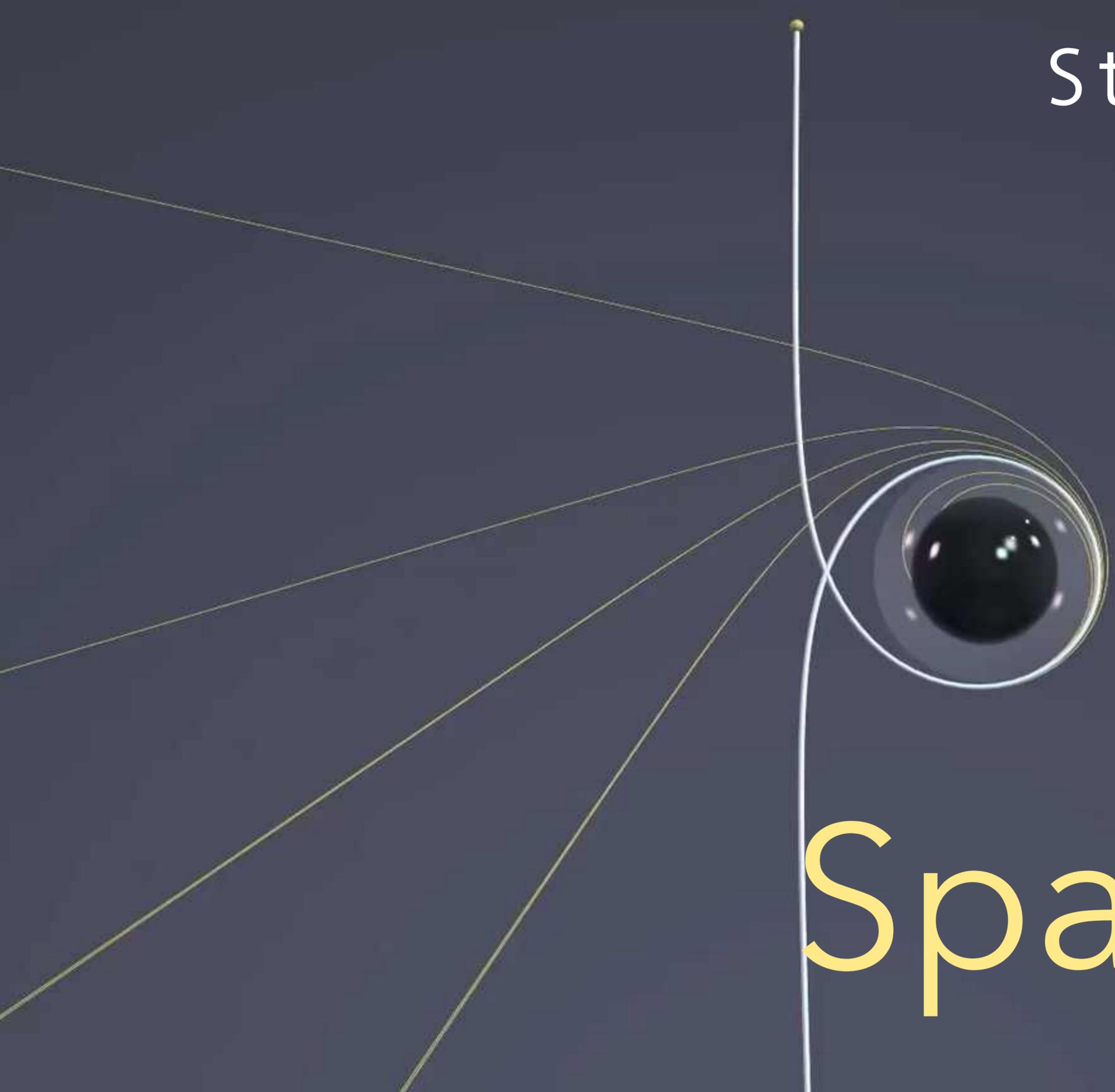


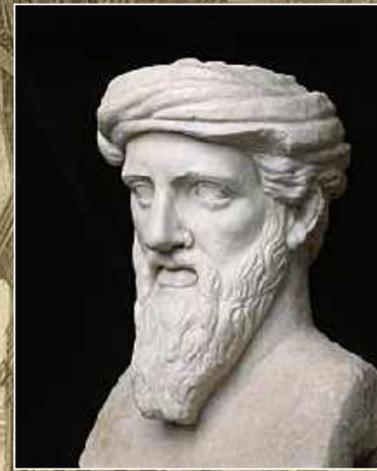
Steve Trettel, USF



Seeing

Space(time)

Geometry of the Greeks



Πυθαγόρας



Εὐκλείδης



Αρχιμήδης



Brunelleschi



Desargues



Pascal



Wantzel



عمر خیّام



Descartes



Newton



Raphael



Copernicus



Barrow



Kepler



Πτολεμαίος



Fermat

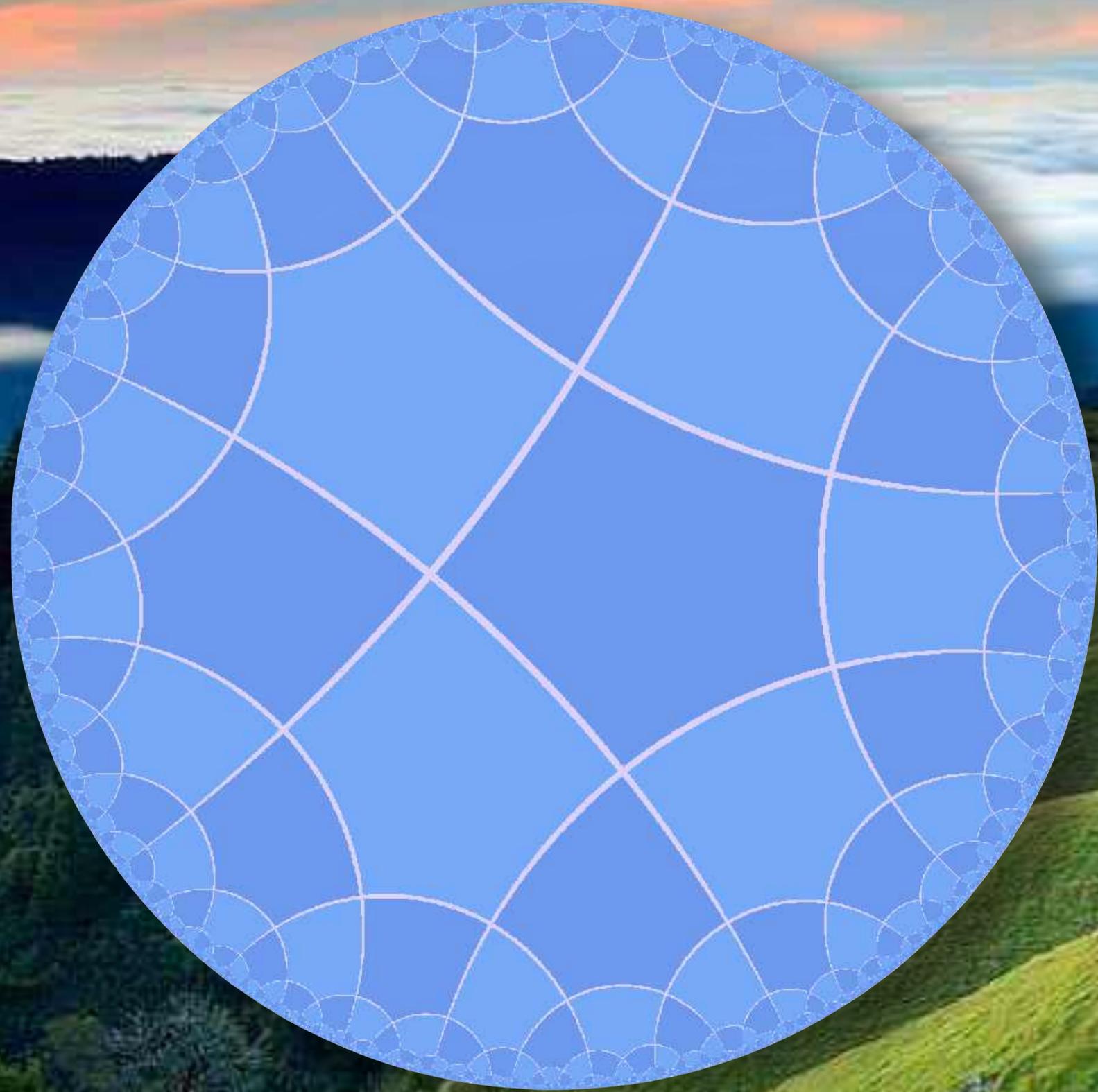


Dürer

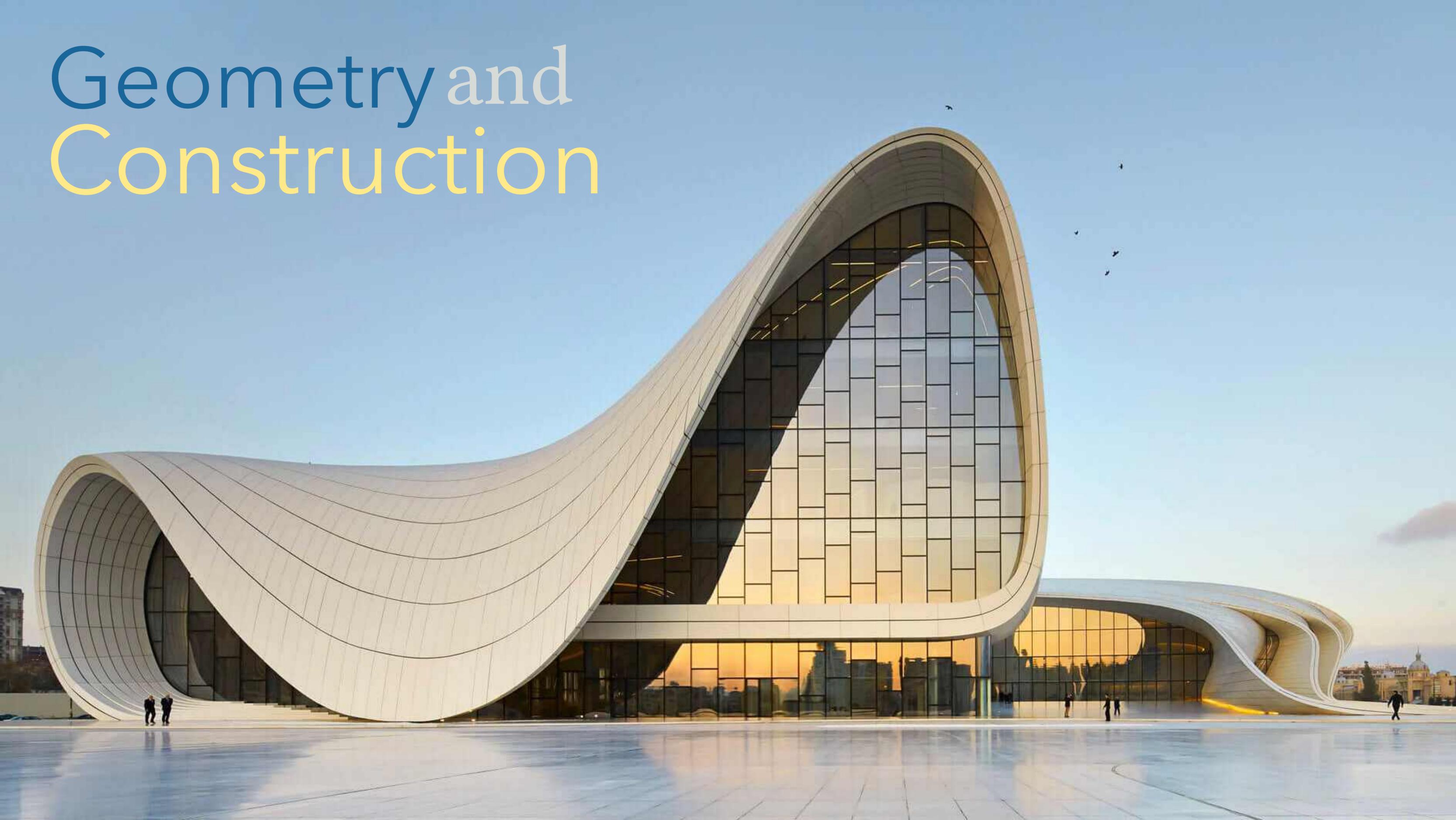


Волюаі Лобачевский Гауџ

**Geometry and
Topography**



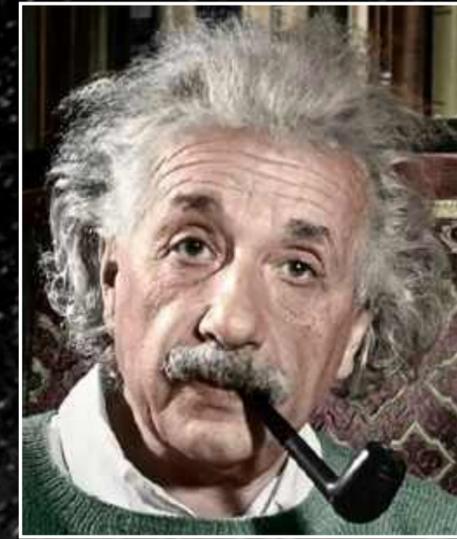
Geometry and Construction



Geometry and Spacetime



Riemann

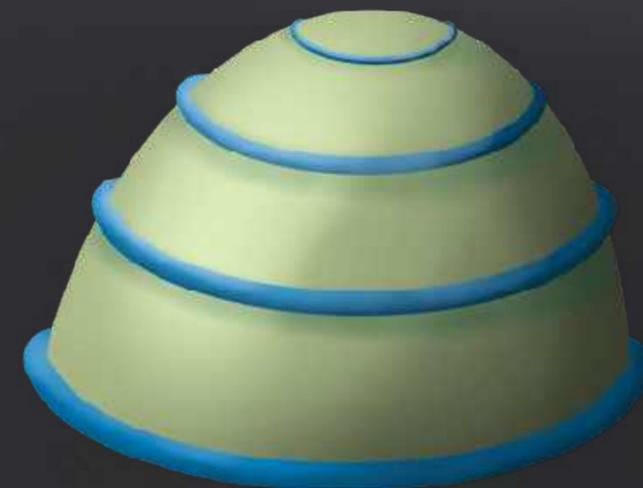
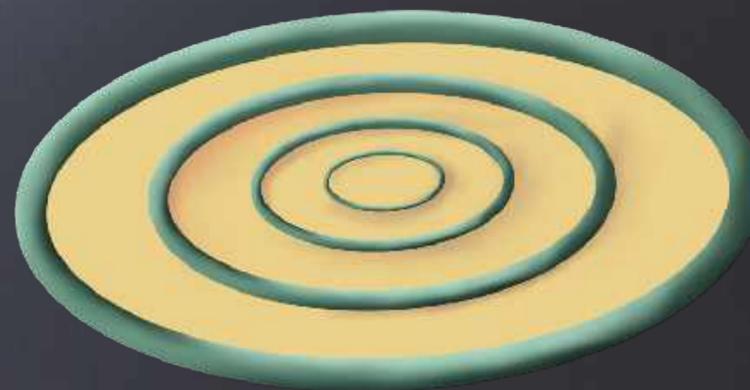
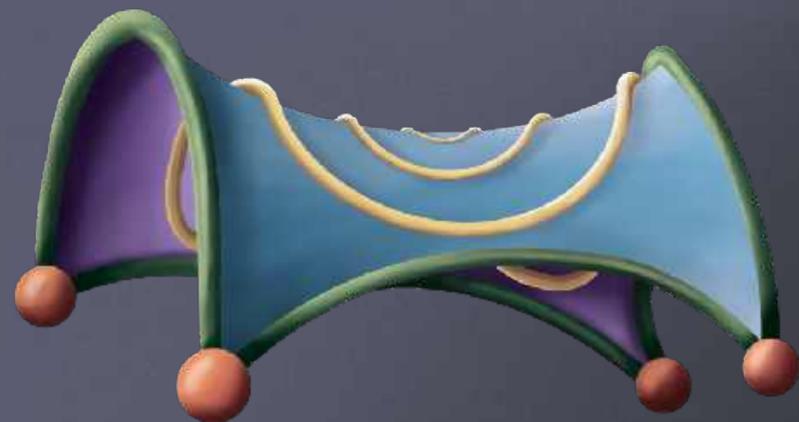


Einstein

What is Curvature?

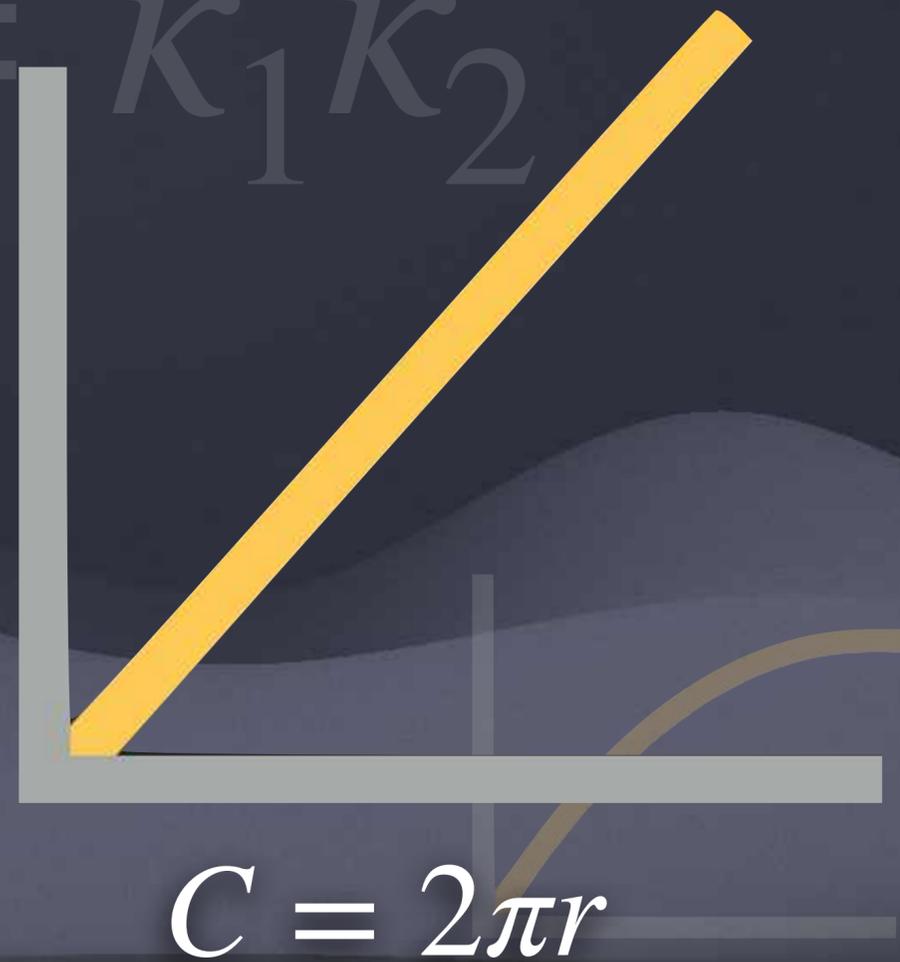
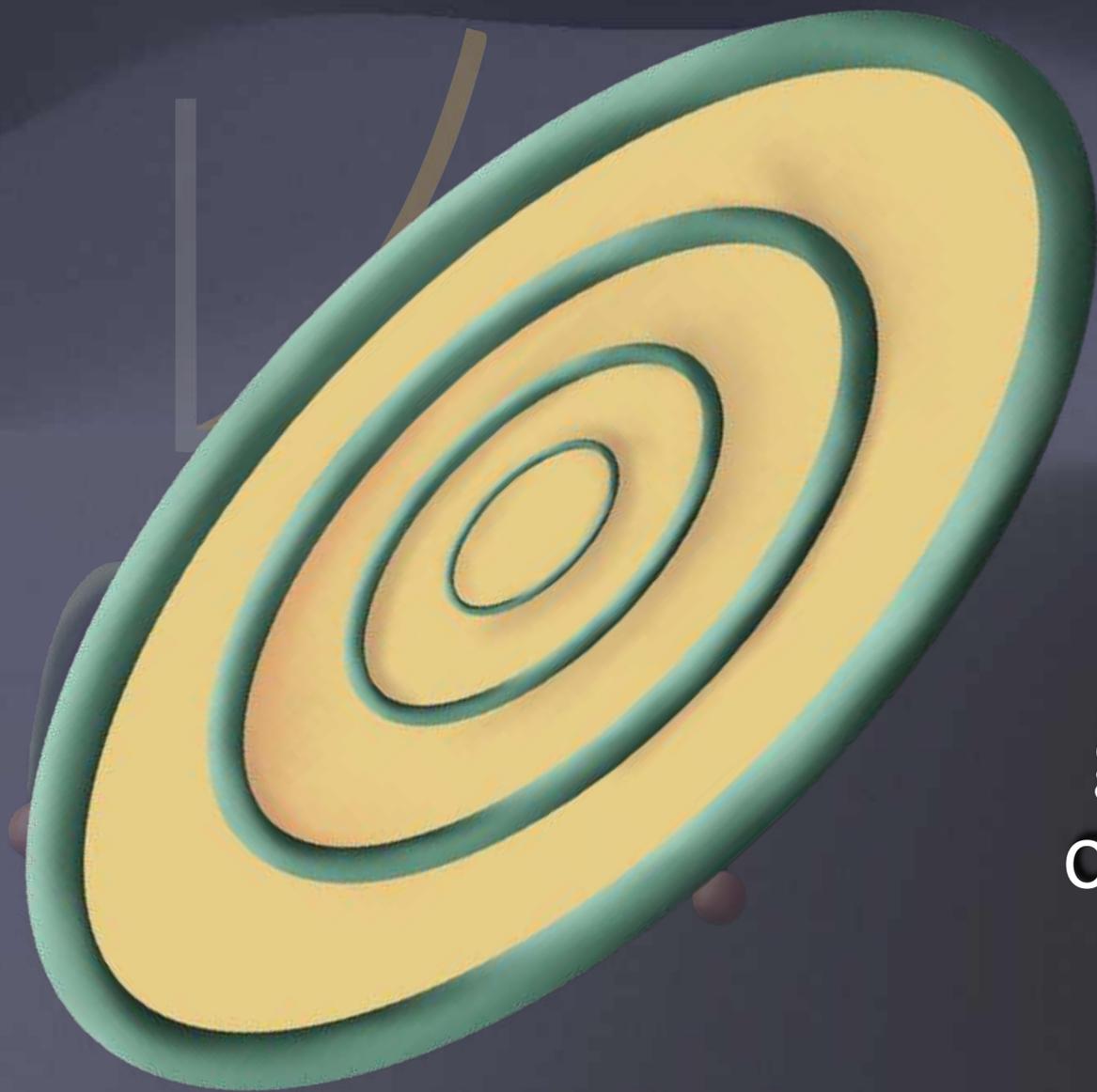
$$K \equiv \kappa \kappa$$

How can we determine if a surface cannot be flattened to the plane?



What is Curvature?

$$\mathcal{K} = \kappa_1 \kappa_2$$



In Euclidean (flat) geometry, the ratio of the circumference of a circle to its radius is constant!

What is Curvature?

$$\mathcal{K} = \mathcal{K}_1 \mathcal{K}_2$$



$$C = 2\pi \sin r$$



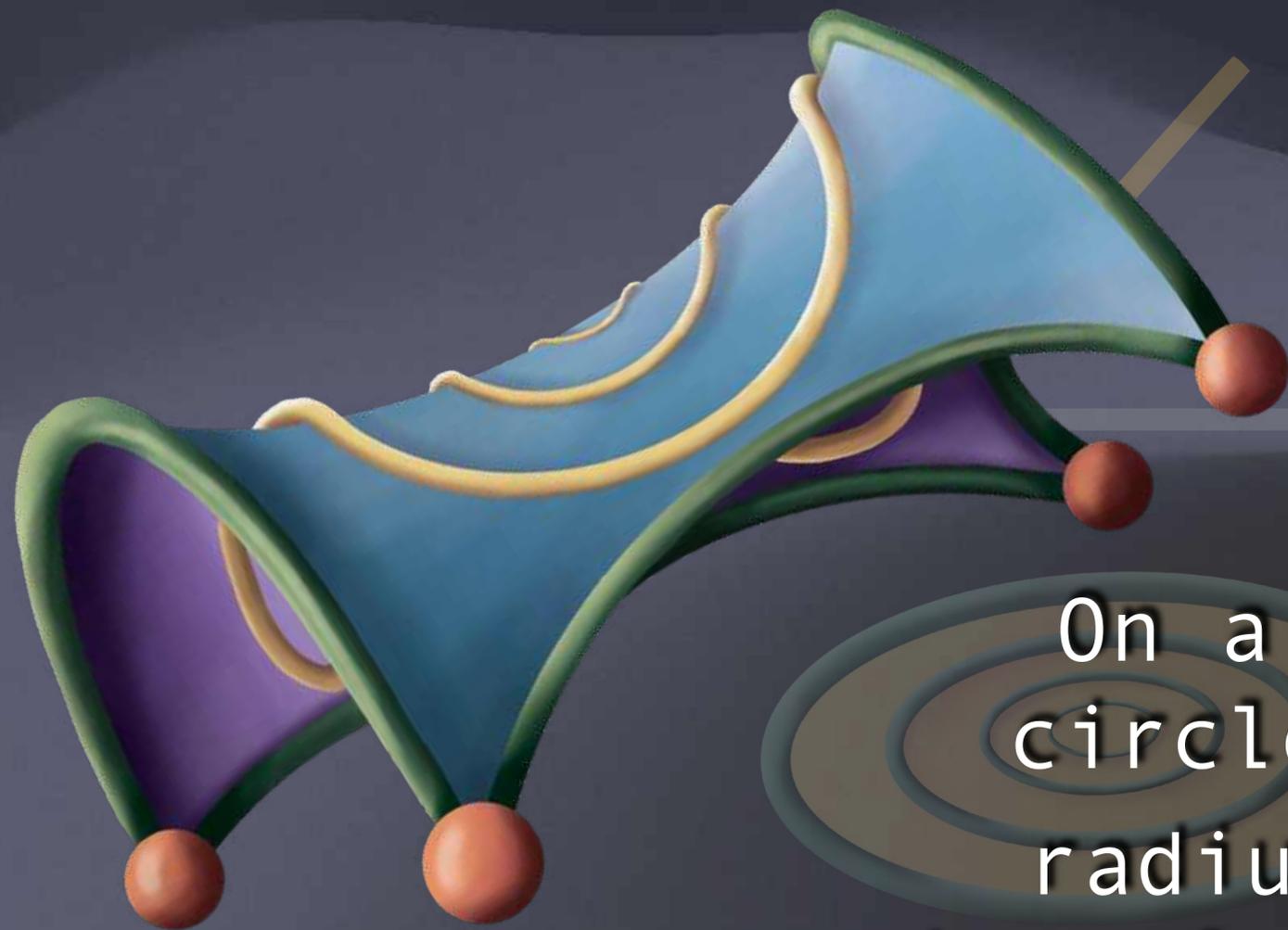
On a **sphere**, circles of a given radius have **smaller** circumference than their Euclidean counterparts.

What is Curvature?

$$\mathcal{K} = \kappa_1 \kappa_2$$



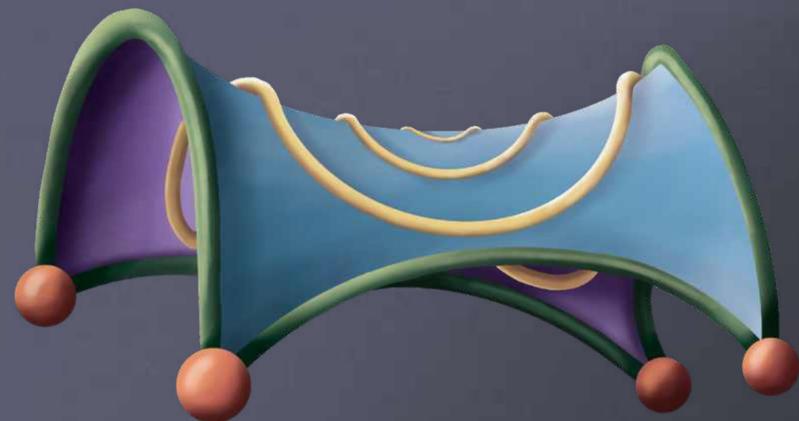
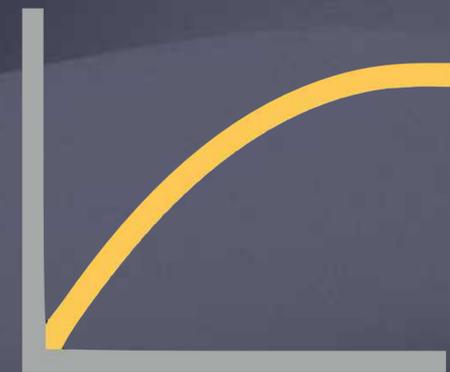
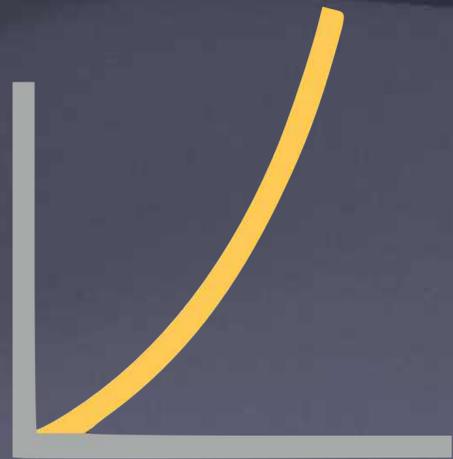
$$C = 2\pi \sinh r$$



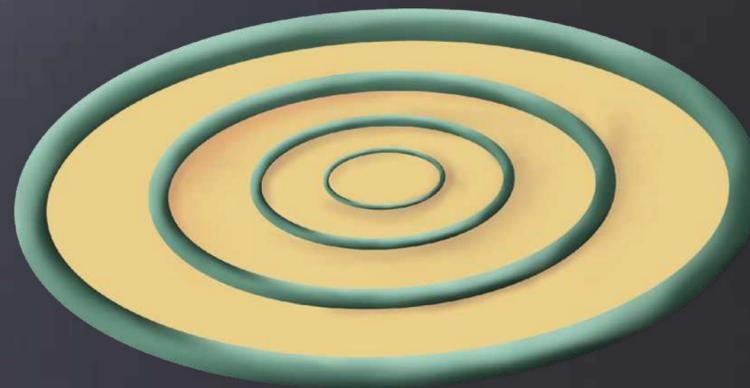
On a hyperboloid, circles of a given radius have **larger** circumference than their Euclidean counterparts.

What is Curvature?

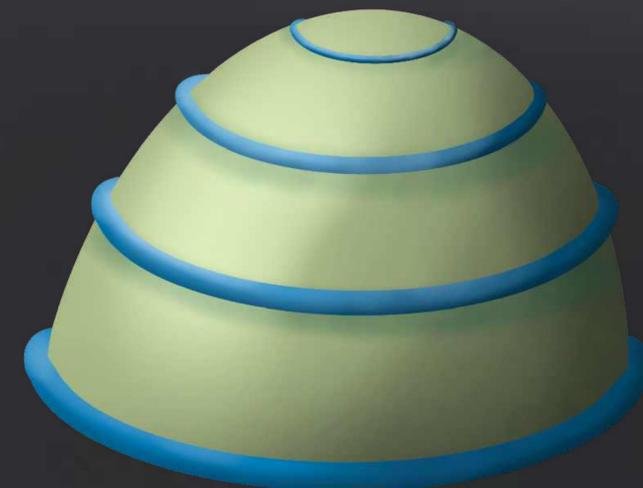
$$\kappa = \kappa_1 \kappa_2$$



$$\kappa < 0$$



$$\kappa = 0$$



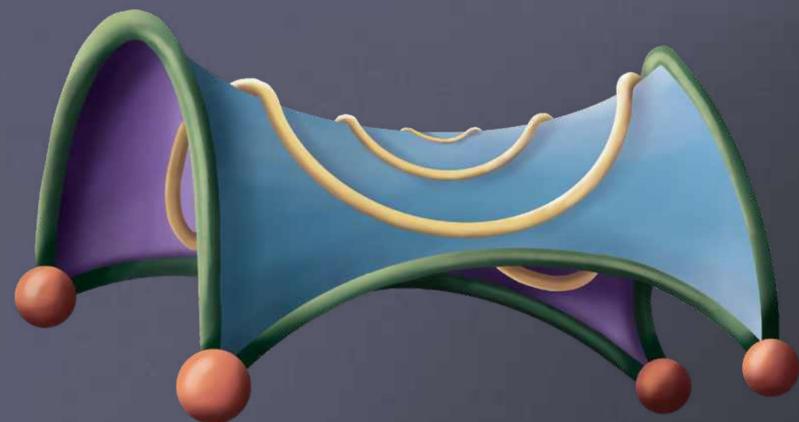
$$\kappa > 0$$

What is Curvature?

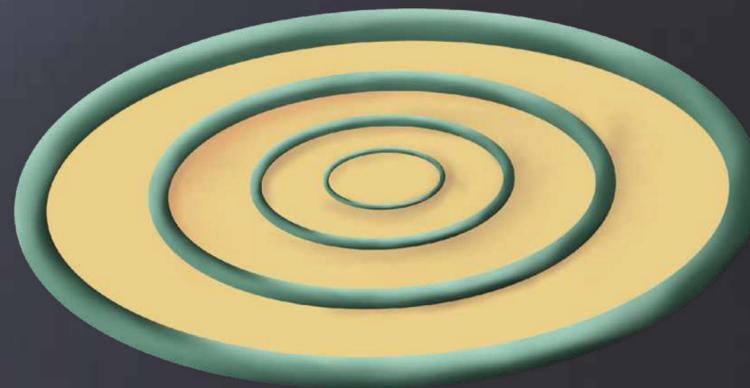
$$\kappa = \kappa_1 \kappa_2$$

Curvature measures this difference.

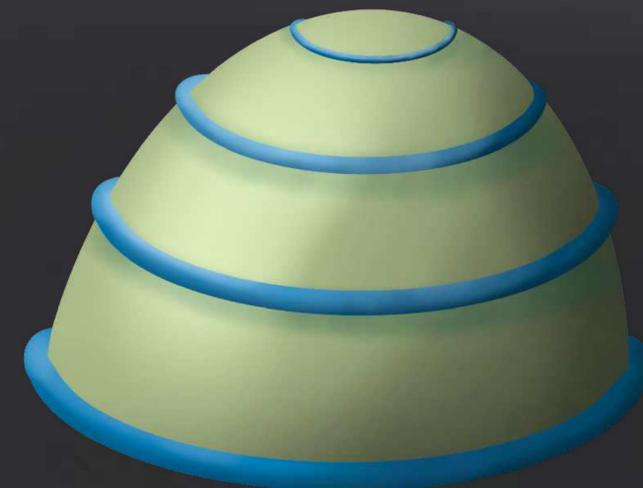
$$\kappa = 3 \lim_{r \rightarrow 0} \frac{2\pi r - C(r)}{\pi r^3}$$



$$\kappa < 0$$

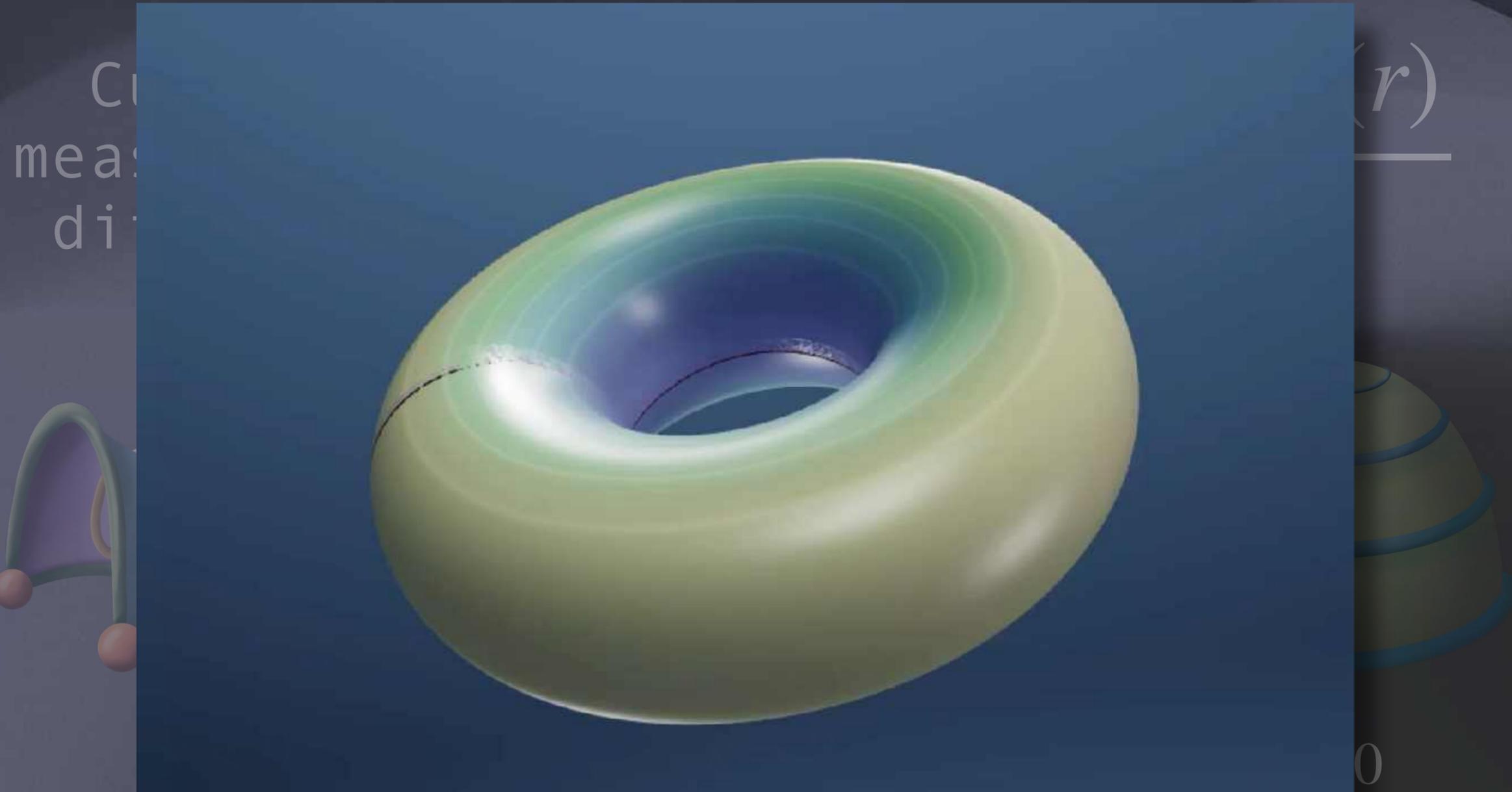


$$\kappa = 0$$



$$\kappa > 0$$

What is $\mathcal{K} = \kappa_1 \kappa_2$ Curvature?



$$S^2$$

$$E^2$$

$$H^2$$



Positive
Curvature

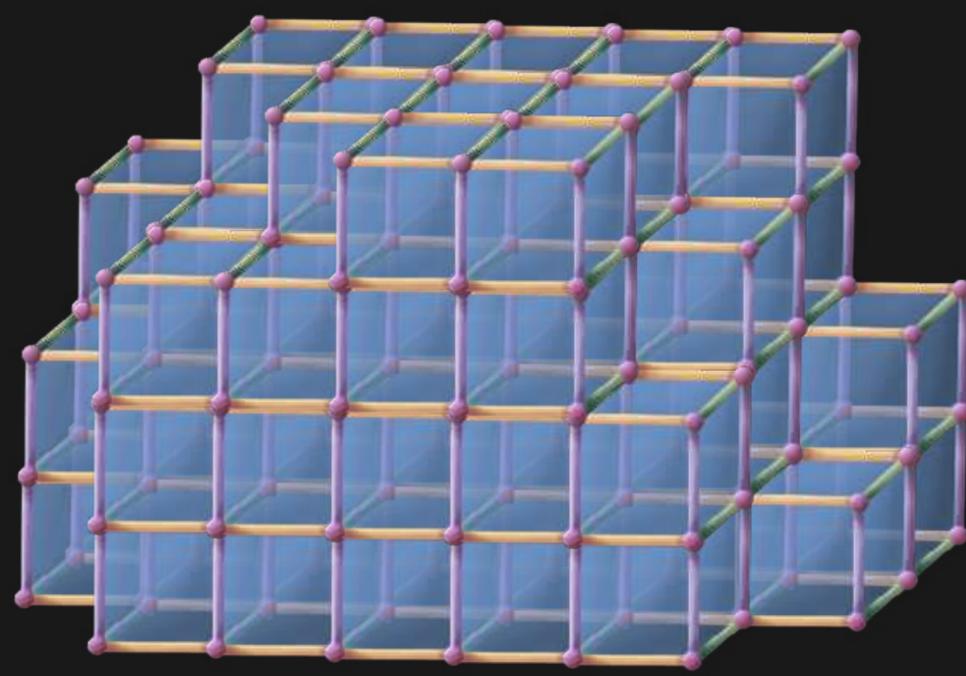


Zero
Curvature



Negative
Curvature

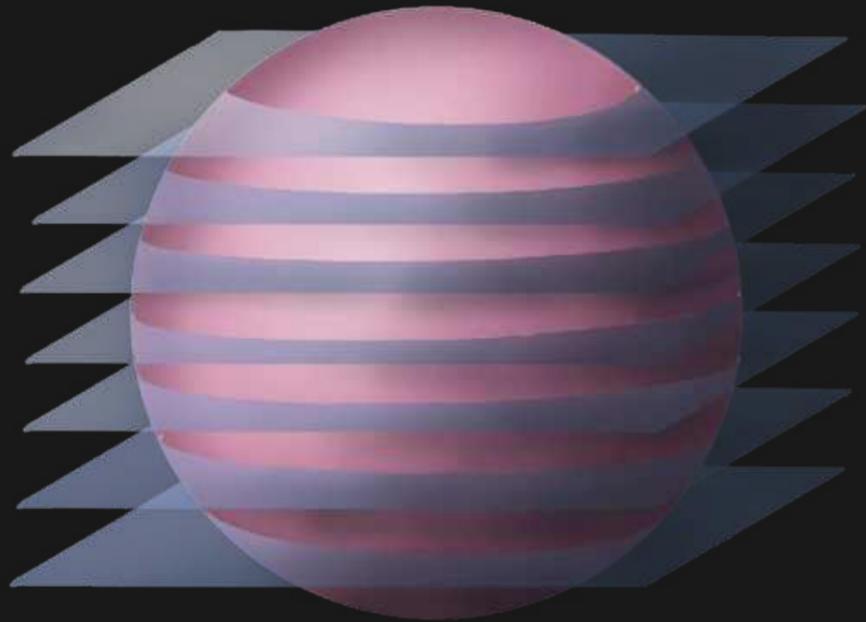
$$\mathbb{E}^3$$



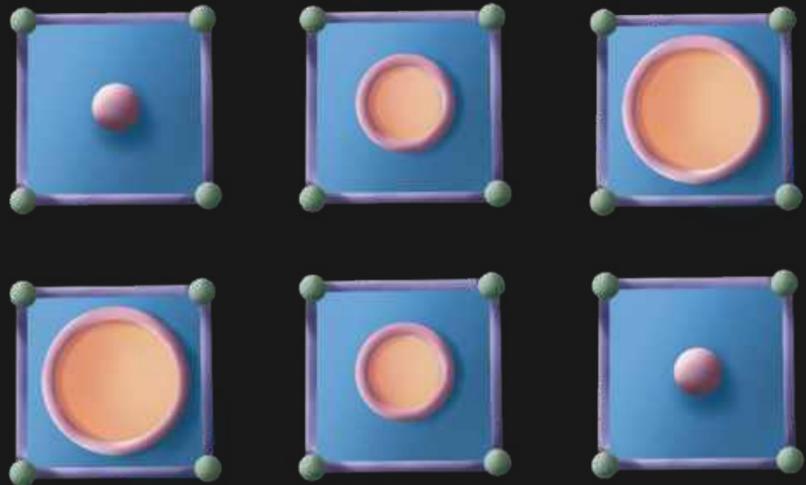
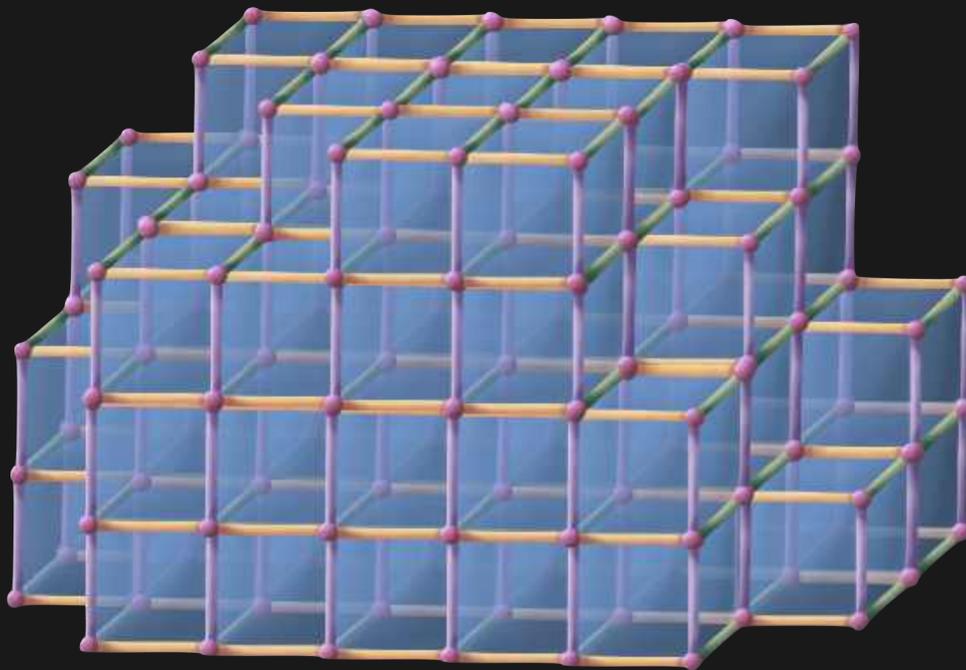
Positive
Curvature

Zero
Curvature

Negative
Curvature



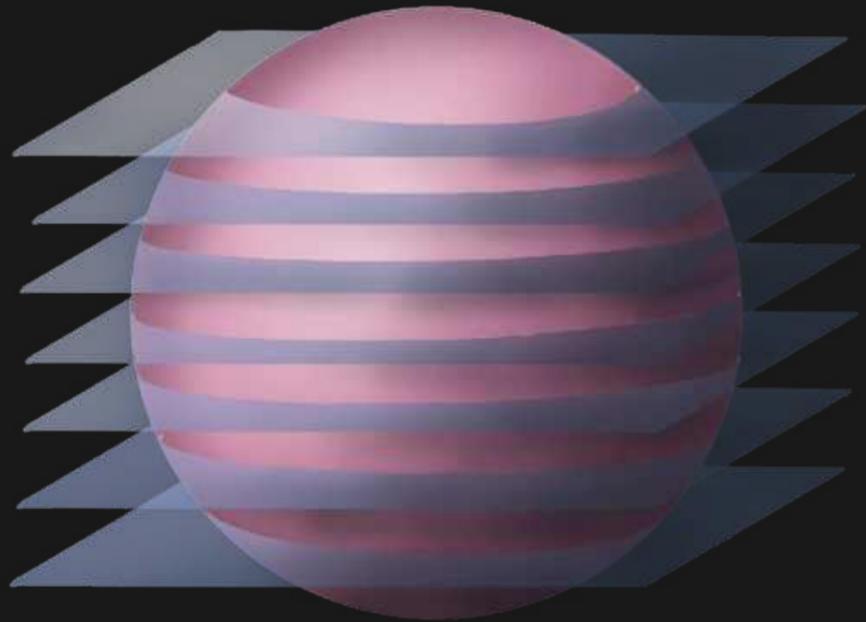
\mathbb{E}^3



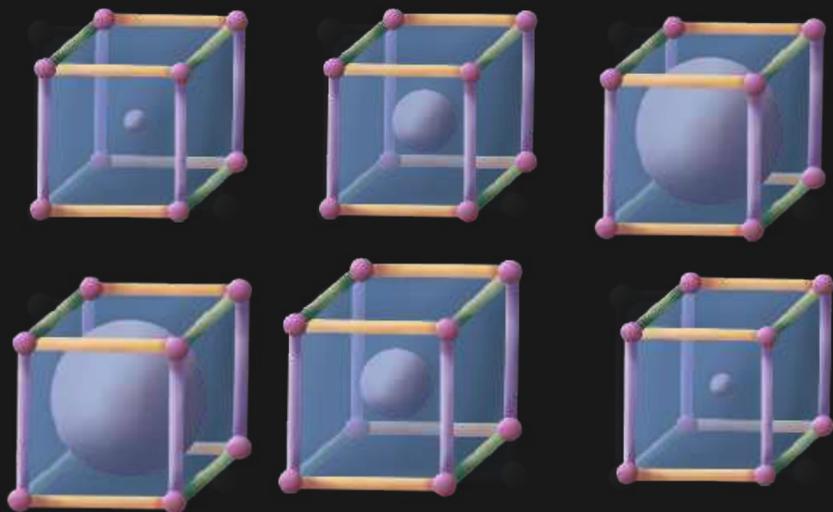
Positive
Curvature

Zero
Curvature

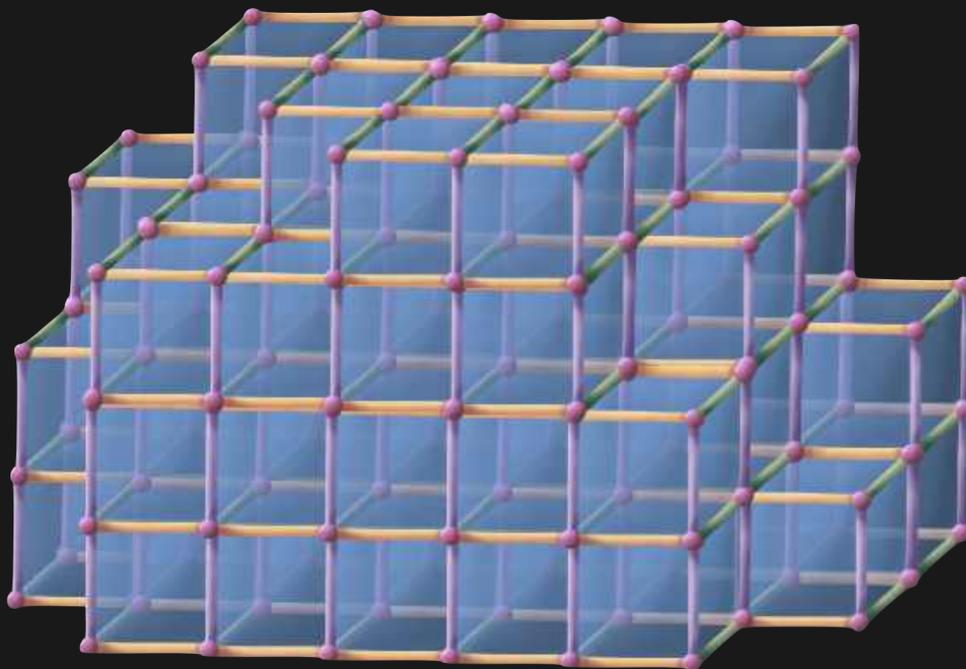
Negative
Curvature



\mathbb{E}^3



Positive
Curvature

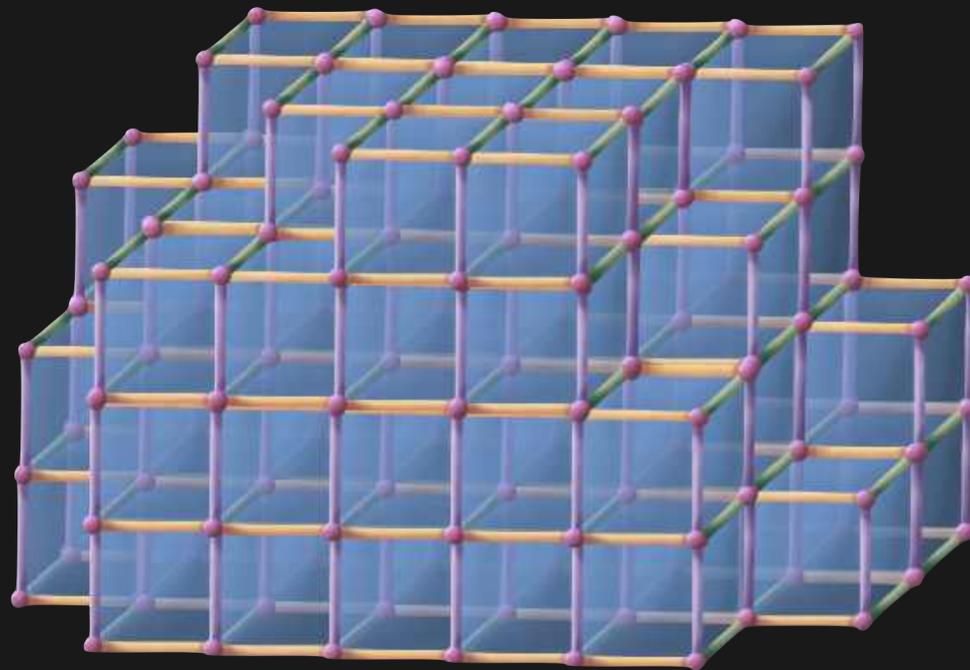
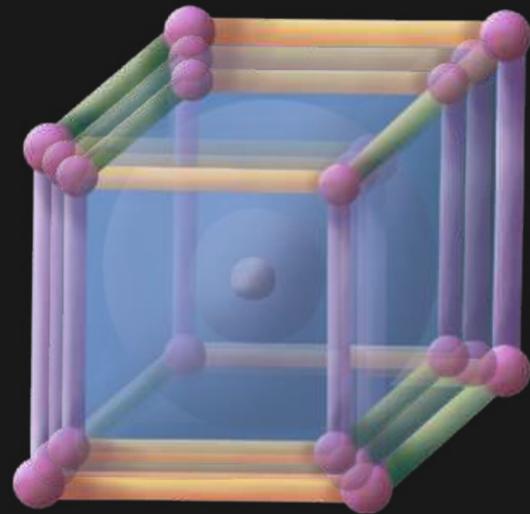


Zero
Curvature

Negative
Curvature

S^3

E^3

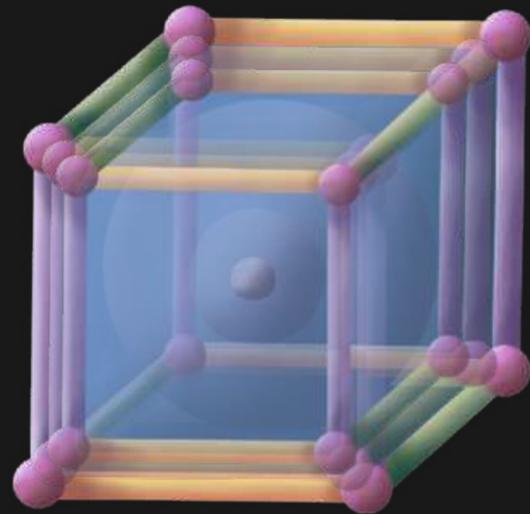


Positive
Curvature

Zero
Curvature

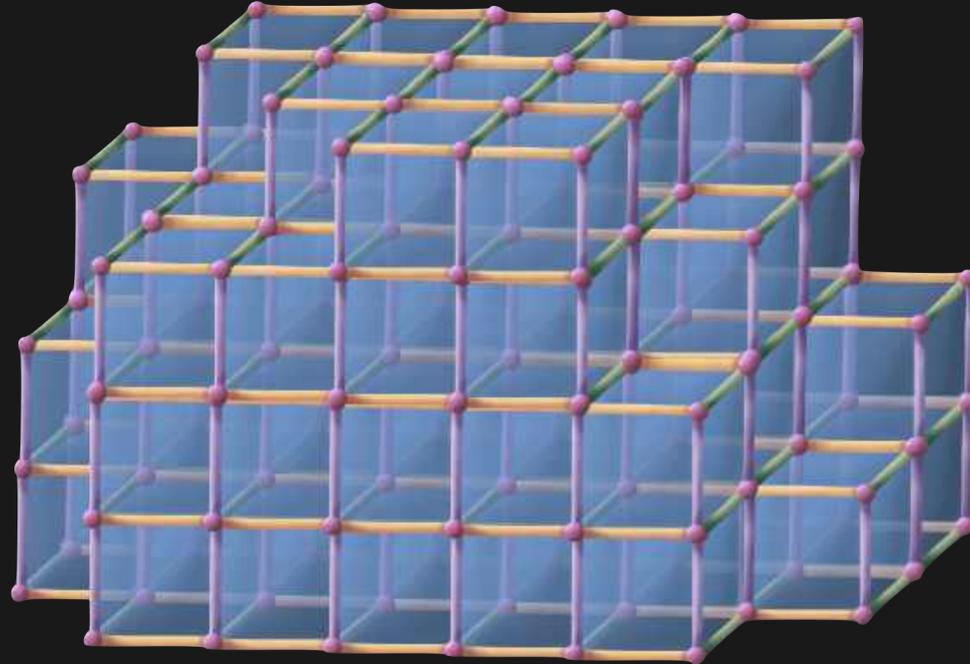
Negative
Curvature

S^3



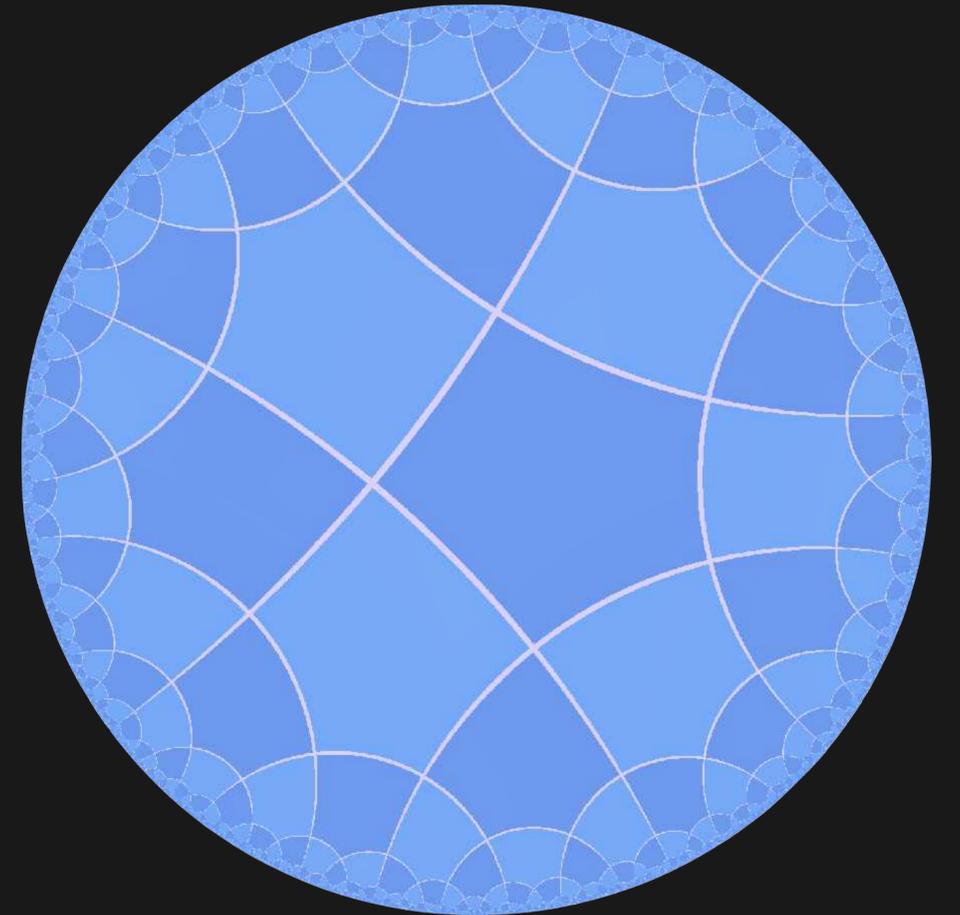
Positive
Curvature

E^3



Zero
Curvature

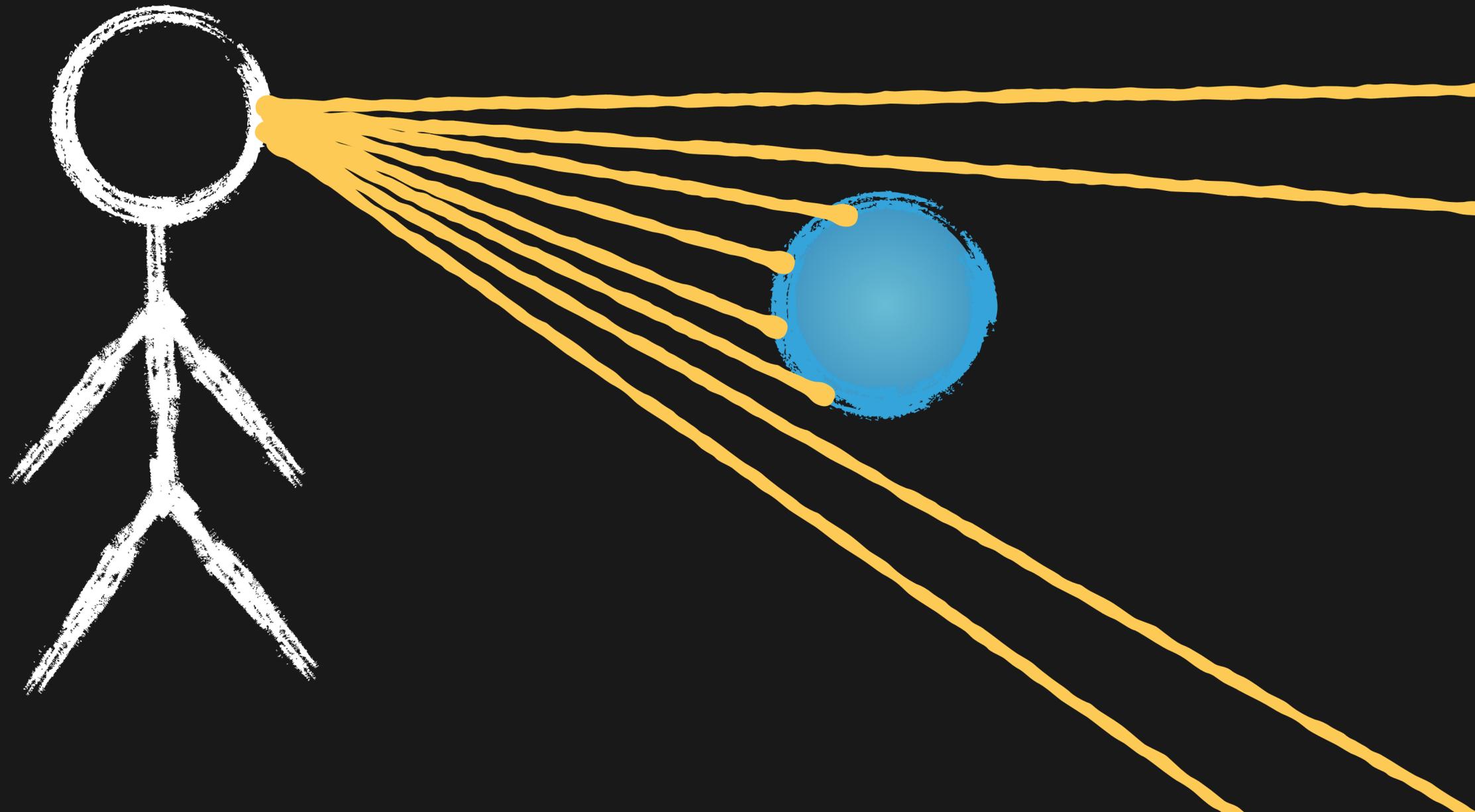
H^3



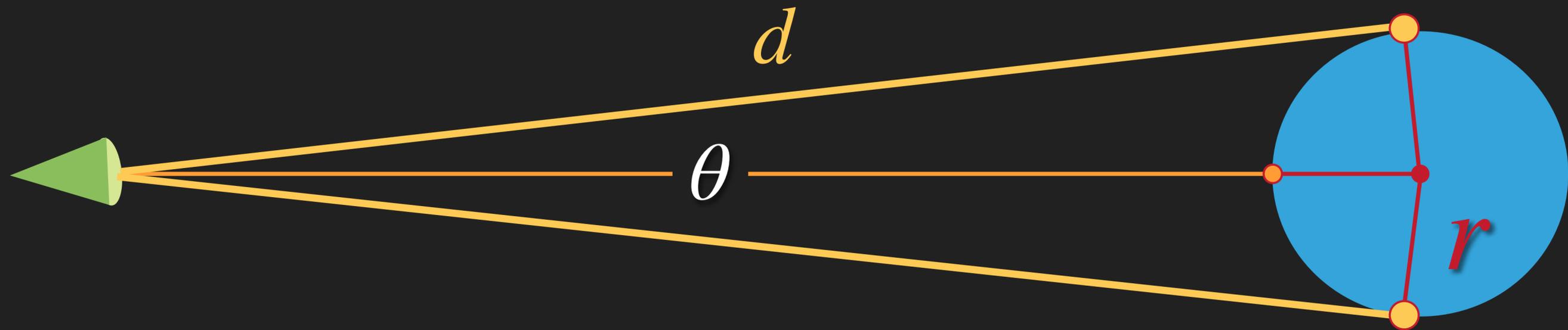
Negative
Curvature

The Geometry of Vision

Light travels on
straight lines or
geodesics.

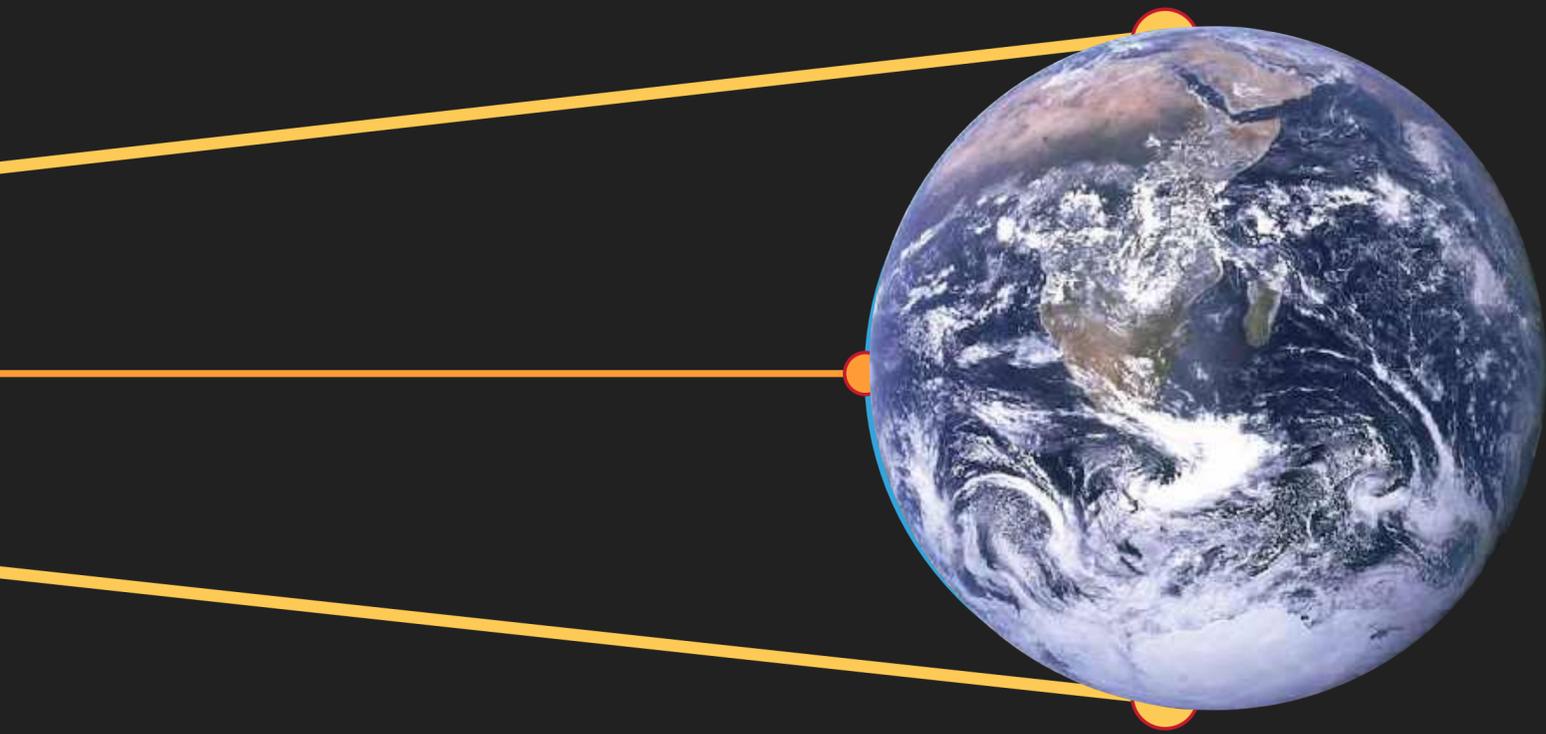


The Geometry of Vision



$$\theta = 2 \tan^{-1} \left(\frac{r}{d} \right) \approx \frac{2r}{d}$$

The Geometry of Vision



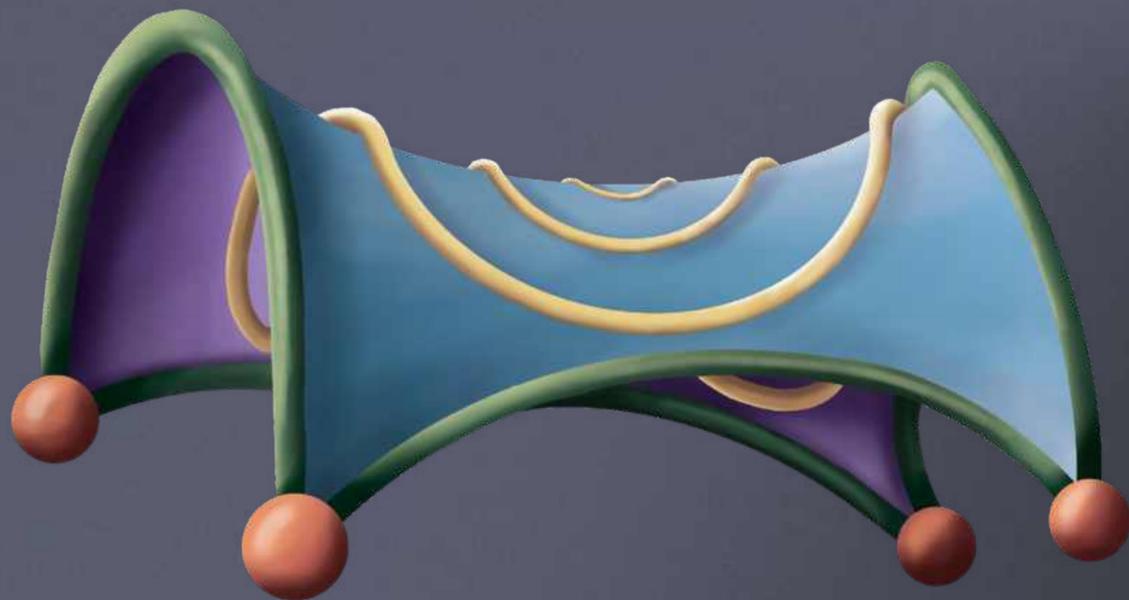
You can see
less than half
the sphere.

$$\theta \approx \frac{2r}{d}$$

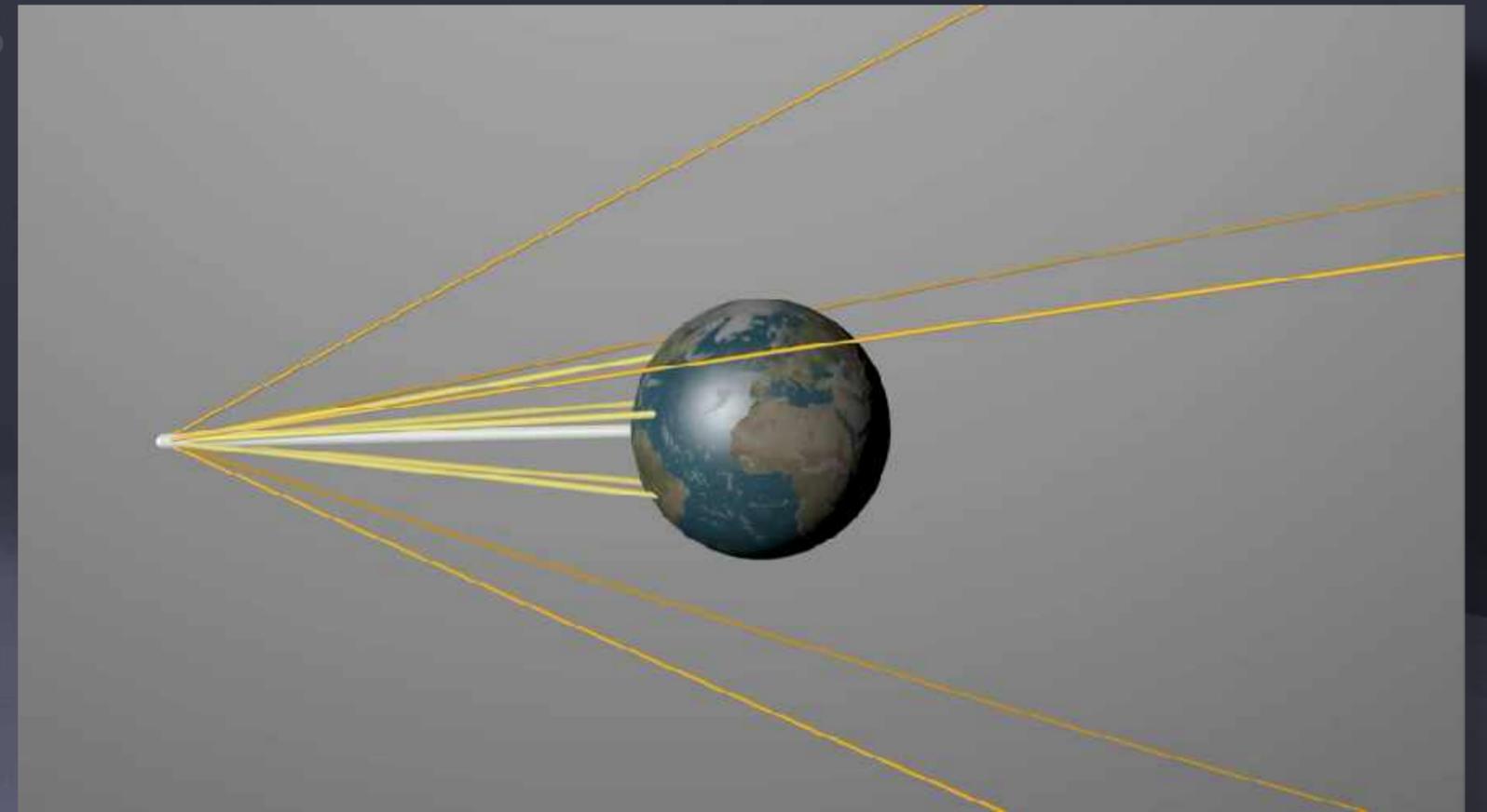
Doubling
distance
halves angular
size.

What about Curvature?

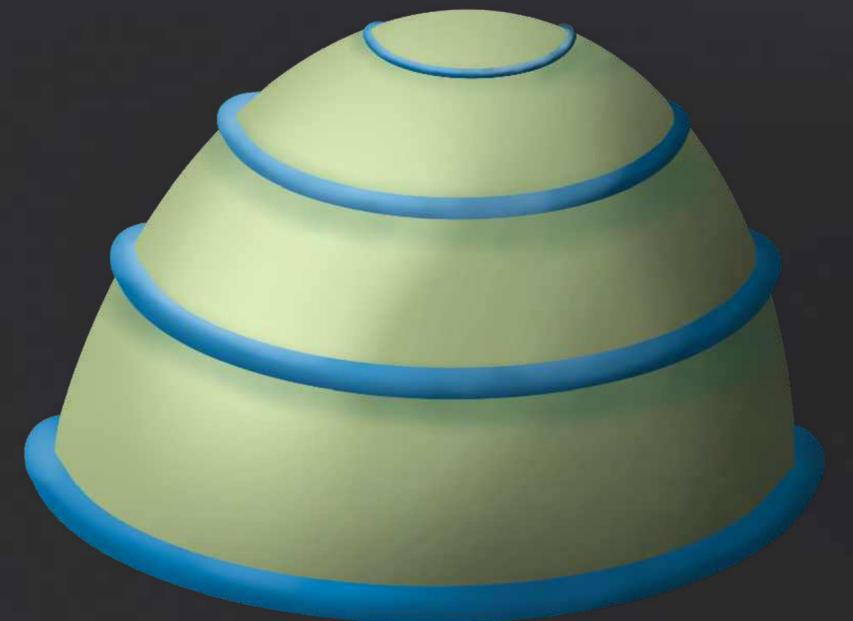
**Negative
Curvature** makes
geodesics spread out.



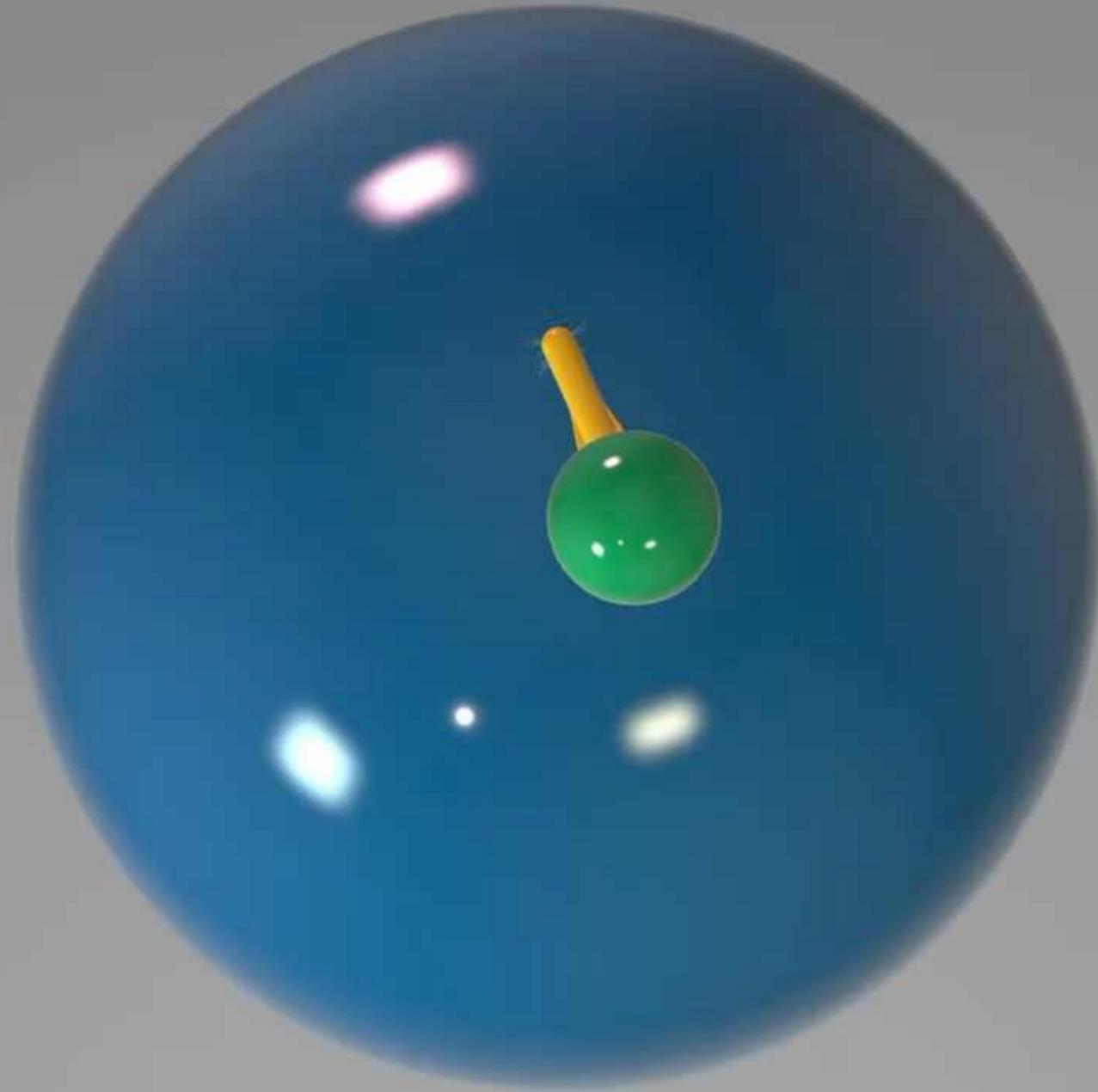
K



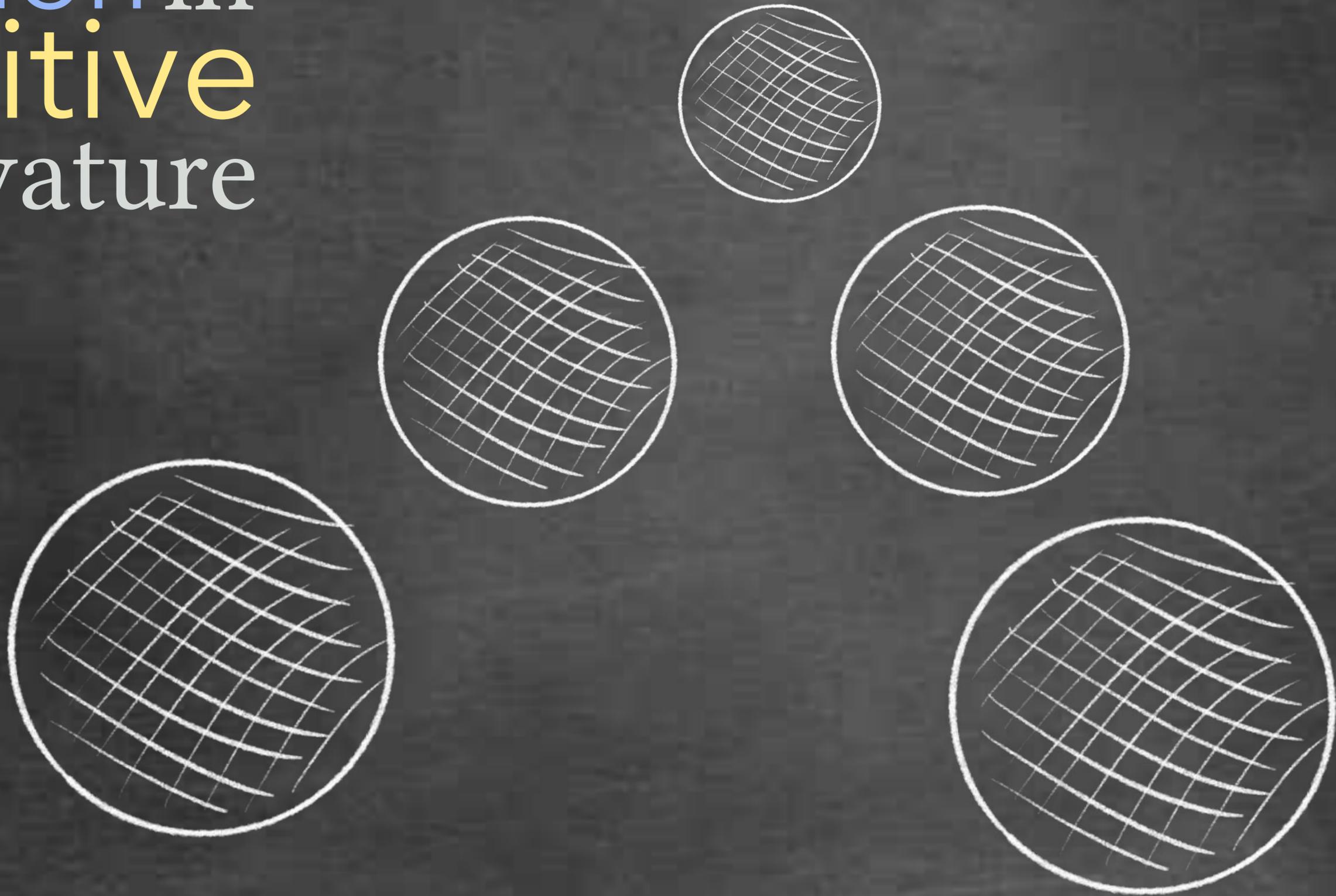
**Positive
curvature**
draws geodesics
together.



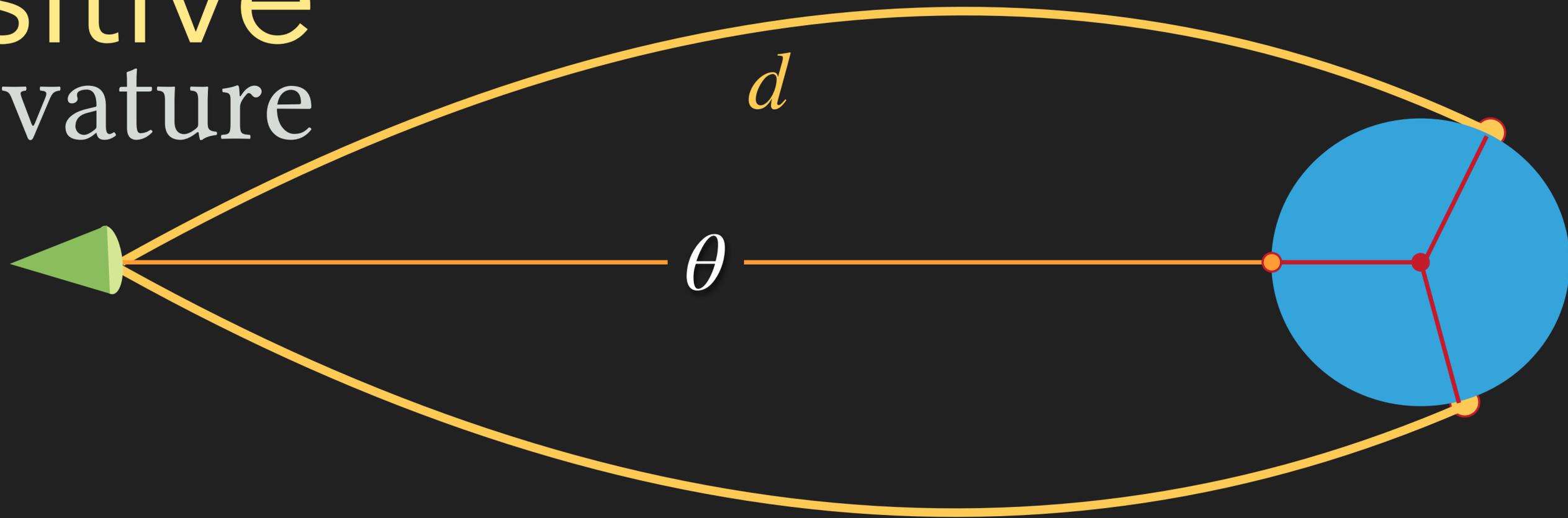
Vision in Positive curvature



Vision in Positive curvature



Vision in Positive curvature



$$\theta \approx 2 \tan^{-1} \left(\frac{\tan r}{\sin d} \right) \approx \frac{2r}{\sin(d)}$$

Vision in
Positive
curvature



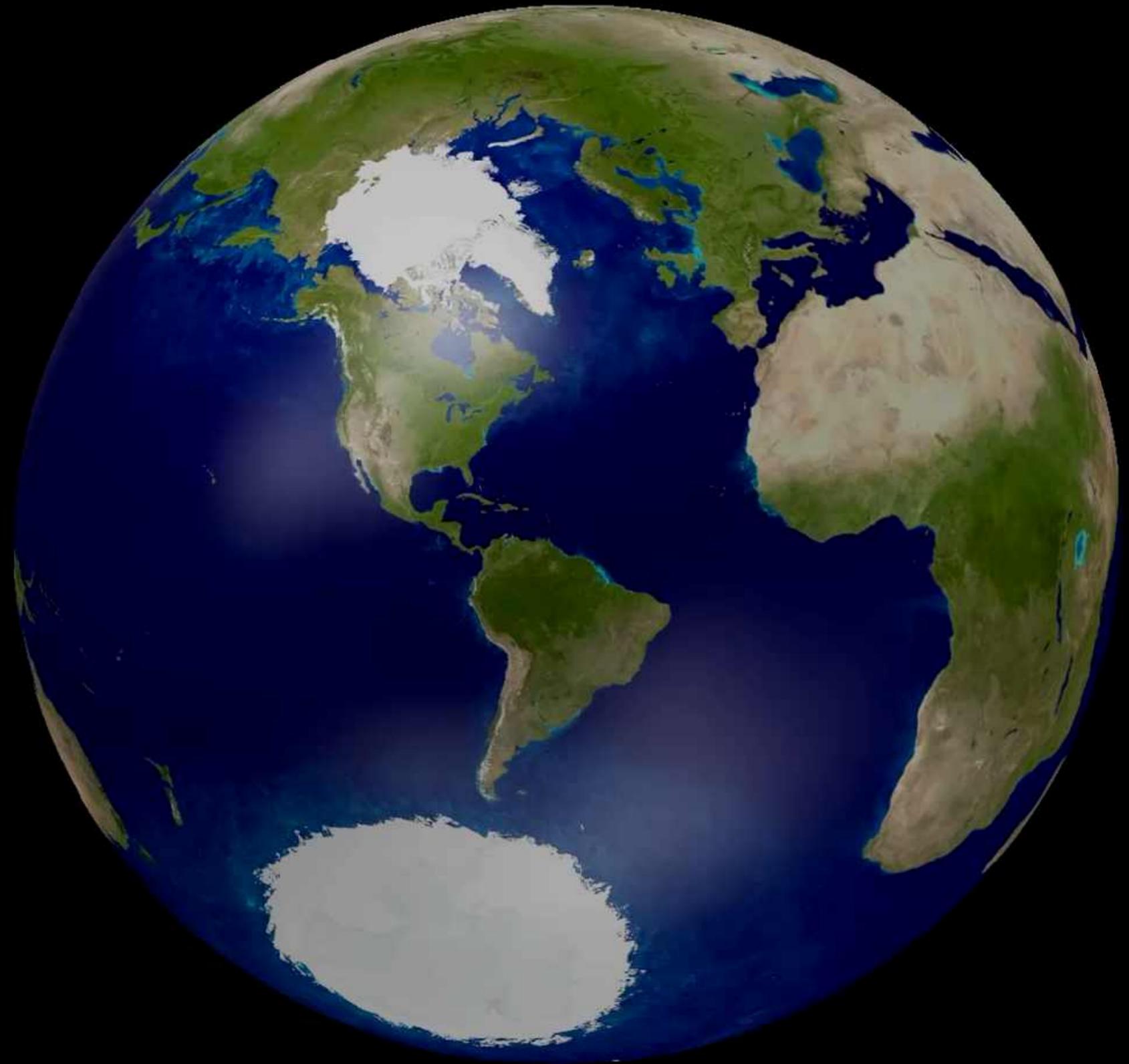
Vision in
Positive
curvature



Vision in
Positive
curvature



Vision in
Positive
curvature



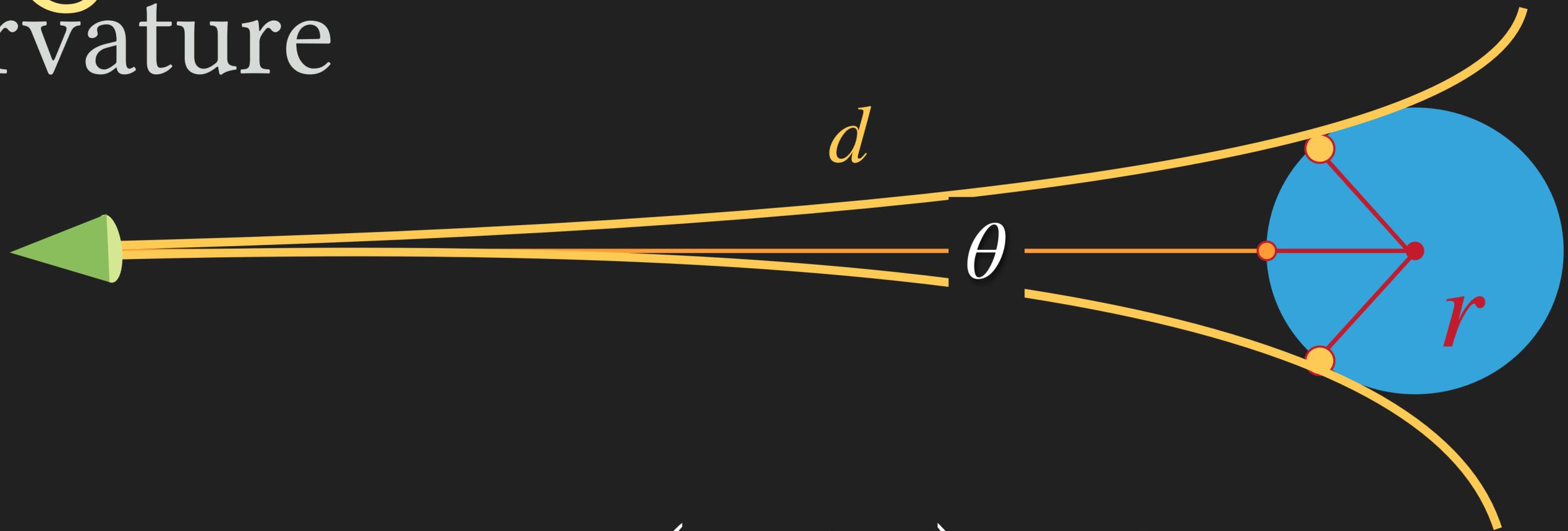


Vision in
Positive
curvature

The image features a dark, textured background densely populated with numerous spheres of various sizes and colors, including red, yellow, cyan, and white. The spheres are scattered across the entire frame, creating a complex, multi-layered visual effect. In the top-left corner, there is a semi-transparent dark grey rectangular box containing the text "Vision in Negative curvature".

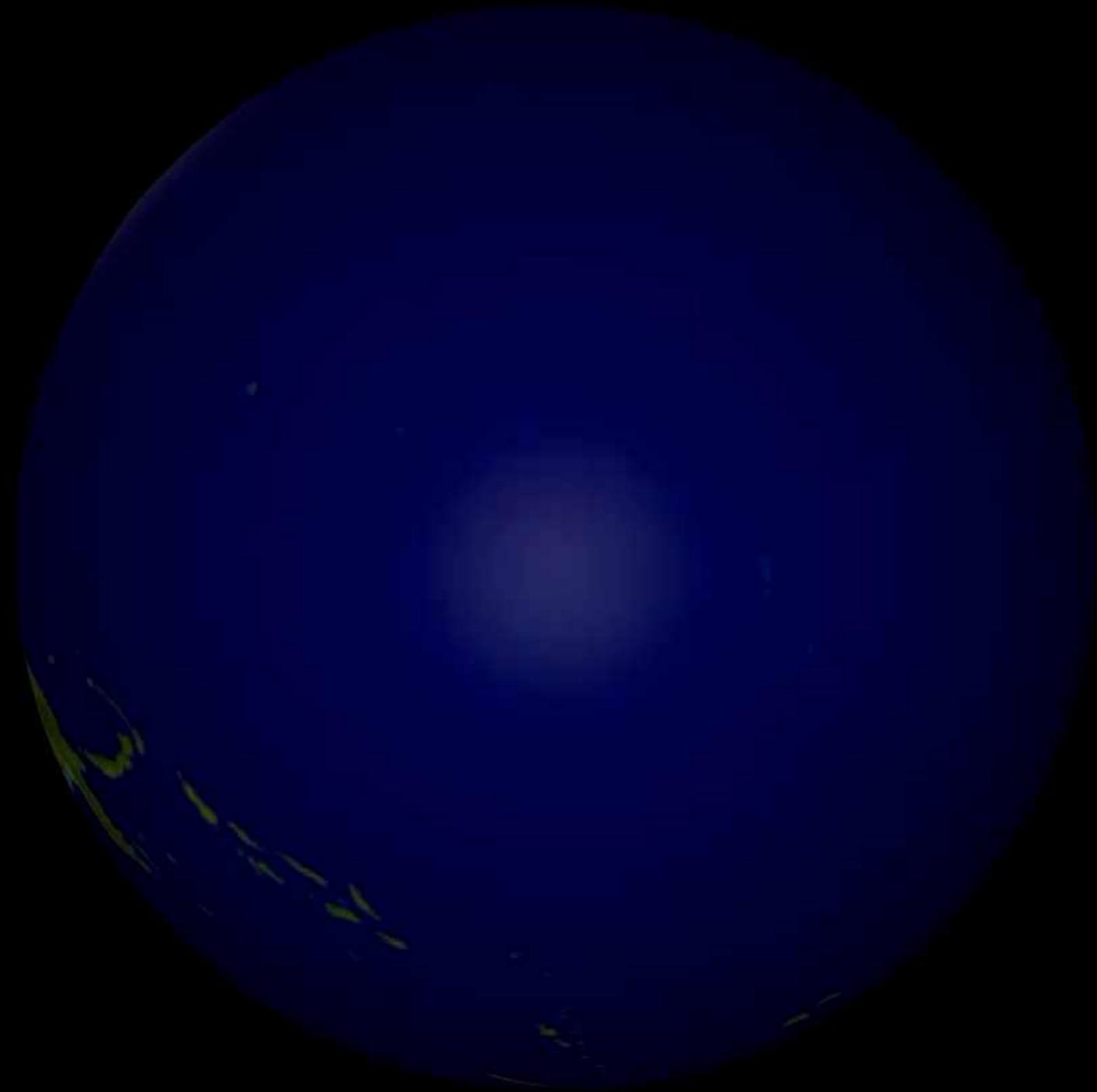
Vision in
Negative
curvature

Vision in Negative curvature



$$\theta \approx 2 \tan^{-1} \left(\frac{\tanh r}{\sinh d} \right) \approx \frac{2r}{e^d}$$

Vision in
Negative
curvature

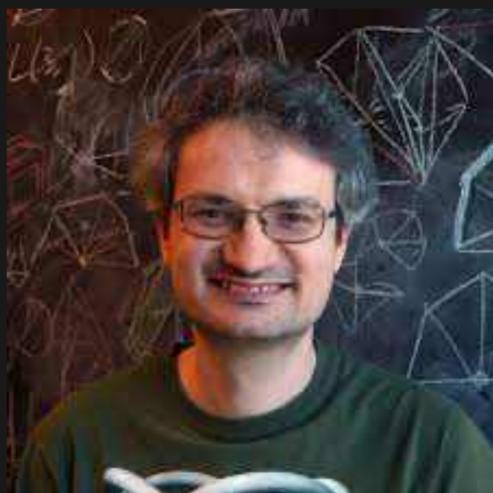




Remi
Coulon



Sabetta
Matsumoto



Henry
Segerman



Steve
Trettel

Raymarching Thurston Geometries

RAY-MARCHING THURSTON GEOMETRIES

RÉMI COULON, ELISABETTA A. MATSUMOTO,
HENRY SEGERMAN, AND STEVE J. TRETTEL

ABSTRACT. We describe algorithms that produce accurate real-time interactive in-space views of the eight Thurston geometries using ray-marching. We give a theoretical framework for our algorithms, independent of the geometry involved. In addition to scenes within a geometry X , we also consider scenes within quotient manifolds and orbifolds X/Γ . We adapt the Phong lighting model to non-euclidean geometries. The most difficult part of this is the calculation of light intensity, which relates to the area density of geodesic spheres. We also give extensive practical details for each geometry.

CONTENTS

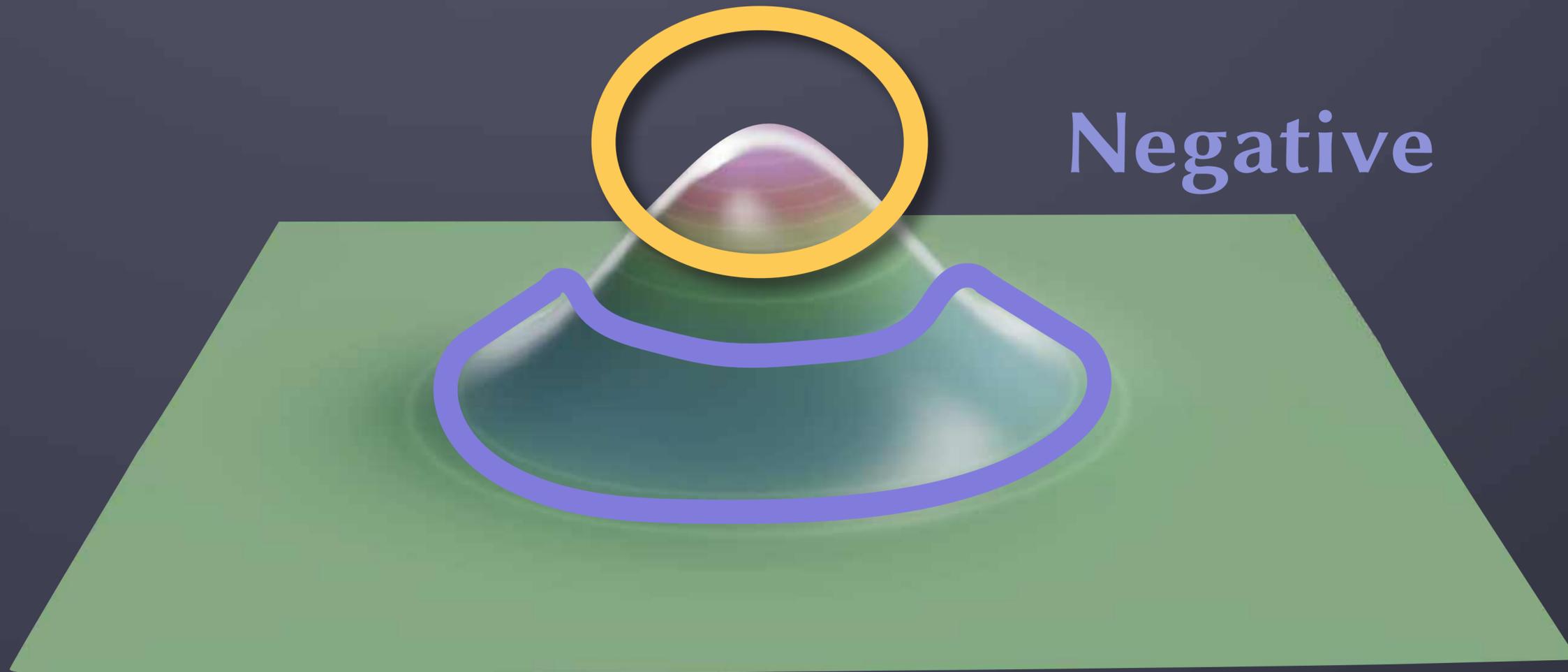
1. Introduction	3
1.1. Thurston's eight geometries	4
1.2. Goals	7
1.3. Related work	8

301v2 [math.GT] 13 Jan 2022

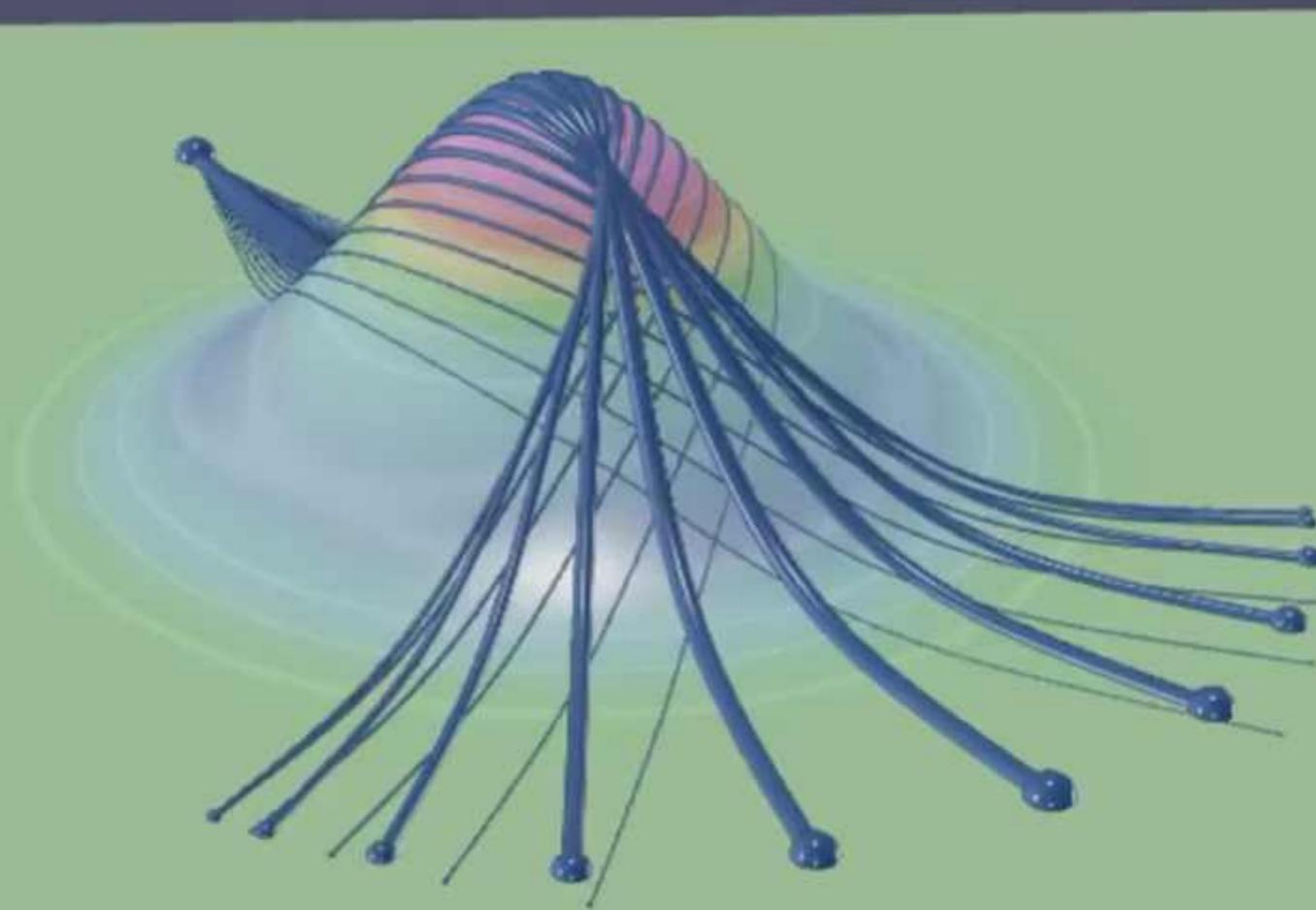
A bump of curvature

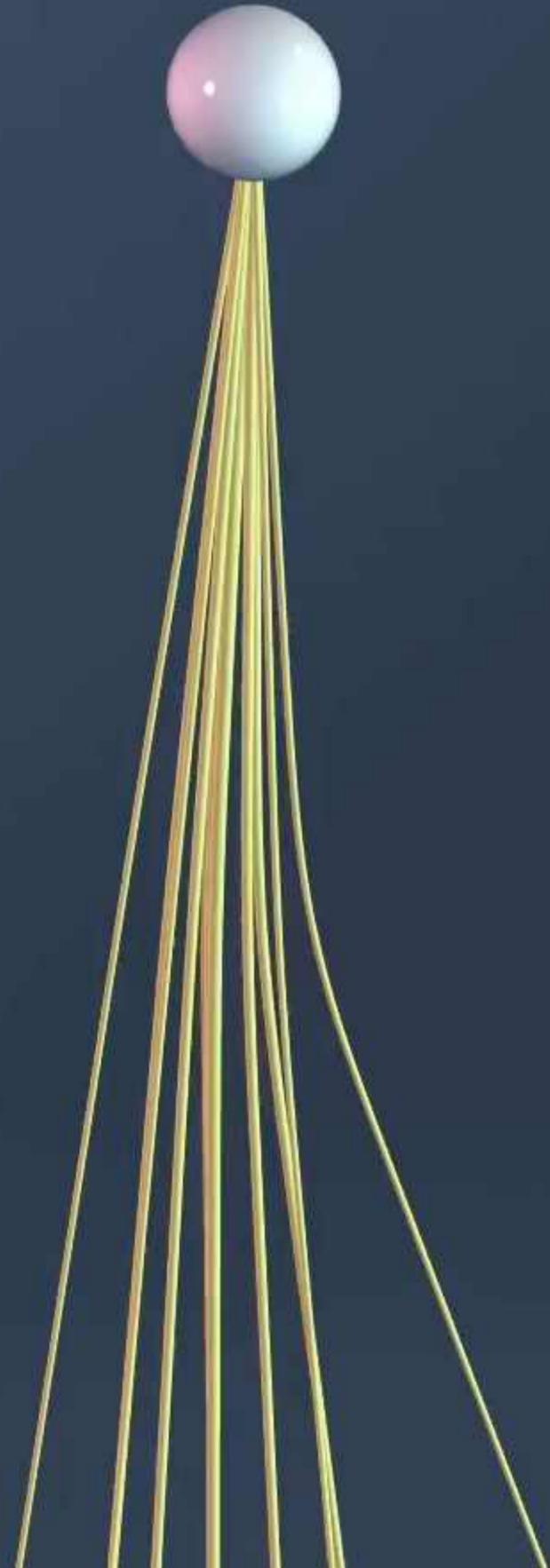
Positive

Negative



A bump of curvature





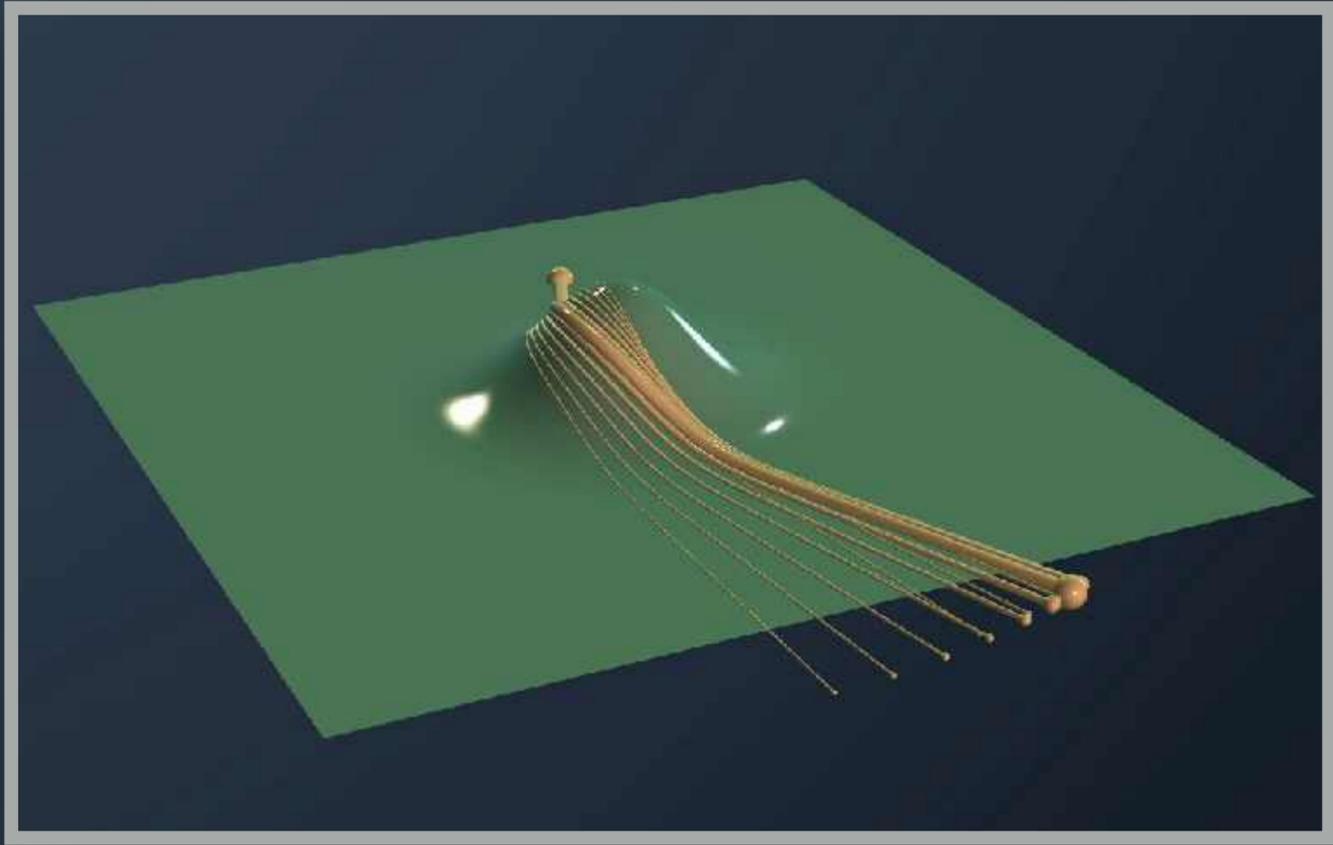
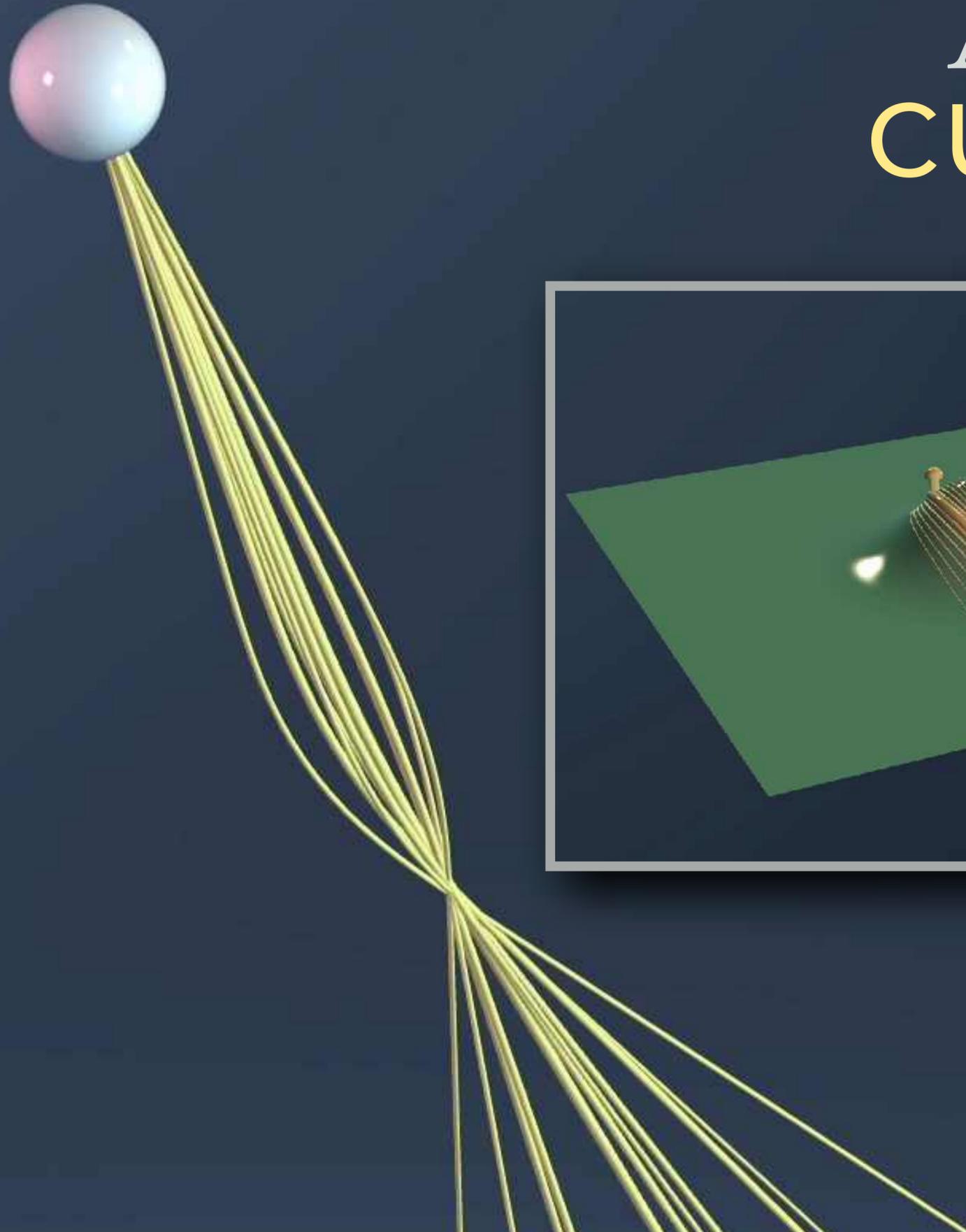
A bump of
curvature



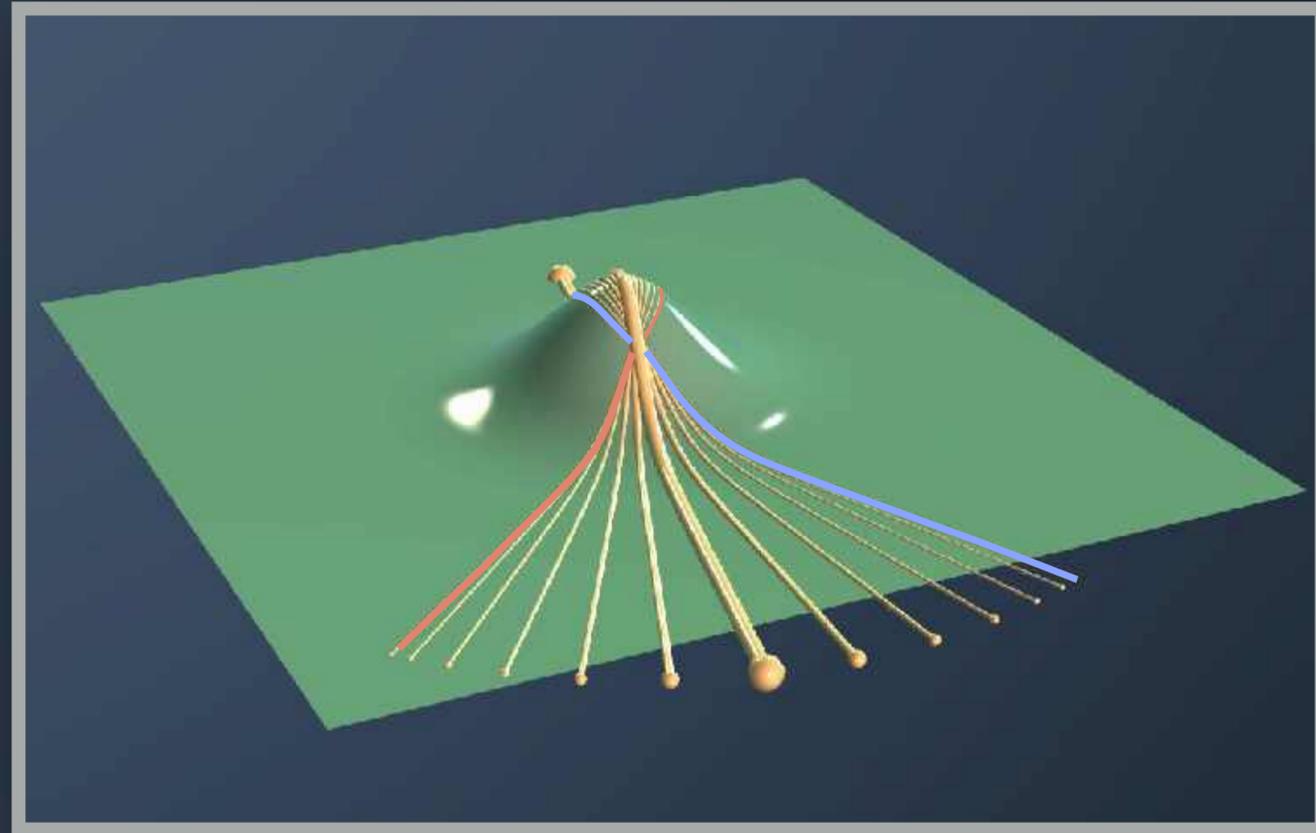
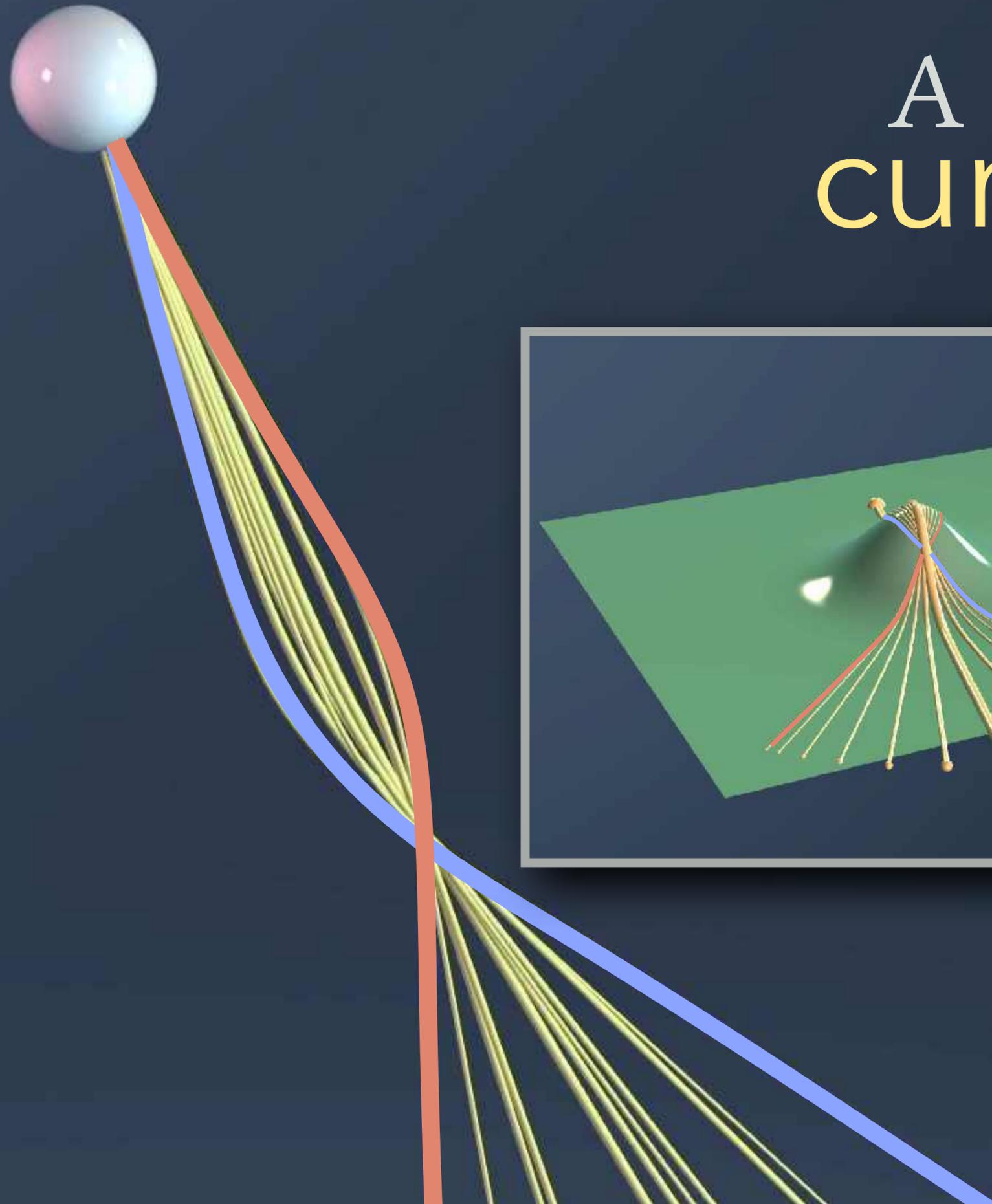


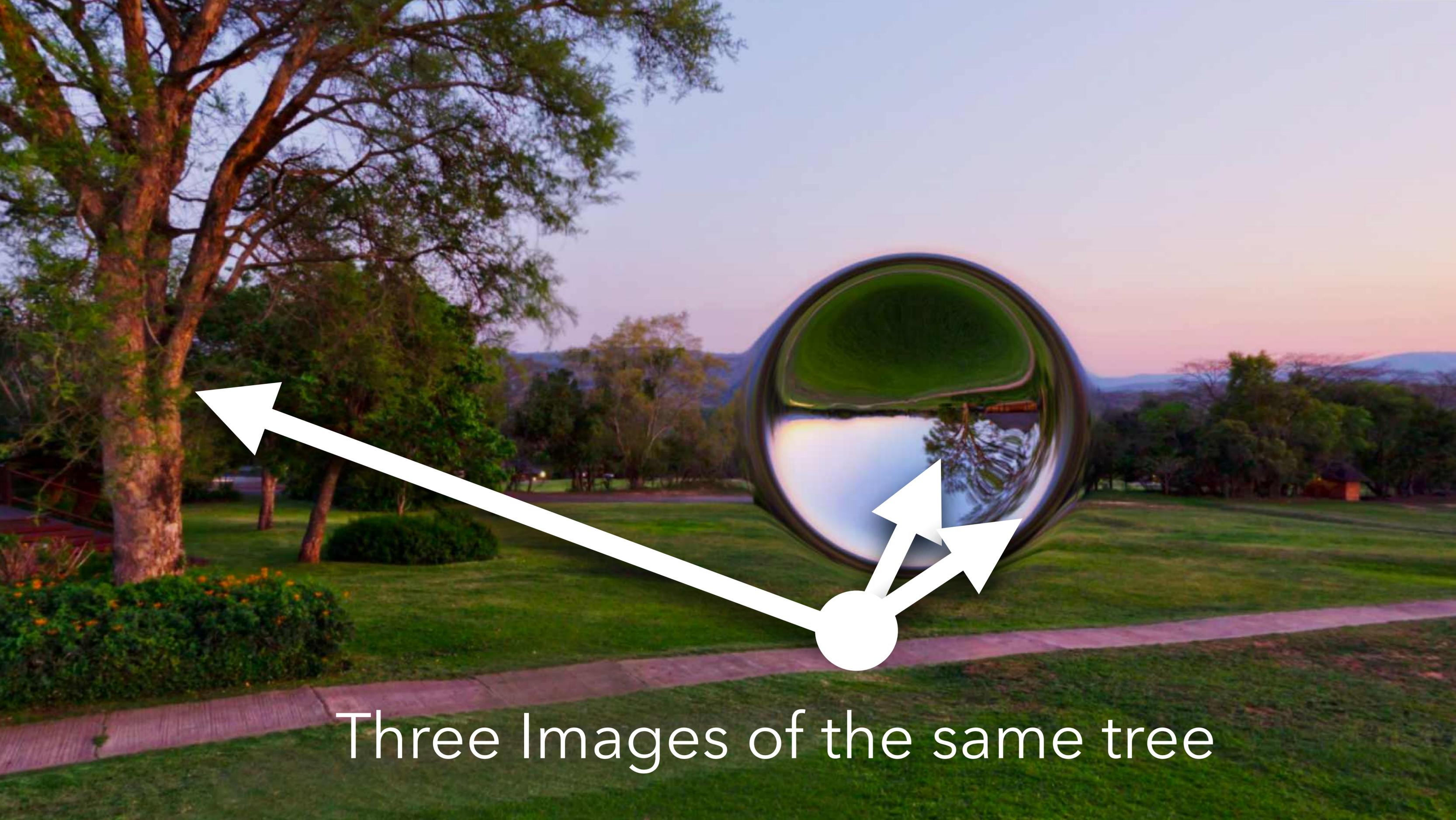
This tree is upside down!

A bump of curvature



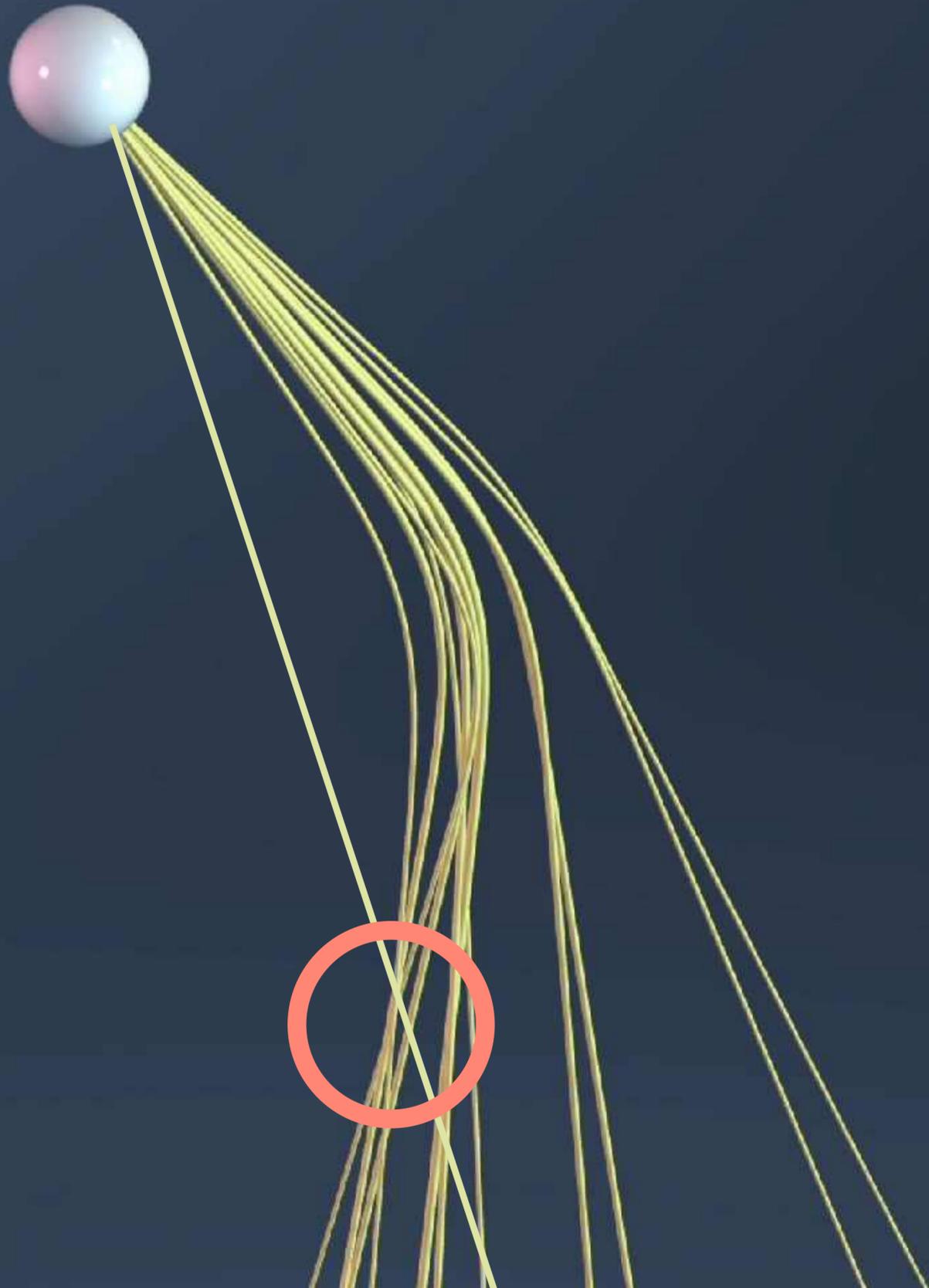
A bump of curvature





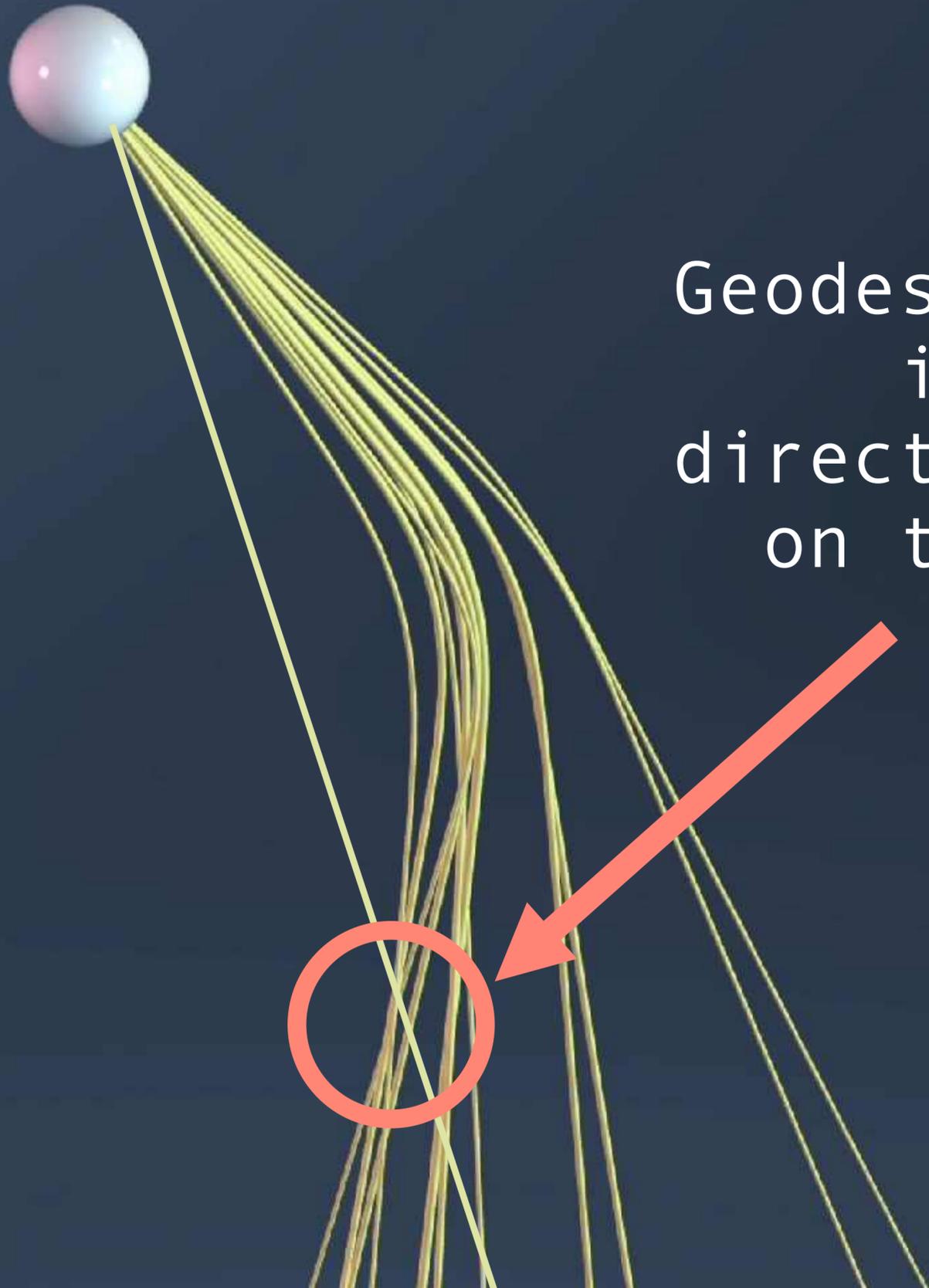
Three Images of the same tree

A bump of
curvature



A bump of curvature

Geodesics starting off
in different
directions, converging
on the same point.



A 3-Dimensional
Geometry called

Nil



What does Earth Look Like in Nil?



What does Earth
Look Like in Nil?



What does Earth Look Like in Nil?

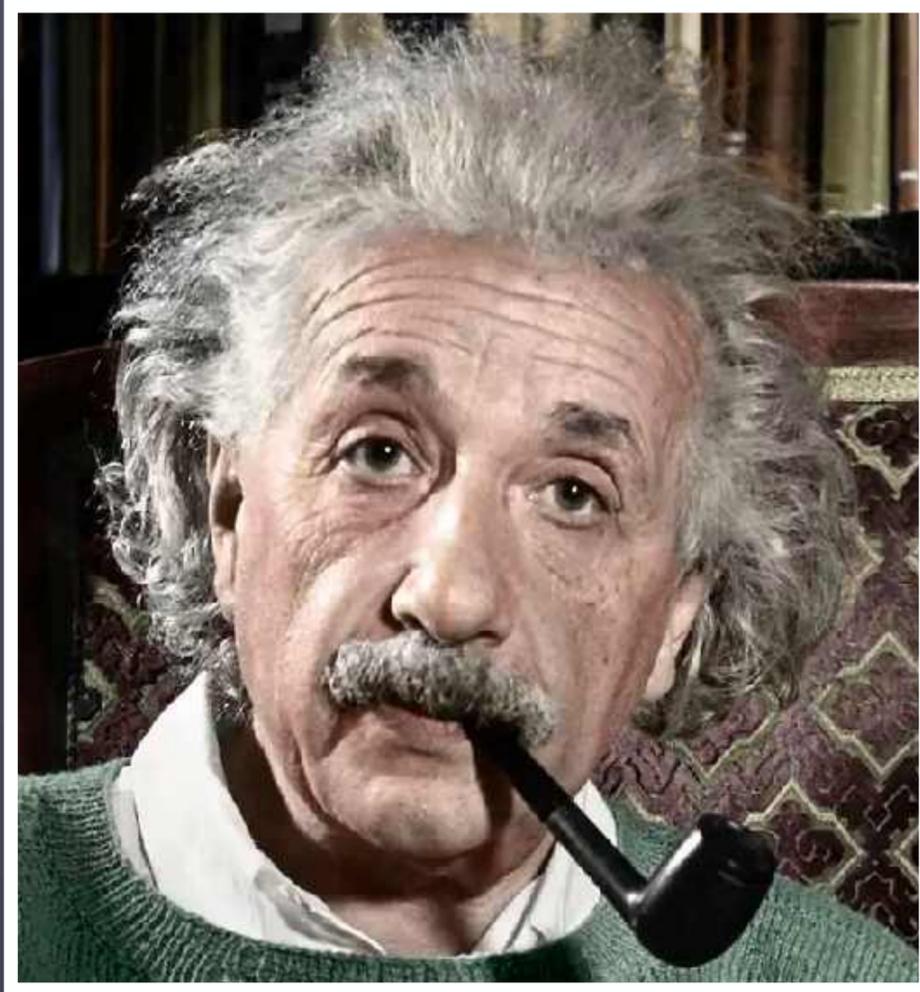


Lessons and Questions

**Learning new math
lets us ask questions
that were previously
inconceivable**

**Is the space around
us curved?**

How could we tell?



Einstein's general relativity

Spacetime is curved!

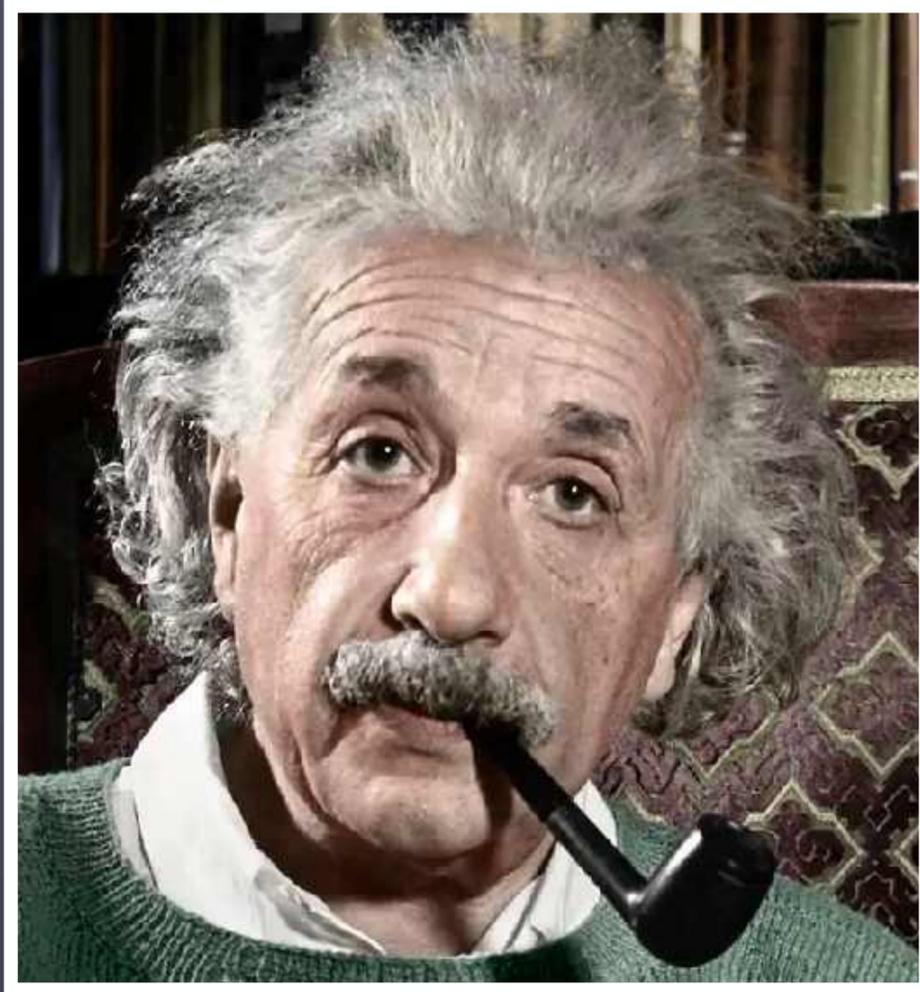
There's a precise
equation to describe
this curvature!

$$\mathbf{Ric} = 8\pi G \left(\mathbf{T} - \frac{1}{2} T \mathbf{g} \right)$$

Space tells matter how
to move. Matter tells
space how to curve.

-John Wheeler

Einstein's general relativity



Einstein's greatest insight:

This curvature is what
we call gravity!

$$\mathbf{Ric} = 8\pi G \left(\mathbf{T} - \frac{1}{2} T \mathbf{g} \right)$$

Space tells matter how
to move. Matter tells
space how to curve.

-John Wheeler

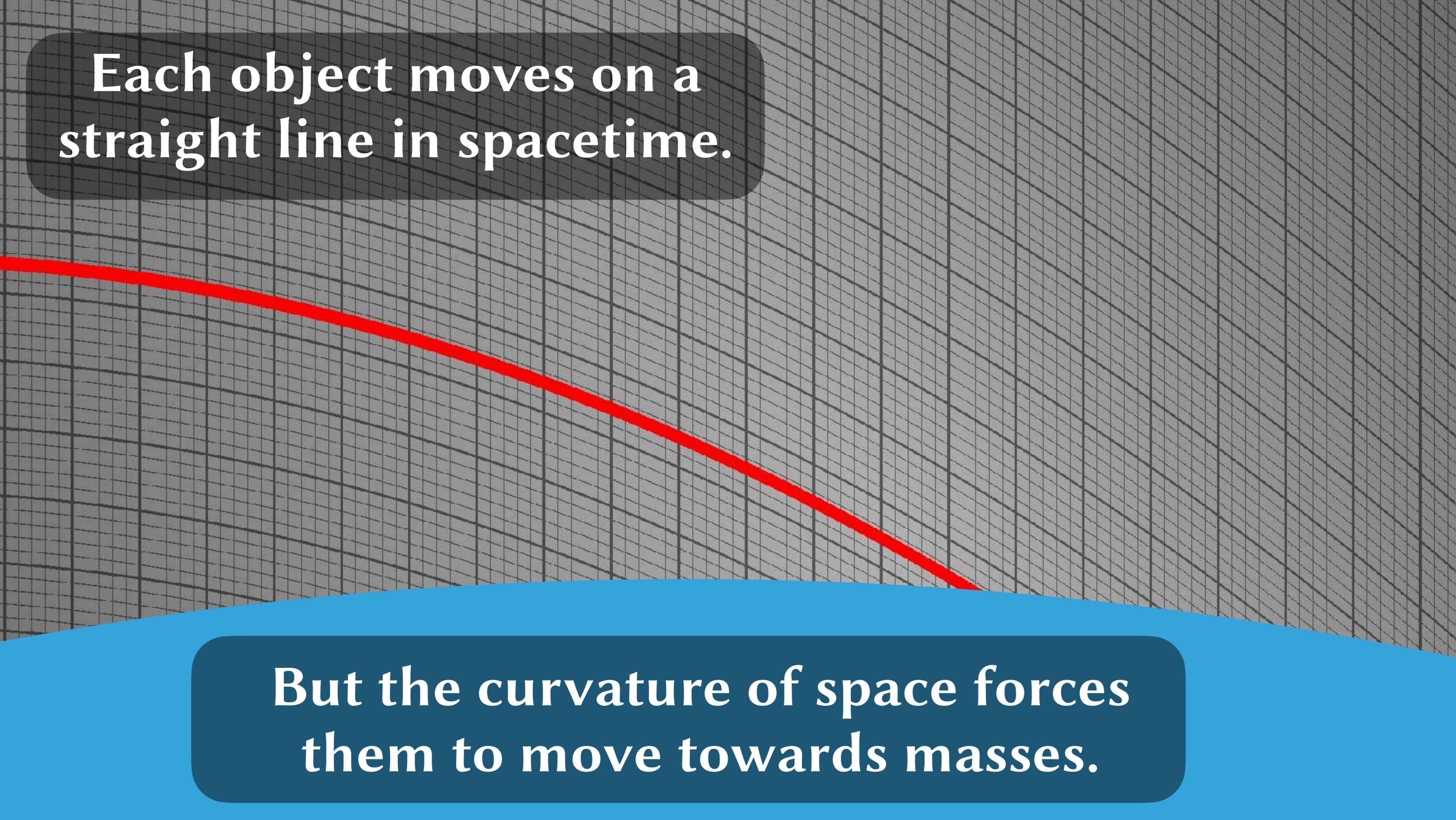
**Each ball moves on a
straight line on the sphere.**



**But the curvature of space forces them to
come together.**

Each object moves on a straight line in spacetime.

But the curvature of space forces them to move towards masses.

A spacetime diagram with a grid of lines. The grid is distorted, with vertical lines curving inward towards the center, representing the curvature of space. A thick red line starts on the left and curves downwards and to the right, following the curvature of the grid. In the top-left corner, there is a dark grey rounded rectangle containing white text. In the bottom-right corner, there is a blue rounded rectangle containing white text.

**Each object moves on a
straight line in spacetime.**

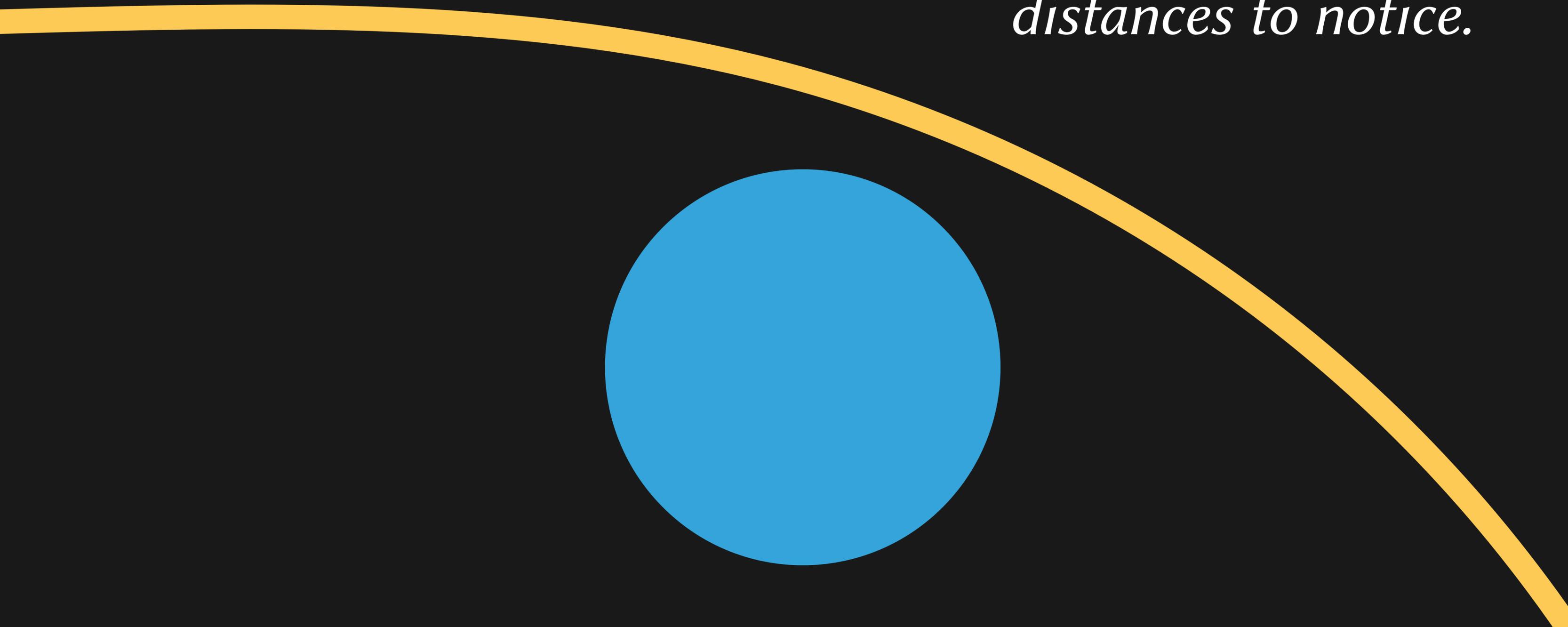
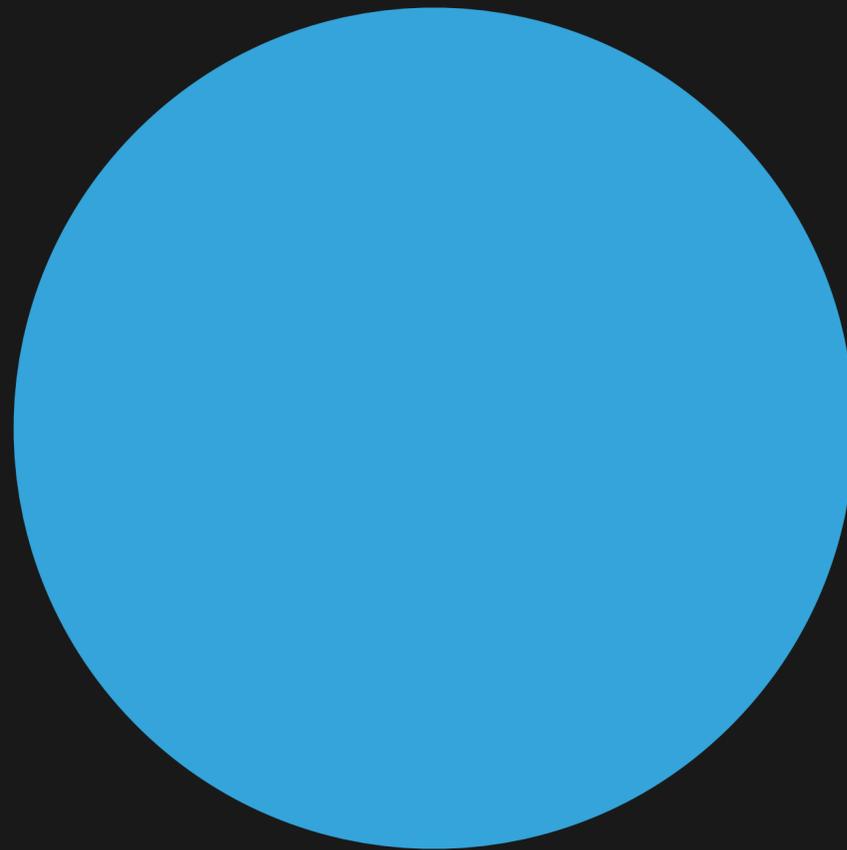
**But the curvature of space forces
them to move towards masses.**

**Even light follows the
curves of spacetime.**



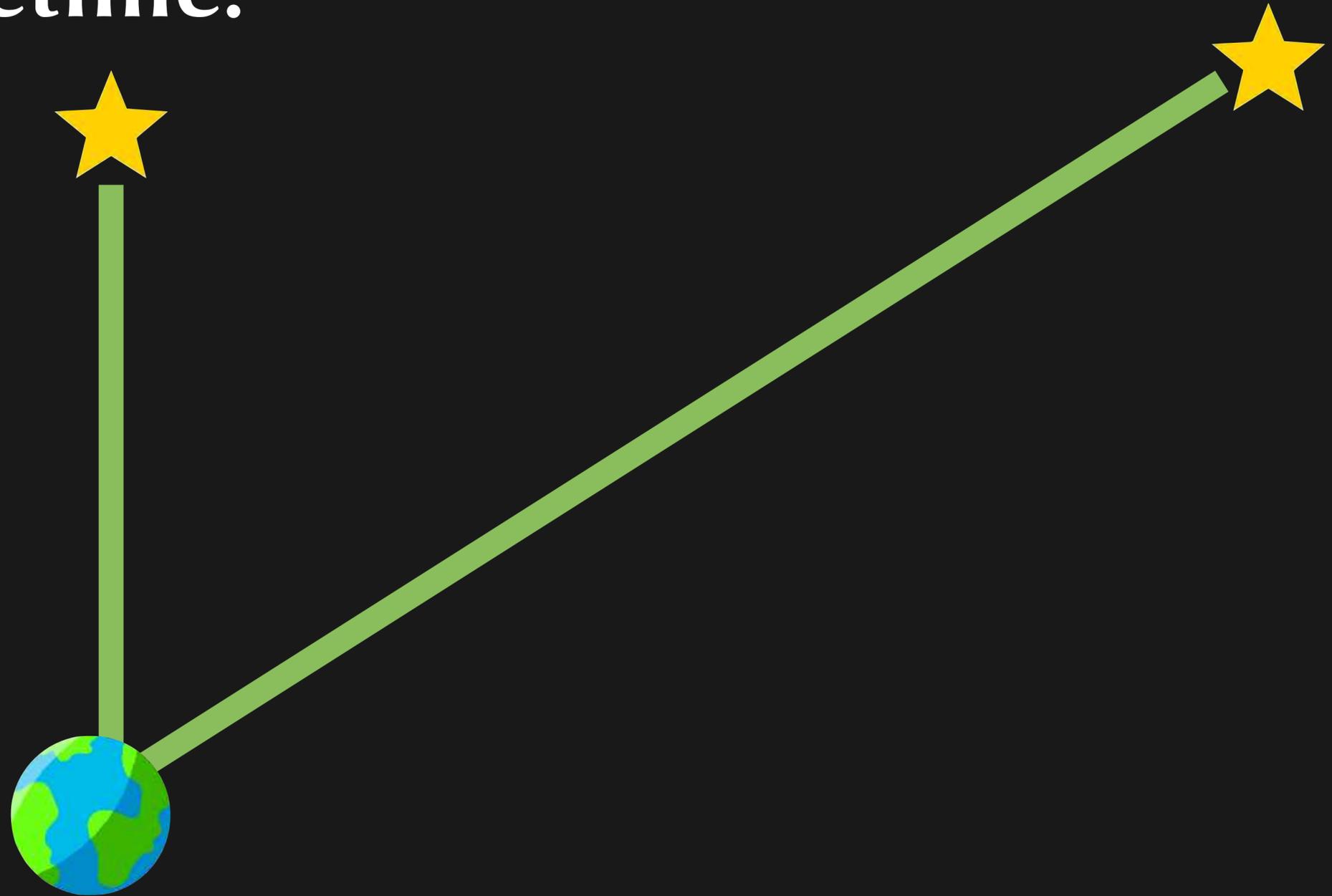
**Even light follows the
curves of spacetime.**

*But you need heavy
objects and big
distances to notice.*



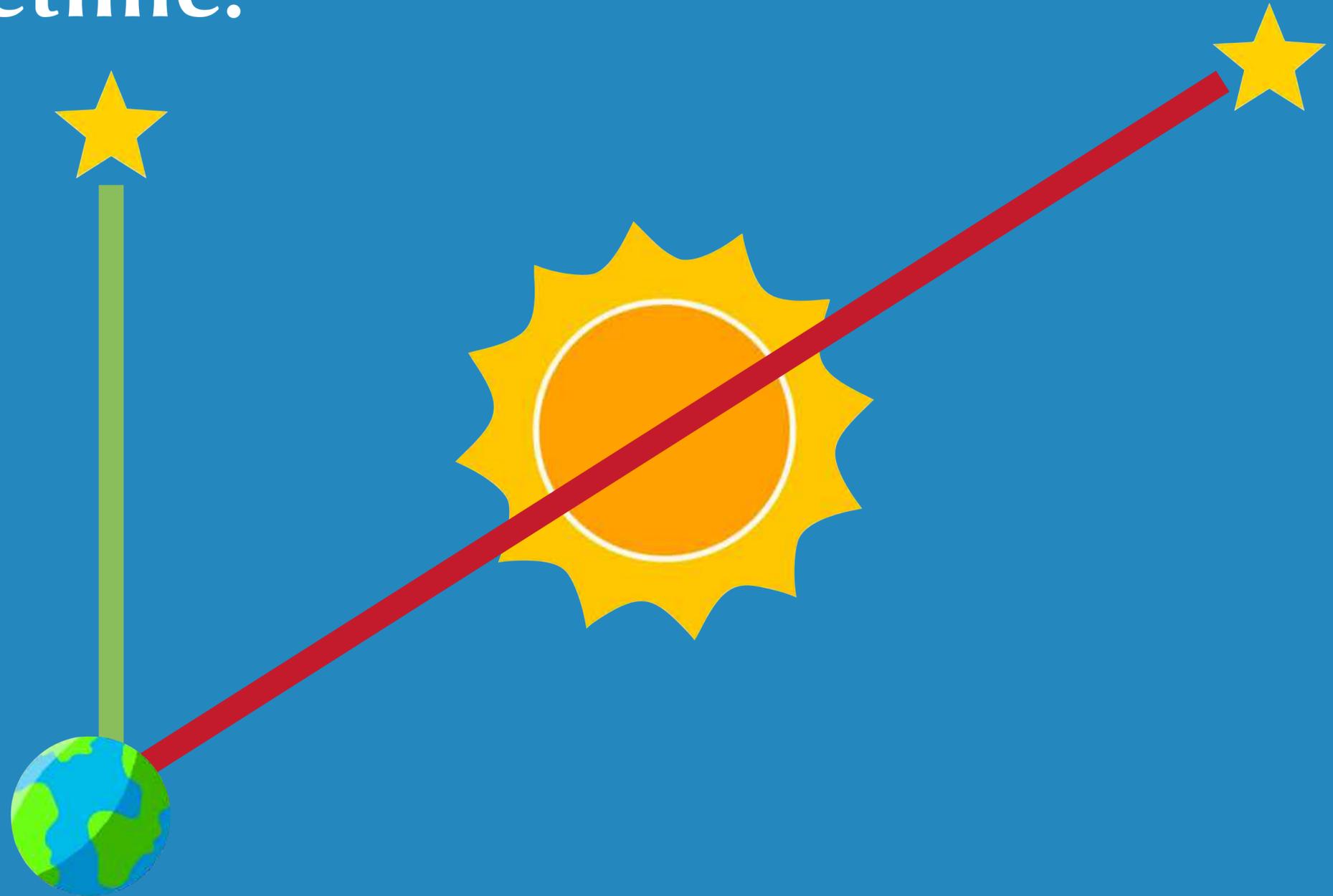
Even light follows the curves of spacetime.

Eddington



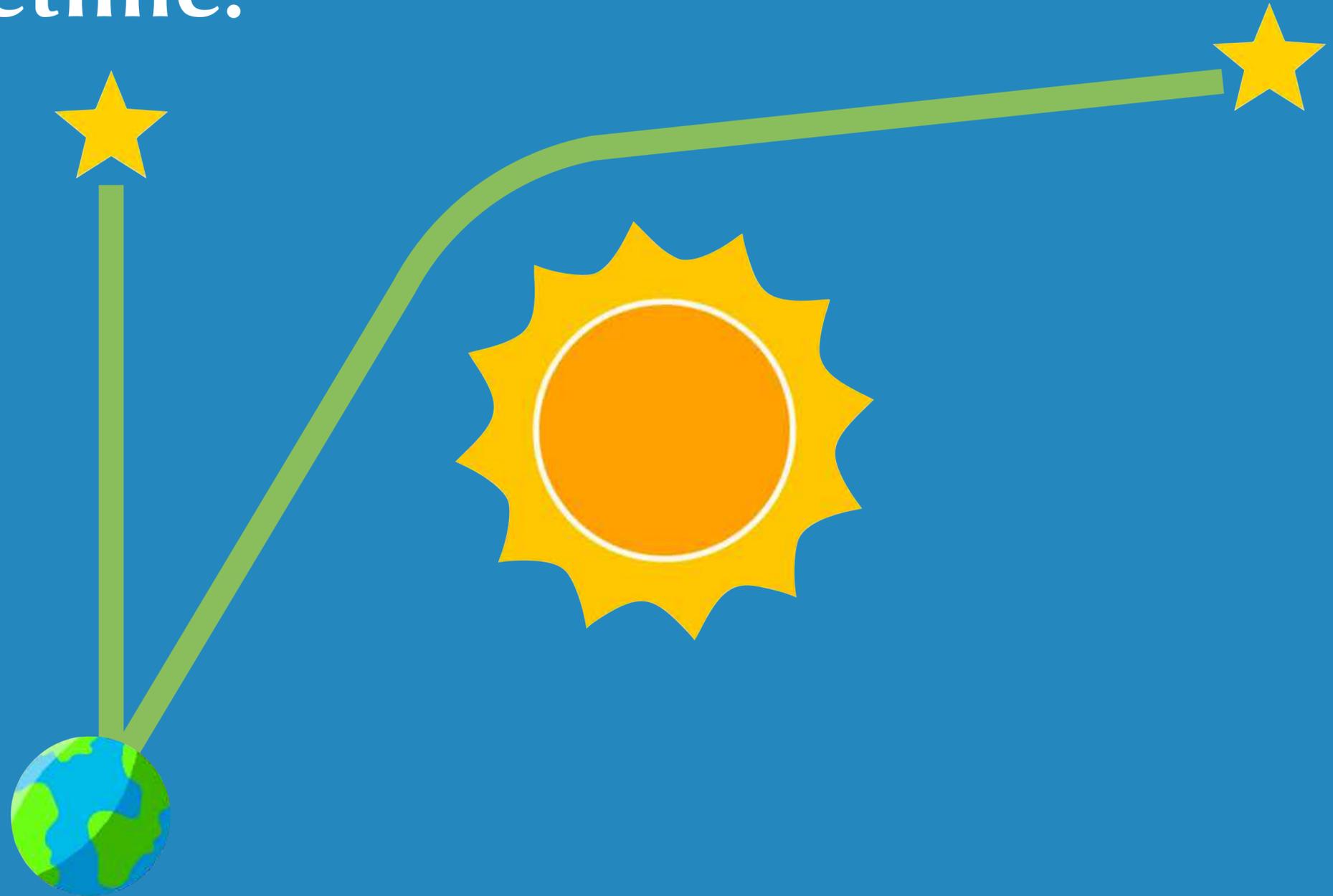
Even light follows the curves of spacetime.

Eddington



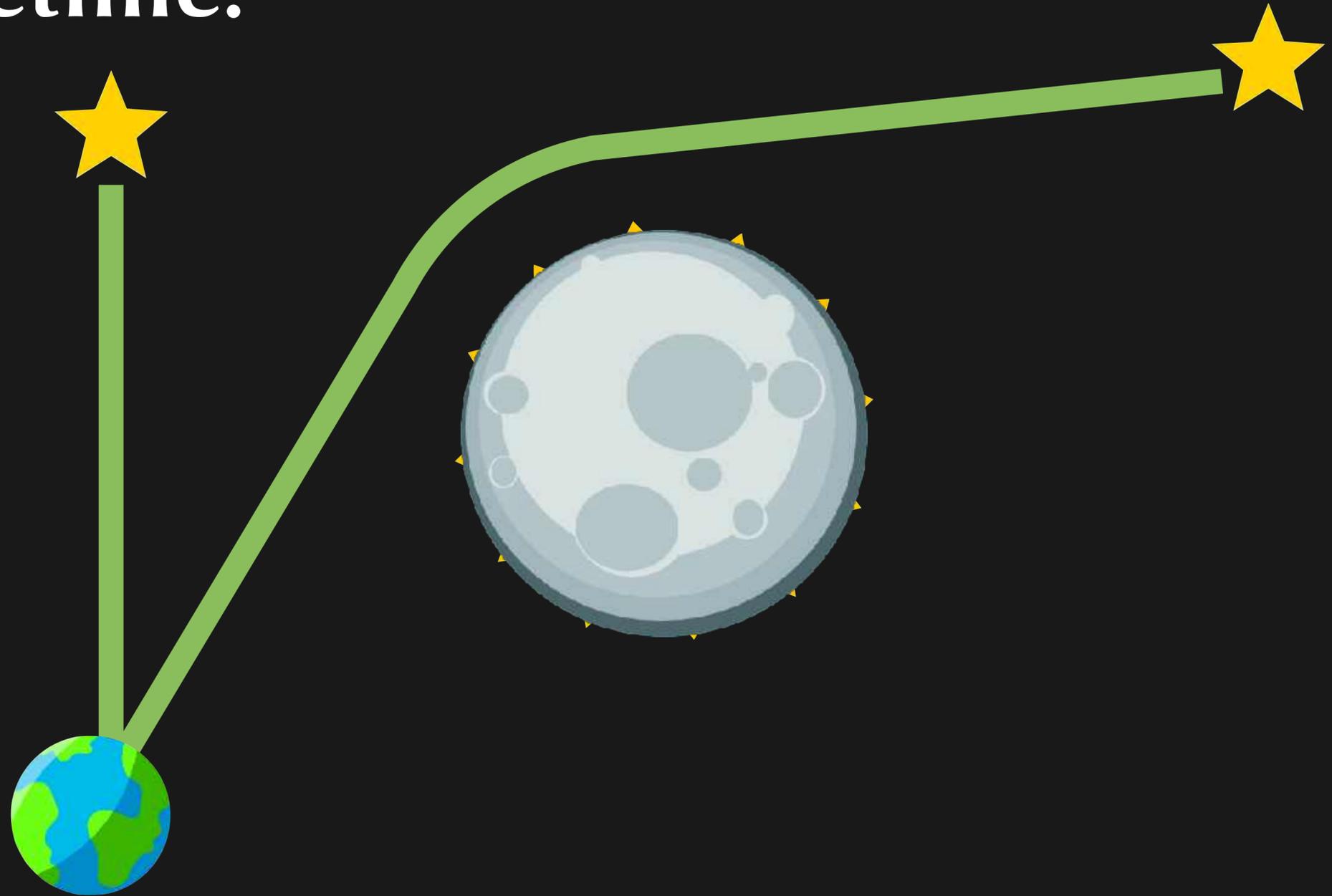
Even light follows the curves of spacetime.

Eddington

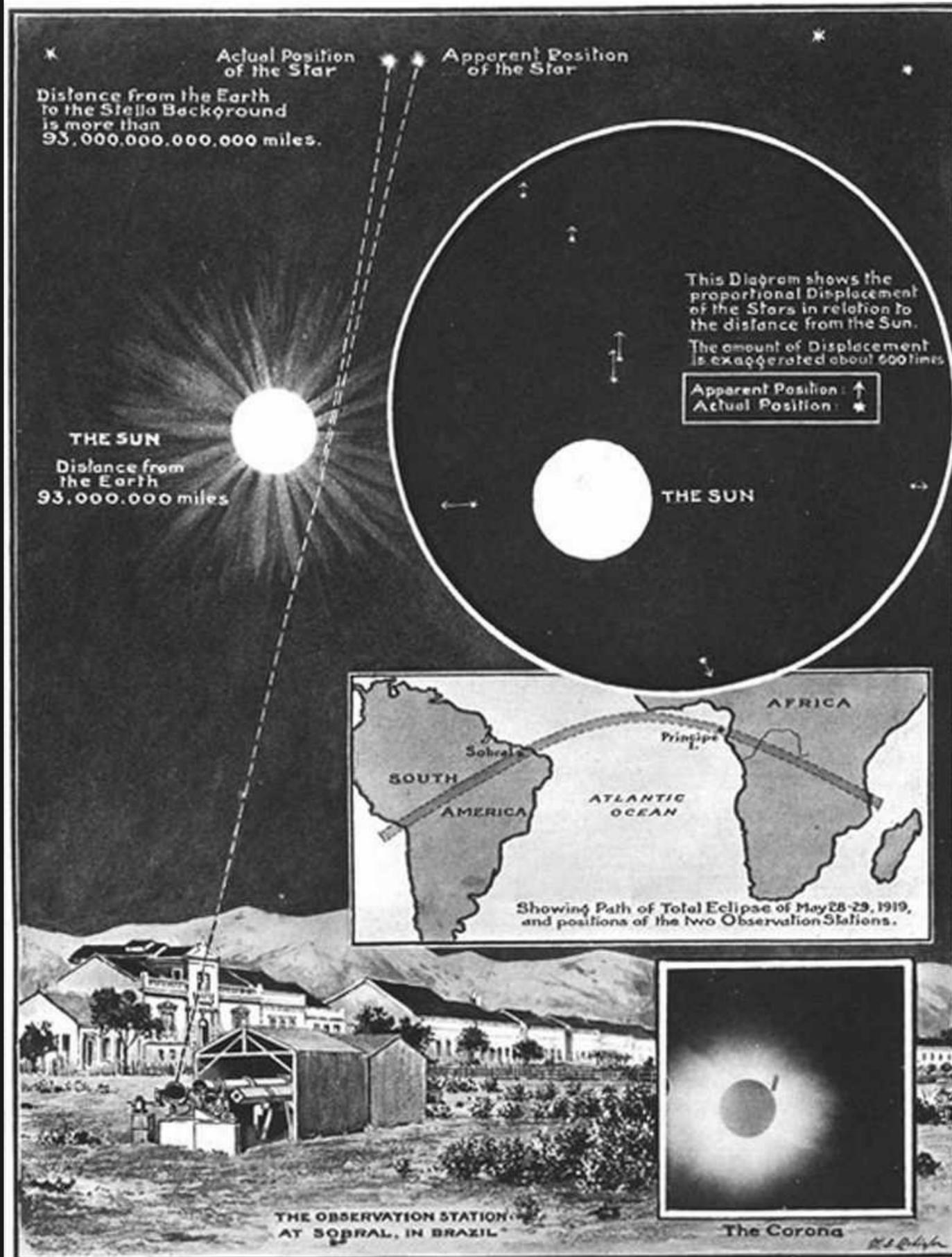


Even light follows the curves of spacetime.

Eddington



May 29, 1919
Eddington



LIGHTS ALL ASKEW IN THE HEAVENS

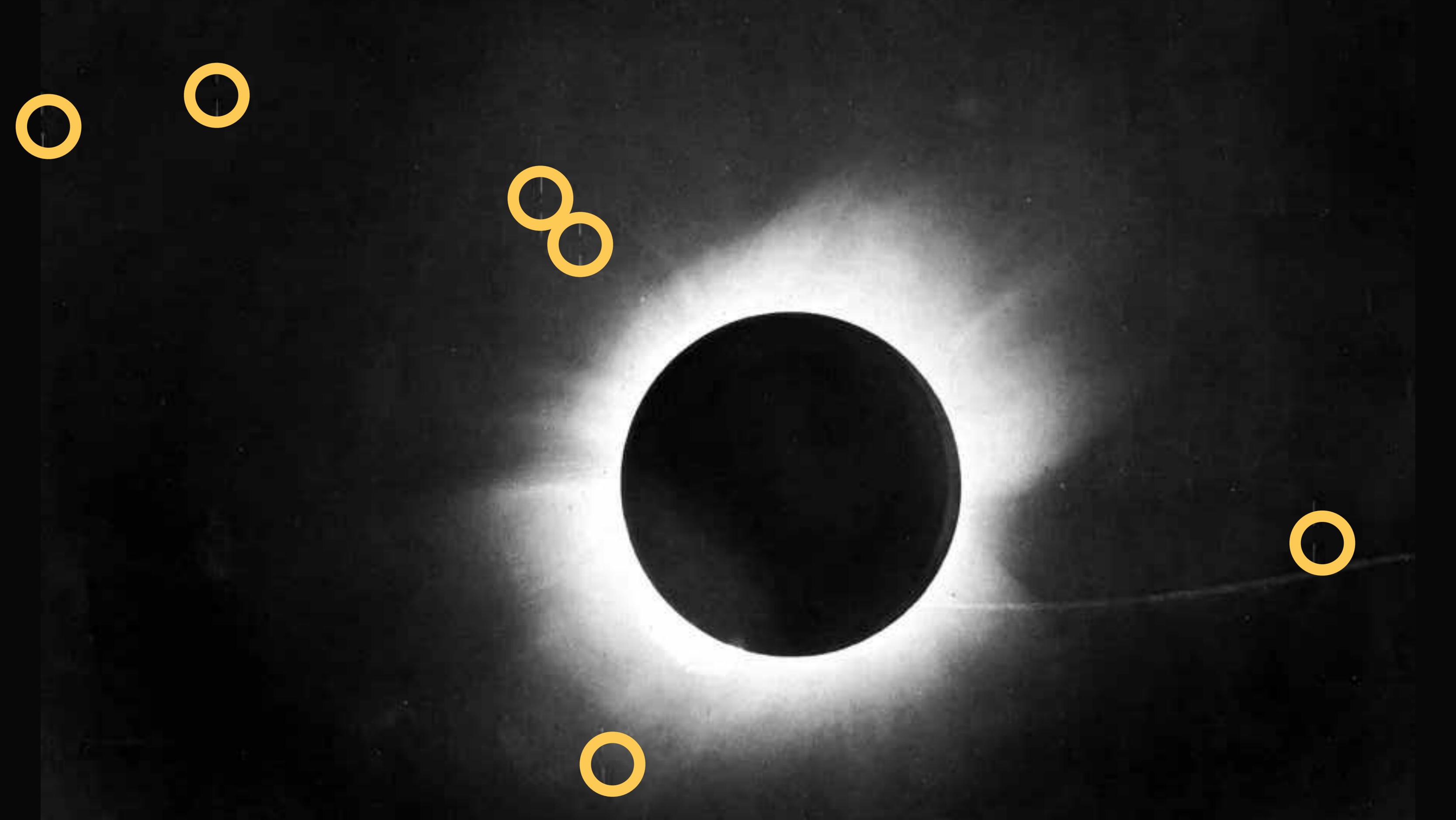
Men of Science More or Less
 Agog Over Results of Eclipse
 Observations.

EINSTEIN THEORY TRIUMPHS

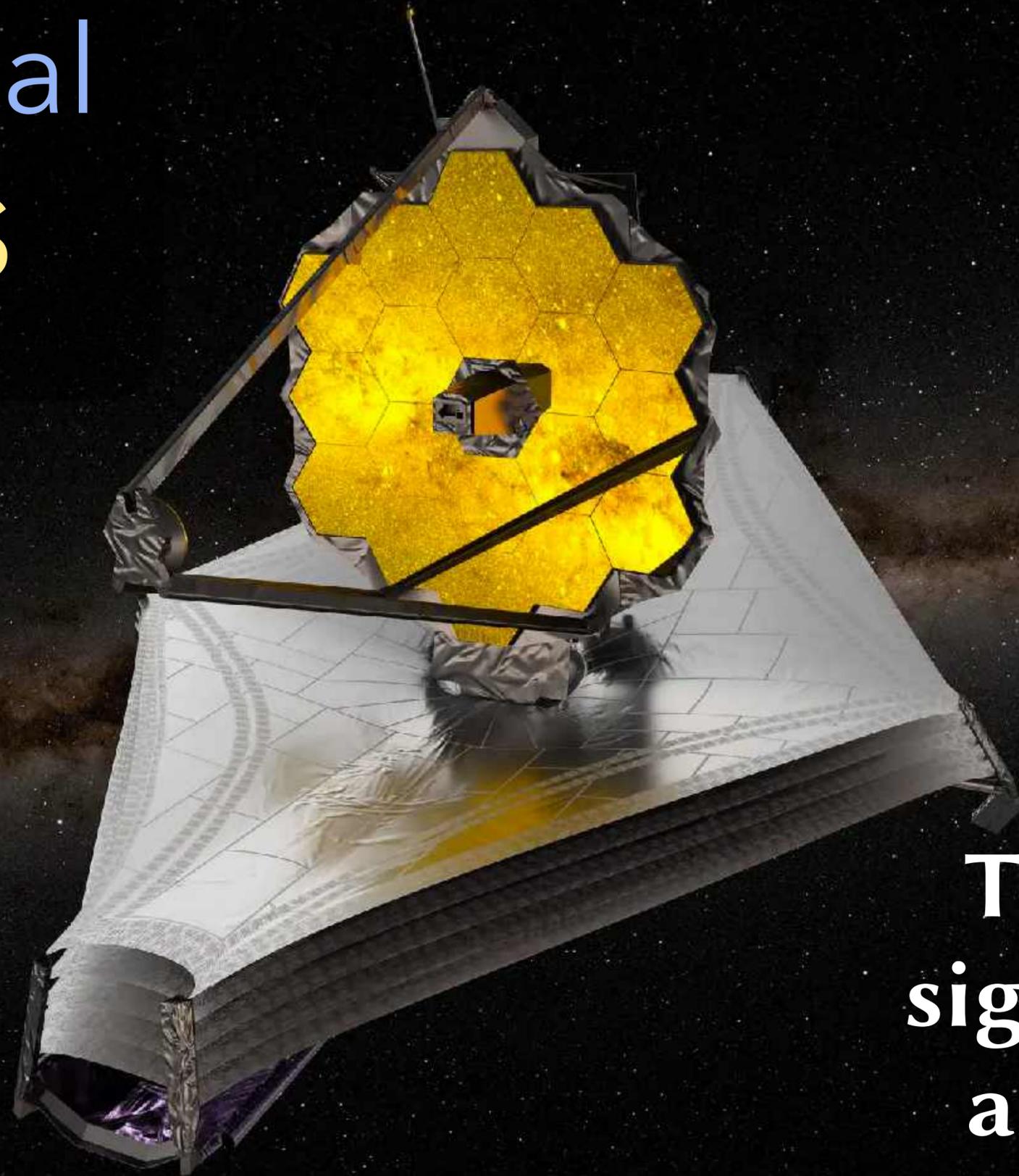
Stars Not Where They Seemed
 or Were Calculated to be,
 but Nobody Need Worry.

A BOOK FOR 12 WISE MEN

No More in All the World Could
 Comprehend It, Said Einstein When
 His Daring Publishers Accepted It.

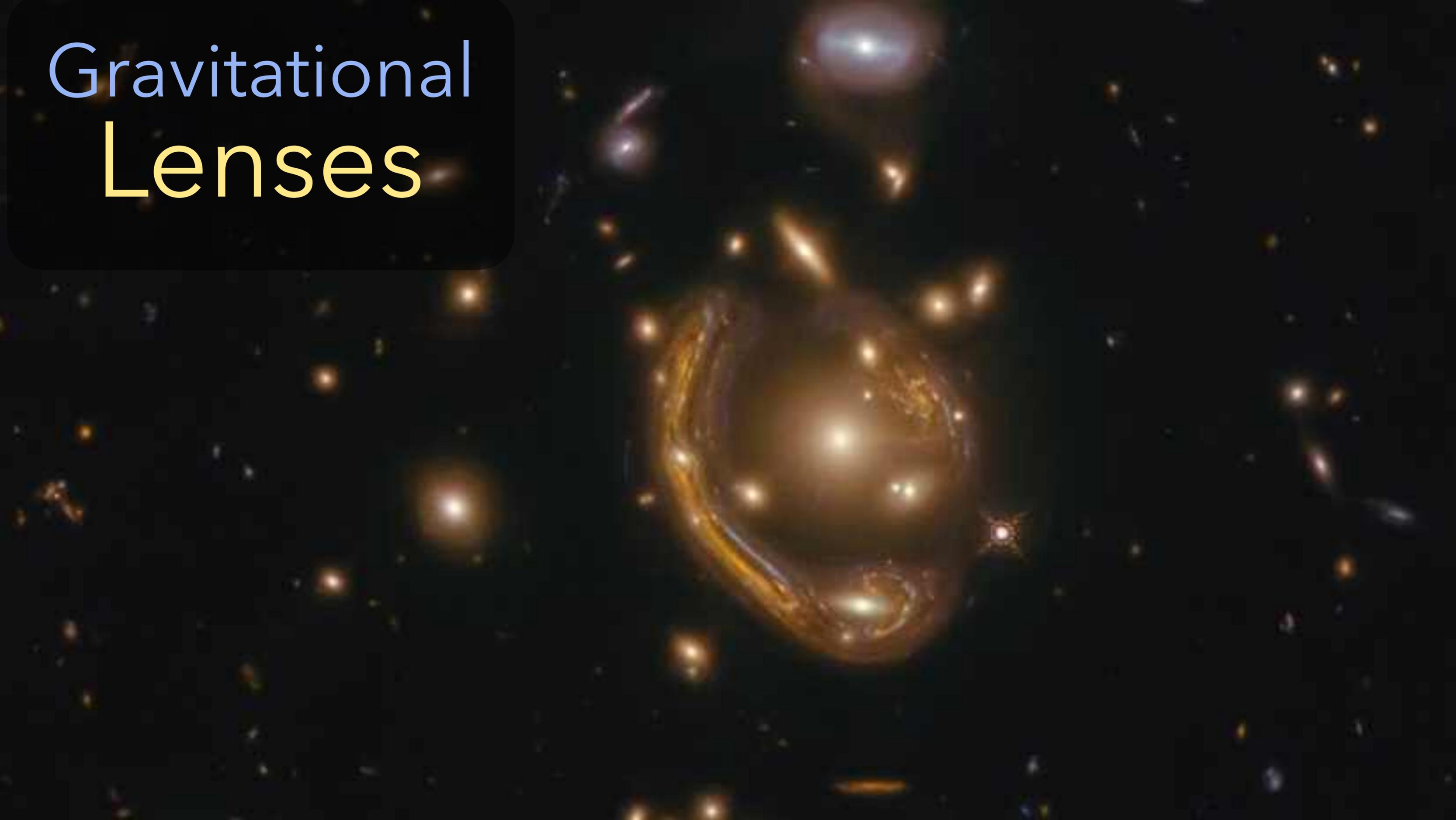


Gravitational Lenses

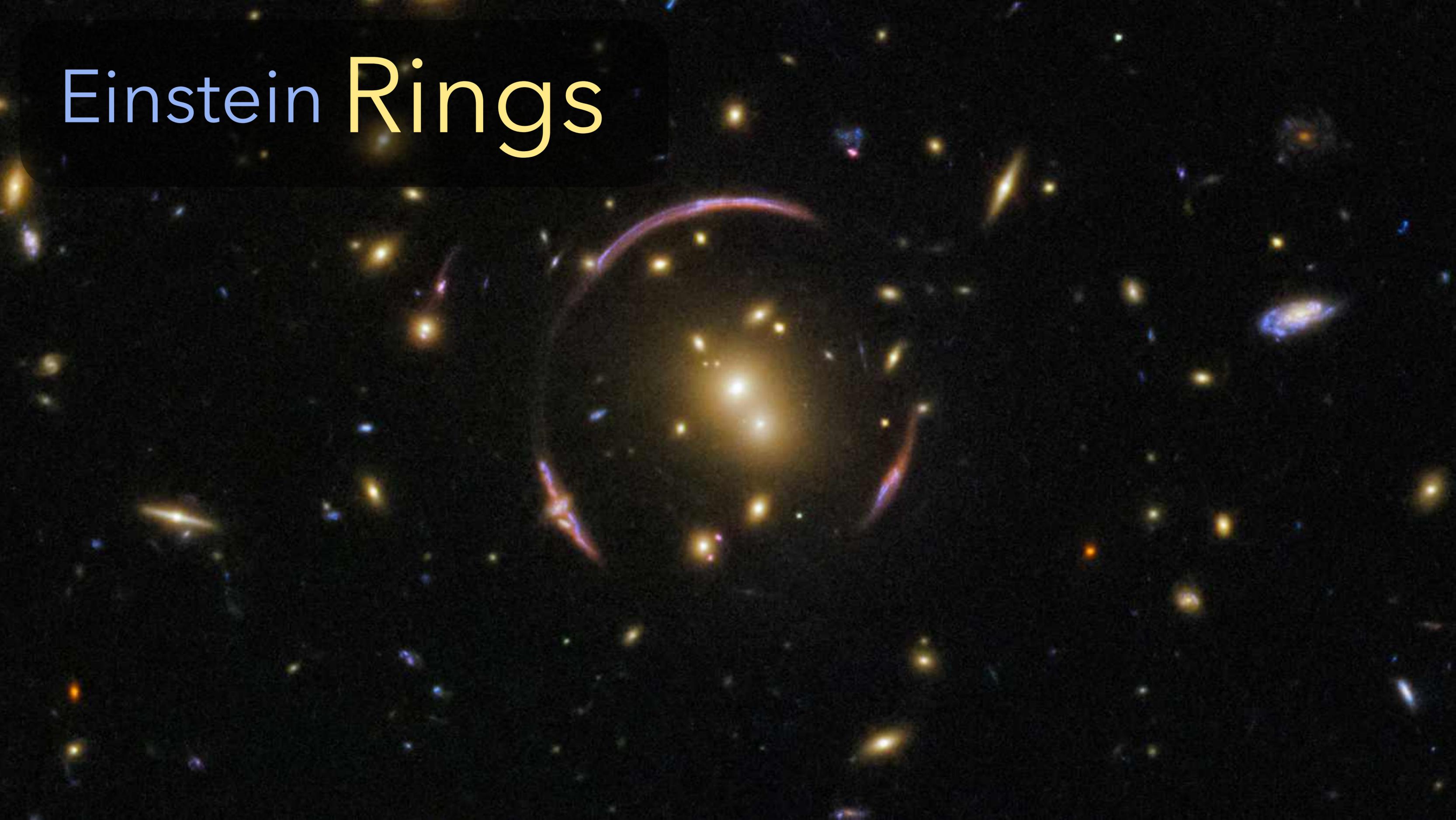


**These days, the
signs of curvature
are much more
visible**

Gravitational Lenses



Einstein Rings



Einstein Rings

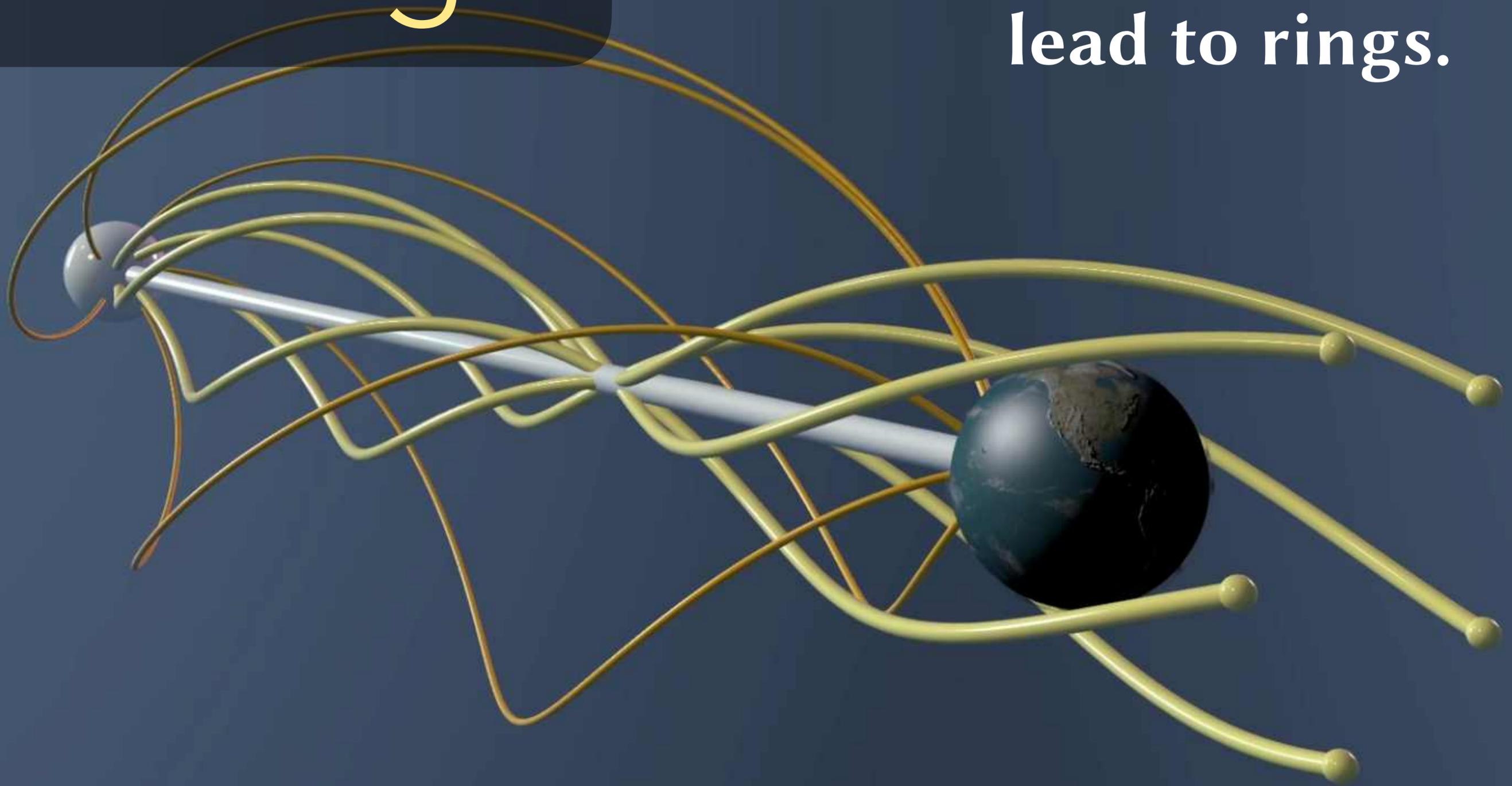


Einstein Rings



Einstein Rings

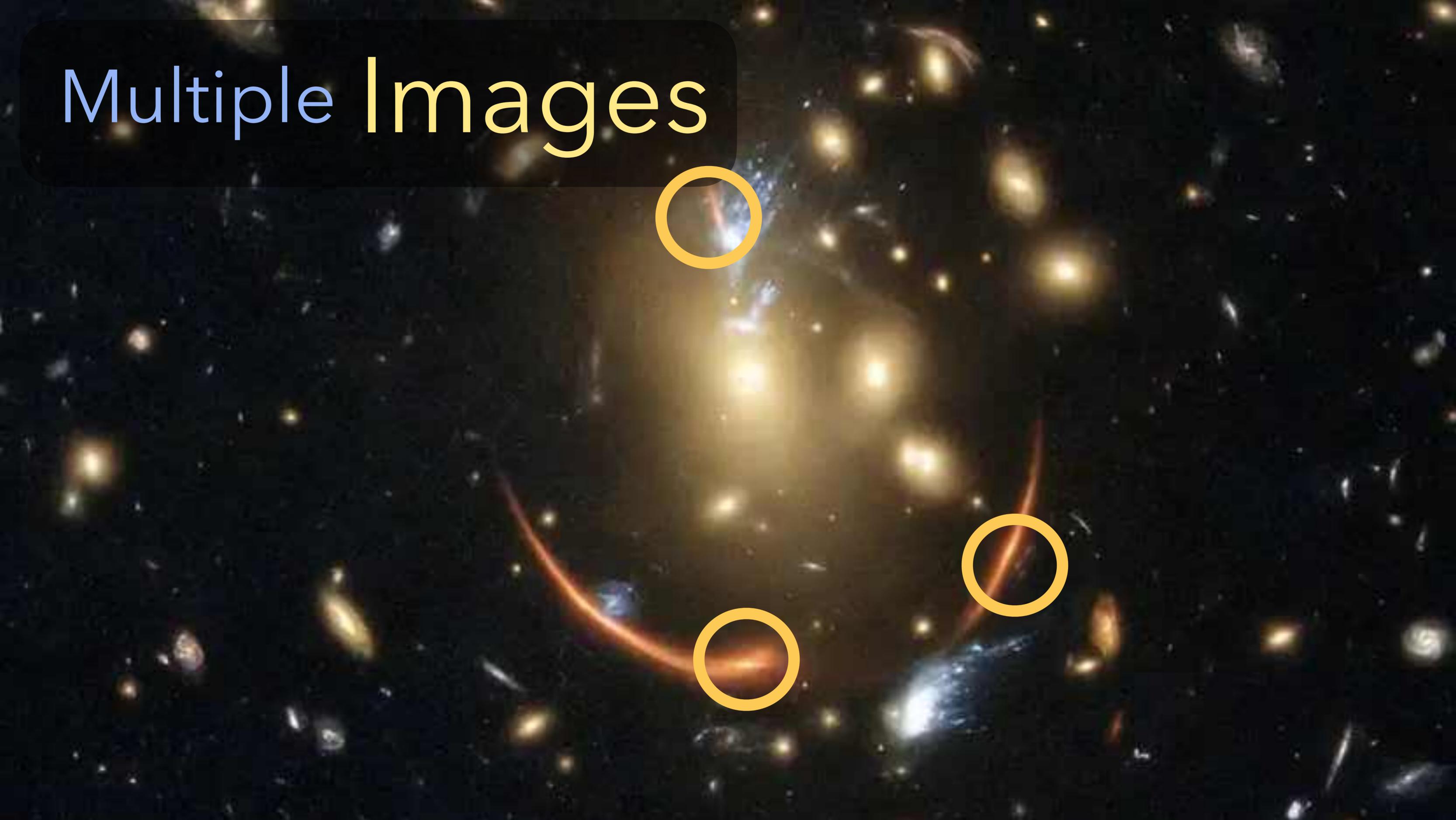
Converging rays
lead to rings.



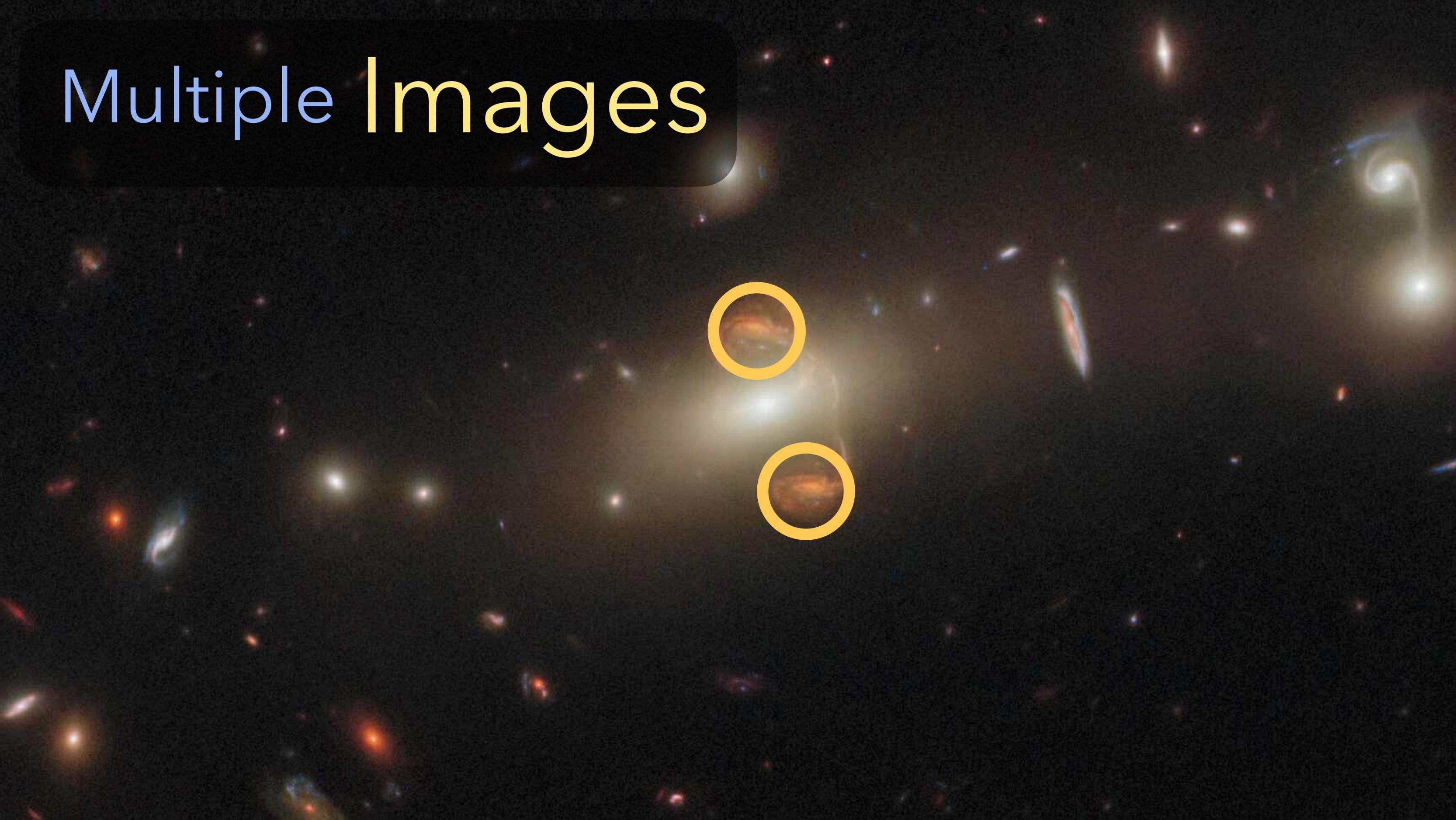
Einstein Rings



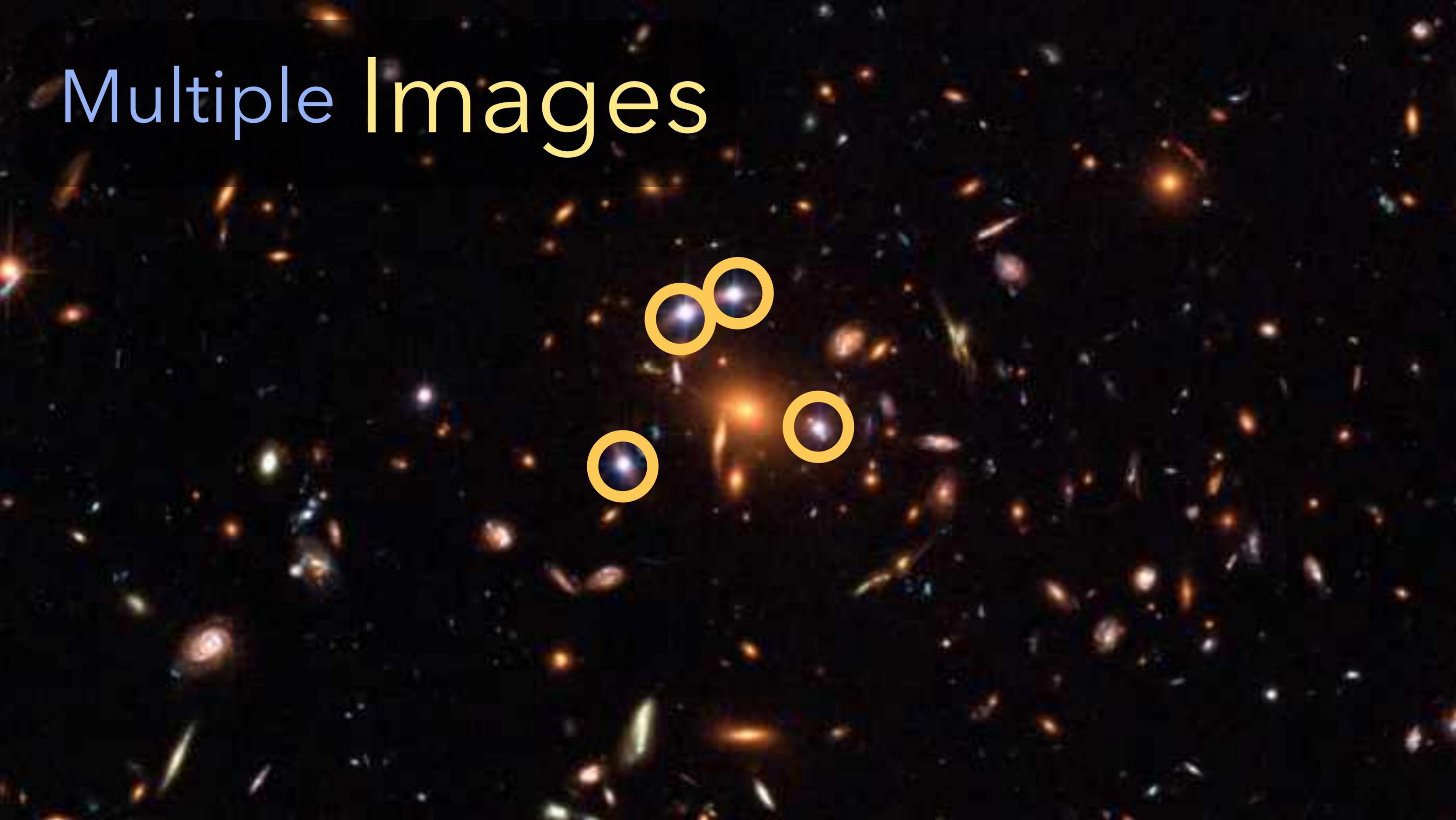
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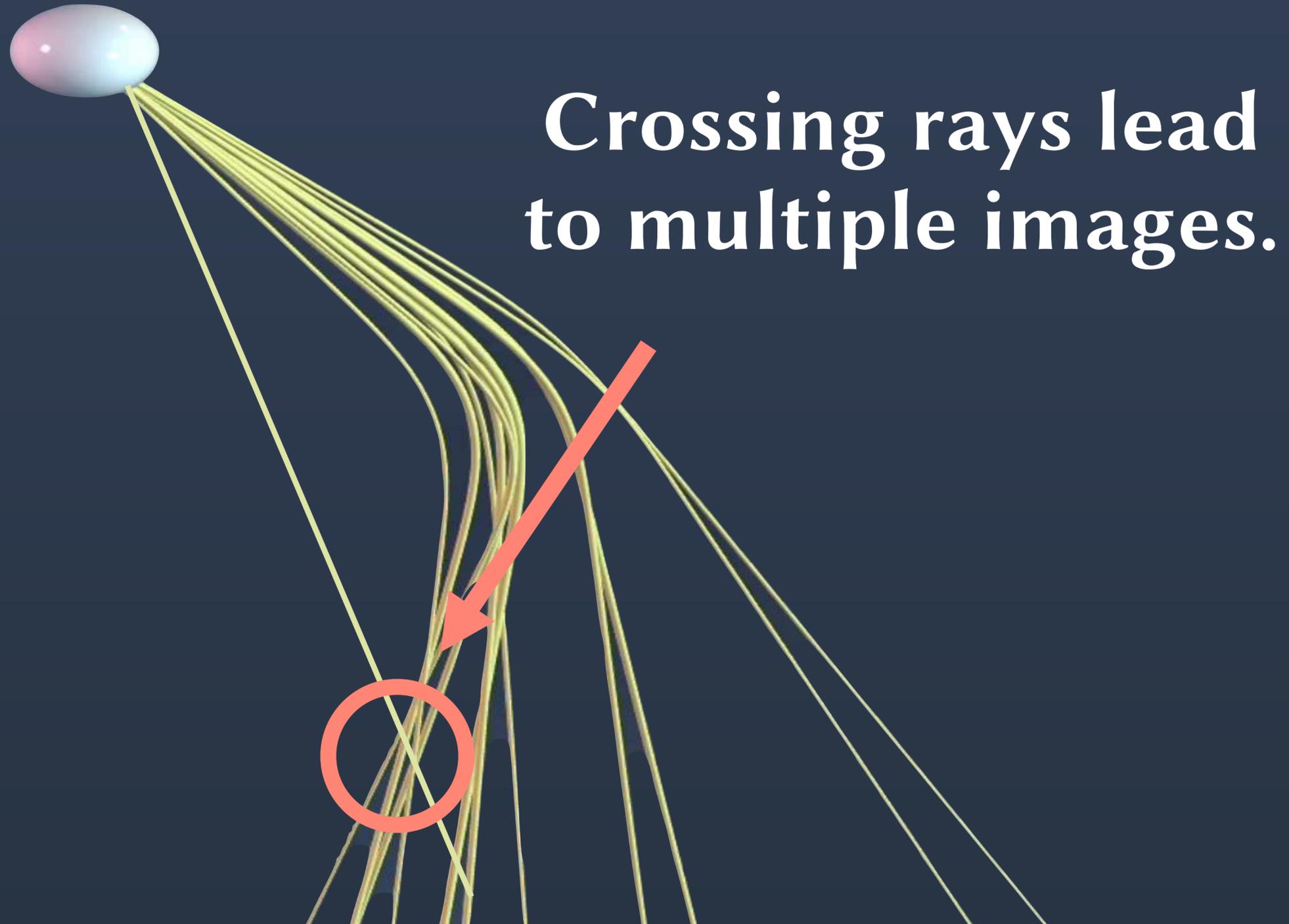
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Multiple Images



Multiple Images



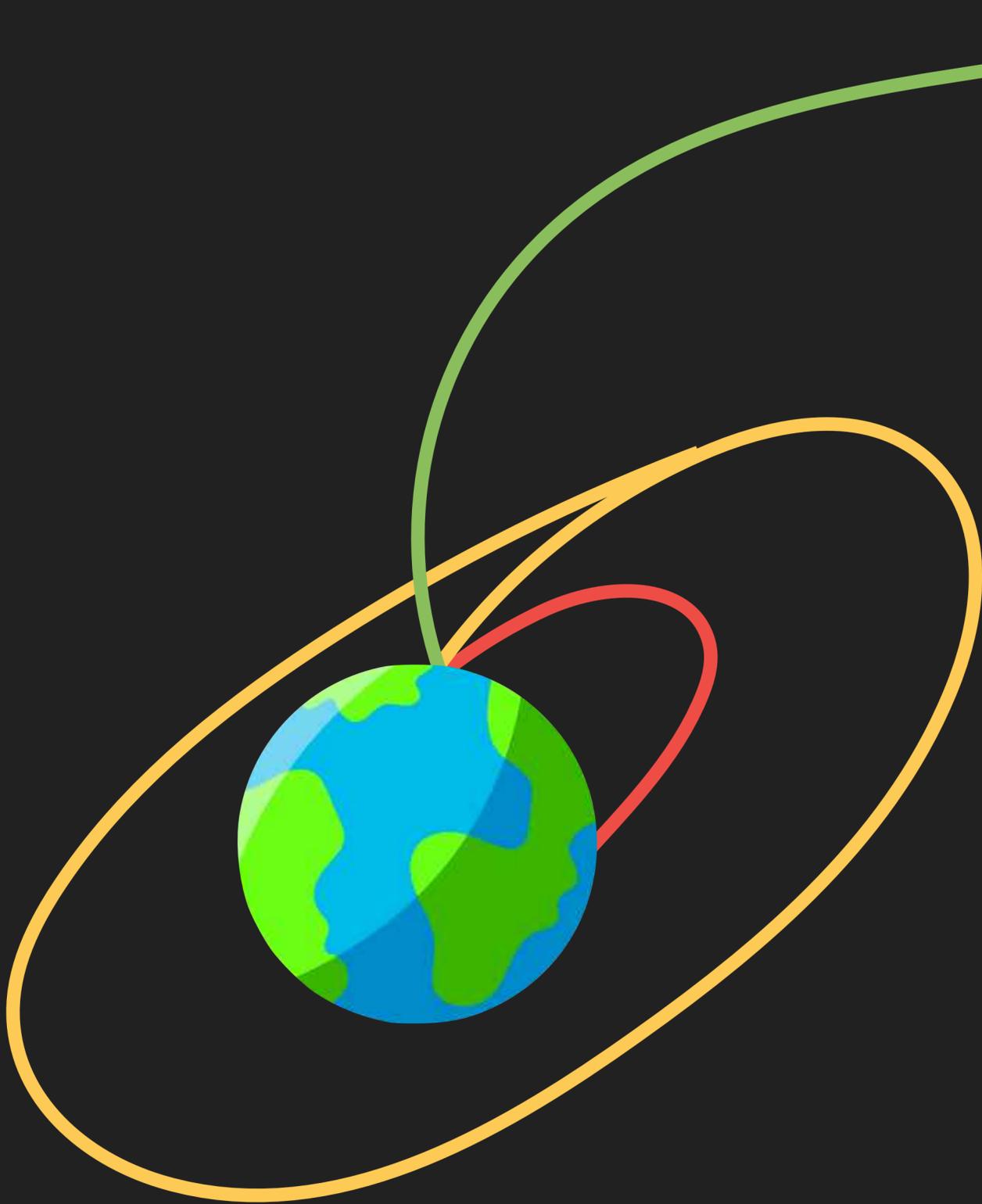
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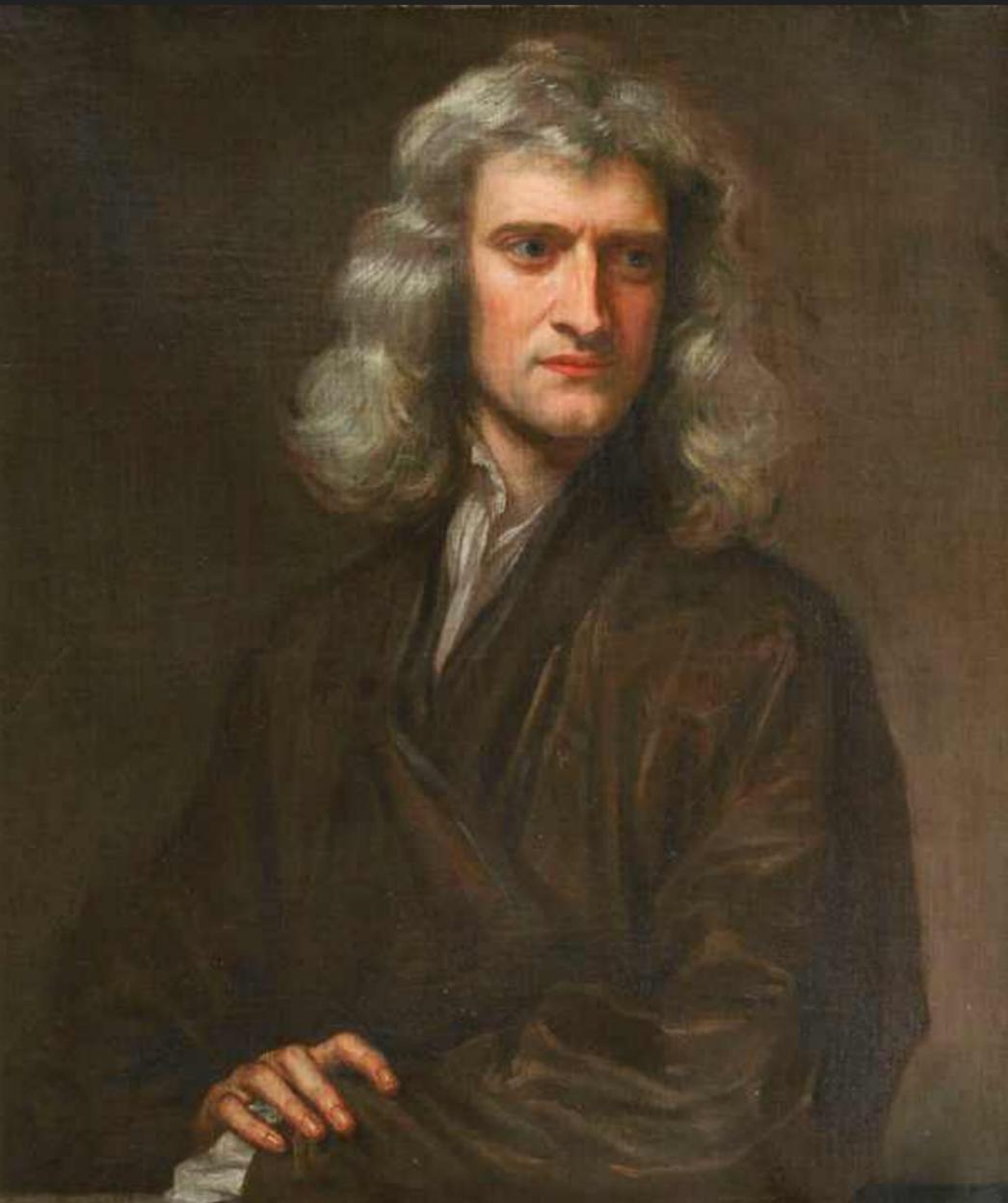
Black

Holes

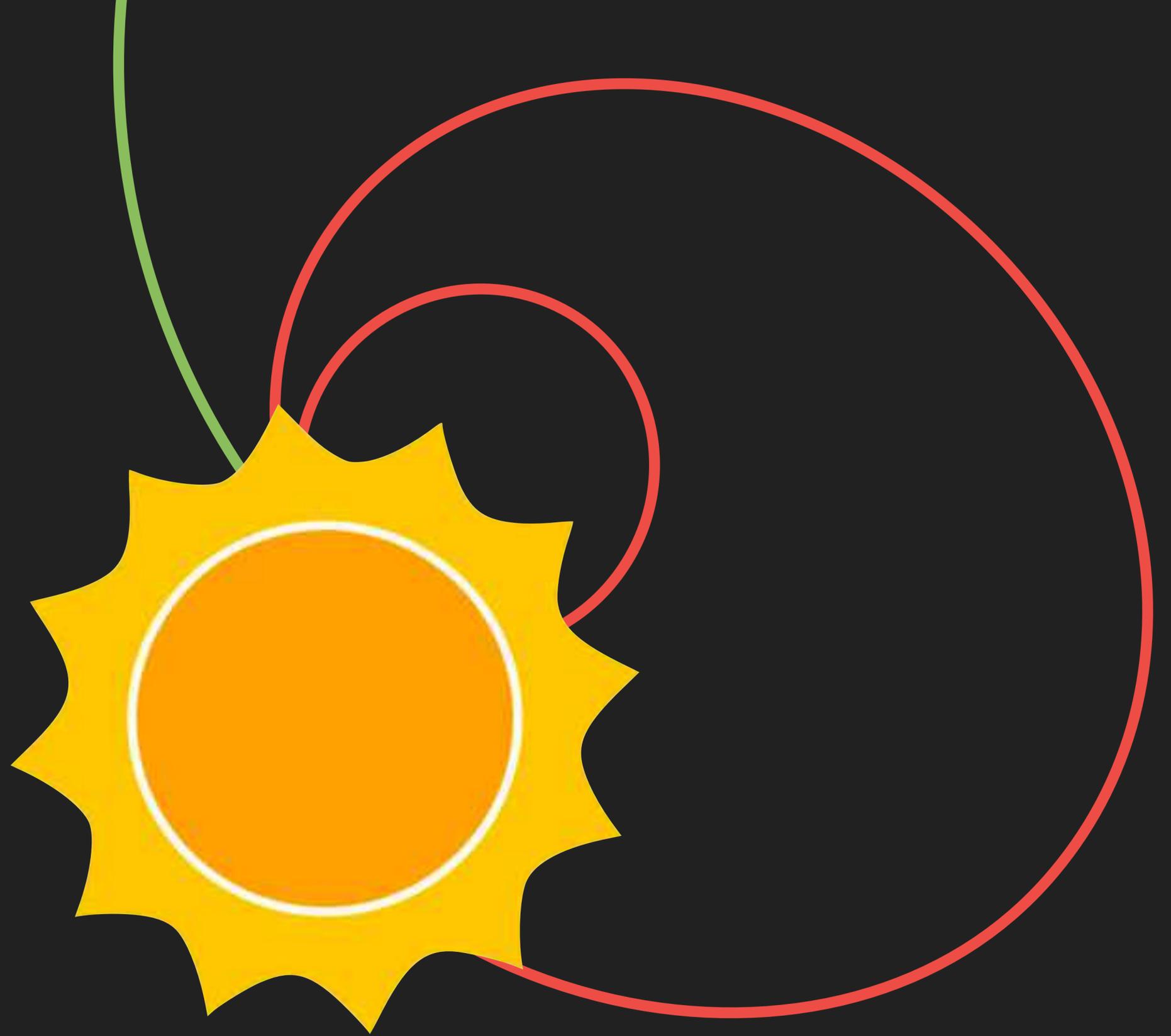
Escape
velocity



1687
Newton



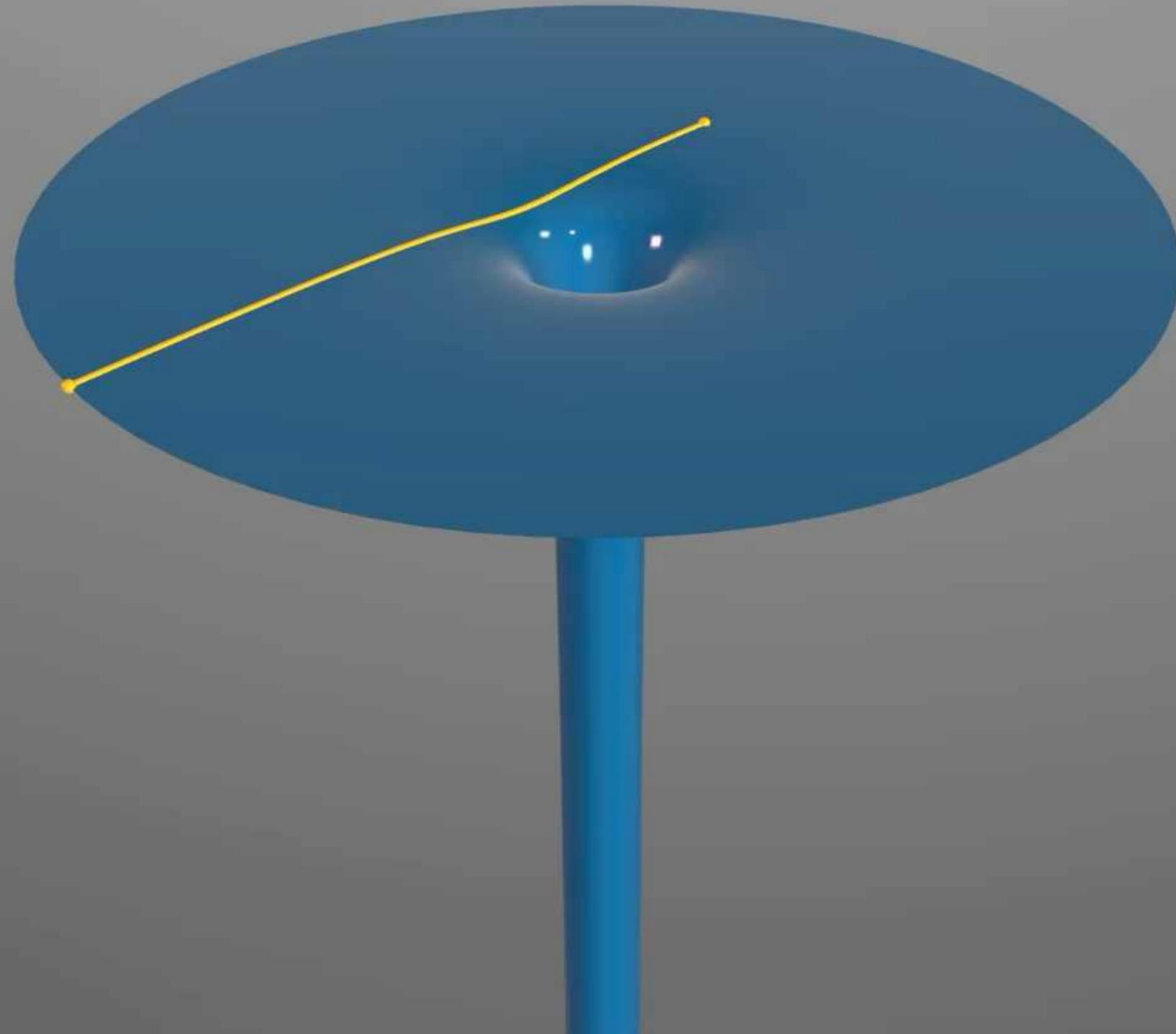
1687
Newton

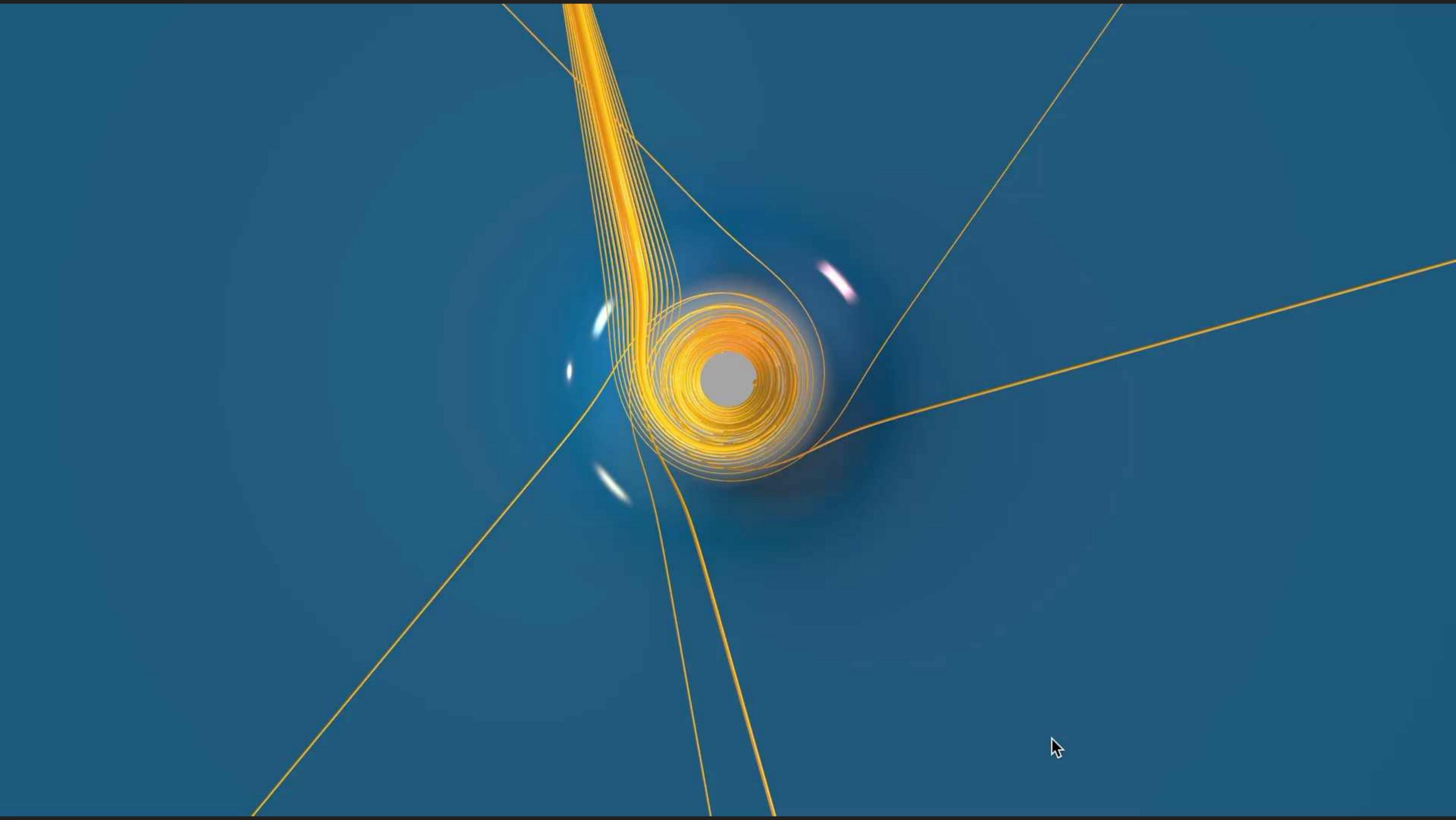


1783
Michell

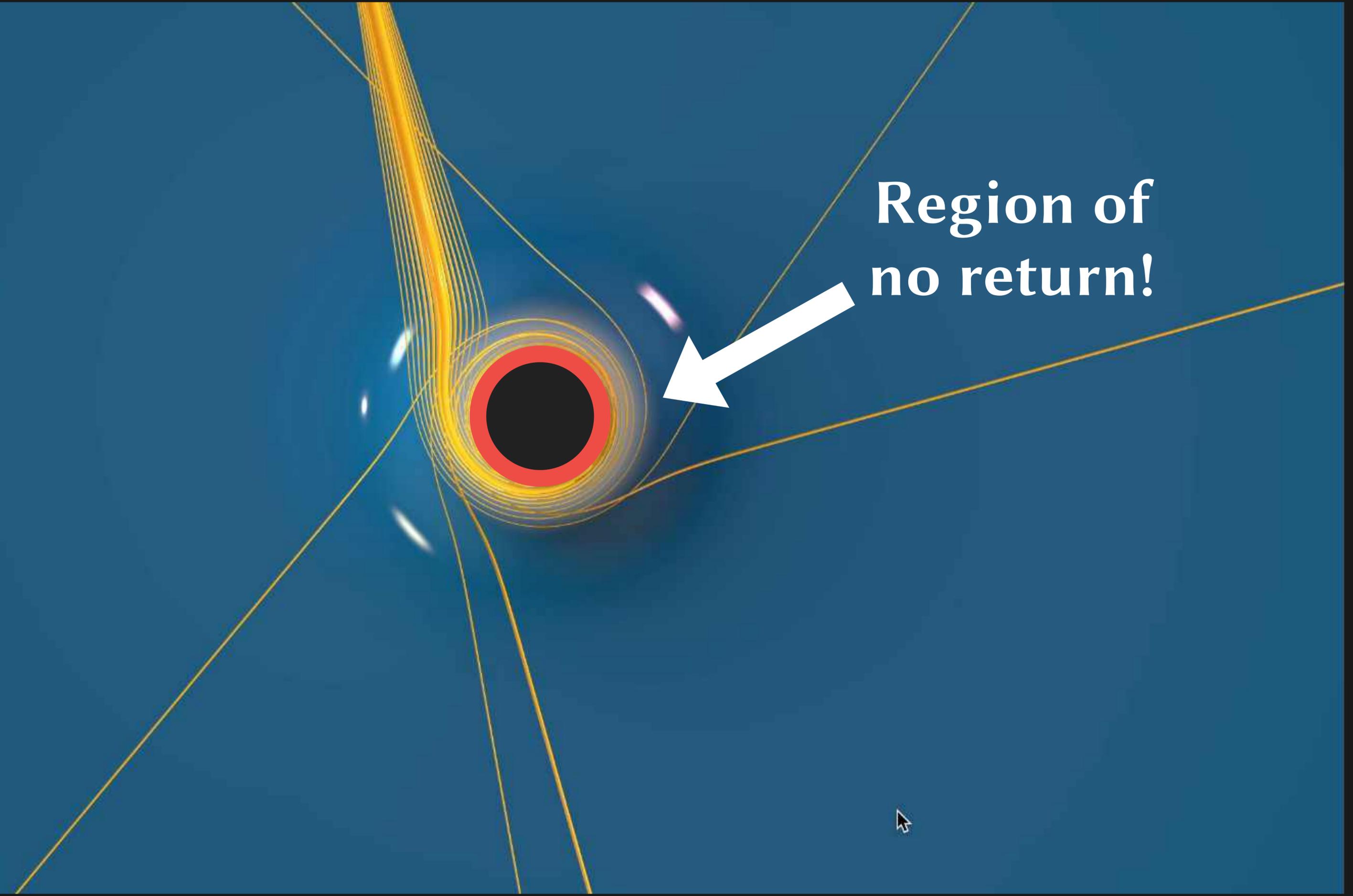


**A space so curved, that
light cannot escape?**

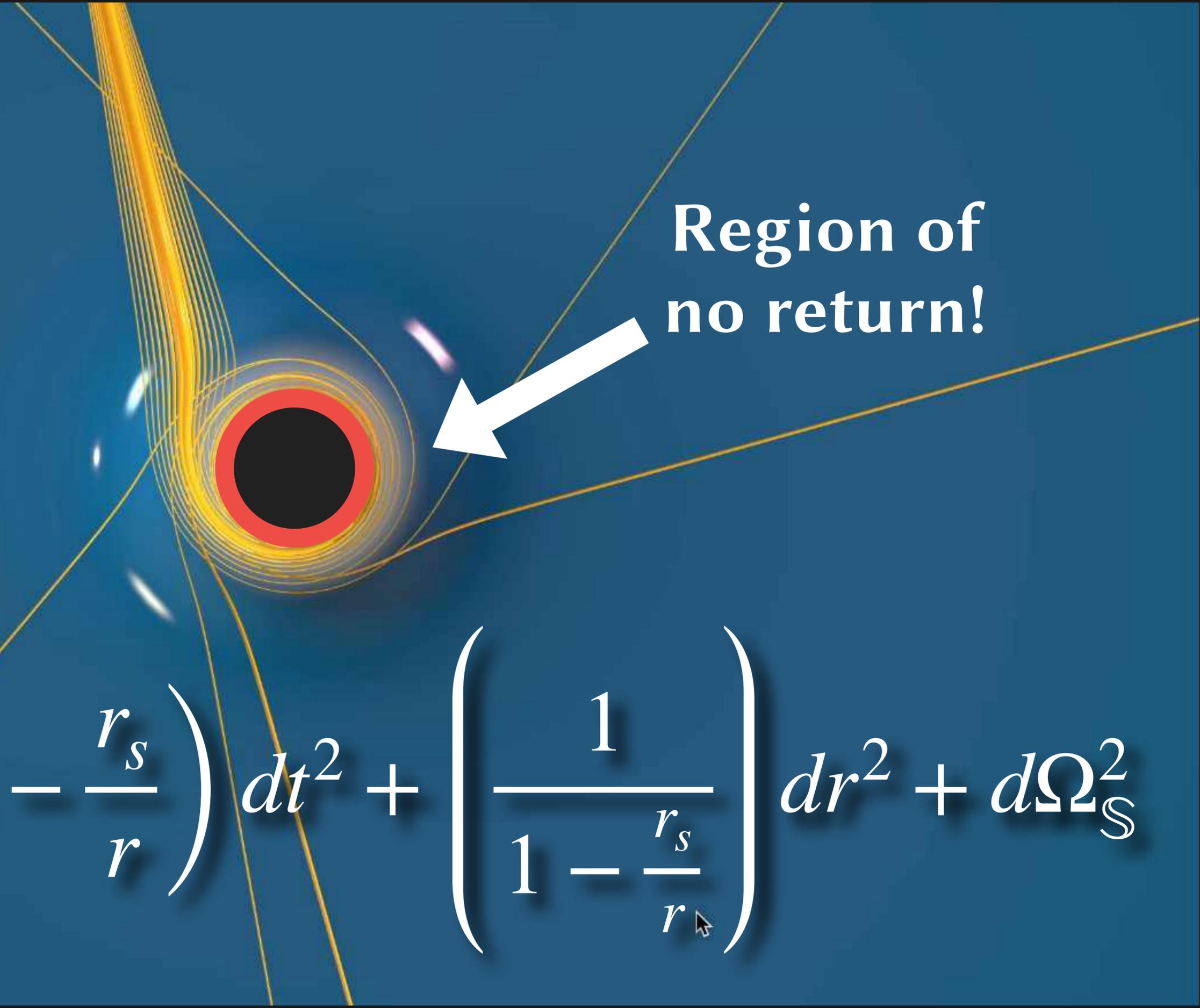
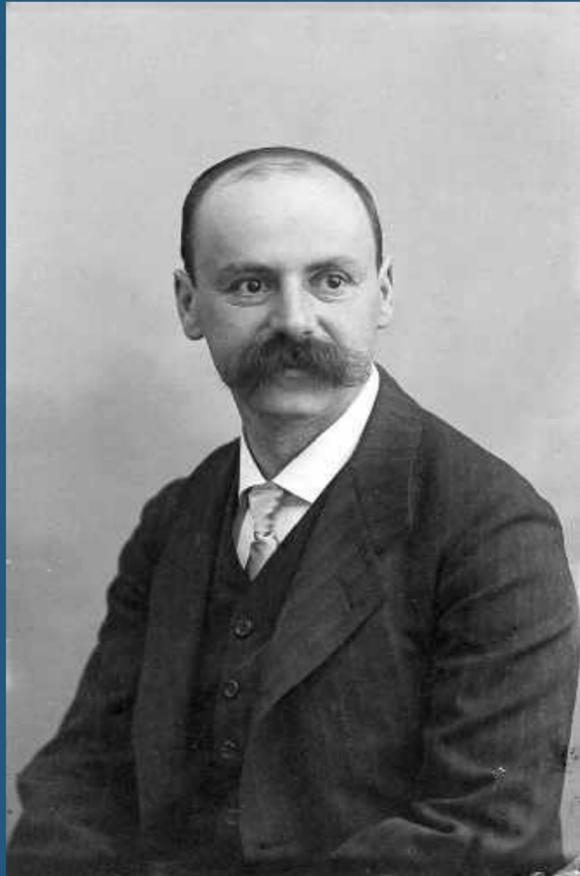




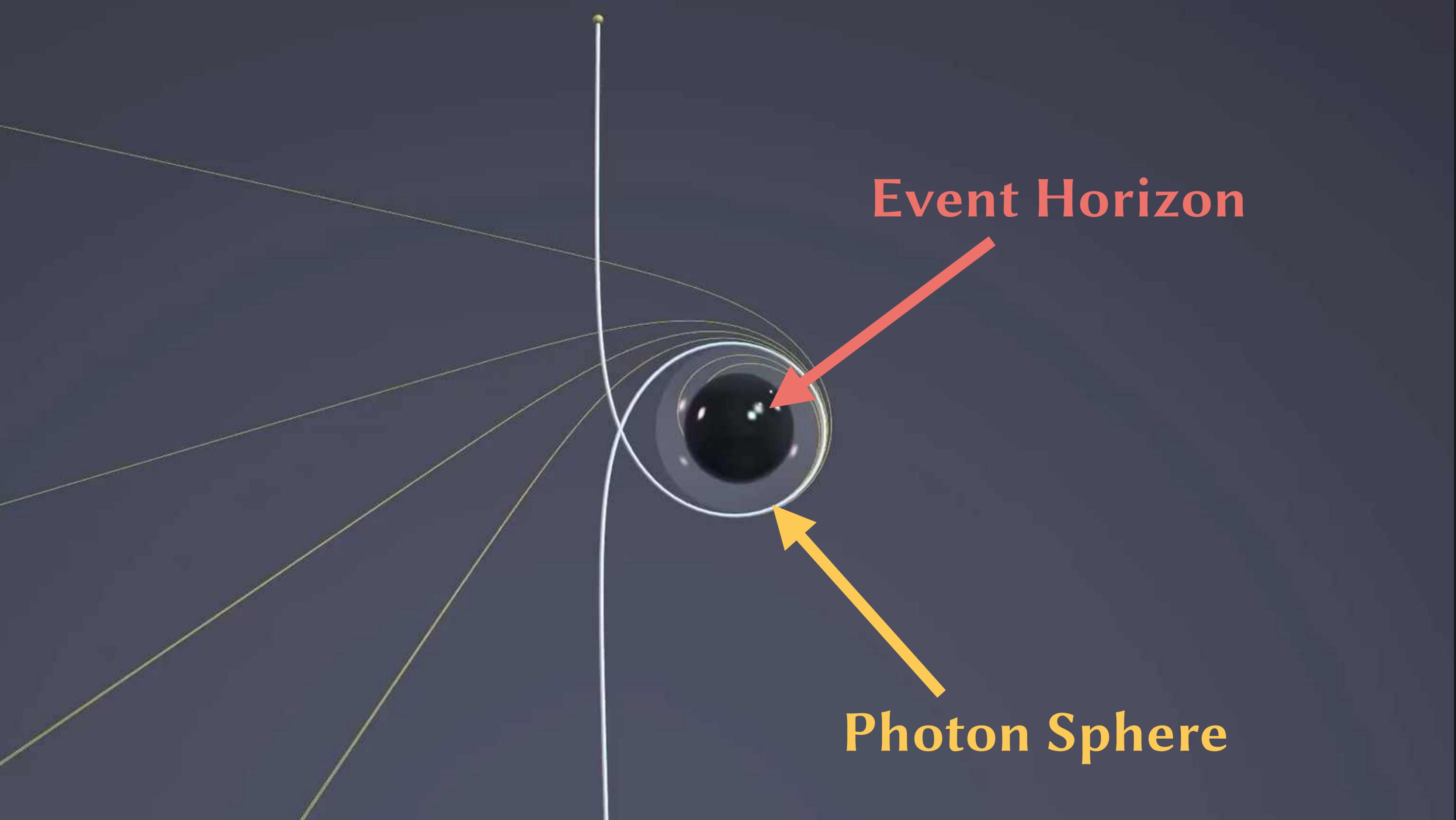
**Region of
no return!**



Schwarzschild



$$ds^2 = -c^2 \left(1 - \frac{r_s}{r} \right) dt^2 + \left(\frac{1}{1 - \frac{r_s}{r}} \right) dr^2 + d\Omega_{\mathbb{S}}^2$$

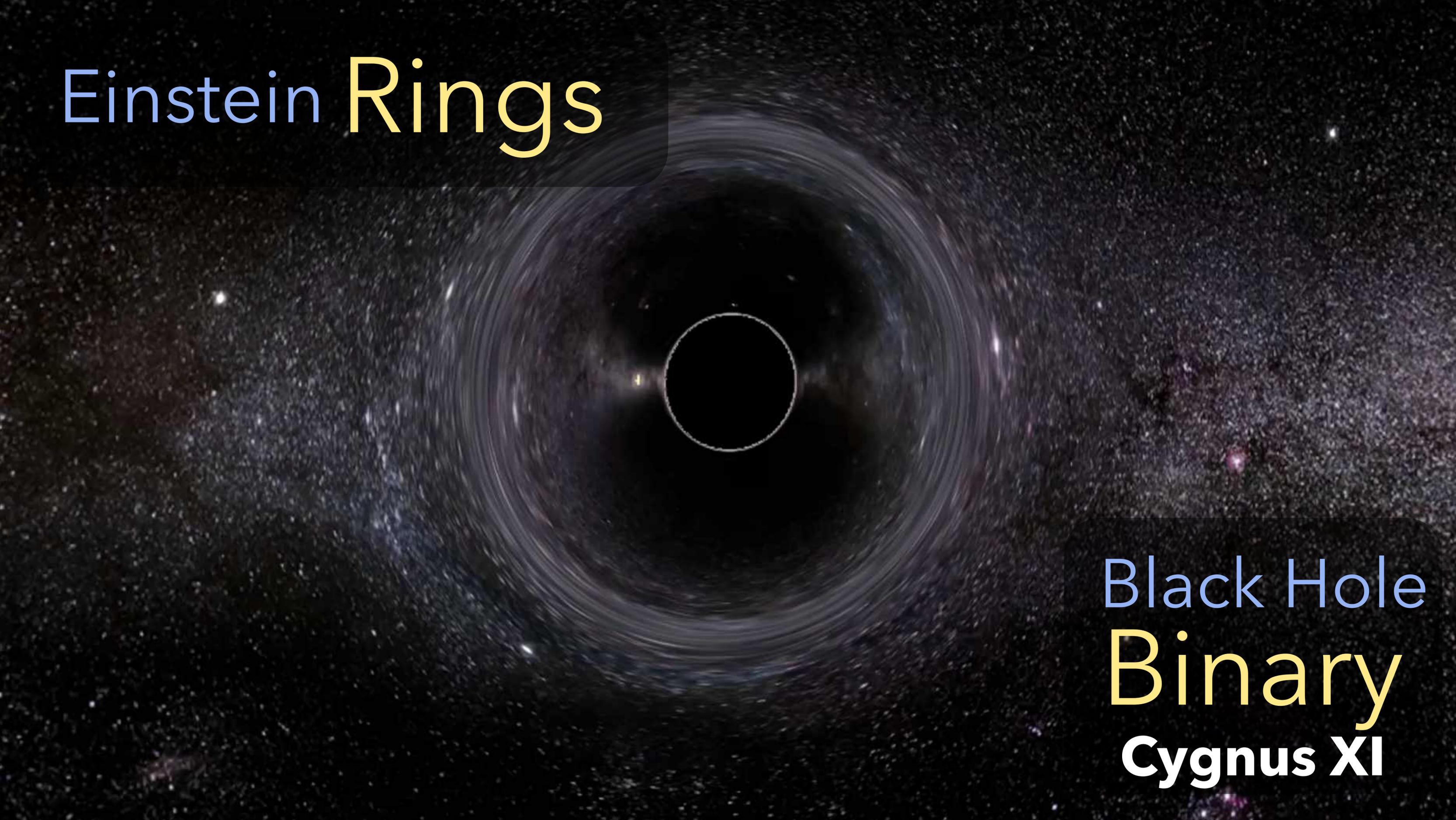


Event Horizon

Photon Sphere



Einstein Rings



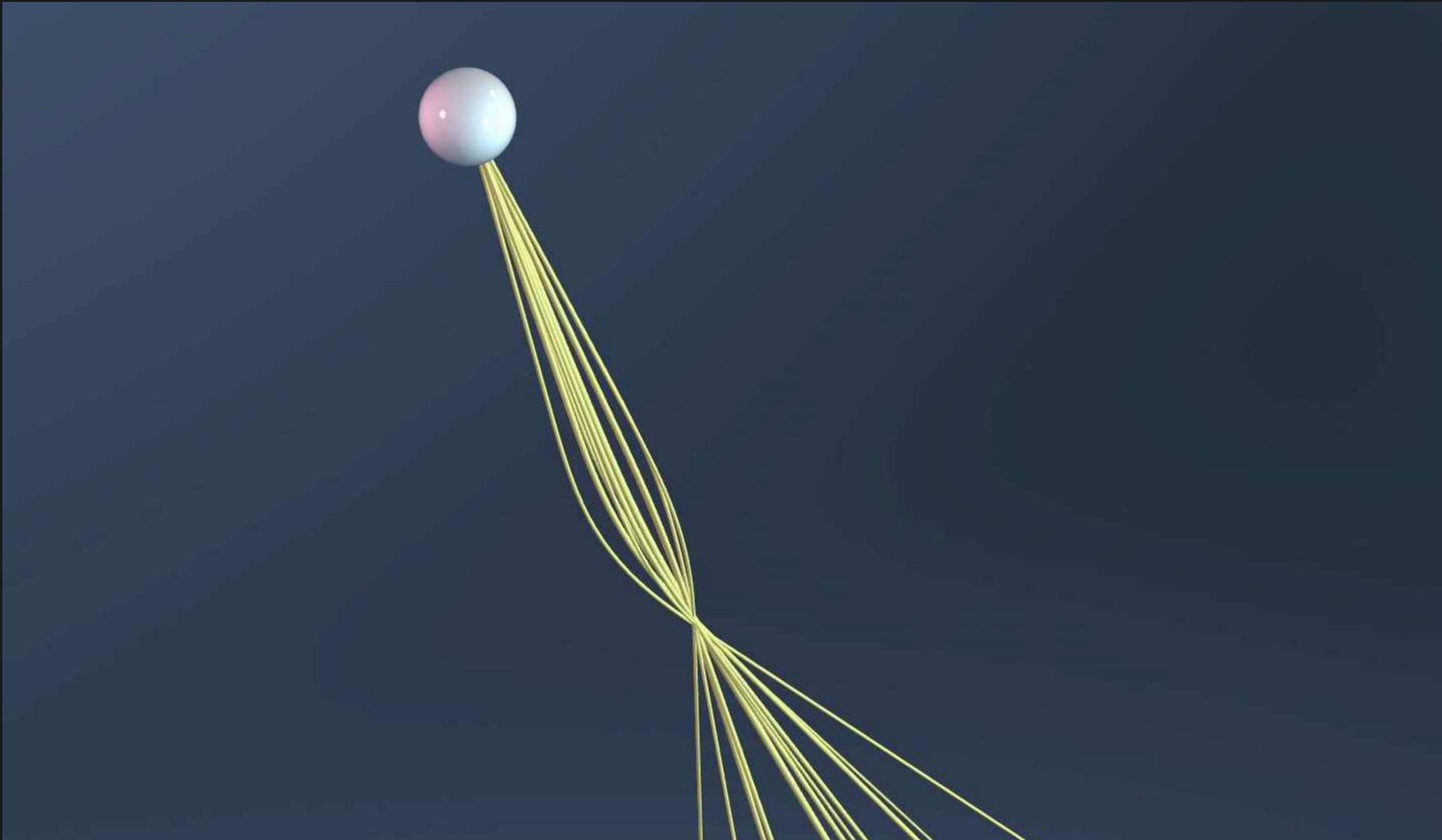
Black Hole
Binary
Cygnus XI

Einstein Rings



Einstein Rings

**Rings require
light to focus...
this should
produce upside
down images
too!**



Einstein Rings



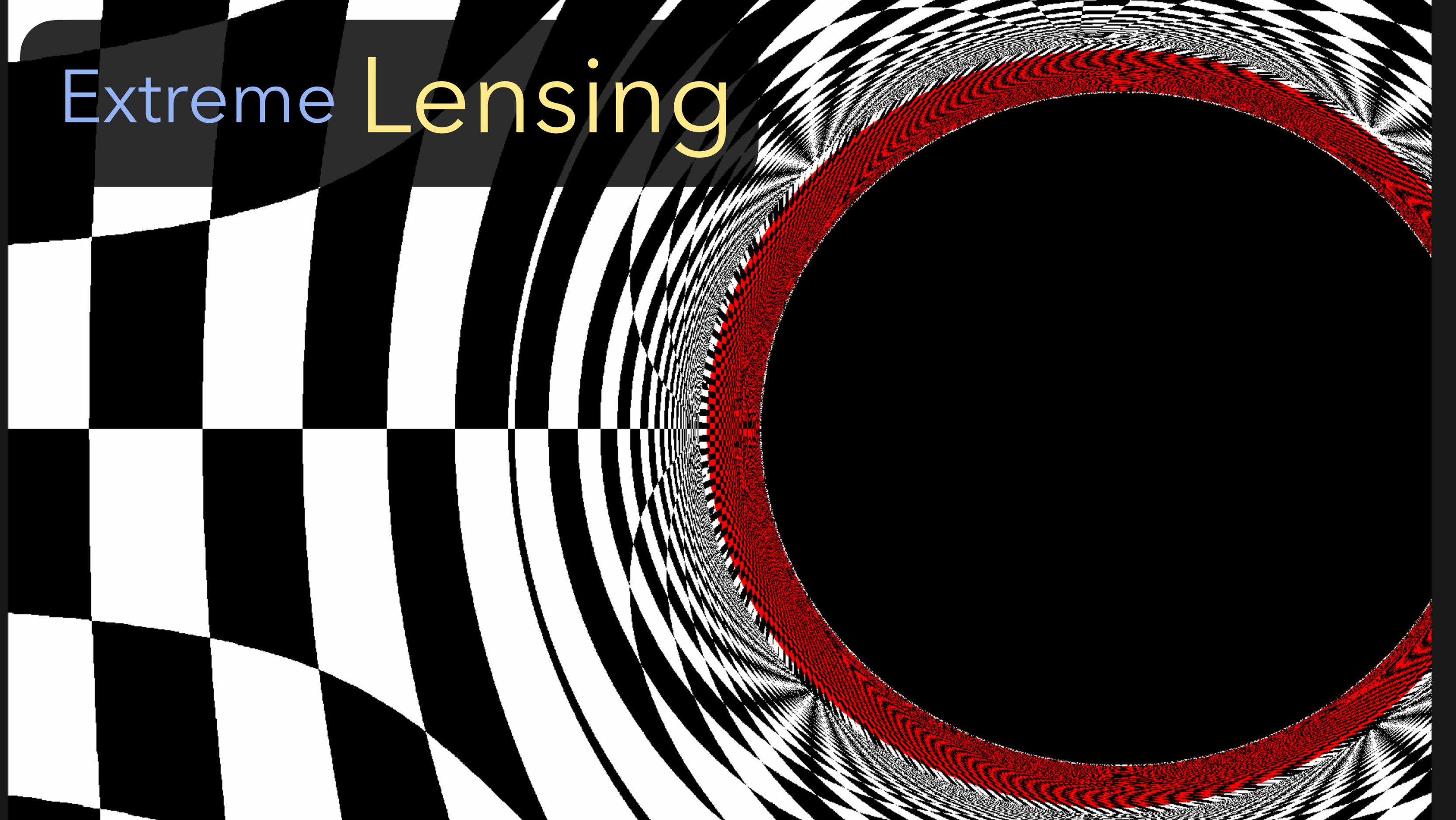
Extreme Lensing



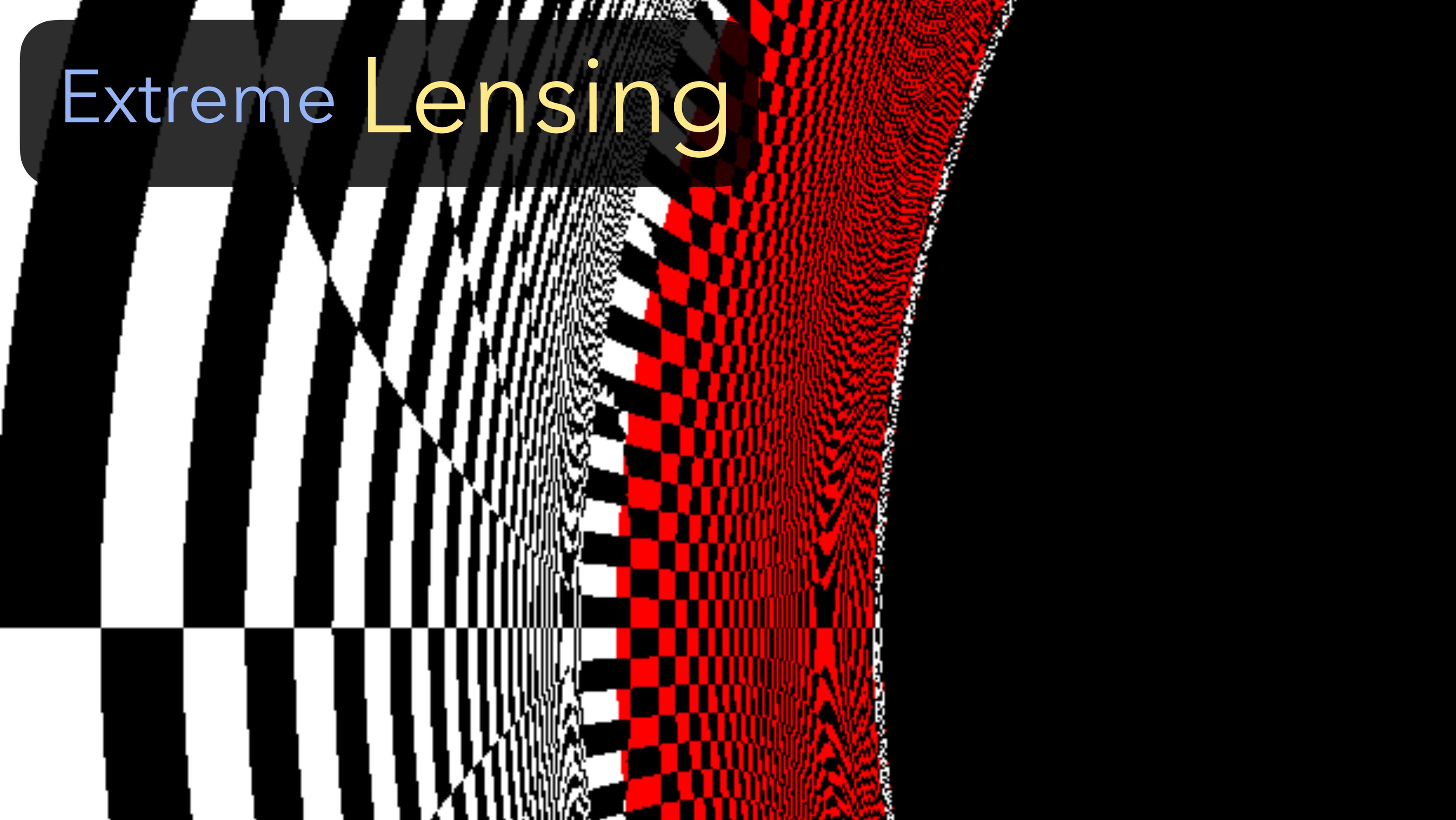
Extreme Lensing



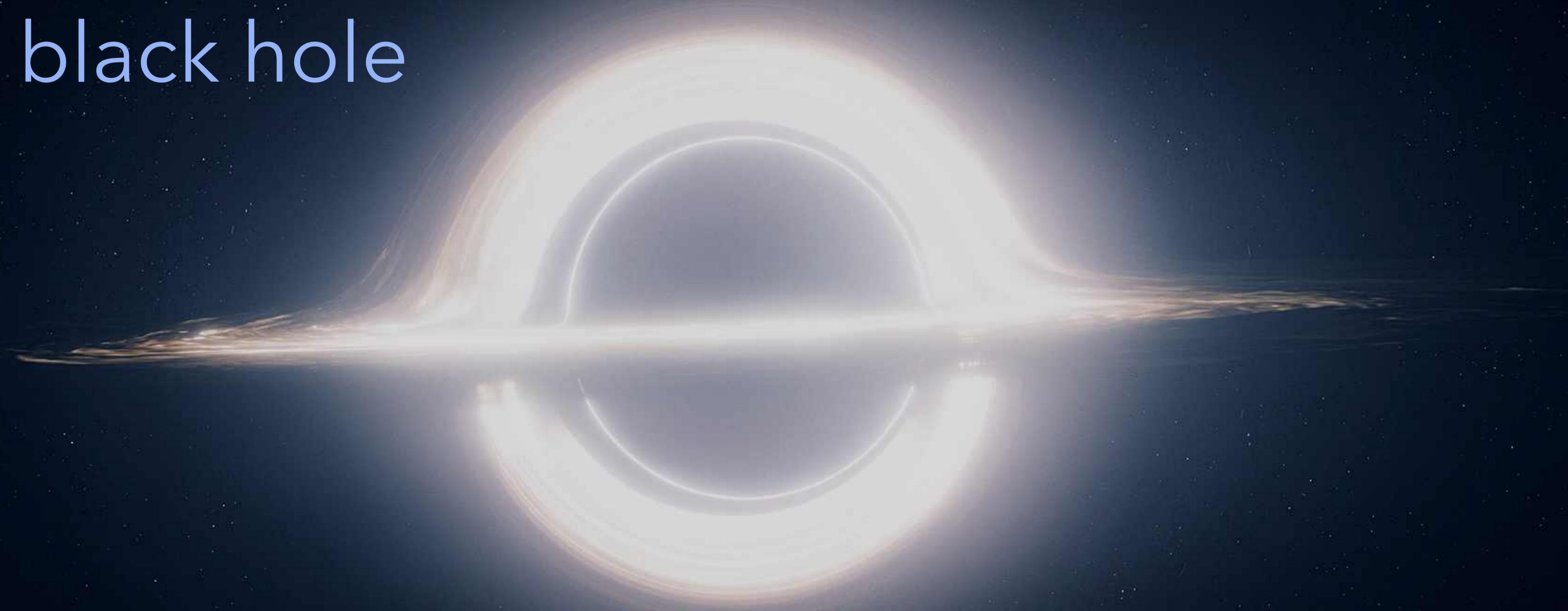
Extreme Lensing



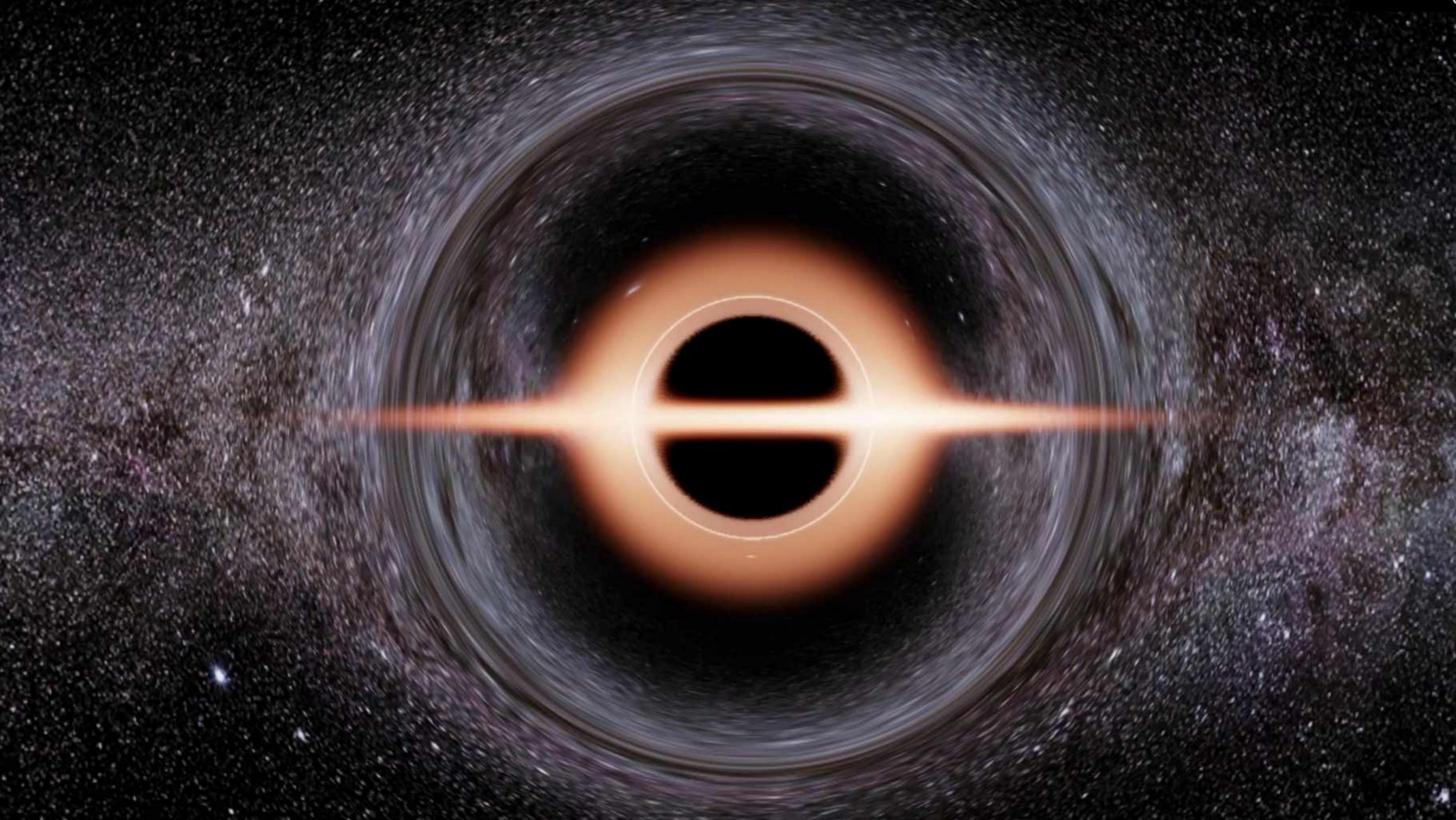
Extreme Lensing



The '*Interstellar*'
black hole







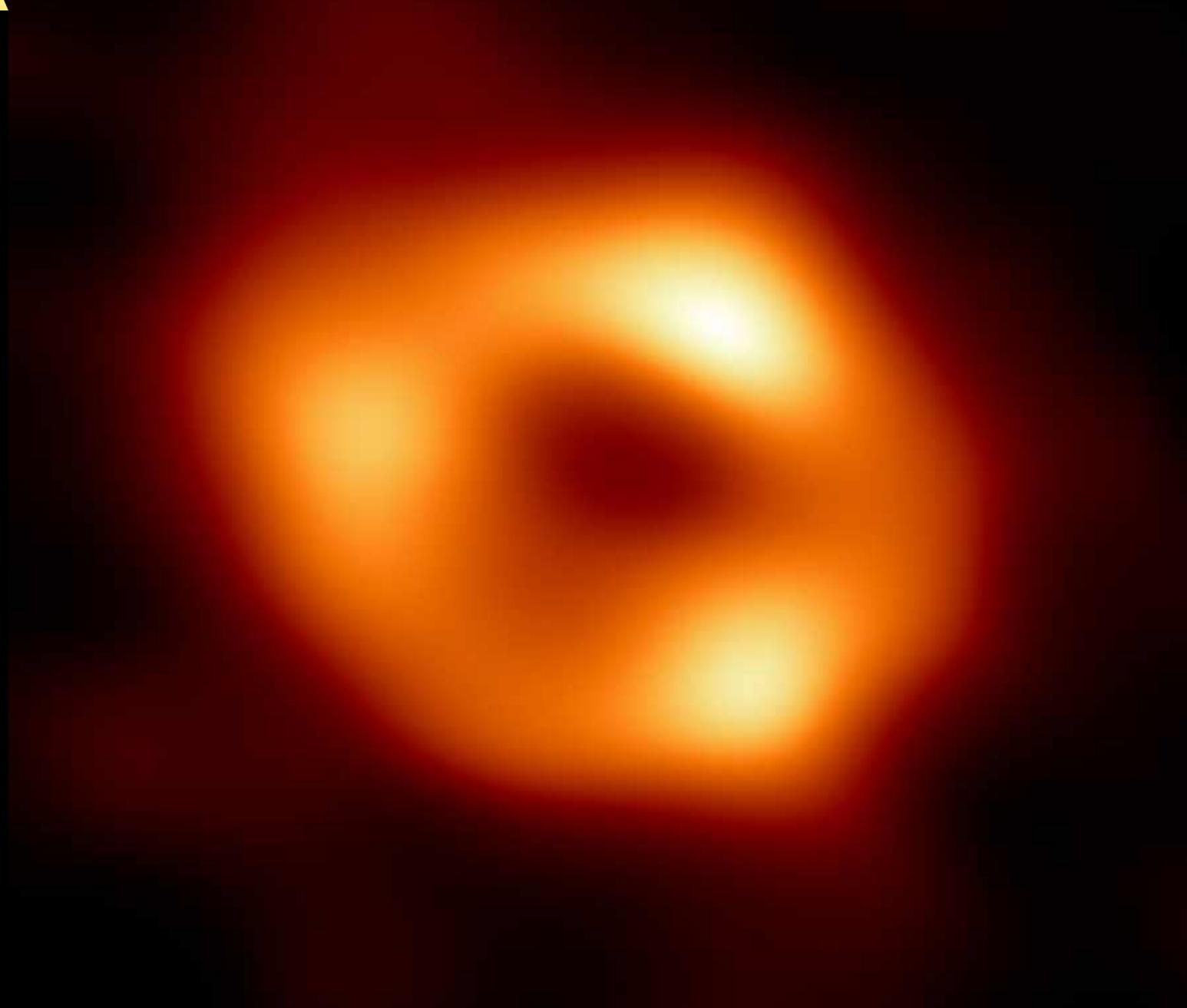
A real black hole

M-87



A real black hole

Sgr A*



Thanks!

