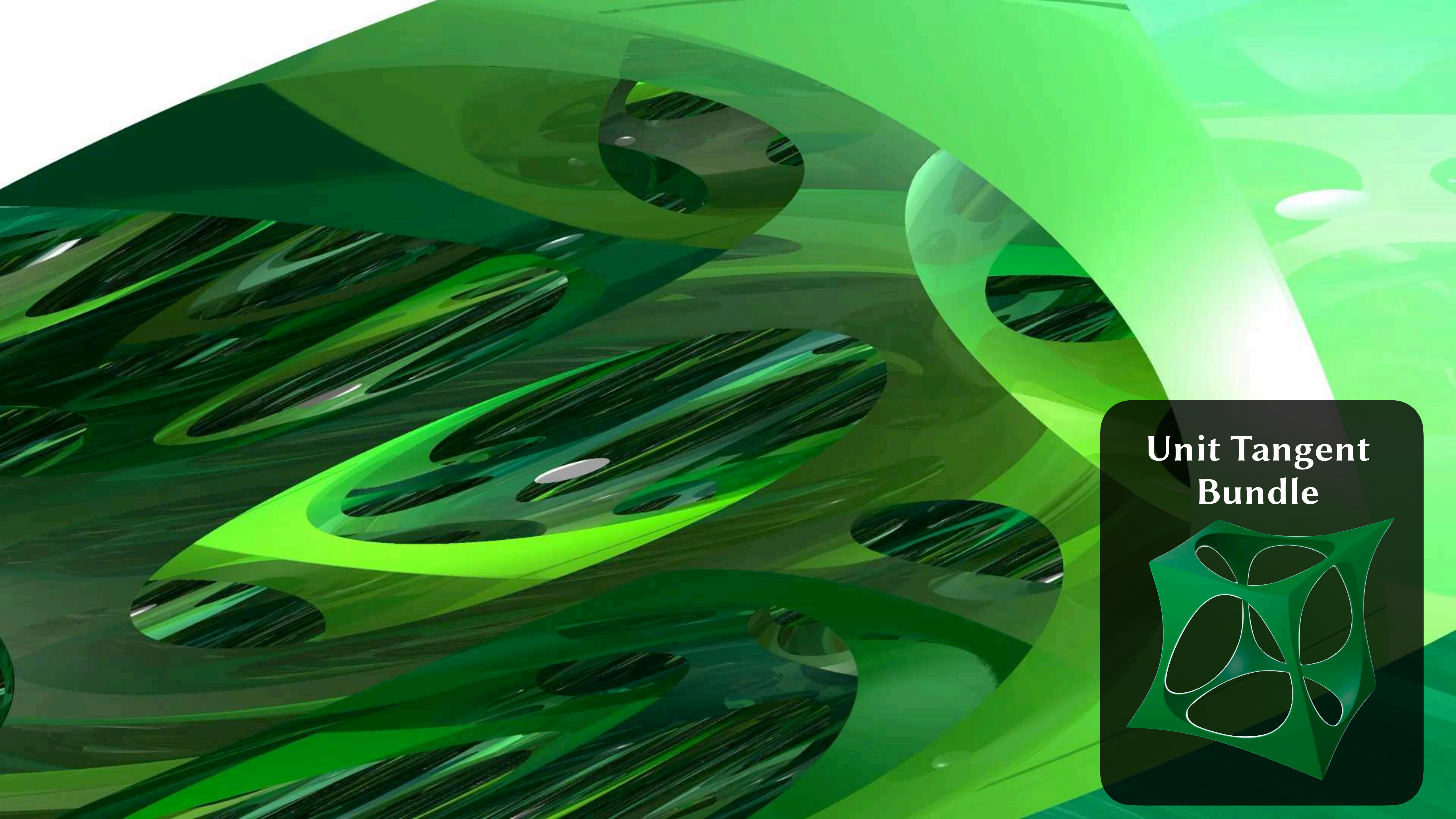




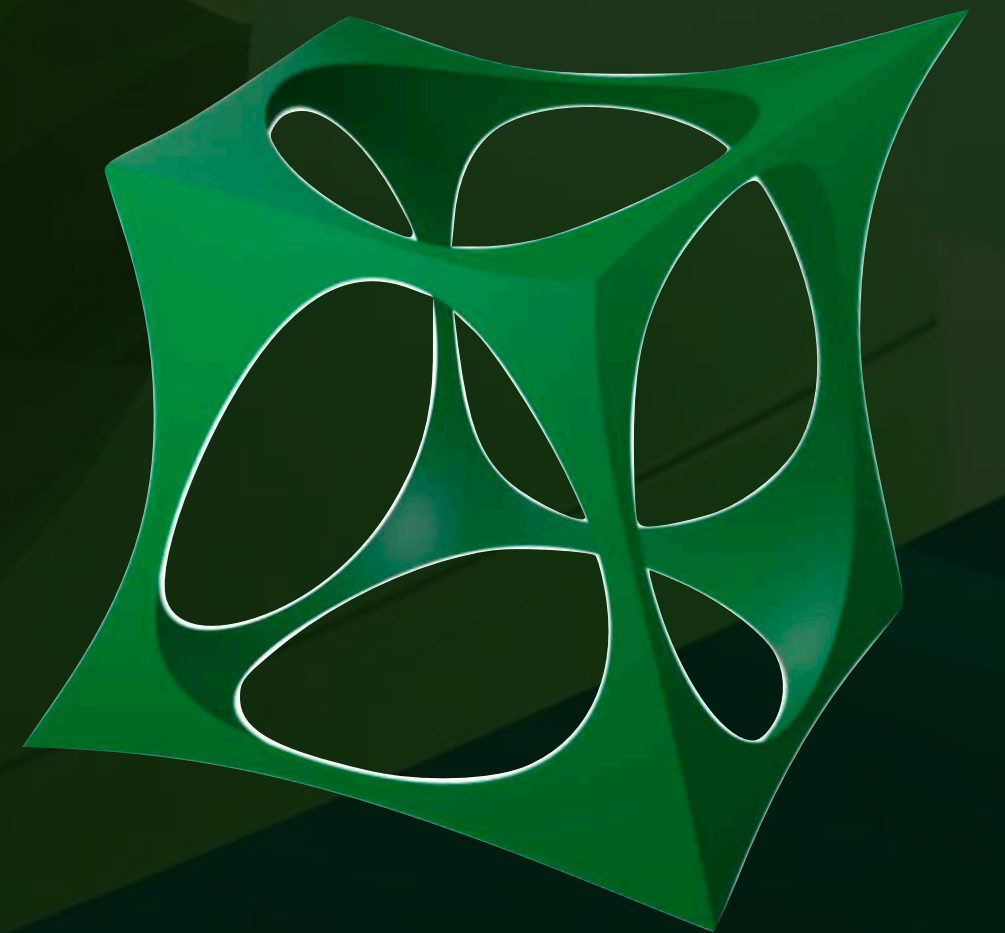
$SL_2\mathbb{R}$

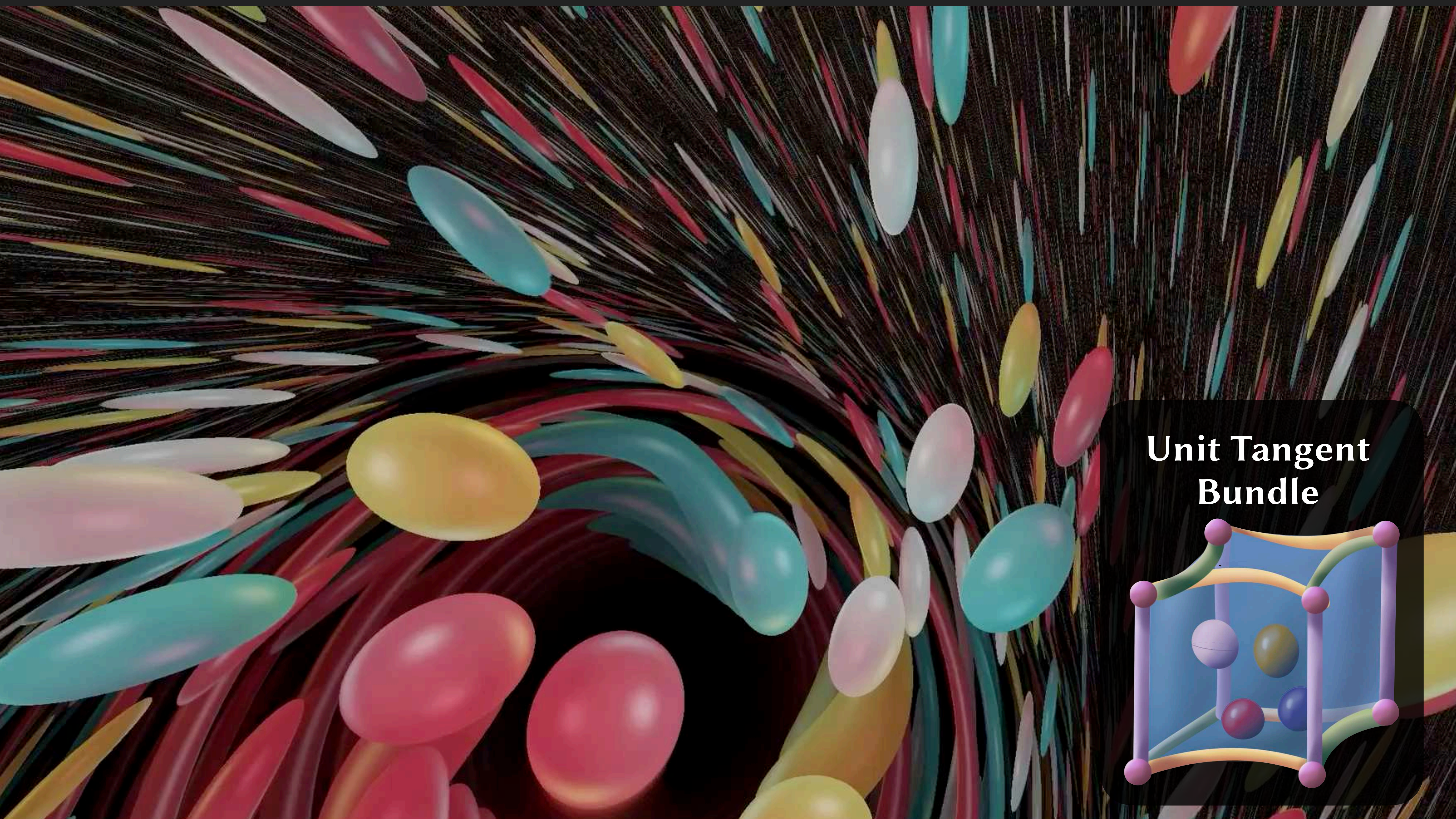
Hyperbolic Unit
Tangent Bundles



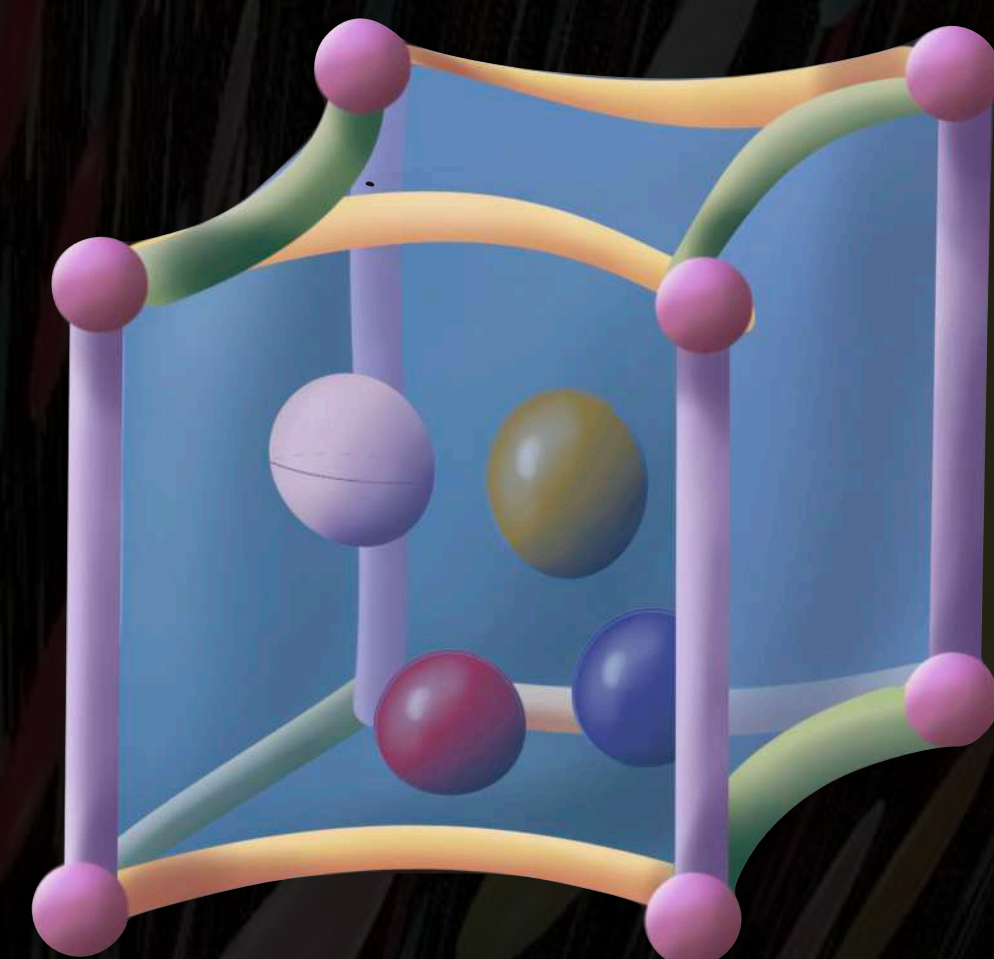


Unit Tangent Bundle

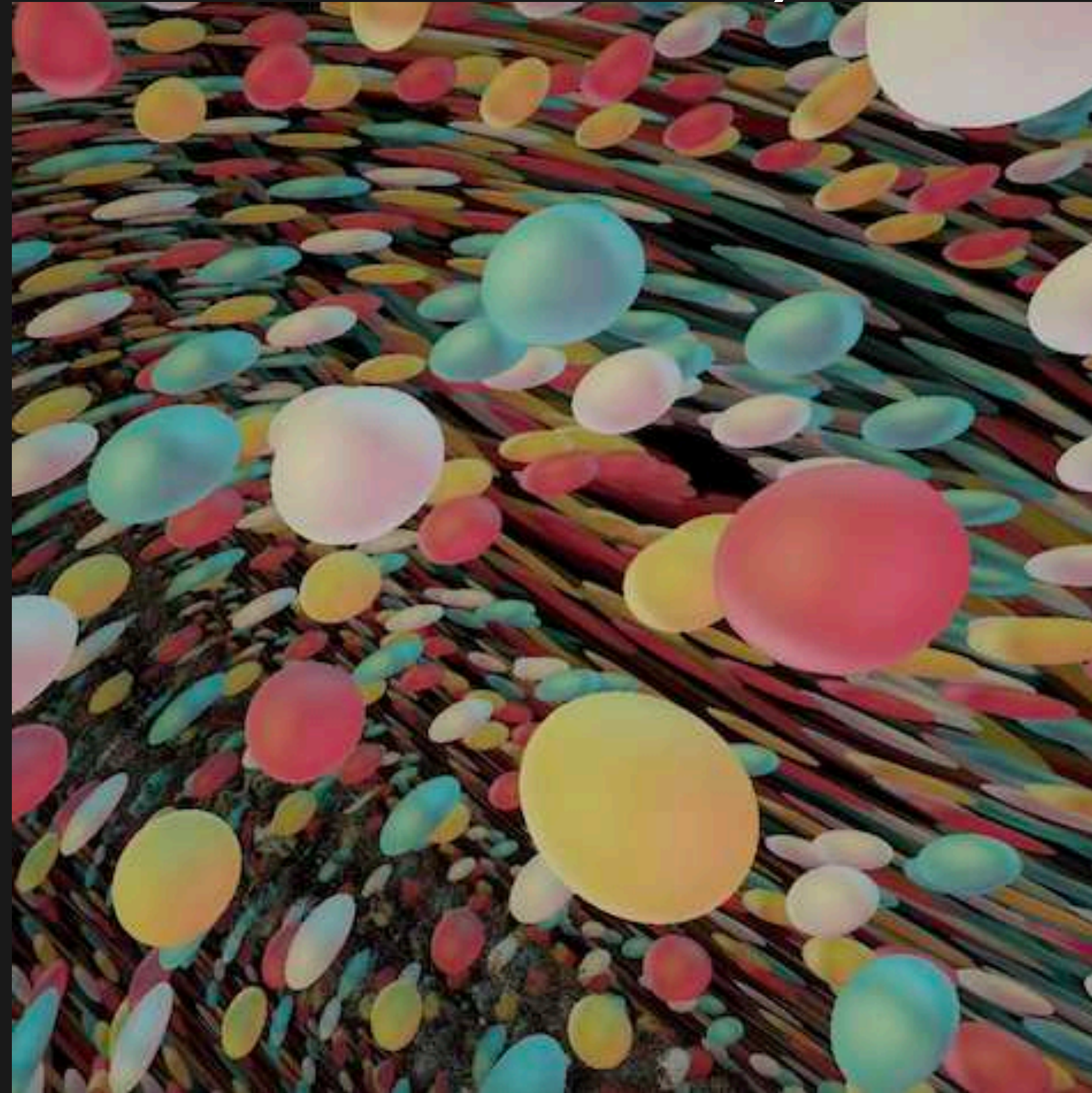




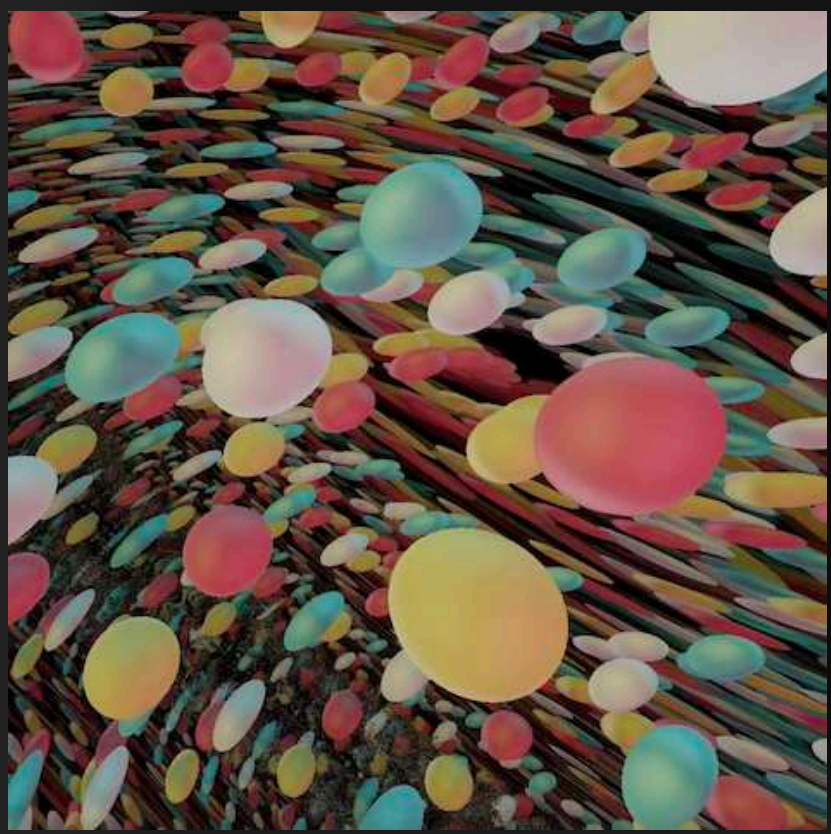
Unit Tangent Bundle



Sol Geometry



Finite quotients of Anosov mapping tori



Sol

$$e^{2z}dx^2 + e^{-2z}dy^2 + dz^2$$

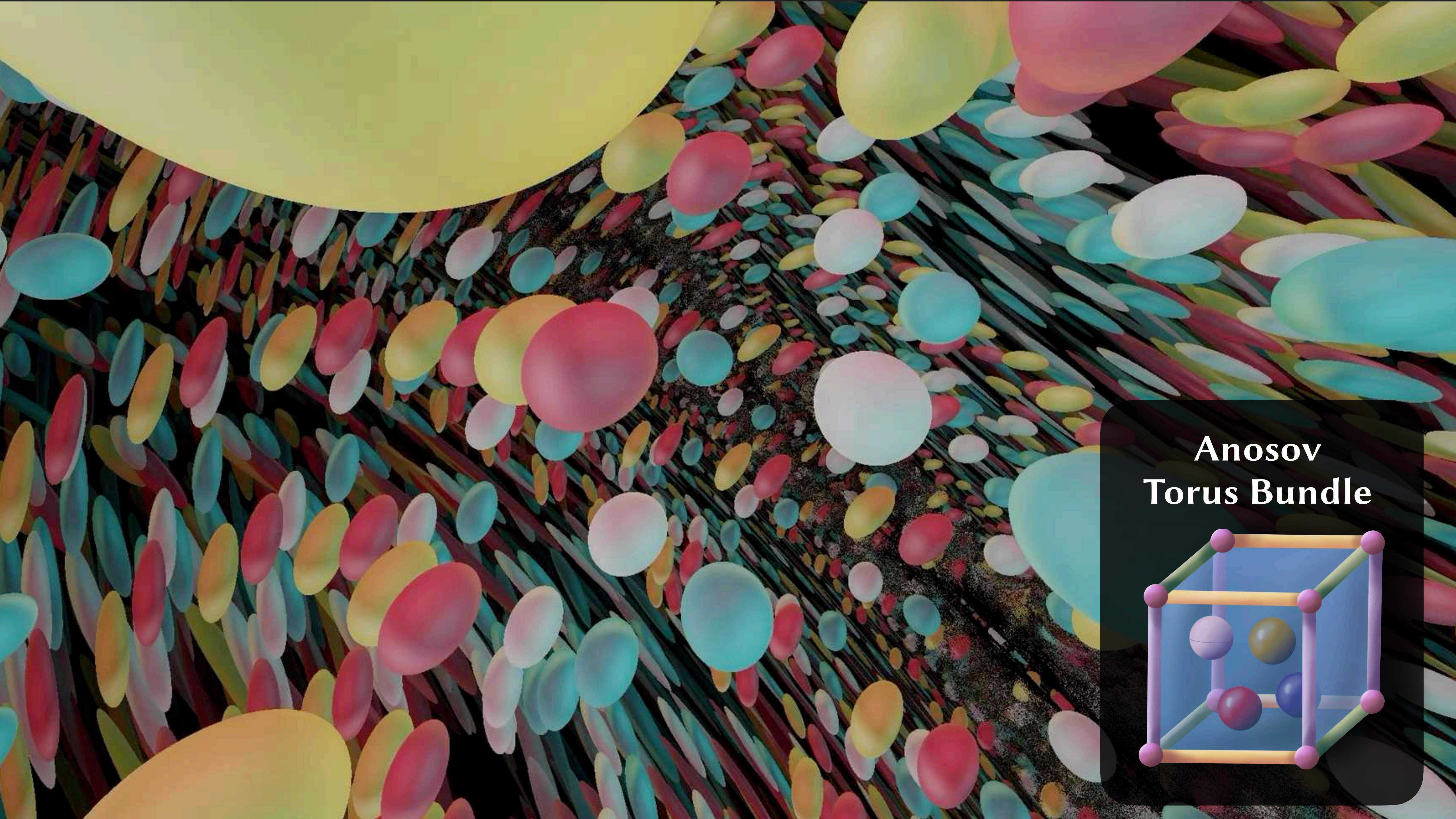
Anosov Mapping Torus



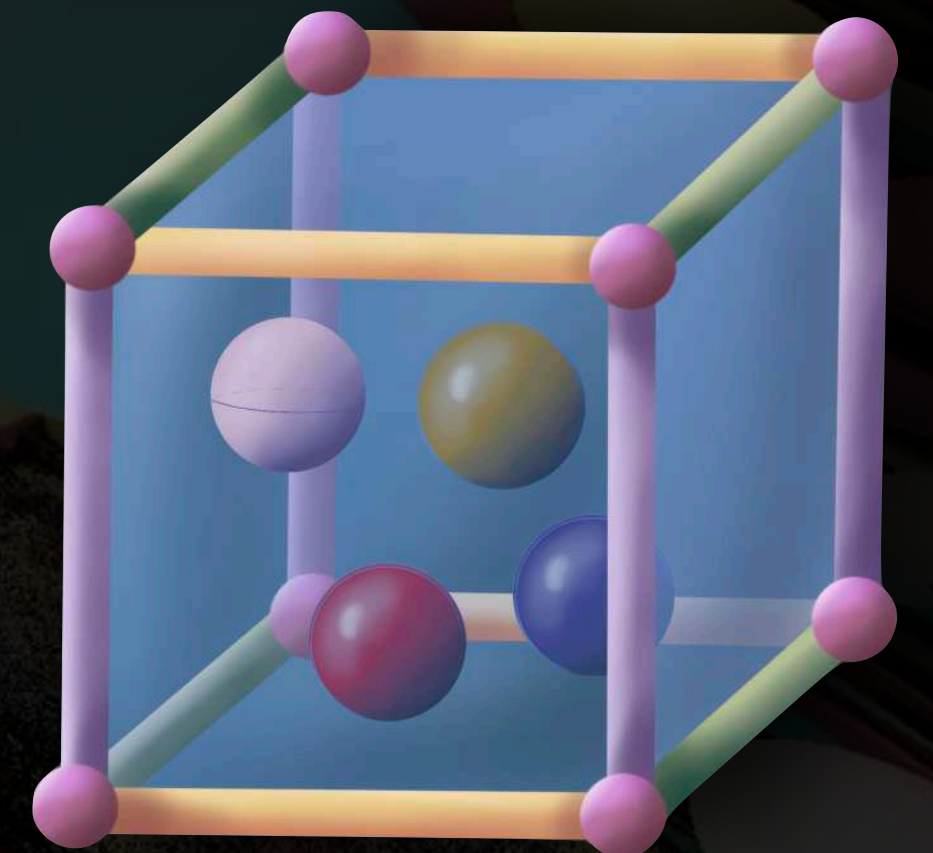
$$f: \pi_1 T^2 \rightarrow \pi_1 T^2 \quad (x, y) \mapsto (2x + y, x + y)$$

Anosov Torus Bundle



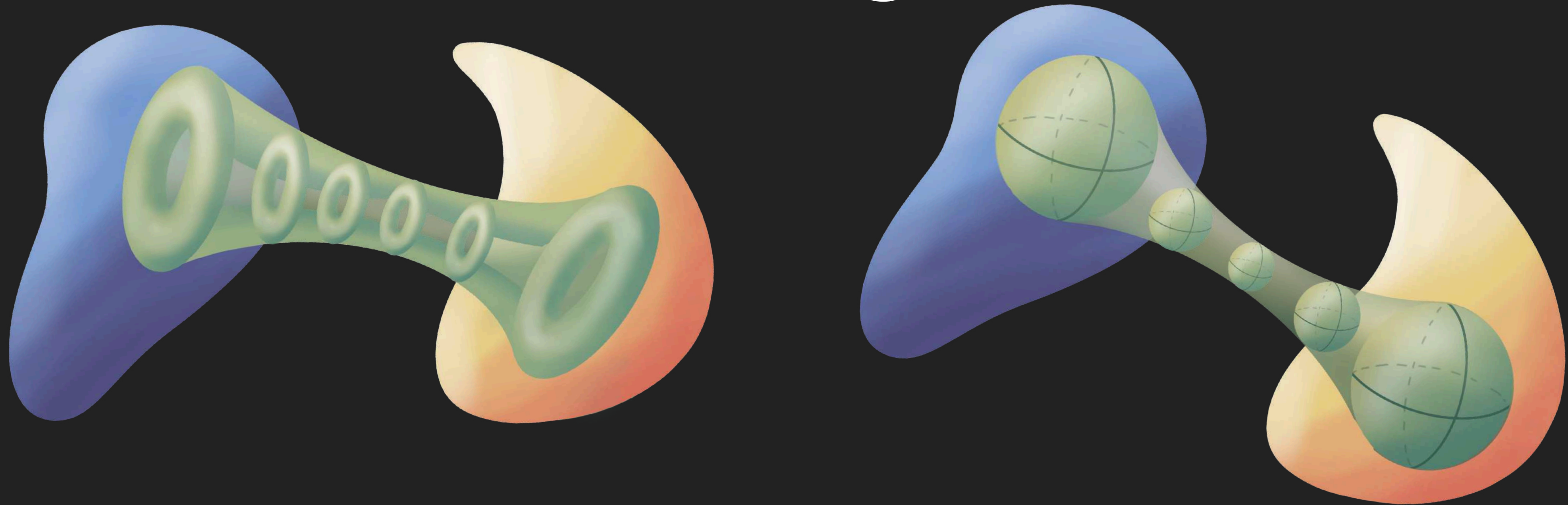


Anosov Torus Bundle



IIIb

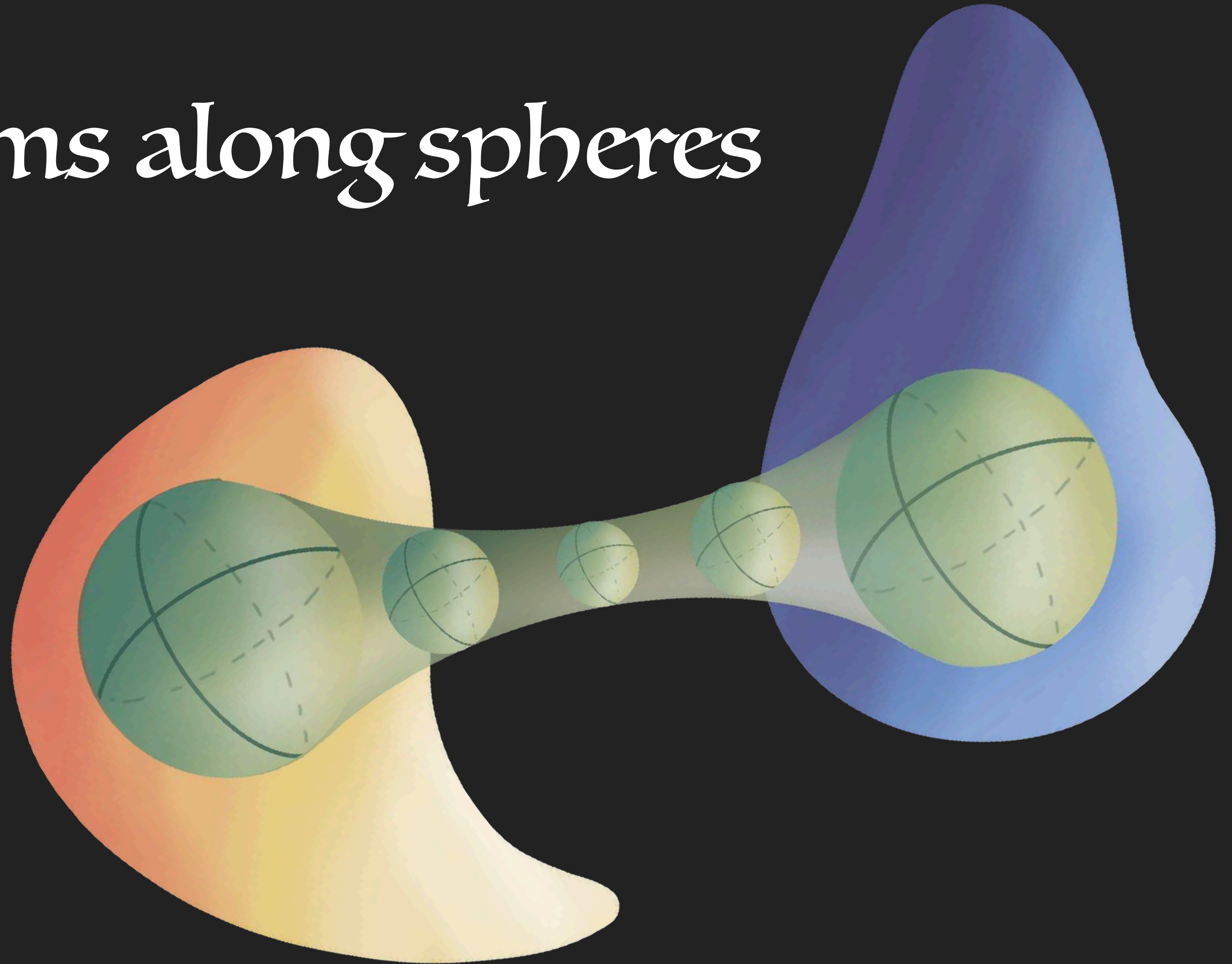
Putting the pieces together



To build a general 3-Manifold, need to connect geometric pieces along spheres and tori.

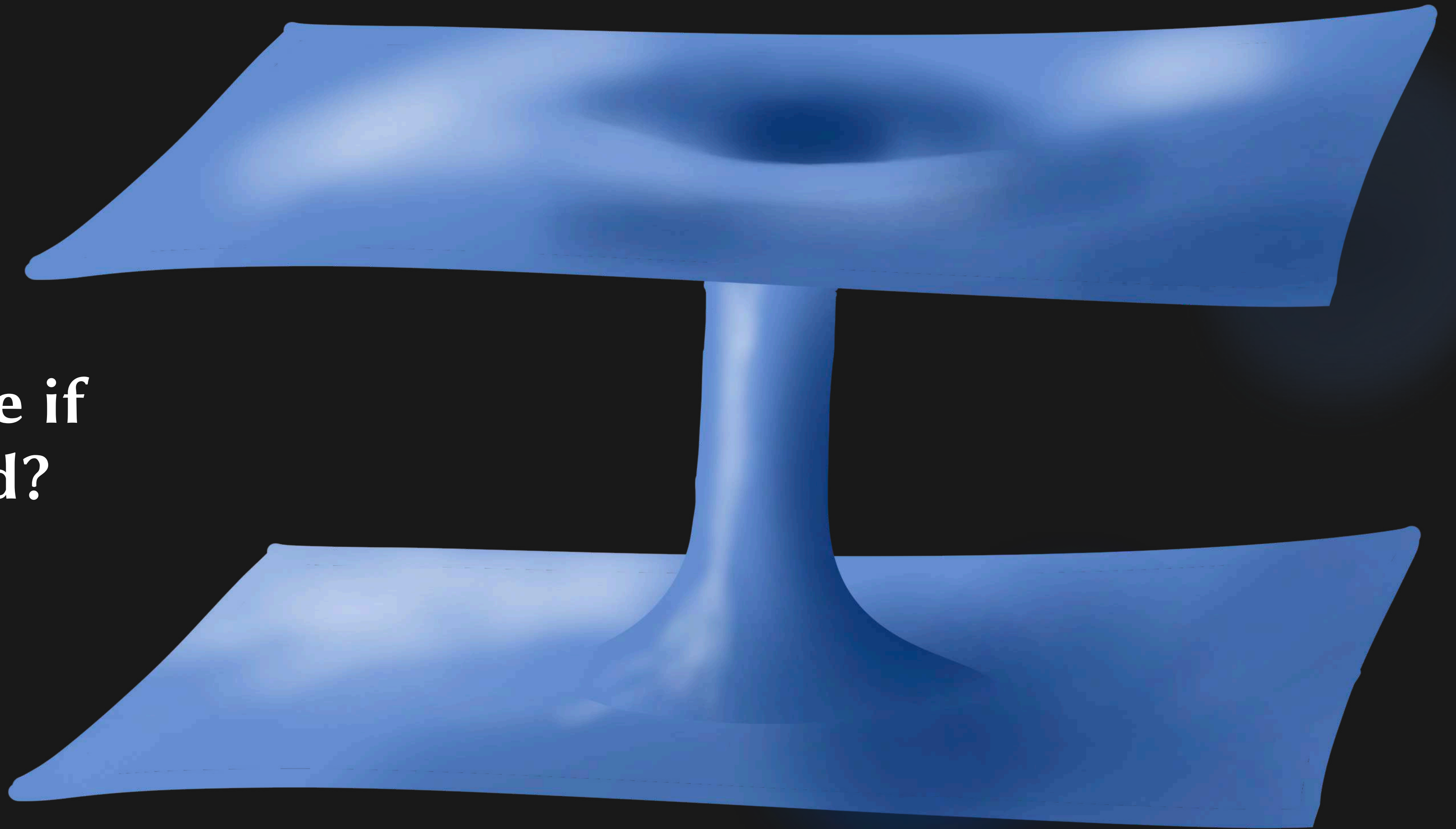
Connect sums along spheres

Connect via
 $\mathbb{S}^2 \times I$ while
minimally
distorting
geometry



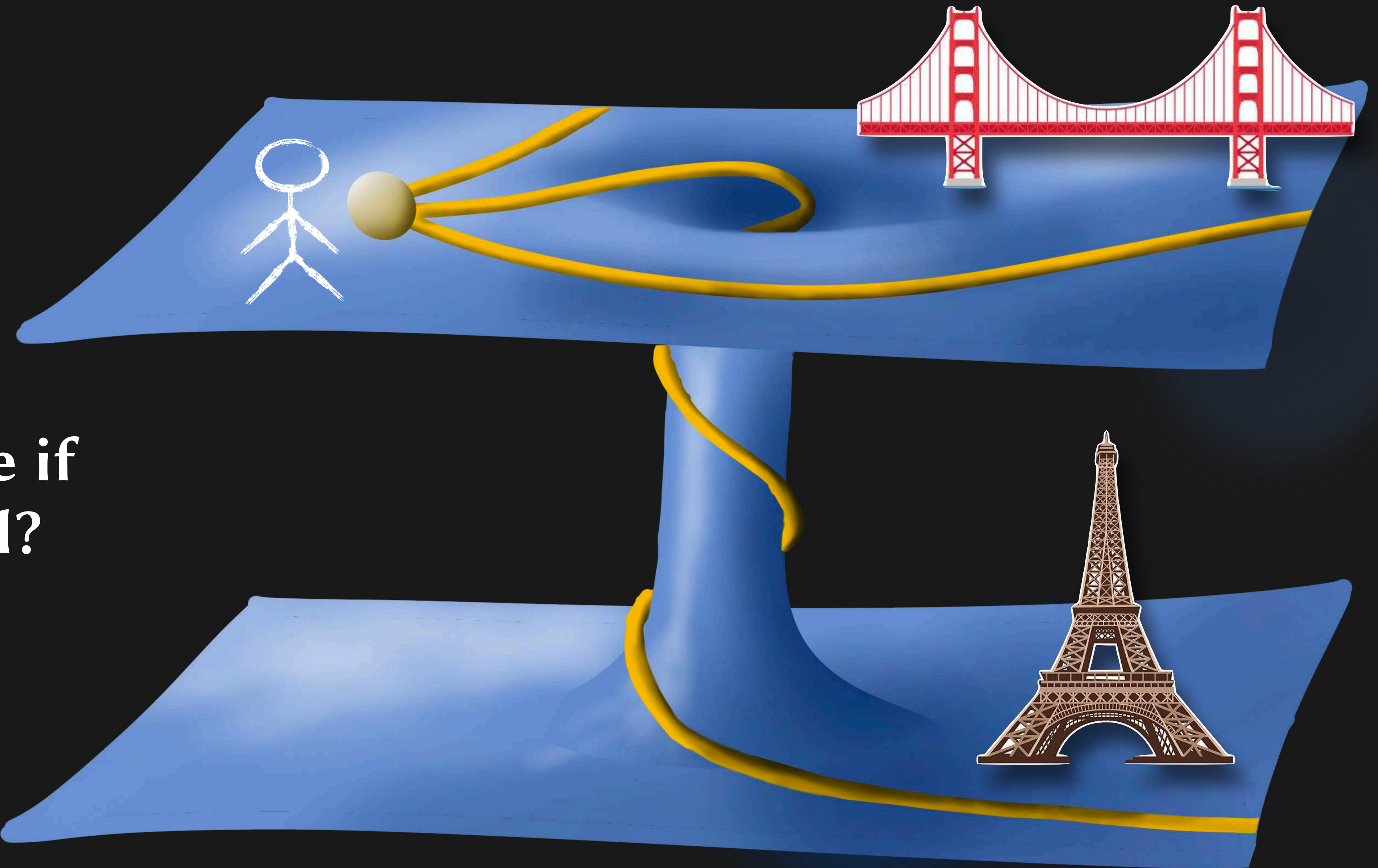
$$\mathbb{E}^3 \# \mathbb{E}^3$$

What do you see if
you look around?



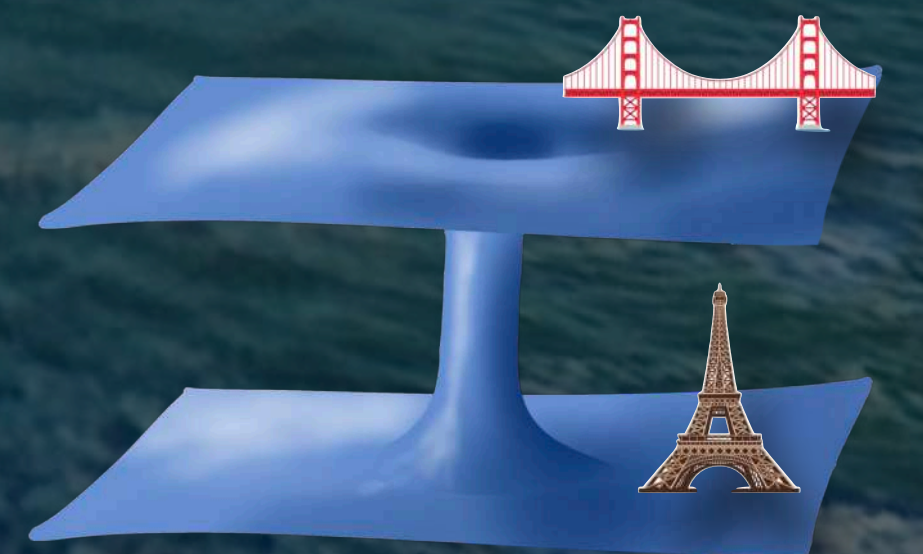
$\mathbb{E}^3 \# \mathbb{E}^3$

What do you see if
you look around?



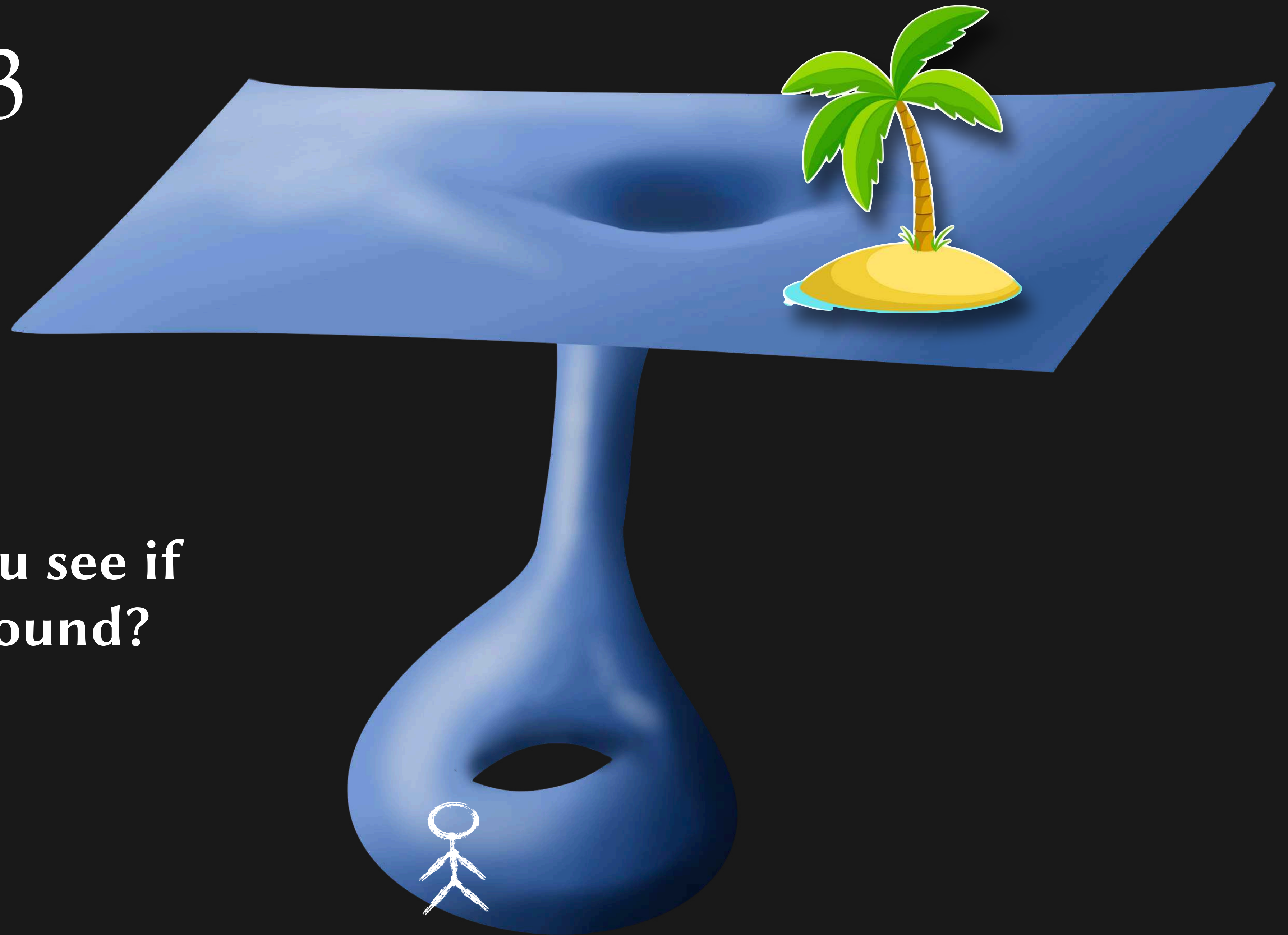


$$\mathbb{E}^3 \# \mathbb{E}^3$$



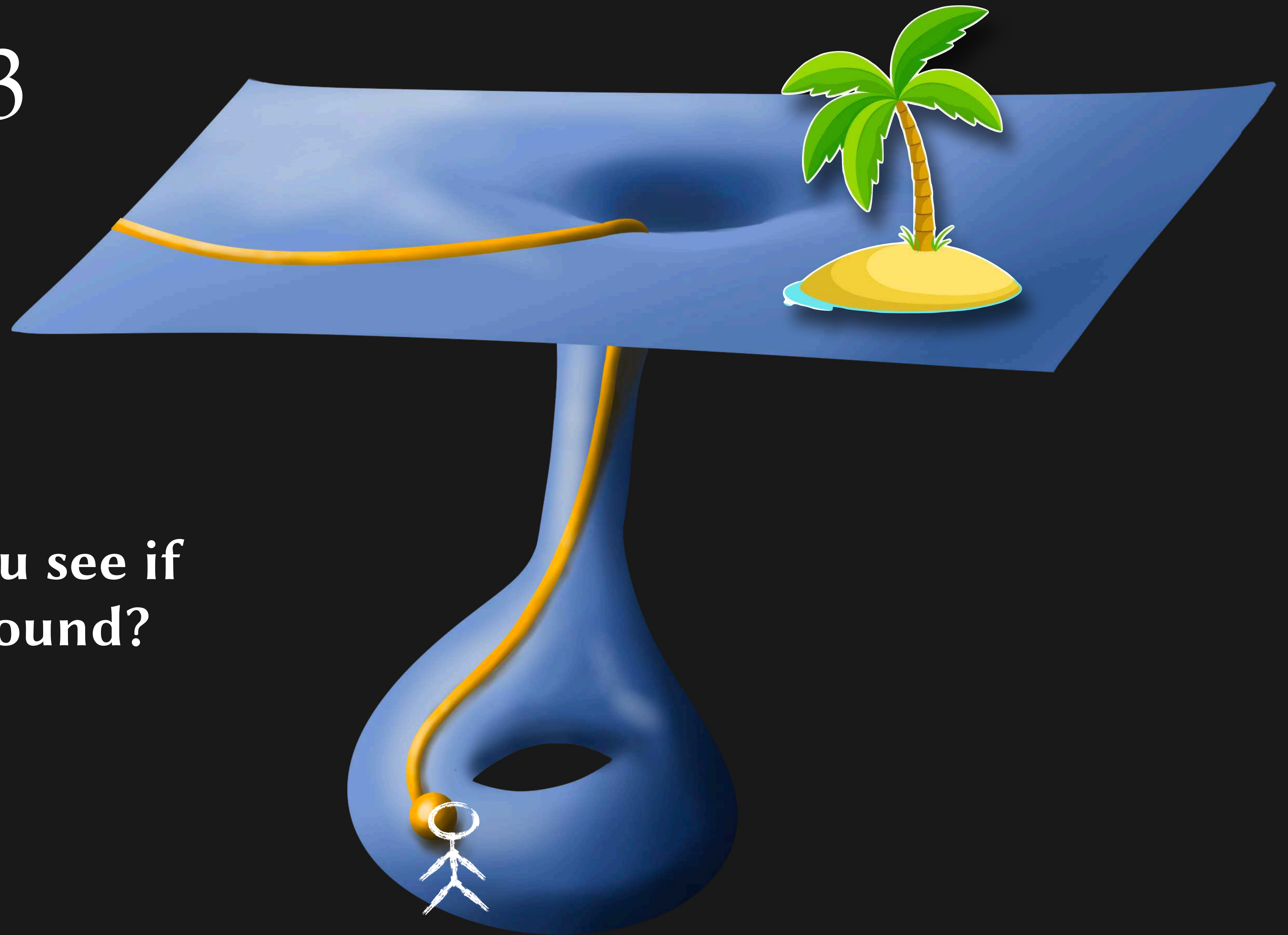
$$\mathbb{E}^3 \# T^3$$

What do you see if
you look around?



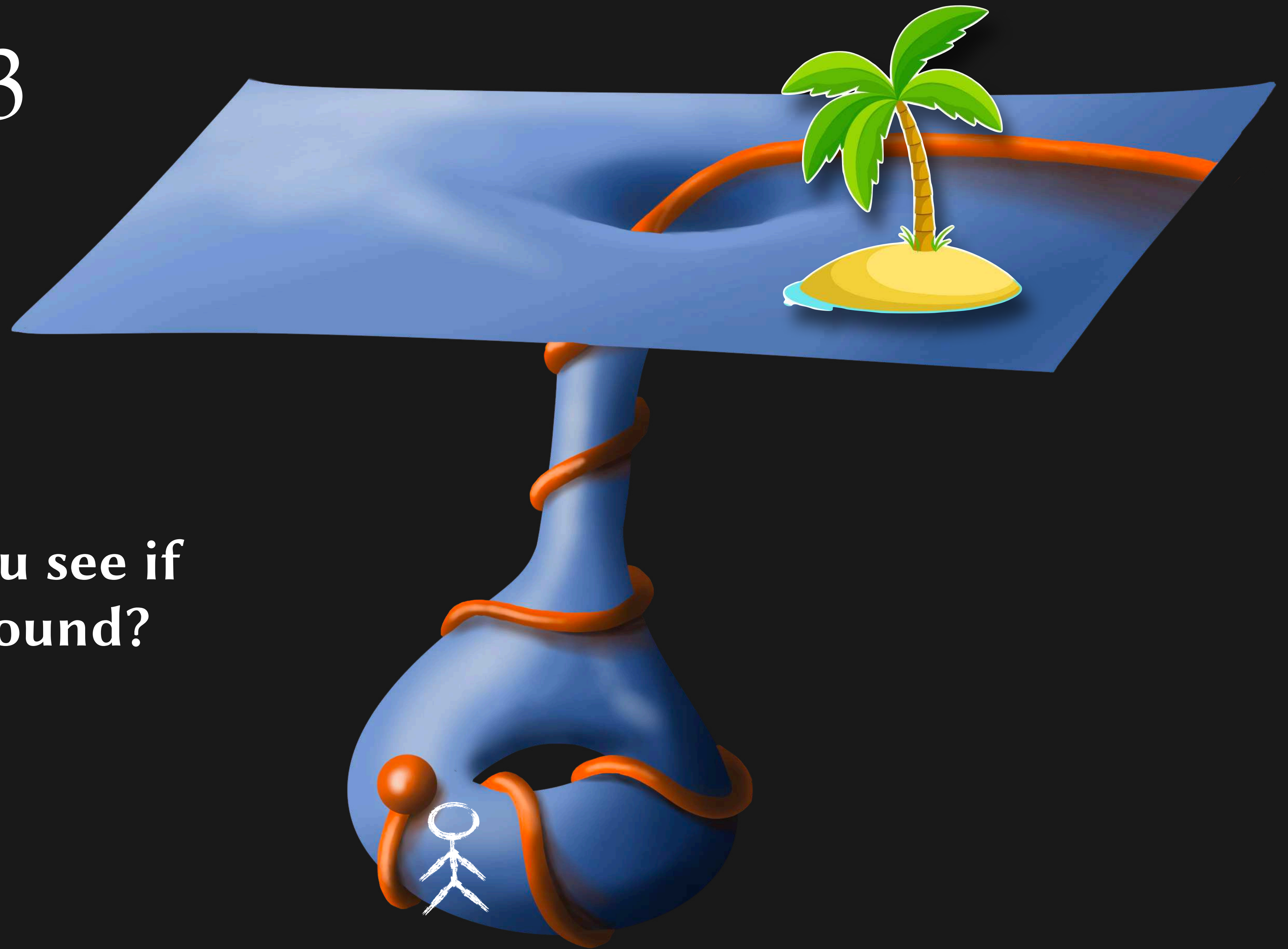
$$\mathbb{E}^3 \# T^3$$

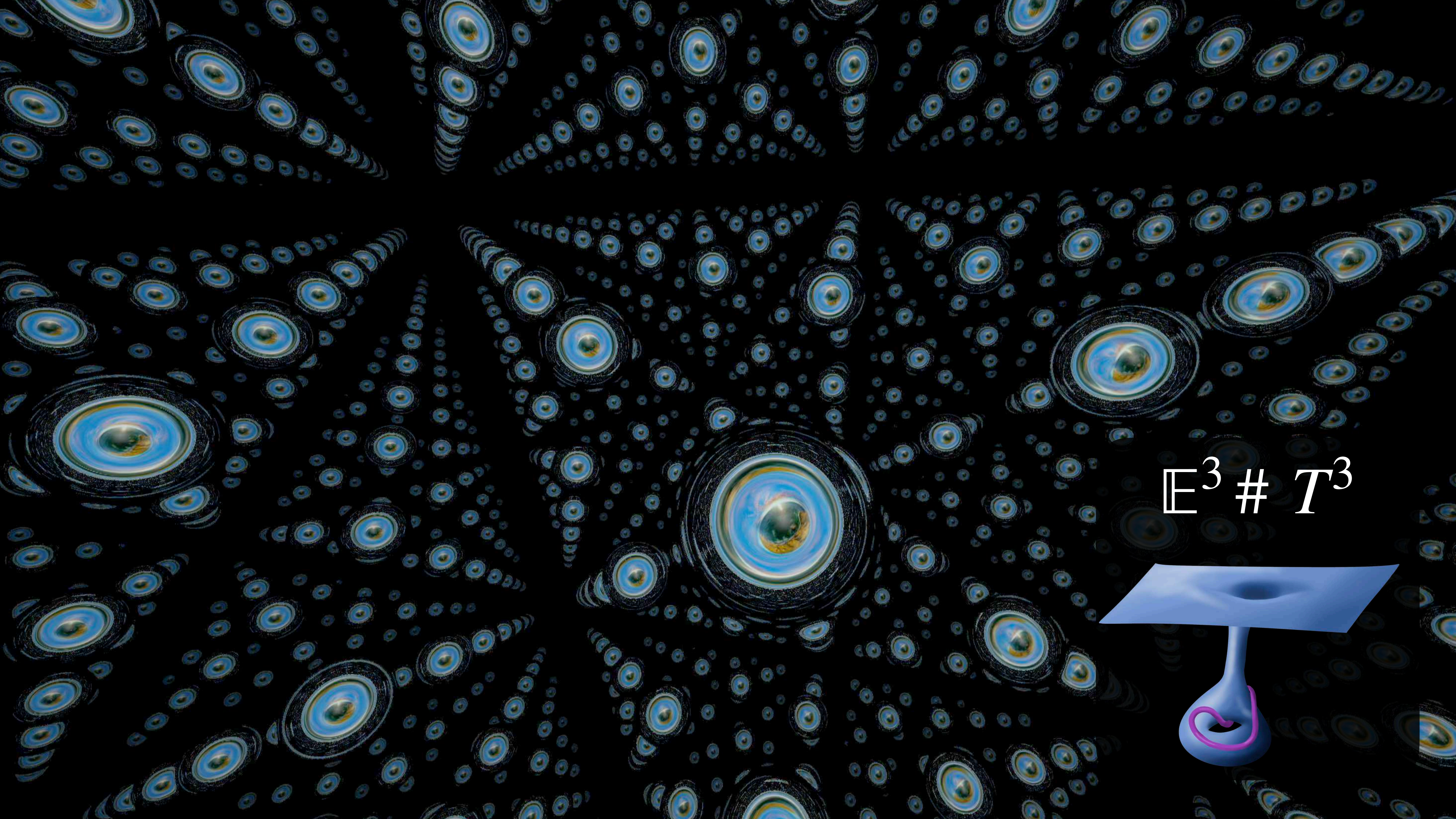
What do you see if
you look around?



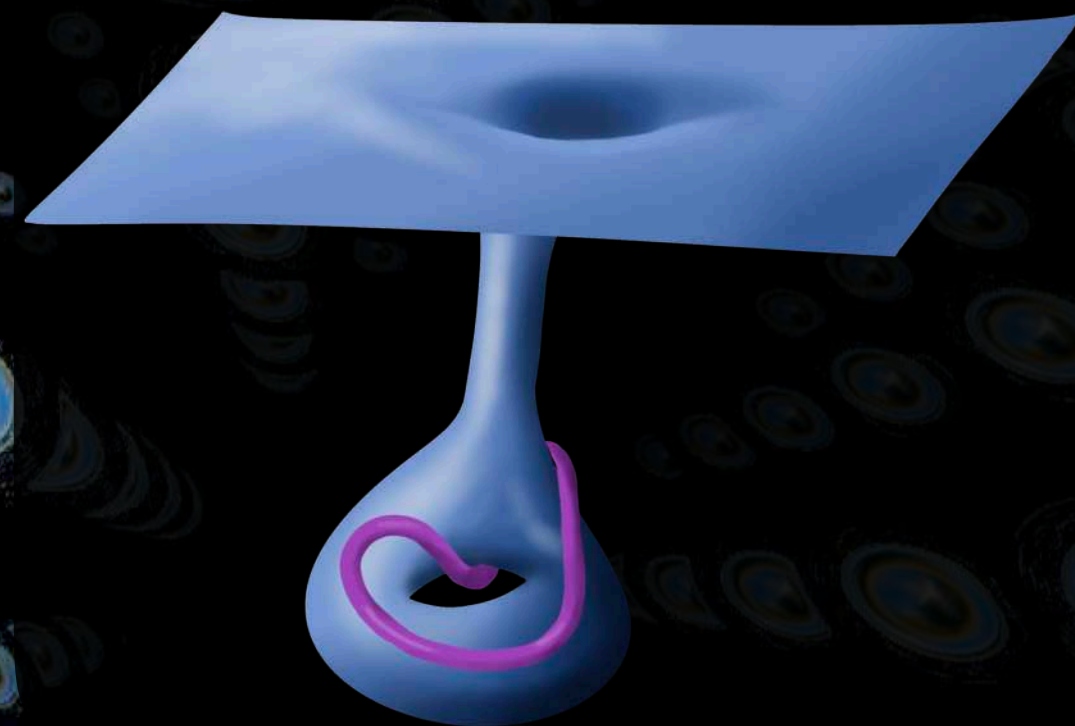
$$\mathbb{E}^3 \# T^3$$

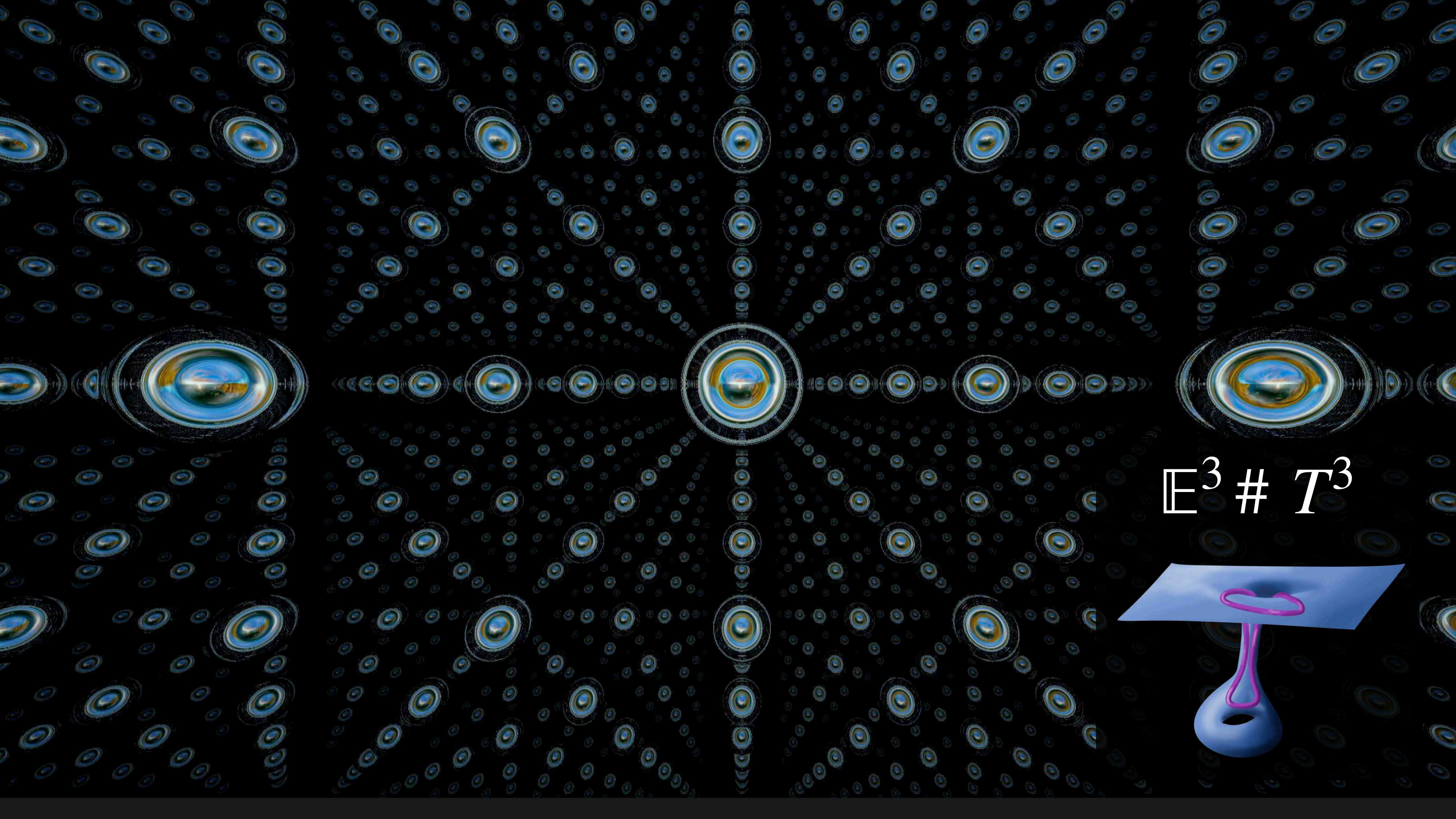
What do you see if
you look around?



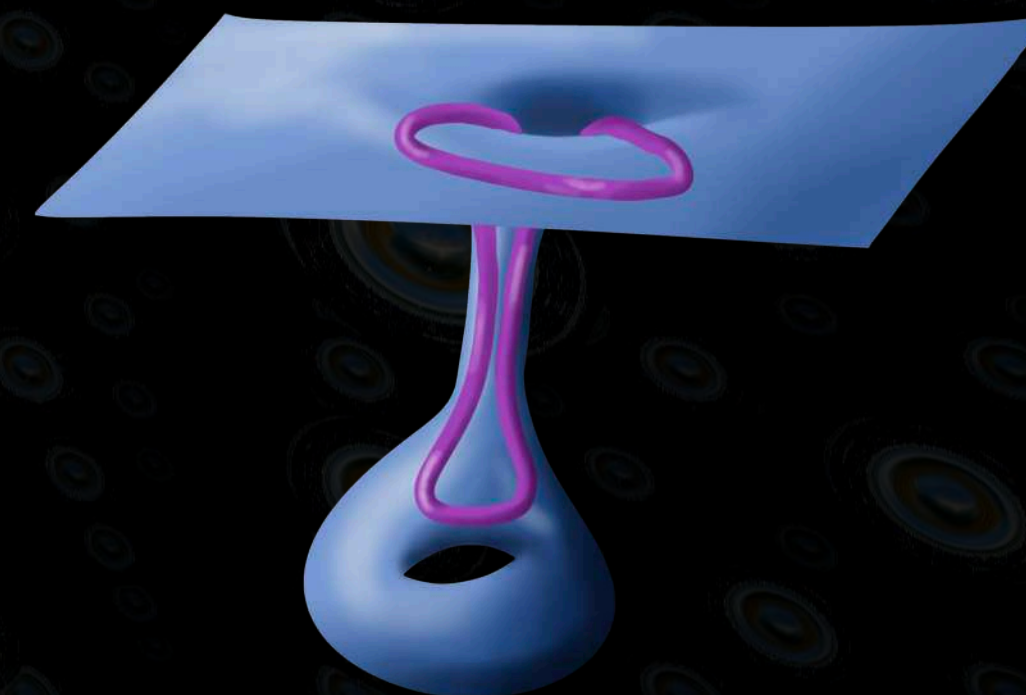


$$\mathbb{E}^3 \# T^3$$

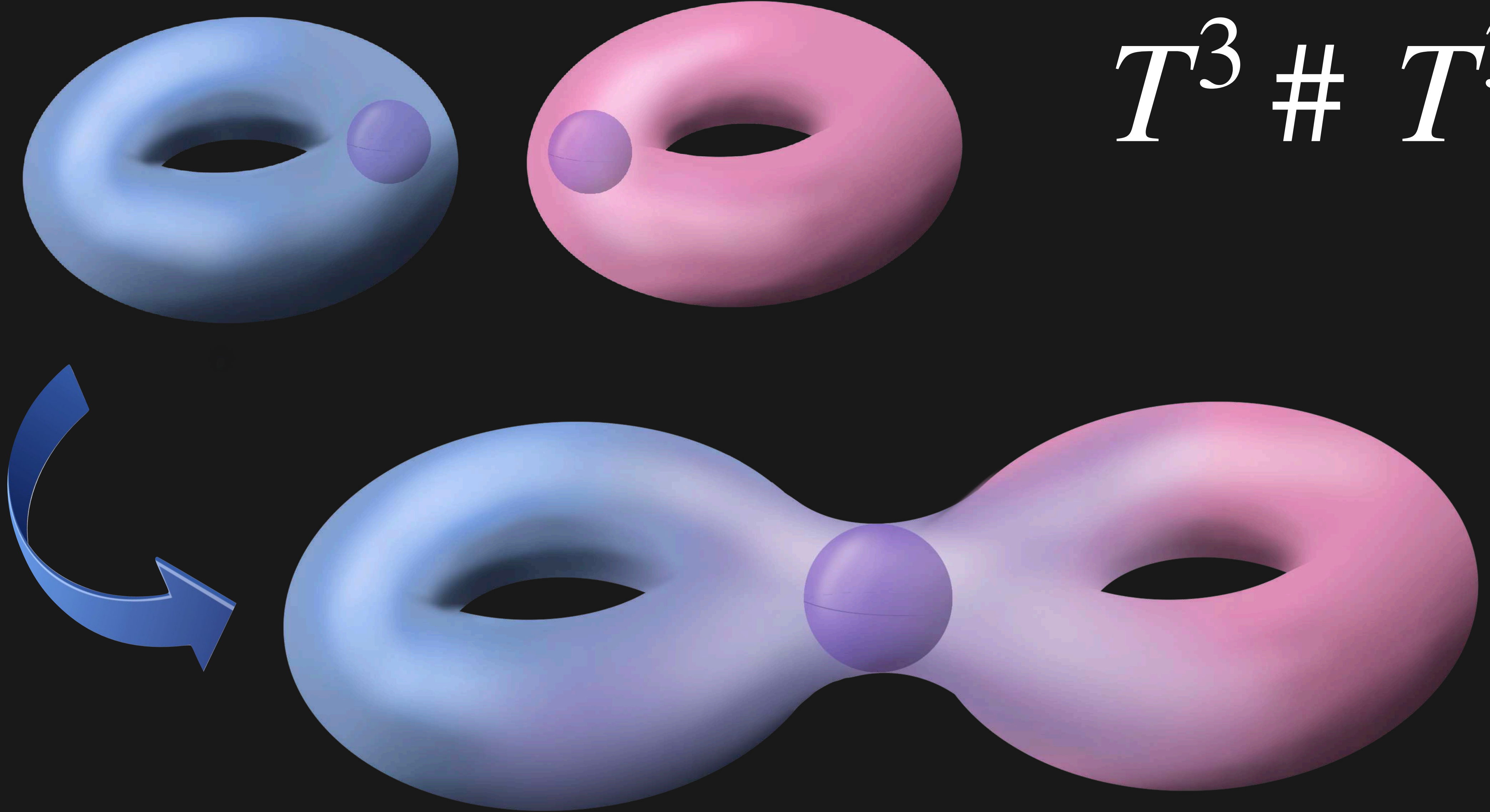


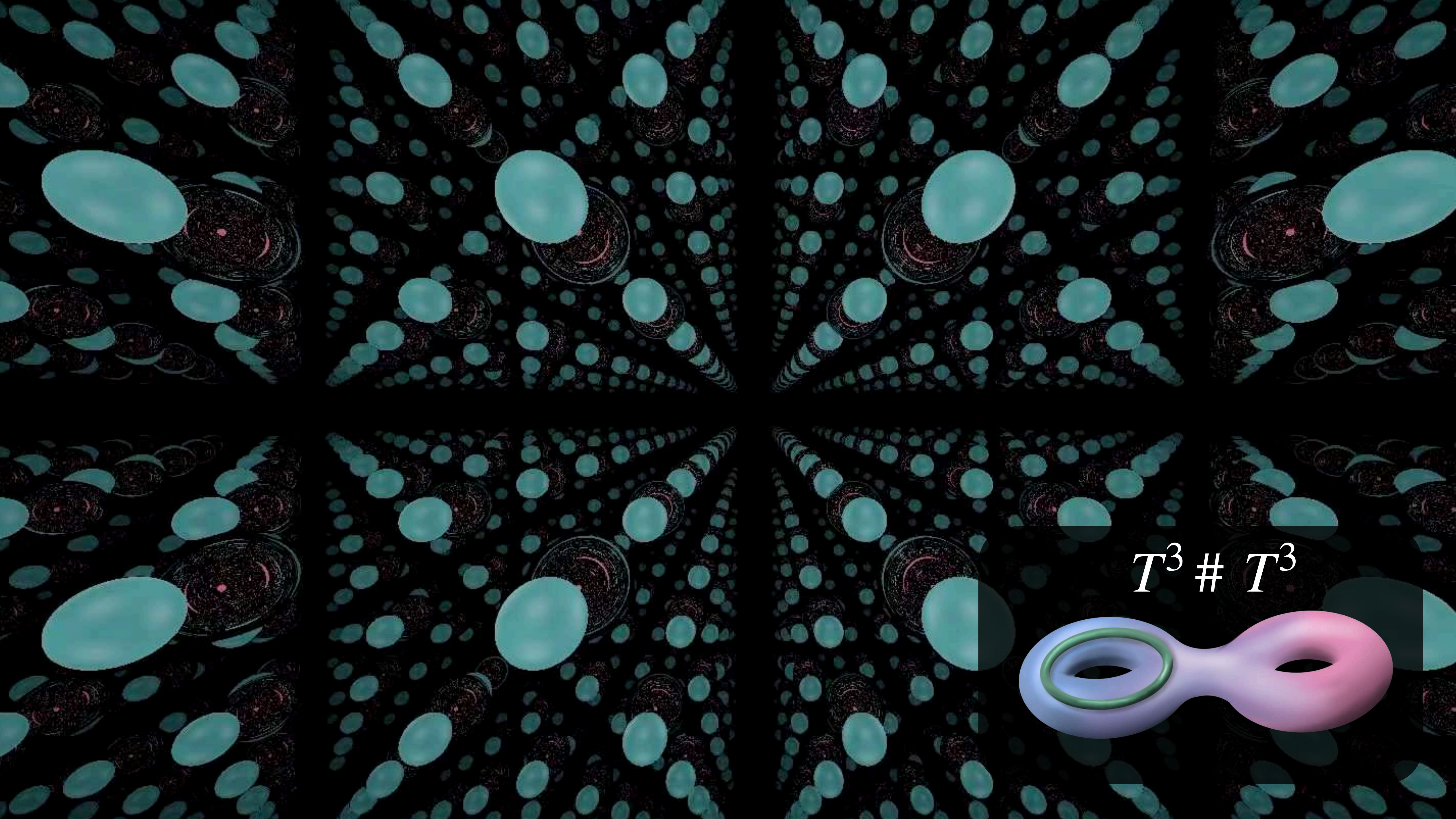


$$\mathbb{E}^3 \# T^3$$



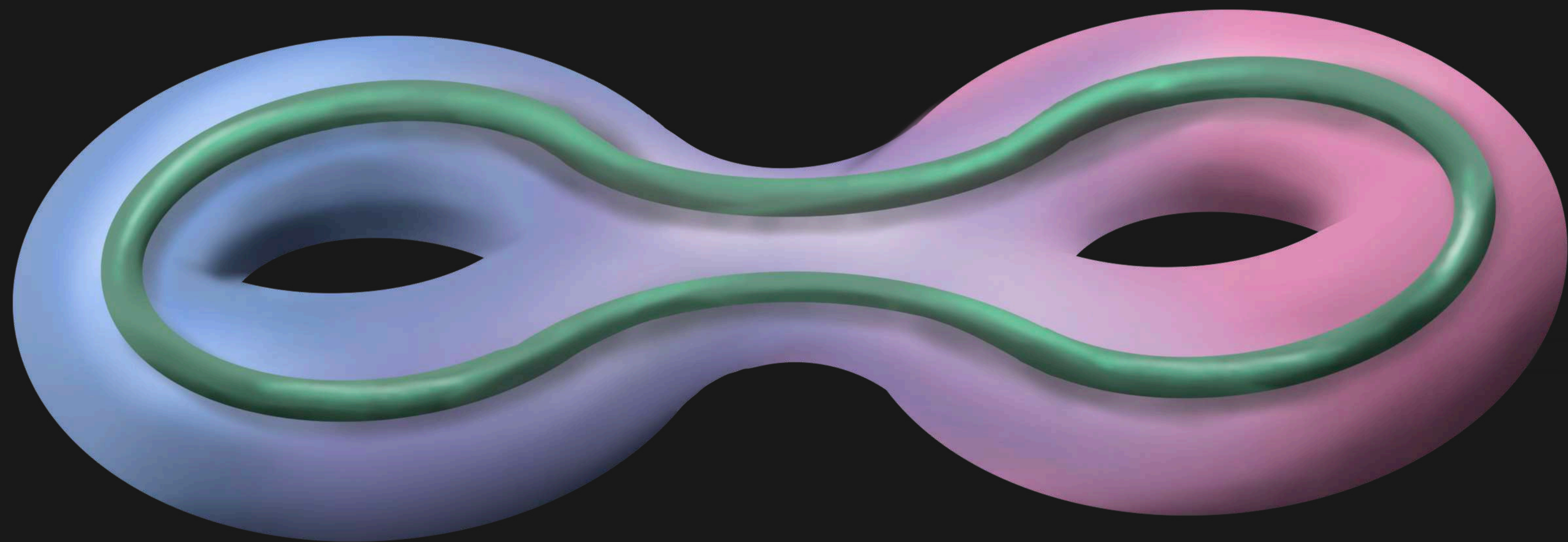
$$T^3 \# T^3$$

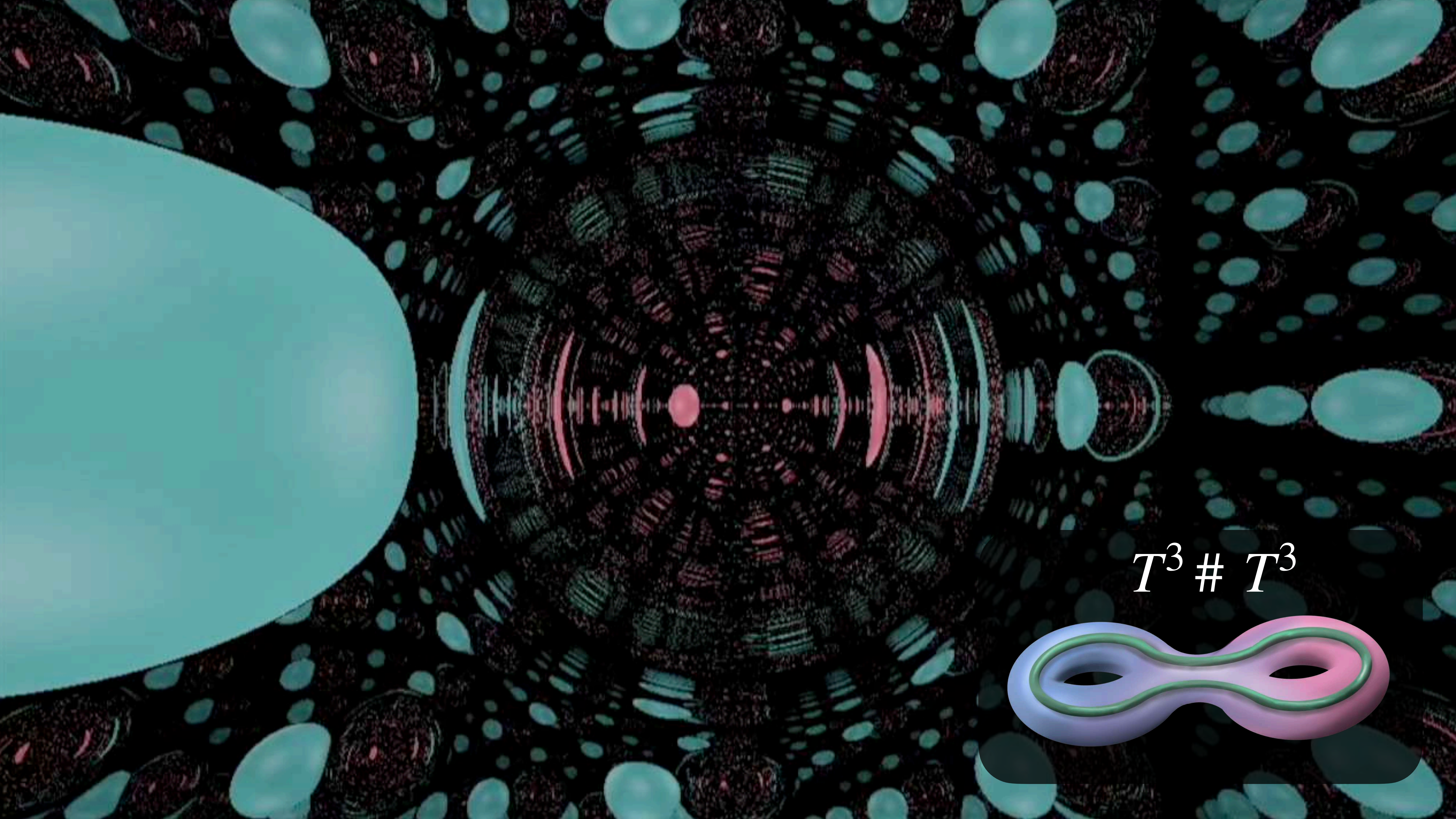




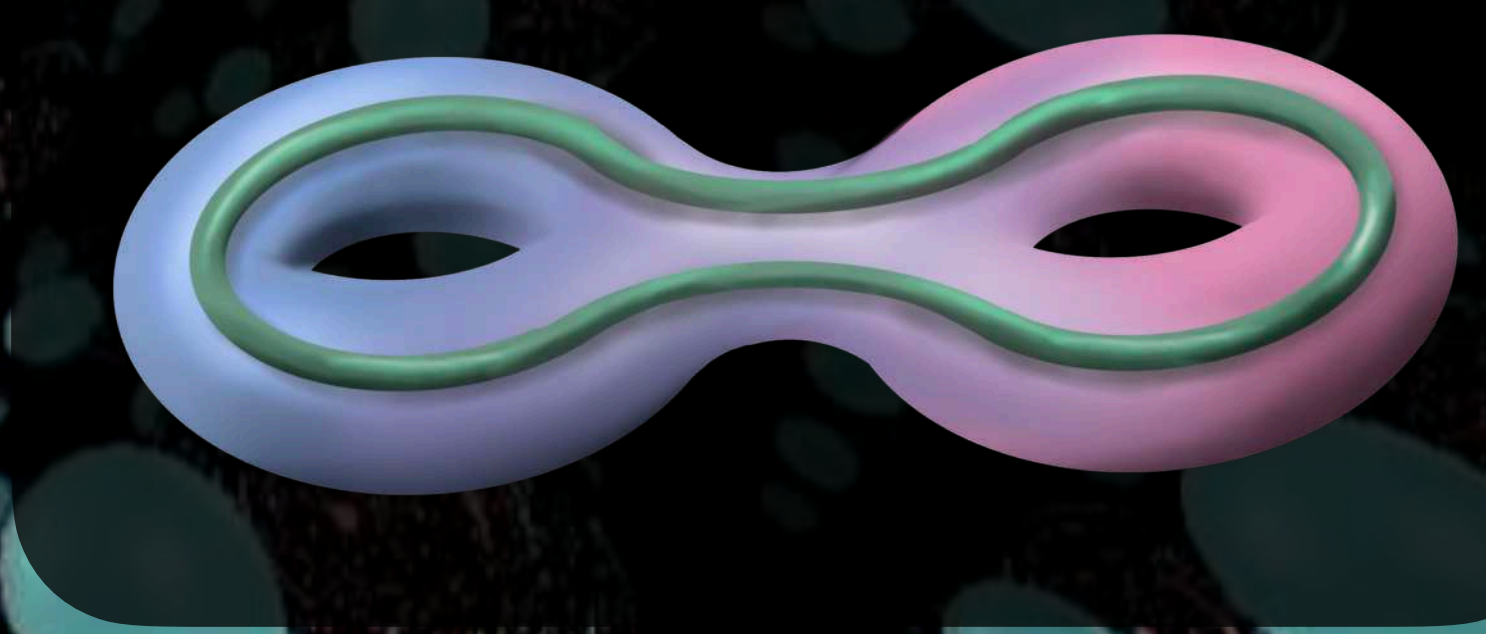
$$T^3 \# T^3$$



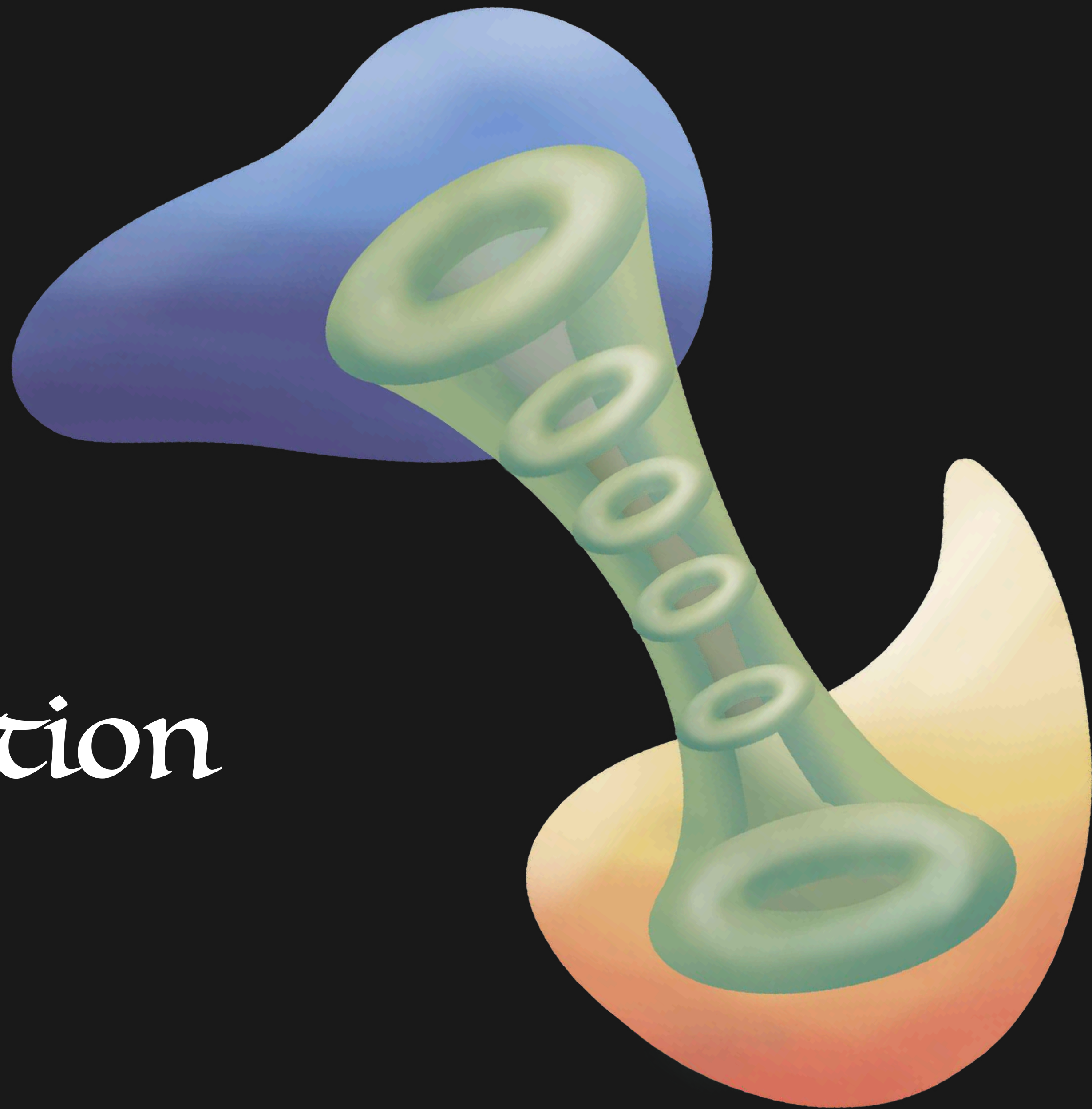


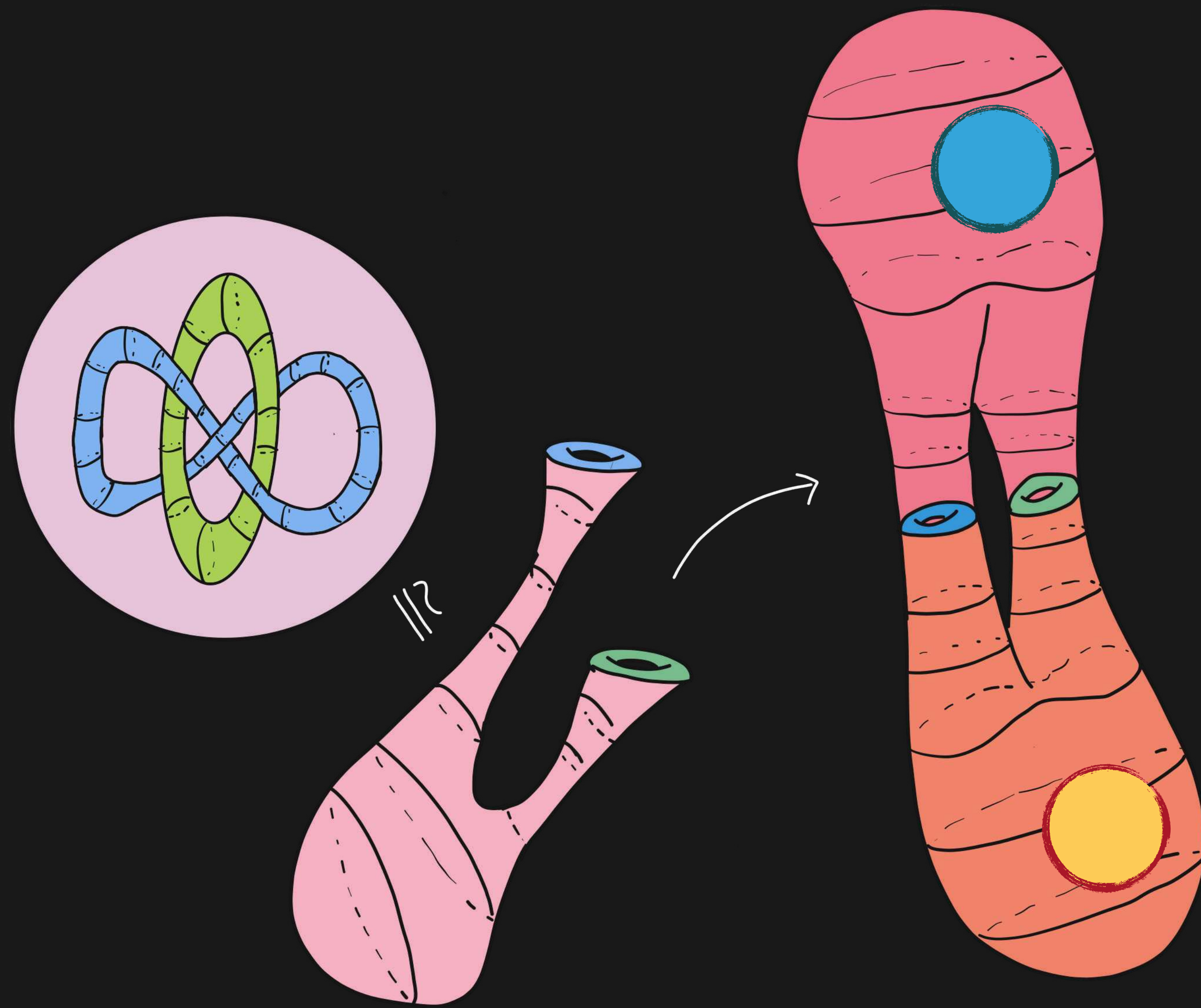


$$T^3 \# T^3$$



Torus decomposition

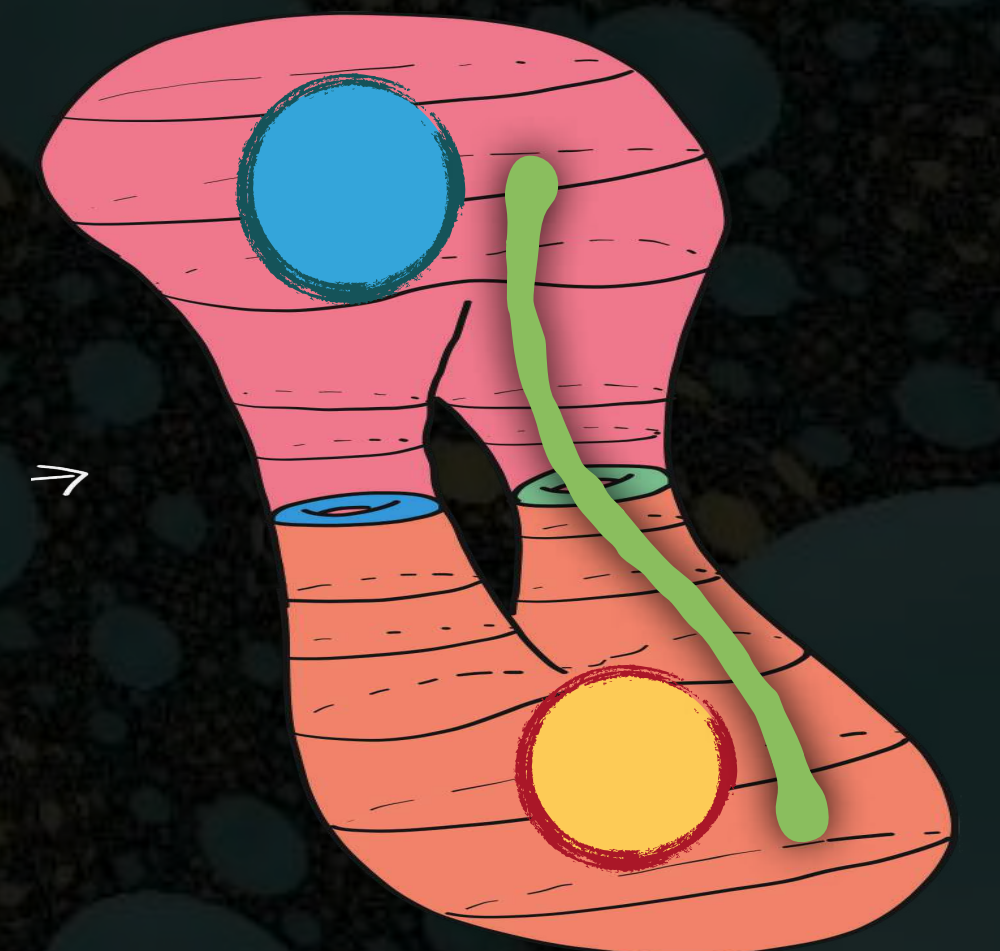




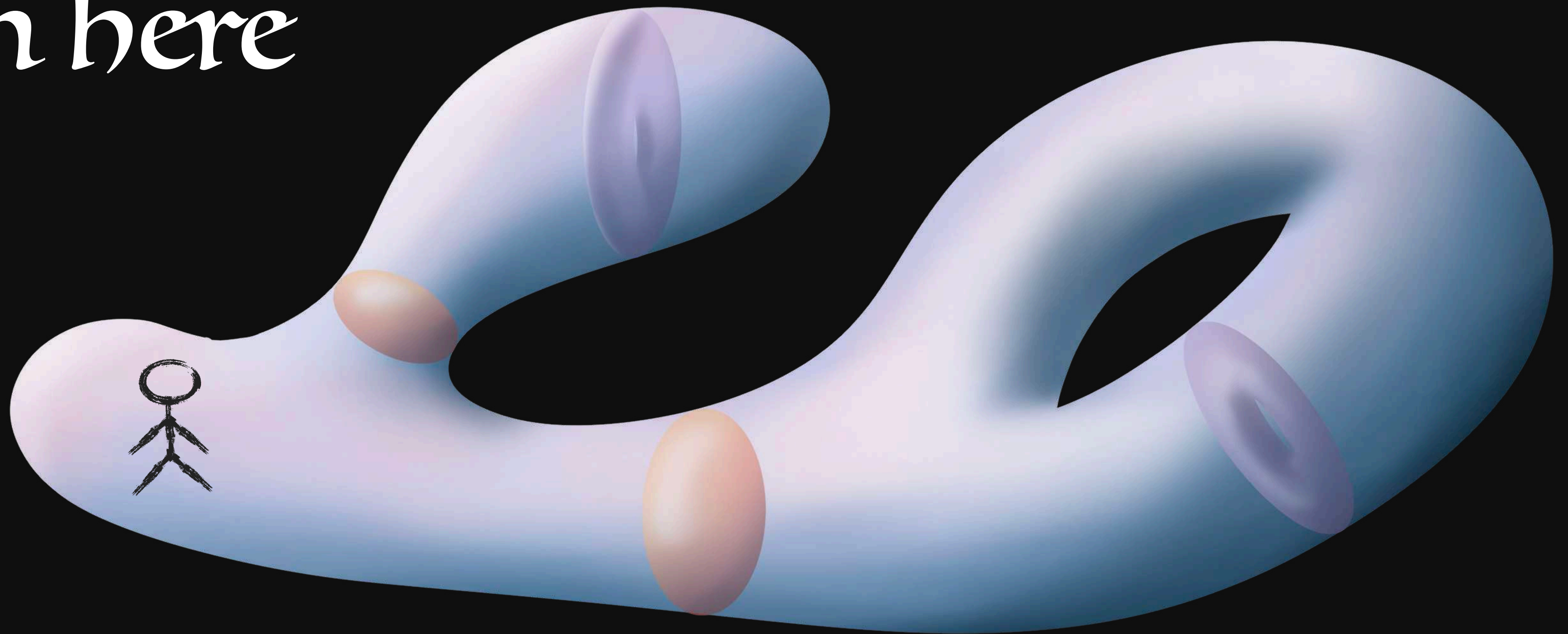
The Double of the Whitehead Link

*Q: What are we
going to see?*

Whitehead Link Double



We can now imagine
What it looks like
to live in here





Thanks!