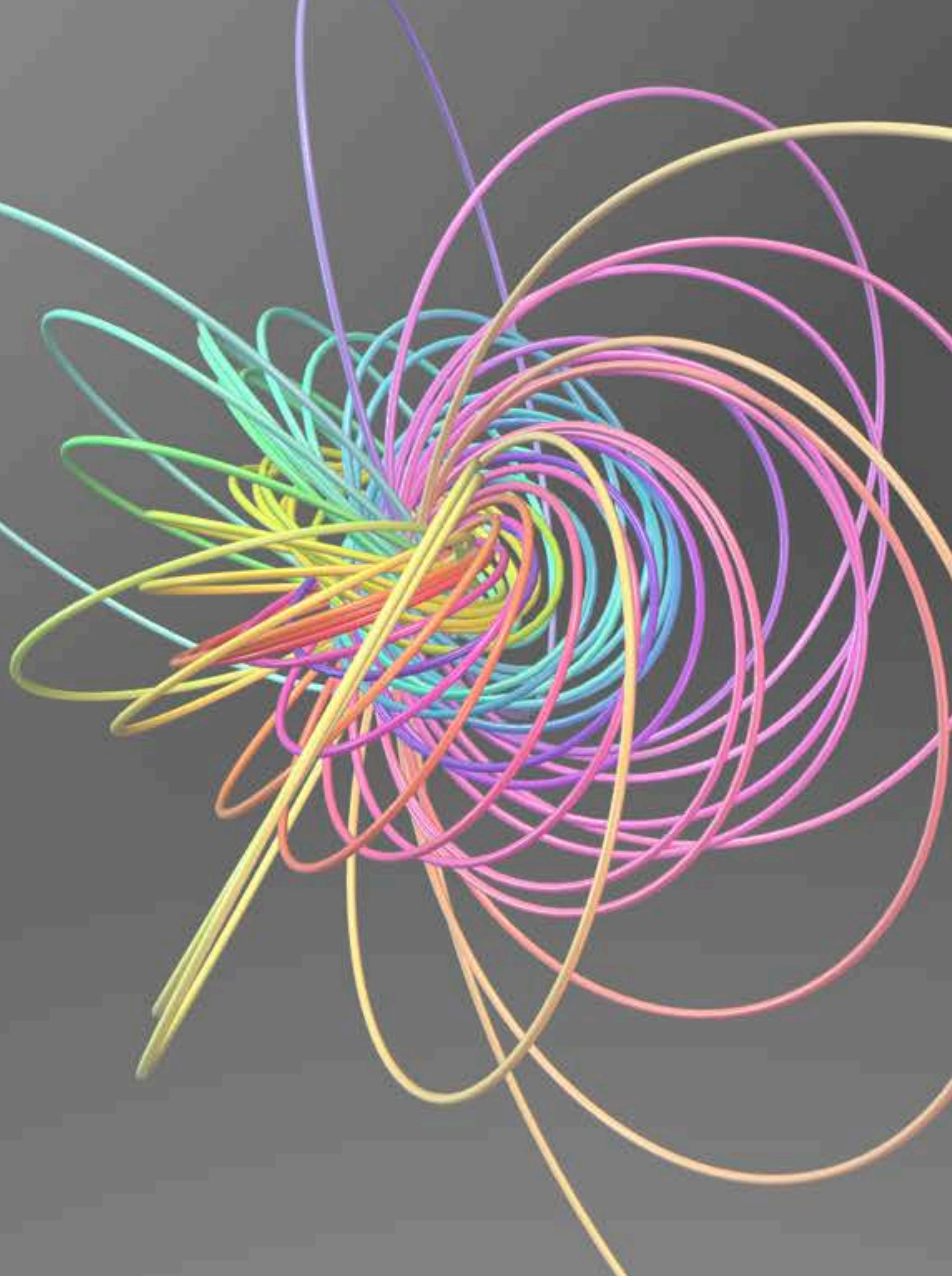
GROUP THEORY

AND

SU(2)



Recall:

$$\mathbb{C}^2 = \{(a, b) \mid a, b \in \mathbb{C}\}$$

Is two-dimensional complex space.

$$GL(2,\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{C} \right\}$$

Is the set of 2x2 complex matrices

A matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is **unitary** if

$$A^\star A = I$$

Where the operation * is conjugate transpose:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^\star = \begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix}$$

The Unitary Group is the set of all unitary matrices:

$$U(n) = \{A \mid A^\star A = I\}$$

The Special Unitary Group is the subgroup with determinant 1.

$$SU(n) = \{A \mid A^\star A = I,\ \det A = 1\}$$

The Unitary groups are analogs of the Orthogonal groups

$$SU(n) = \left\{ A \;\middle|\; \begin{array}{c} A^{\star}A = I \\ \det A = 1 \end{array} \right\}$$

$$SO(n) = \left\{ A \;\middle|\; \begin{array}{c} A^{T}A = I \\ \det A = 1 \end{array} \right\}$$

Our goal is to study the simplest example: the 2x2 special unitary matrices:

Last time: saw we can try to understand this group exactly the same way you did for SO(2) on the homework.

$$SU(2) = \left\{ \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} \middle| \, \alpha\bar{\alpha} + \beta\bar{\beta} = 1 \right\}$$

$$SU(2) = \left\{ \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} \middle| \, \alpha\bar{\alpha} + \beta\bar{\beta} = 1 \right\}$$

Special unitary matrices are uniquely specified by their first column.

$$\begin{aligned} \alpha\bar{\alpha} + \beta\bar{\beta} &= (x+iy)\overline{(x+iy)} + (z+iw)\overline{(z+iw)} \\ &= x^2 + y^2 + z^2 + w^2 \\ &= 1 \end{aligned}$$

And this column is actually a unit vector in $\mathbb{R}^4$!

Just like SO(2) is a circle, SU(2) is the three dimensional sphere in 4-space!

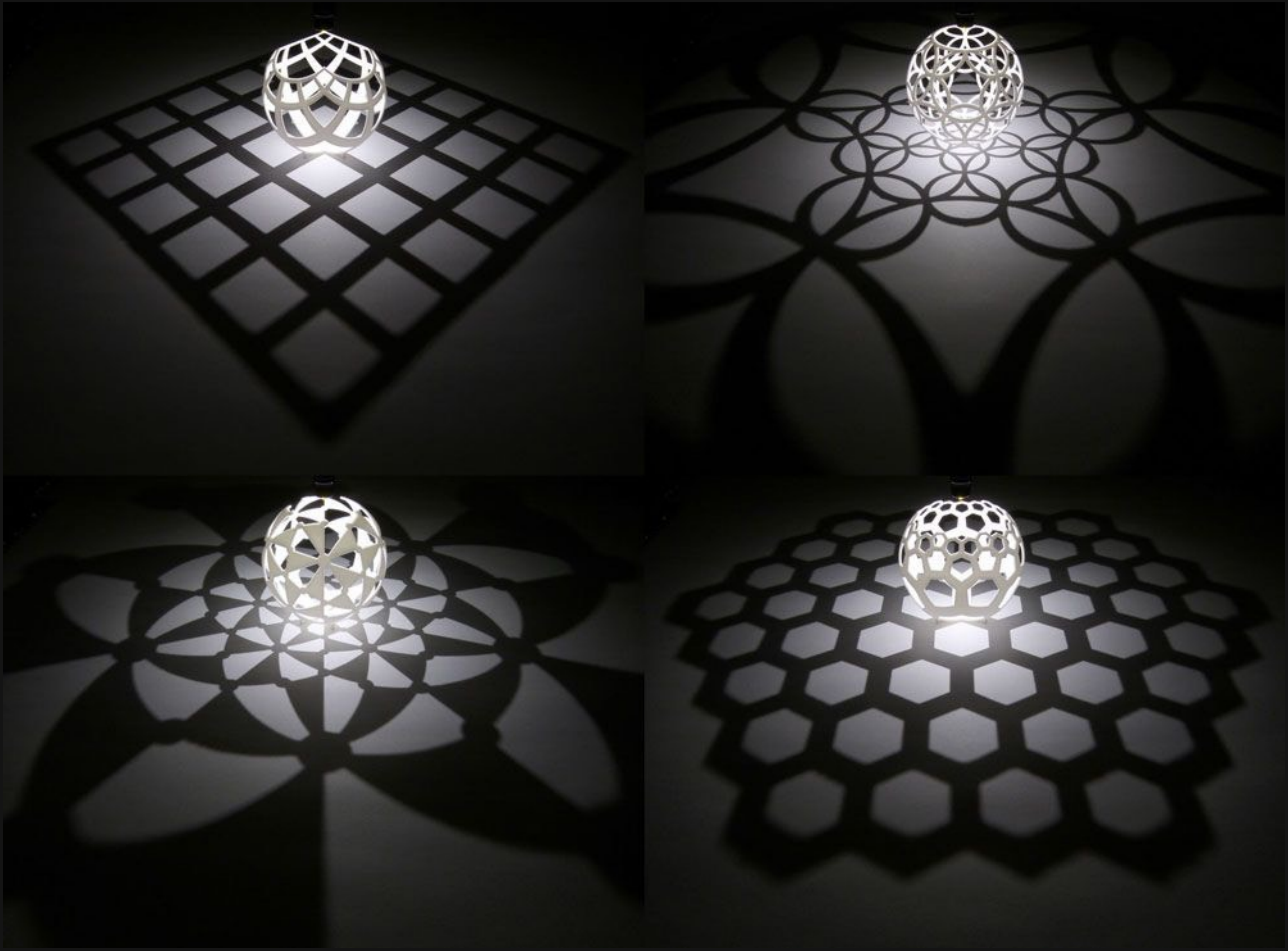
$$SU(2) = \left\{ \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} \middle| \ \alpha\bar{\alpha} + \beta\bar{\beta} = 1 \right\}$$

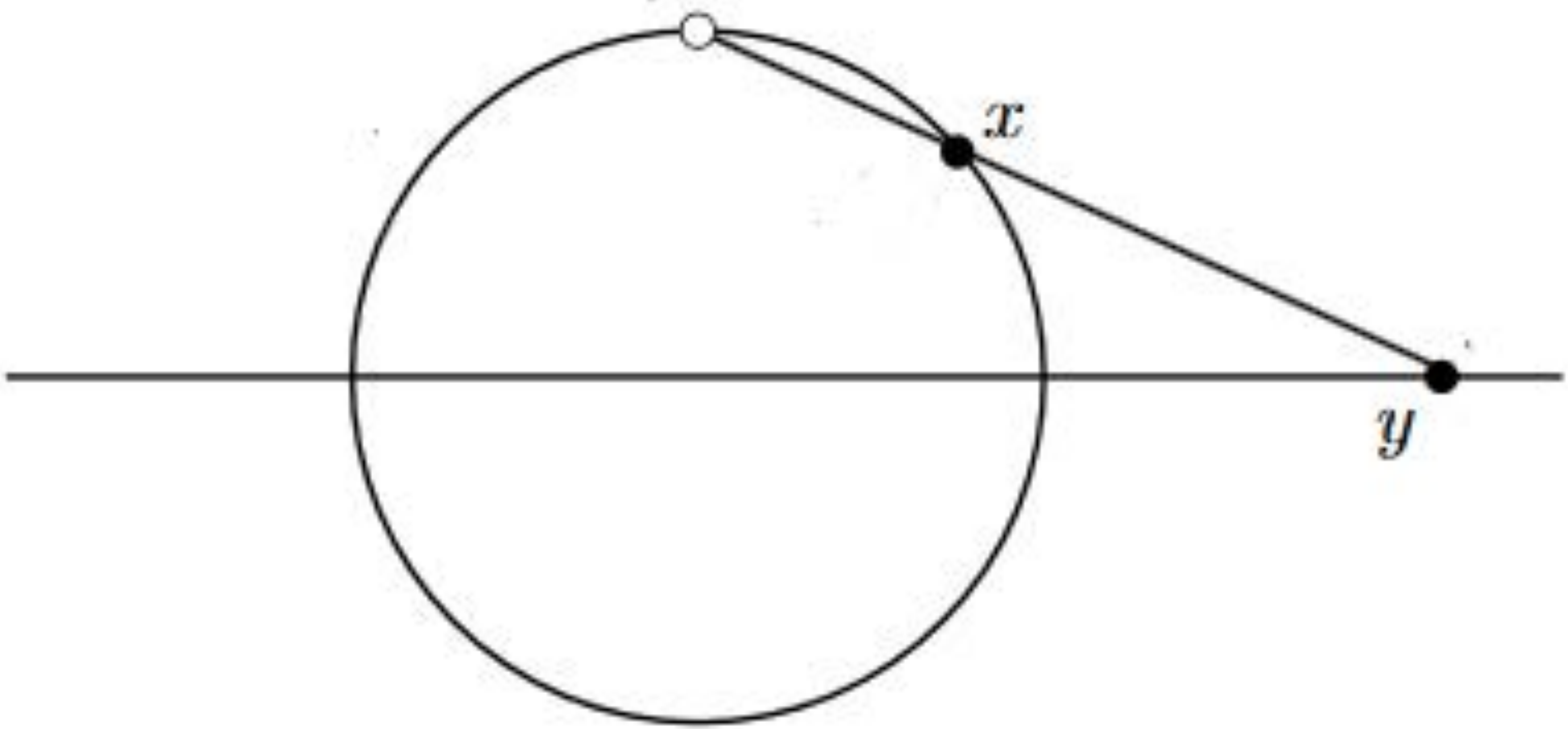
$$\cong$$

$$\mathbb{S}^3 = \{(x, y, z, w) \in \mathbb{R}^4 \mid x^2 + y^2 + z^2 + w^2 = 1\}$$

HOW DO WE THINK ABOUT

THE THREE DIMENSIONAL SPHERE?





x
y

https://www.youtube.com/watch?v=pWOMDm6ejlw

Thus, we can visualize SU(2) as being $\mathbb{R}^3$, with a couple rules:

There is one missing point: (1,0,0,0) is “at infinity”

Sizes are distorted: things far from the origin are MUCH smaller than they appear.

Half of the 3-sphere is inside the unit ball, the other half is everything else!

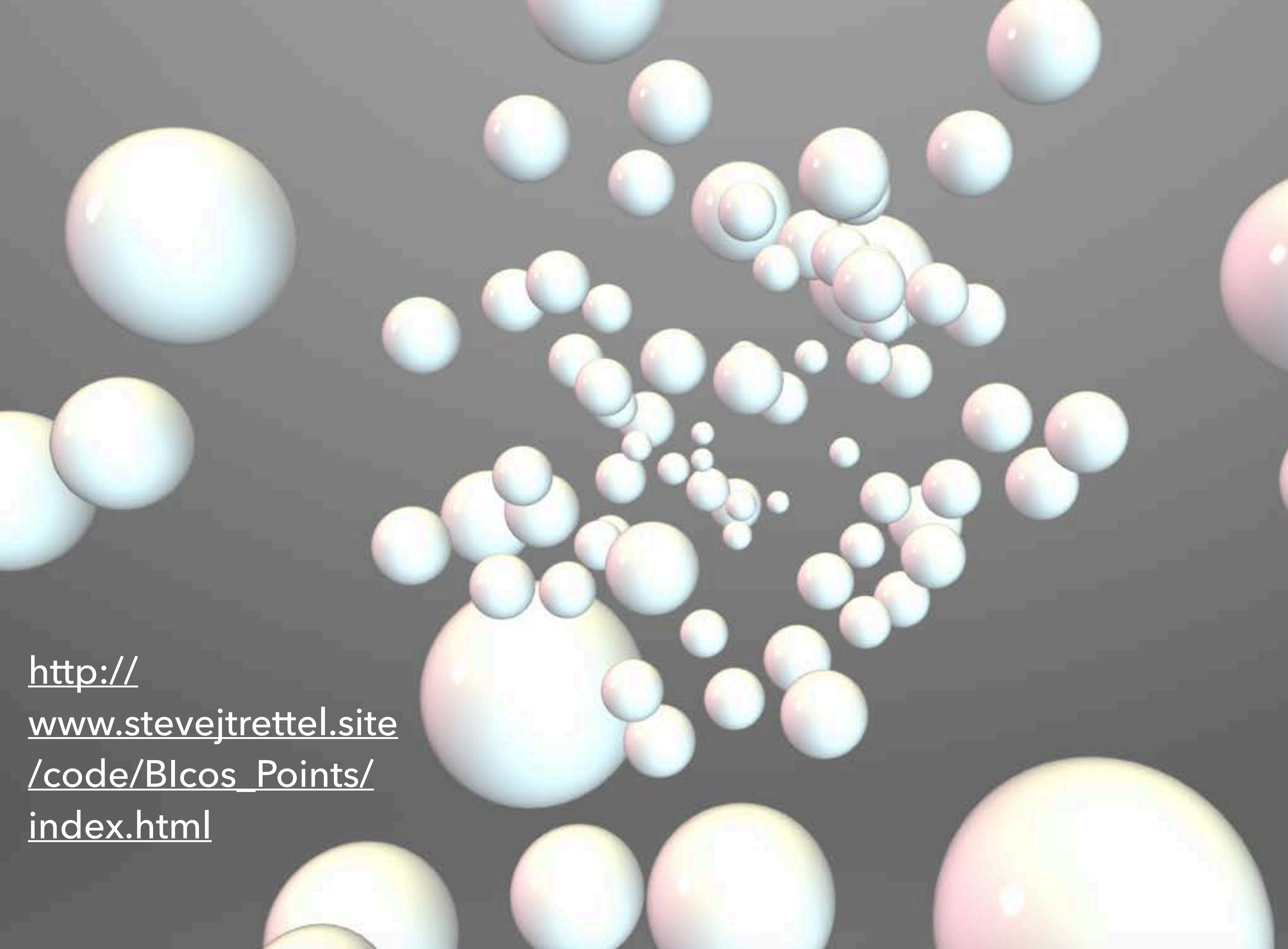
Example: Visualizing a finite subgroup of SU(2):

$$G = \left\langle s, t \mid (st)^2 = s^3 = t^5 = e \right\rangle$$

$$s = \frac{1}{2}\begin{pmatrix} 1+i & 1+i \\ -1+i & 1-i \end{pmatrix} \qquad t = \frac{1}{2}\begin{pmatrix} \phi + \frac{i}{\phi} & 1 \\ -1 & \phi - \frac{i}{\phi} \end{pmatrix}$$

Where ϕ is the golden ratio.

G has 120 elements.



http://www.stevejtrettel.site/code/BIcos_Points/index.html

SU(2), COSETS OF SO(2), AND

THE HOPF FIBRATION

Again recall from last time:

$$SU(2) = \left\{ A \;\middle|\; \begin{array}{l} A^{\star}A = I \\ \det A = 1 \end{array} \right\} \quad SO(2) = \left\{ A \;\middle|\; \begin{array}{l} A^{T}A = I \\ \det A = 1 \end{array} \right\}$$

SO(2) is a subgroup of SU(2), so SU(2) is a union of cosets of SO(2).

Interpret this geometrically: the 3-sphere is a disjoint union of circles!

This filling of the 3-sphere by 1-spheres (circles) is a fundamental result in topology.

Called the “Hopf Fibration” after Heinz Hopf.

Usual construction involves complex analysis, or quaternions, projective geometry….

But its existence is a direct consequence of a simple fact in group theory!

One more thing!

What is the set of cosets $SU(2)/SO(2)$?

Using the orbit stabilizer theorem: just need to find a space X that SU(2) acts on, with stabilizer SO(2).

Complex analysis: SU(2) acts on the Riemann sphere by Mobius transformations.

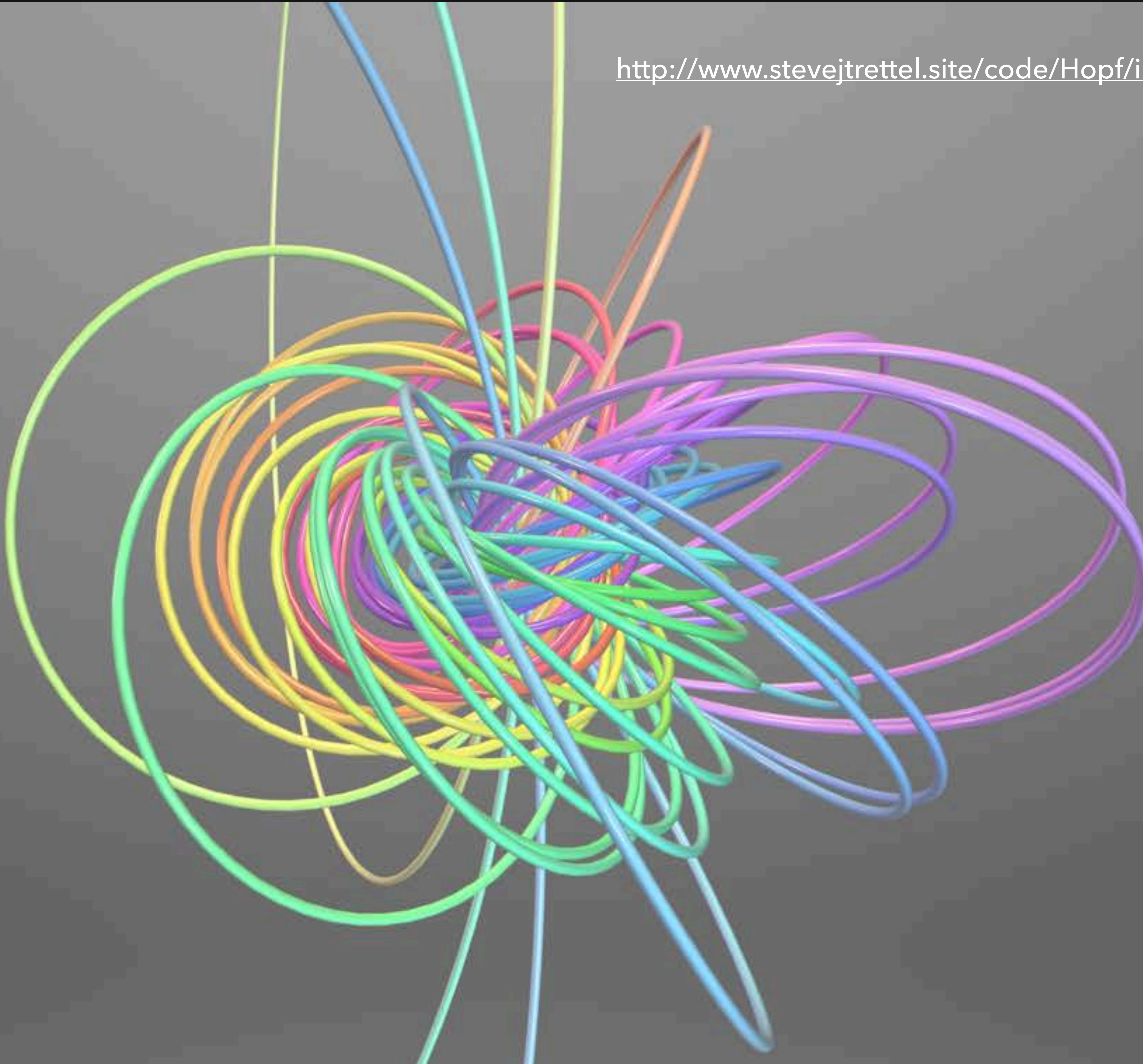
$$SU(2)/SO(2) \cong \mathbb{S}^2$$

The Hopf Fibration:

A way to fill the 3-sphere (that is, SU(2)) with 1-spheres (that is, SO(2)): one for each point in the 2-sphere.

$$\begin{array}{ccc} SO(2) & < & SU(2) \\ & & \downarrow \\ & & \mathbb{S}^2 \end{array}$$

http://www.stevejtrettel.site/code/Hopf/index.html

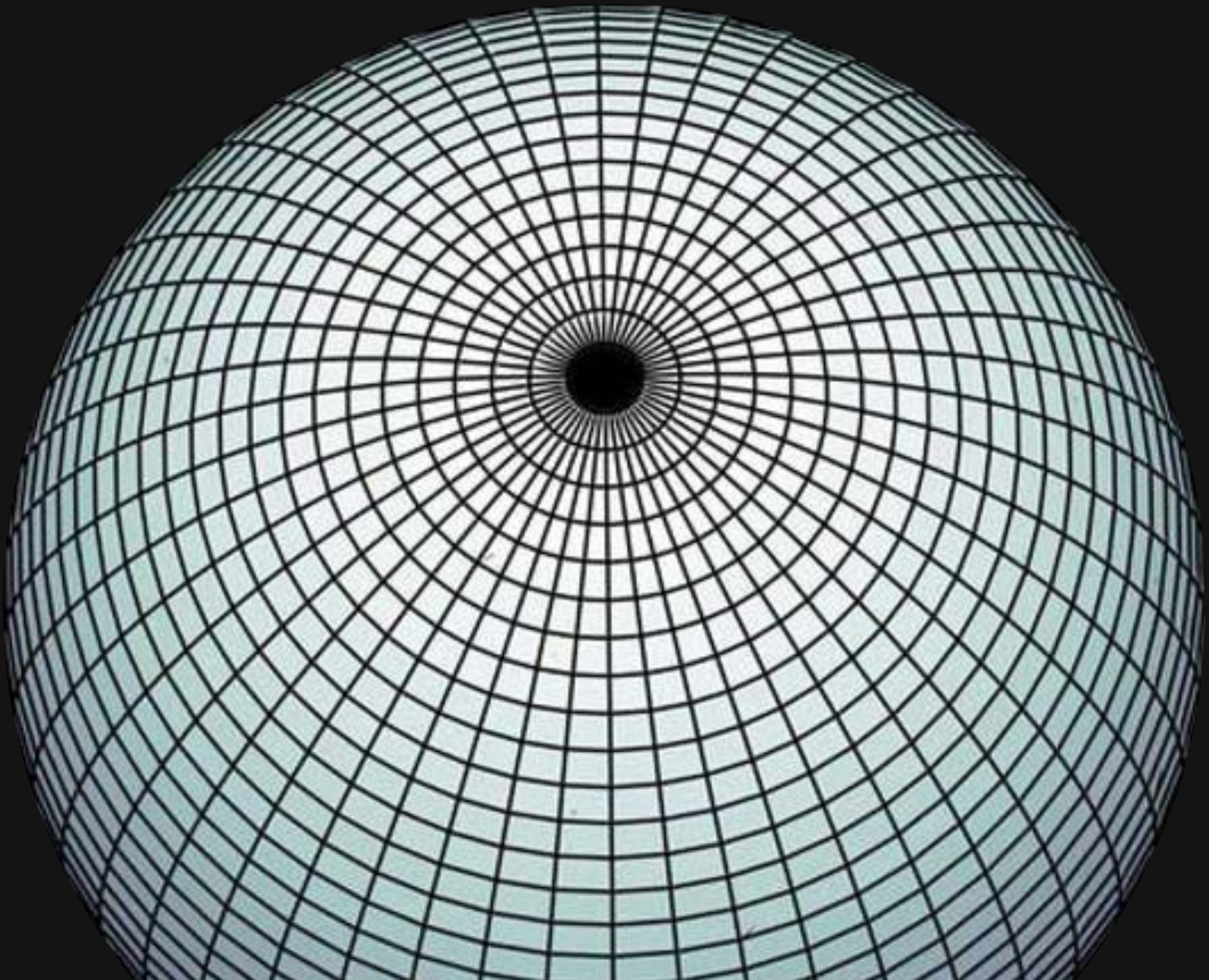


SIDE TANGENT!

MATHEMATICAL THOUGHT AS A CONNECTION TO THE PAST

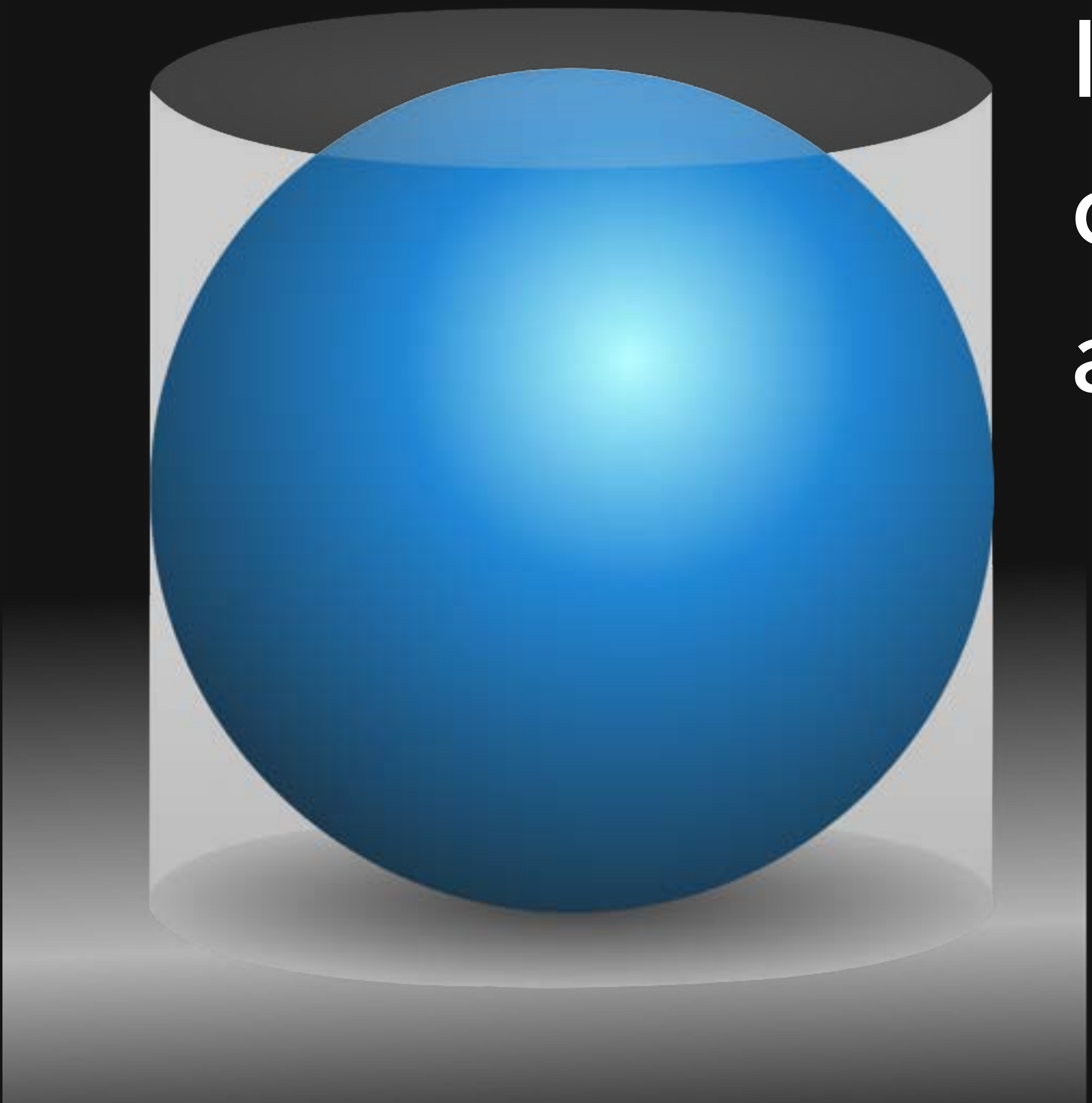
In order to be able to draw a nice uniform looking collection of cosets, one needs to be able to choose random points uniformly from the space of cosets, here SU(2)/SO(2)=the sphere.

How does one choose a point uniformly at random from the surface of a sphere?



Can't use random spherical coordinates!

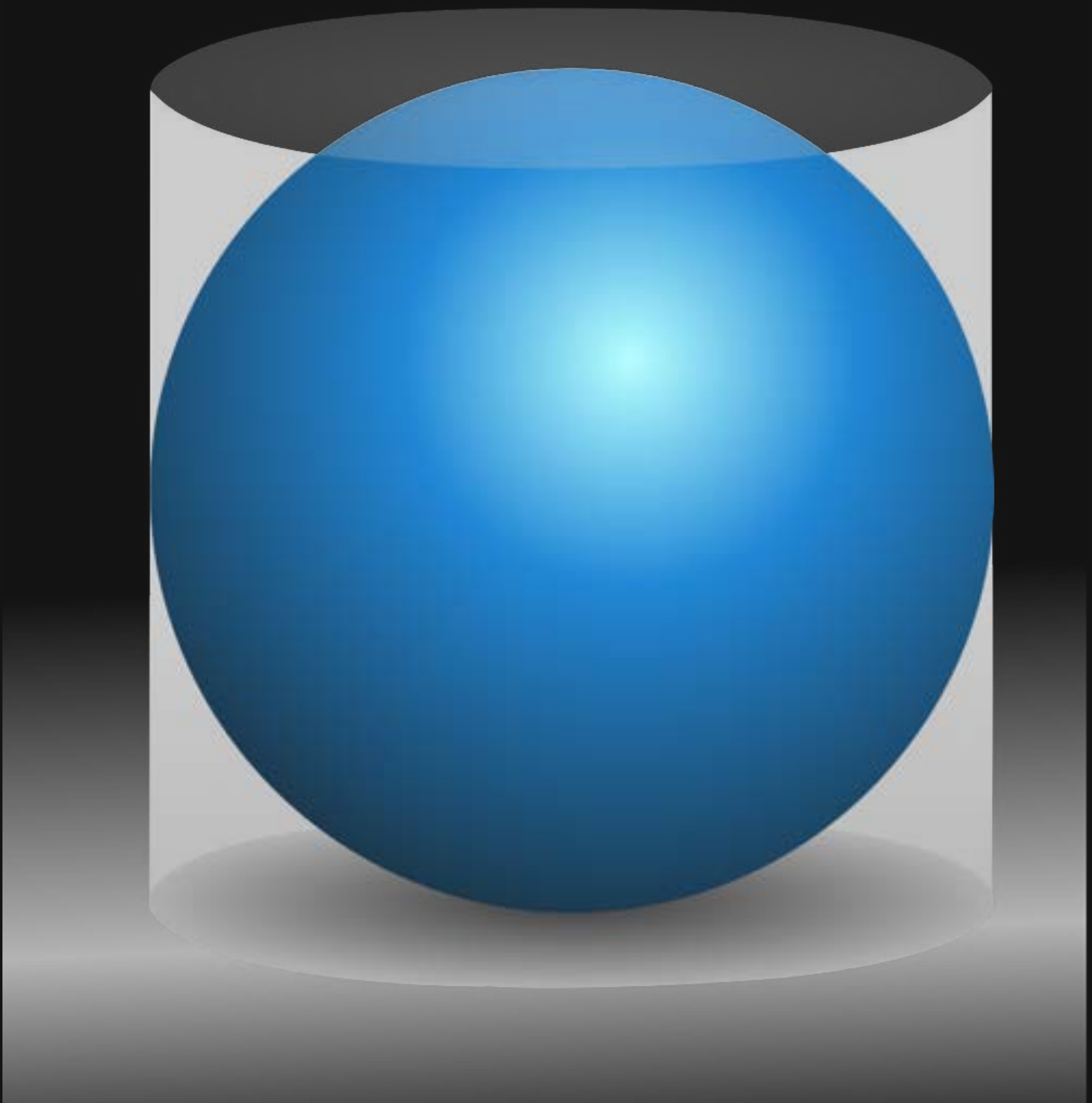
THE SPHERE AND THE CYLINDER



In 225BC, Archimedes calculated the surface area of a sphere

$$A = 4\pi r^2$$

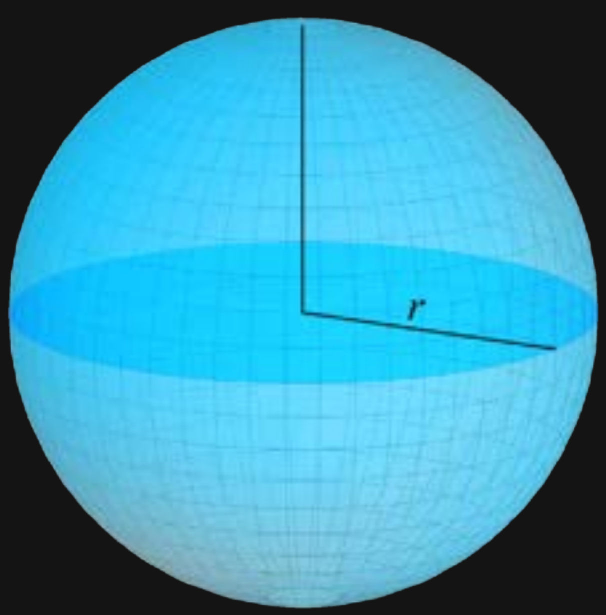
THE SPHERE AND THE CYLINDER



He did so by proving the horizontal projection outwards onto a cylinder preserves area.

THE SPHERE AND THE CYLINDER

Thus:



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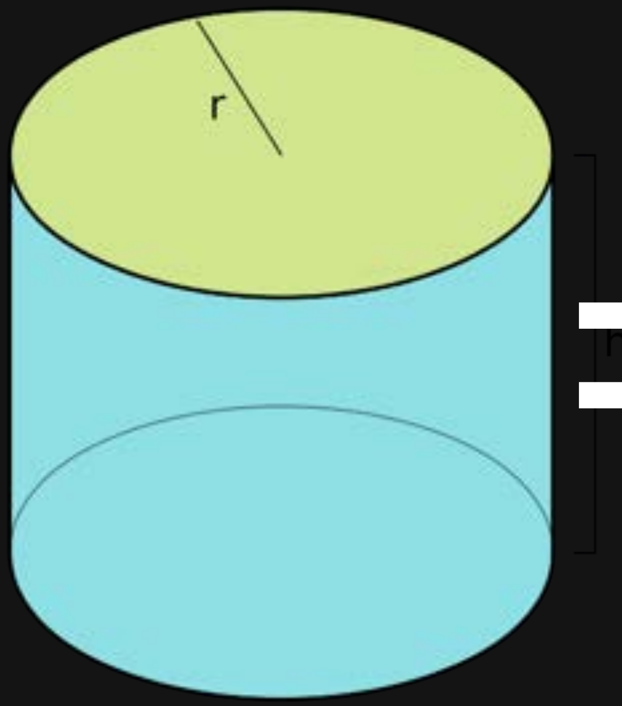

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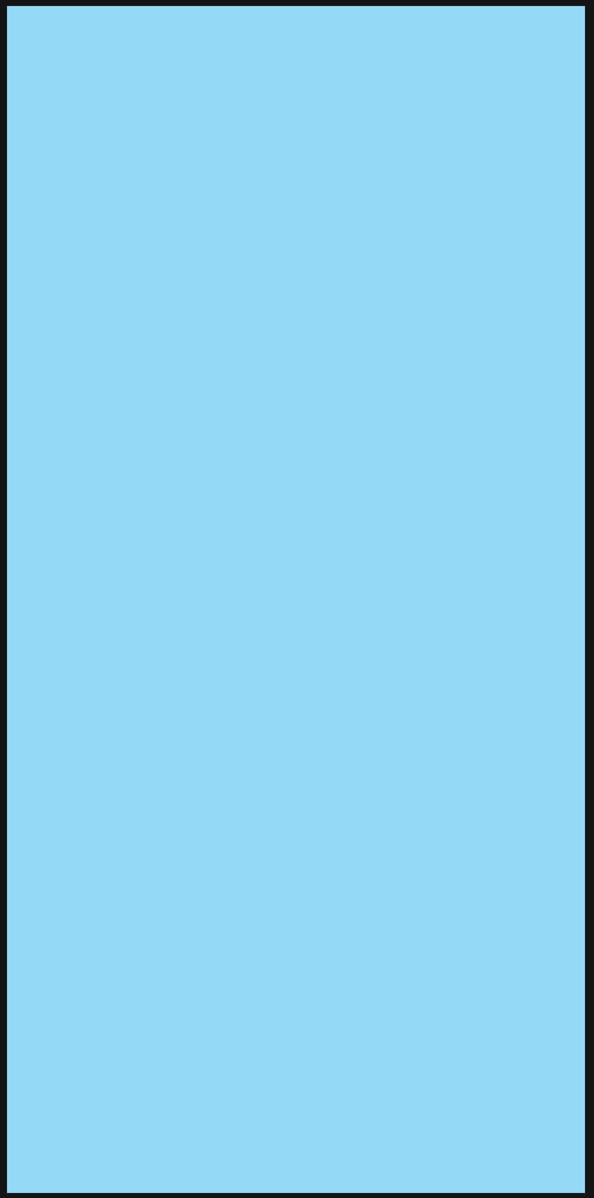
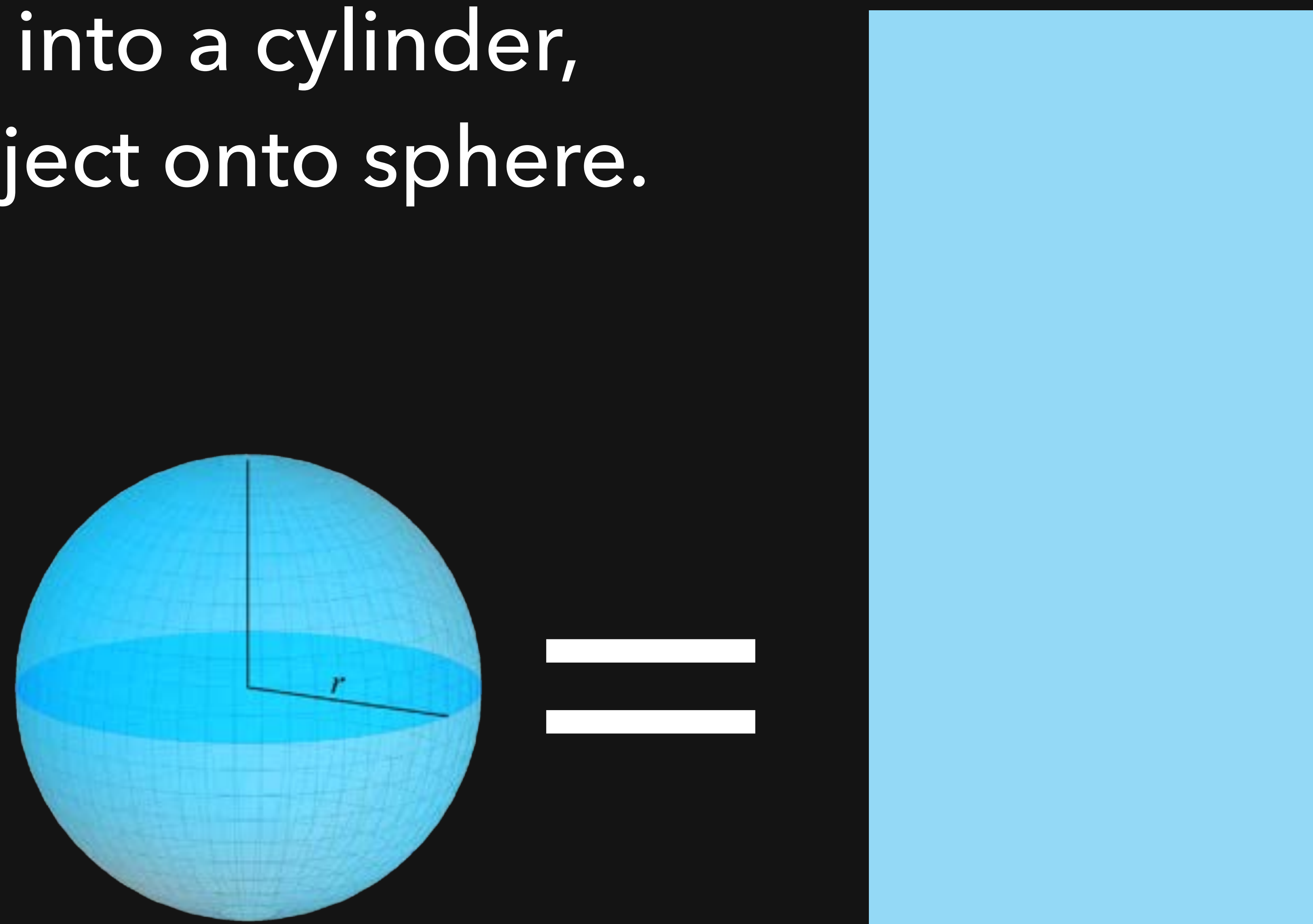


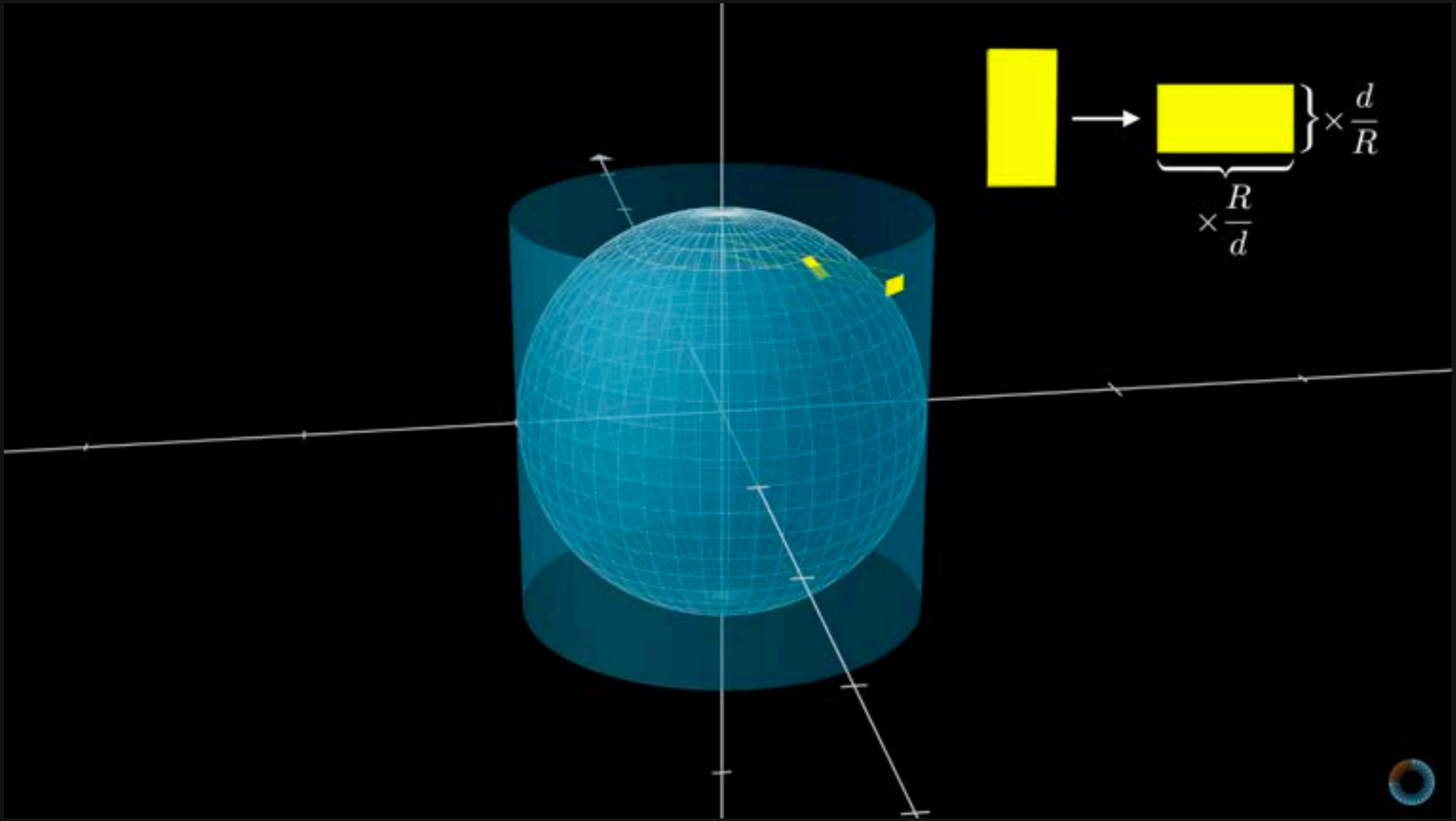
The random sampling in the animation:

Directly use Archimedes' idea: uniformly sample from a rectangle, roll into a cylinder, horizontally project onto sphere.



THE SPHERE AND THE CYLINDER

3 Blue 1 Brown
Video on the proof



https://youtu.be/GNcFjFmqEc8?t=530

OK……

BACK TO SU(2)

Remember the finite group we took a look at?

$$G = \langle s, t \mid (st)^2 = s^3 = t^5 = e \rangle$$

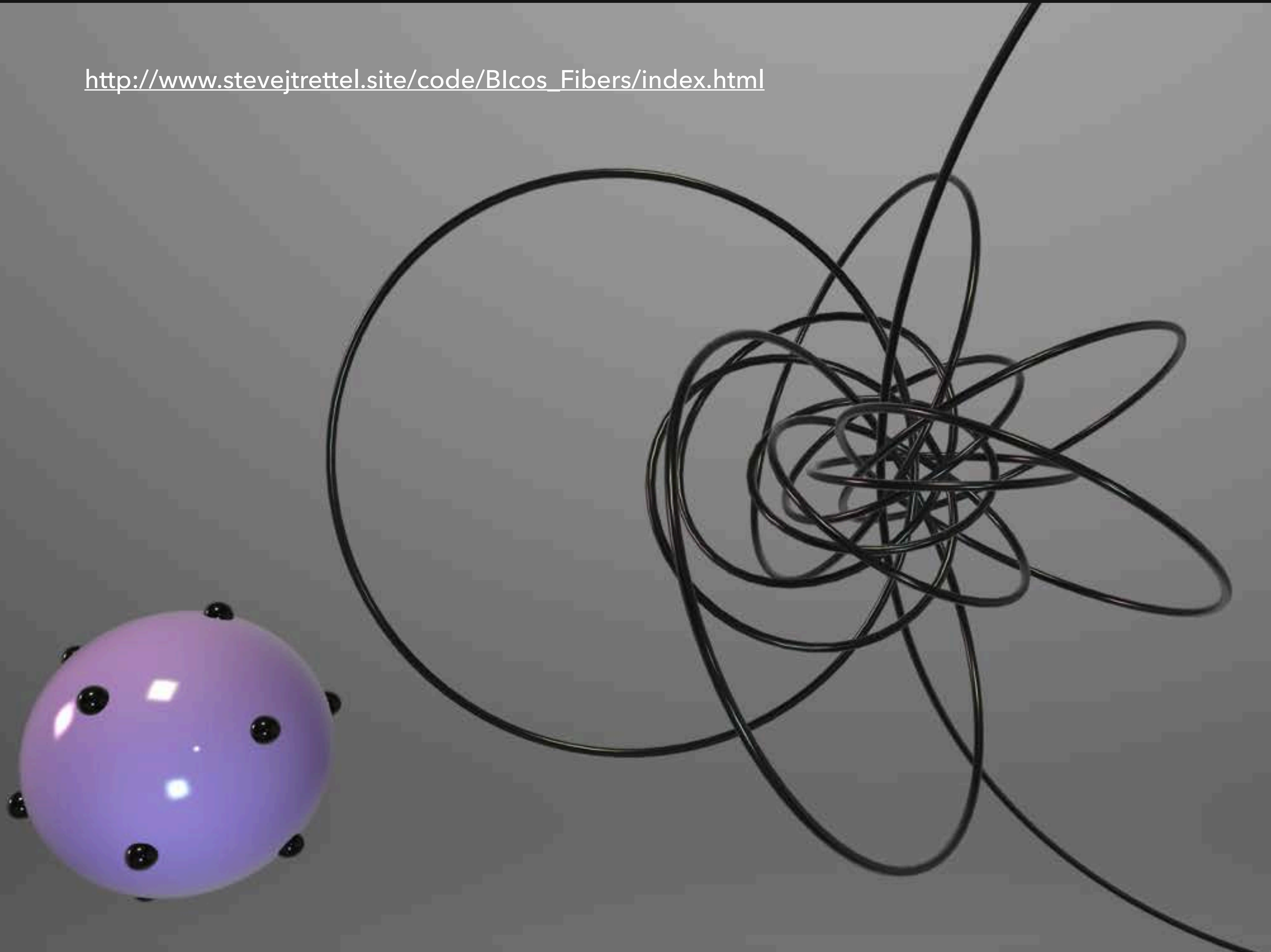
Remember the finite group we took a look at?

$$G = \langle s, t \mid (st)^2 = s^3 = t^5 = e \rangle$$

The points look like a jumbled mess - 120 is too many for our brains to keep track of visually!

But there is a relationship of this group to symmetries of the 2-sphere!

http://www.stevejtrettel.site/code/BIcos_Fibers/index.html



THE SURPRISING CONNECTION OF

SU(2), SO(3) AND SO(4)

SU(2) acts on itself by multiplication.

(Not surprising, remember all groups do this!)

But SU(2) is the three-sphere!

Thus in fact, SU(2) acts on R4 by rotations.

That is, SU(2) is isomorphic to a subgroup of SO(4)!

If we had a day, we are not far from an incredible theorem:

There are two actions of SU(2) on the 3-sphere:

Multiply on the left

Multiply on the right.

These commute with each other: which gives a homomorphism

$$SU(2) \times SU(2) \to SO(4)$$

If we had a day, we are not far from an incredible theorem:

$$SU(2) \times SU(2) \to SO(4)$$

What is the kernel?

If we had a day, we are not far from an incredible theorem:

$$SU(2) \times SU(2) \to SO(4)$$

What is the kernel?

Since $(-I)(-I) = I$, we have that $-I$ is in the kernel. But this is it!

If we had a day, we are not far from an incredible theorem:

$$SU(2) \times SU(2) \to SO(4)$$

What is the kernel?

$$\ker = \{\pm I\}$$

If we had a day, we are not far from an incredible theorem:

The first Isomorphism Theorem!

$$\frac{SU(2) \times SU(2)}{\{\pm I\}} \cong SO(4)$$

We can understand rotations of 4 dimensional space as pairs of special unitary transformations!

But SU(2) is ALSO closely related to rotations in 3-dimensions

We said, via complex analysis, SU(2) acts on the Riemann sphere.

We can use this action of SU(2) on a sphere, to make an action of it on all of R3!

This action on R3 is by rotations!

$$SU(2) \to SO(3)$$

New situation, same question:

$$SU(2) \to SO(3)$$

What is the kernel?

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$$SU(2) \to SO(3)$$

What is the kernel?

$$\ker = \{\pm I\}$$

The first Isomorphism Theorem!

$$\frac{SU(2)}{\{\pm I\}} \cong SO(3)$$

$$\frac{SU(2) \times SU(2)}{\{\pm I\}} \cong SO(4)$$

Understanding SU(2) lets you understand both 3 and 4 dimensional rotations.

$$\frac{SU(2)}{\{\pm I\}} \cong SO(3)$$

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Used in math, theoretical physics

Understanding SU(2) lets you understand both 3 and 4 dimensional rotations.

$$\frac{SU(2)}{\{\pm I\}} \cong SO(3)$$

Used in math, computer programming

APPLICATIONS PART I:

SU(2) AND VIDEO GAMES

‘Quaternions’ in graphics programming are elements of SU(2)!

Just like complex numbers are pairs of real numbers

Quaternions are pairs of complex numbers

$$q = z + jw \qquad z, w \in \mathbb{C}$$

‘Quaternions’ in graphics programming are elements of SU(2)!

Quaternions are pairs of complex numbers

$$q = z + wj \qquad z, w \in \mathbb{C}$$

$$z = a + bi \quad w = c + di$$

‘Quaternions’ in graphics programming are elements of SU(2)!

Quaternions are pairs of complex numbers

$$q = z + wj$$
$$= (a + bi) + (c + di)j$$
$$= a + bi + cj + dij$$

‘Quaternions’ in graphics programming are elements of SU(2)!

Quaternions are pairs of complex numbers

$$q = z + wj$$

$$= (a + bi) + (c + di)j$$

$$= a + bi + cj + dk$$

‘Quaternions’ in graphics programming are elements of SU(2)!

But remember: SU(2) is pairs of complex numbers!

$$SU(2) = \left\{ \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} \middle| \, \alpha\bar{\alpha} + \beta\bar{\beta} = 1 \right\}$$

‘Quaternions’ in graphics programming are elements of SU(2)!

The quaternion $\alpha + \beta j$

Can be identified with the matrix $\begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} \in SU(2)$

‘Quaternions’ in graphics programming are elements of SU(2)!

.setFromAxisAngle (axis : Vector3, angle : Float) : Quaternion

Sets this quaternion from rotation specified by axis and angle.
Adapted from the method here.
Axis is assumed to be normalized, **angle** is in radians.

.setFromEuler (euler : Euler) : Quaternion

Sets this quaternion from the rotation specified by Euler angle.

.setFromRotationMatrix (m : Matrix4) : Quaternion

m - a Matrix4 of which the upper 3×3 of matrix is a pure rotation matrix (i.e. unscaled).
Sets this quaternion from rotation component of m.
Adapted from the method here.

This command just means take a matrix in SO(3) and return an element of SU(2) which projects to it under the quotient!

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Sets this quaternion from rotation component of m.
Adapted from the method here.

And this one says to take an element of SU(2) and return the element of SO(3) from the quotient

.setFromQuaternion (q : Quaternion, order : String) : Euler

q - a normalized quaternion.

order - (optional) a string representing the order that the rotations are applied.

Sets the angles of this euler transform from a normalized quaternion based on the orientation specified by order.

GROUP THEORY AND SYMMETRY IN

MODERN PHYSICS

Emmy Noether and the relationship of symmetry to differential equations.

Revolutionized modern algebra:
Group Theory
Ring Theory

Emmy Noether and the relationship of symmetry to differential equations.

Found the crucial link between group theory and physics

Emmy Noether and the relationship of symmetry to differential equations.

Every continuous symmetry of a physical system

Corresponds to a conservation law.

Emmy Noether and the relationship of symmetry to differential equations.

Every matrix group action on a system

Corresponds to a conservation law.

Emmy Noether and the relationship of symmetry to differential equations.

Physics should be invariant under the Euclidean group!

What does Emmy Noether's insight tell us?

The Euclidean Group

$$\begin{pmatrix} A & v \\ 0 & 1 \end{pmatrix}$$

$$A \in SO(3)$$

$$v \in \mathbb{R}^3$$

Two natural subgroups: rotations and translations. These symmetries imply conservation of momentum!

The Euclidean Group

$$\begin{pmatrix} I & v \\ 0 & 1 \end{pmatrix}$$

Translations in Space

Conservation of Linear Momentum

$$p = mv$$

The Euclidean Group

$$\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$$

Rotations of Space

Conservation of Angular Momentum

$$p = m(r \times v)$$

This story continues.....

Invariance in time

Conservation of Energy

This story continues.....

Einstein combined the space and time symmetries into the Poincare group

Invariance in spacetime

Conservation of Energy-Momentum

This story continues…..

Einstein combined the space and time symmetries into the Poincare group

Invariance in spacetime

Conservation of Energy-Momentum

$$E^2 - p^2 = m^2c^4$$

This story continues.....

Einstein combined the space and time symmetries into the Poincare group

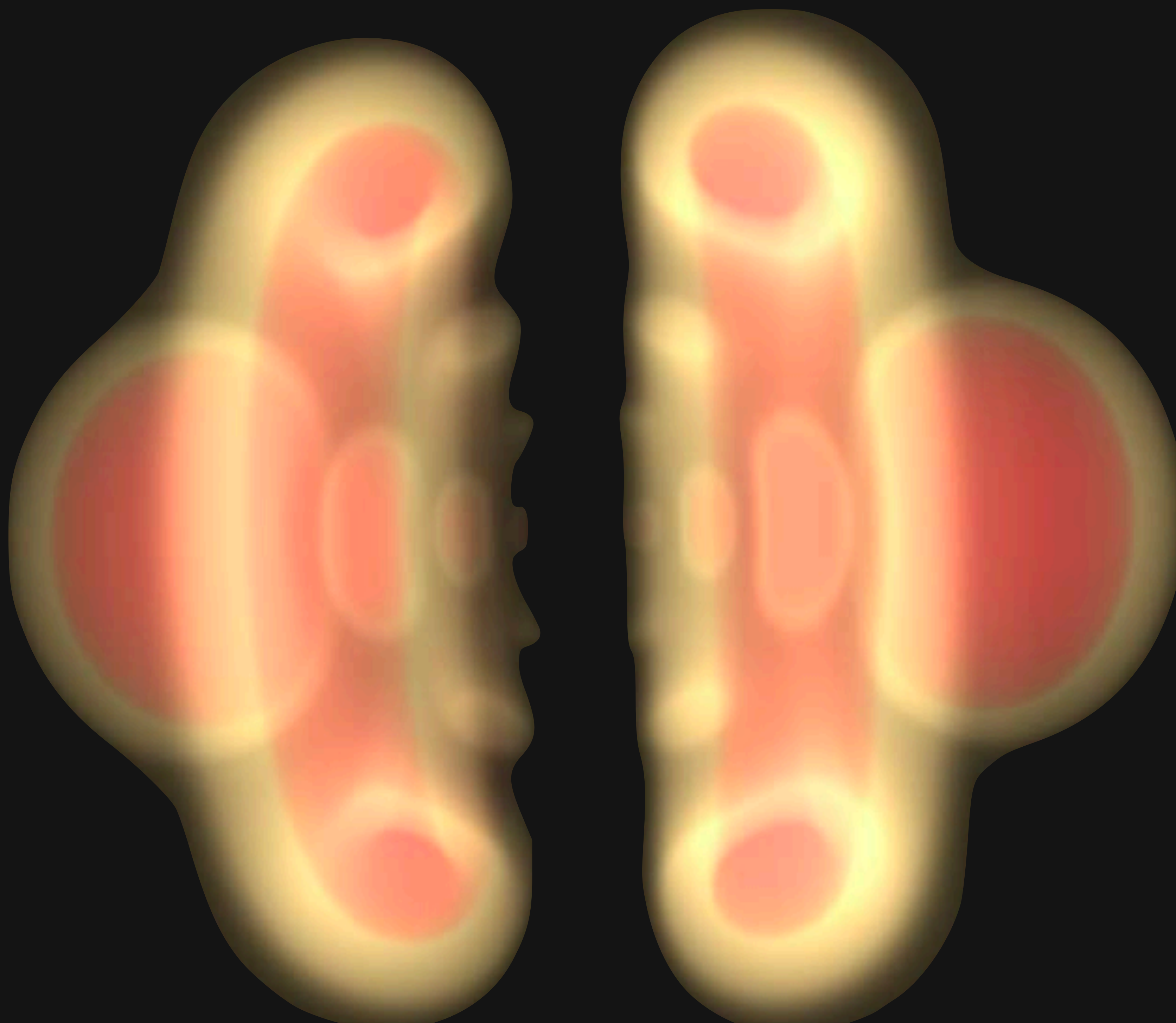
Invariance in spacetime

Conservation of Energy-Momentum

$$E = mc^2$$

SO(3), SU(2) AND QUANTUM MECHANICS

In quantum mechanics, particles are modeled by wave functions.



The Euclidean Group

$$\begin{pmatrix} A & v \\ 0 & 1 \end{pmatrix}$$

$$A \in SO(3)$$
$$v \in \mathbb{R}^3$$

Symmetries of the Euclidean group should correspond to properties of these wave functions.

Something magical happens for rotations

The symmetry of rotations actually extends…from SO(3) to SU(2)

$$SO(3) \curvearrowright \mathscr{H}$$

Something magical happens for rotations

The symmetry of rotations actually extends…from SO(3) to SU(2)

$$SU(2) \curvearrowright \mathcal{H}$$

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Something magical happens for rotations

The symmetry of rotations actually extends…from SO(3) to SU(2)

$$SU(2) \curvearrowright \mathscr{H}$$

This SU(2) conserves a quantity called ‘spin’ (since it comes from rotations)

Something magical happens for rotations

The symmetry of rotations actually extends…from SO(3) to SU(2)

$$SU(2) \curvearrowright \mathcal{H}$$

Because it is SU(2) and not SO(3), there are **twice as many symmetries**

Something magical happens for rotations

The symmetry of rotations actually extends…from SO(3) to SU(2)

$$SU(2) \curvearrowright \mathscr{H}$$

Because it is SU(2) and not SO(3), there are twice as many symmetries

This is what allows there to be two types of particles: fermions and bosons!

Thus, you can thank SU(2) that

Lasers Exist, and

That you don’t fall through the floor!

SO(3), SO(4) AND PLANETARY ORBITS

A *central force* is a force field on $\mathbb{R}^3$ which only depends on the distance from the origin:

$$F(r, \theta, \phi) = f(r)$$

A *central force* is a force field on $\mathbb{R}^3$ which only depends on the distance from the origin:

$$F = \frac{GMm}{r^2}\hat{r}$$

A *central force* is a force field on $\mathbb{R}^3$ which only depends on the distance from the origin:

$$F = \frac{q_1 q_2}{4\pi\epsilon_0 r^2}\hat{r}$$

A *central force* is a force field on $\mathbb{R}^3$ which only depends on the distance from the origin:

$$F = \frac{q_1 q_2}{4\pi\epsilon_0 r^2}\hat{r}$$

We can phrase this purely using group theory: a *central force* is a vector field which is invariant under the action of SO(3)

$$F = \frac{1}{r^n}\hat{r}$$

We can phrase this purely using group theory: a *central force* is a vector field which is invariant under the action of SO(3)

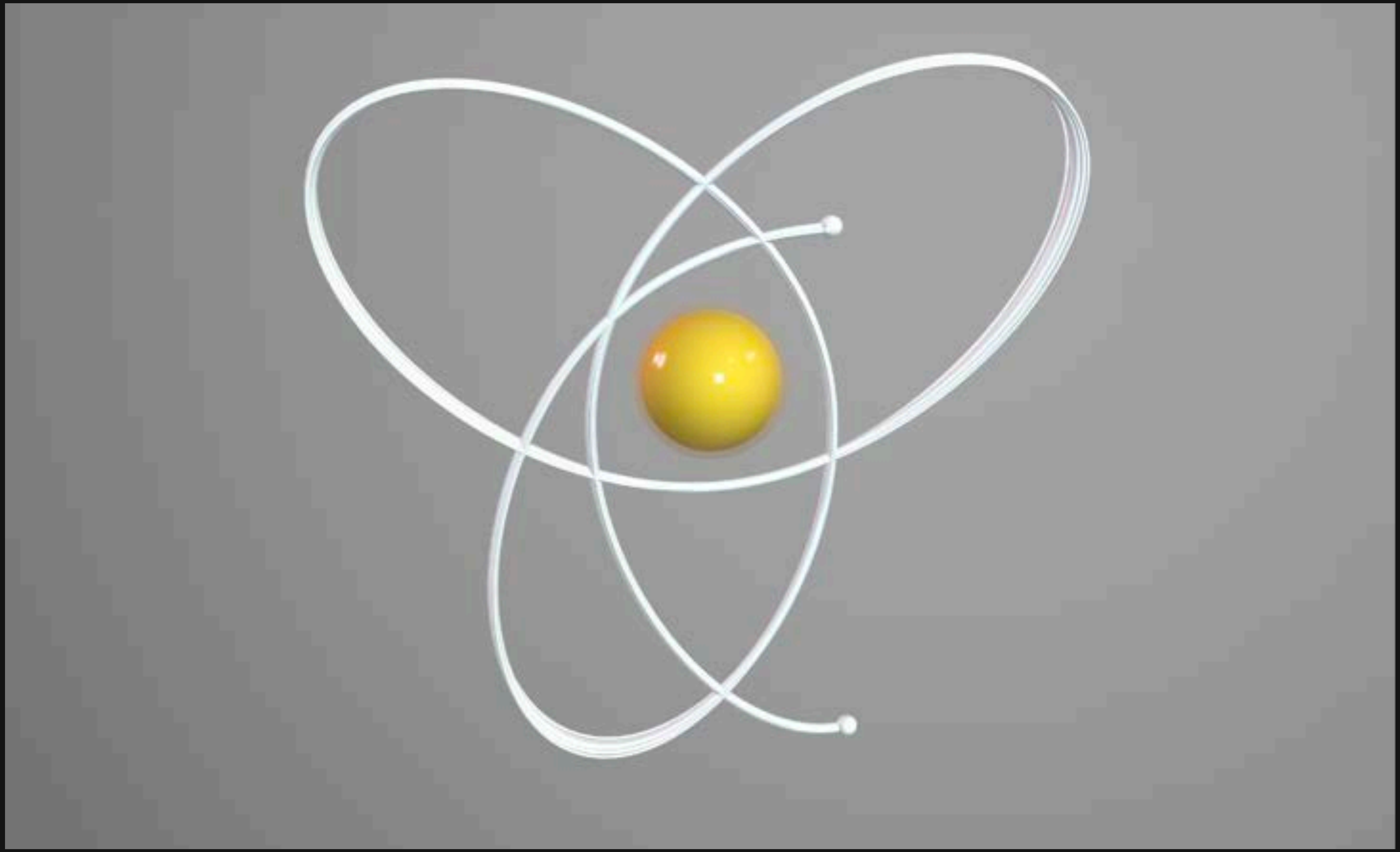
$$F = \frac{1}{r^n}\hat{r}$$

This symmetry gives a conserved quantity (angular momentum) makes the equations solvable.

http://www.stevejtrettel.site/code/PlanetOrbits_CentralForce/index.html

The magic of the inverse square law:

If we calculate the orbits for every central force 1/r^n…we see some interesting behavior



The magic of the inverse square law:

For $n<2$, the orbits are not periodic but precess. For $n>2$ the orbits are unstable.

For $n=2$, there are perfect periodic stable solutions (ellipses).

The magic of the inverse square law:

For $n<2$, the orbits are not periodic but precess. For $n>2$ the orbits are unstable.

For $n=2$, there are perfect periodic stable solutions (ellipses).

Why?

The magic of the inverse square law:

We know the symmetry group of every central force law *contains SO(3)*.

But is this all the symmetries? Or are there potentially *hidden symmetries that we did not notice?*

The magic of the inverse square law:

Theorem:

For all $n \neq 2$ the symmetry group of the force law $1/r^n$ is SO(3).

If $n = 2$ the symmetry group is SO(4).