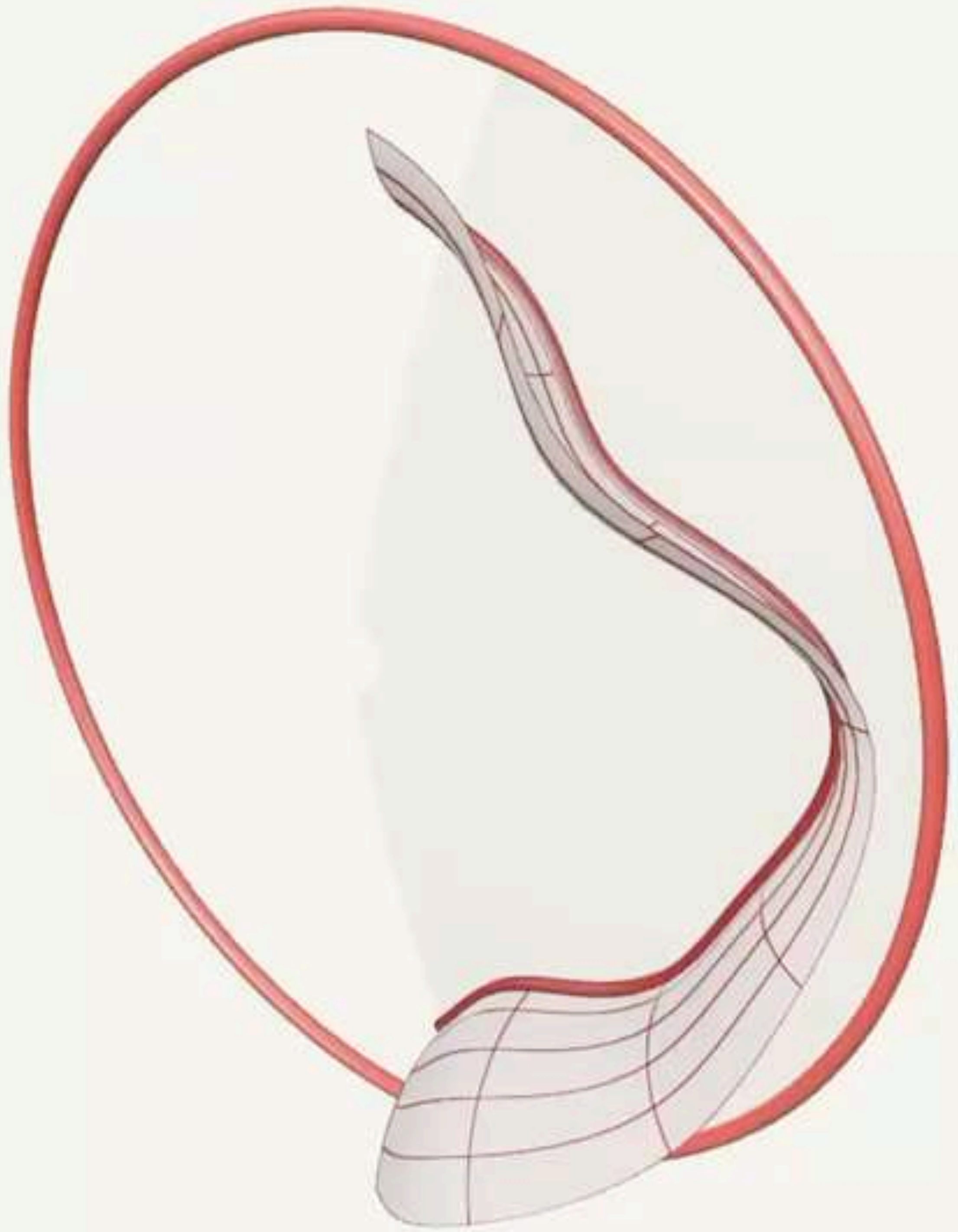
Surfaces Of Revolution In Homogeneous 3 Manifolds



Steve Trettel

University of San Francisco

A story of playful curiosity leading to surprising patterns (and too many inequalities)

Joint work with

Fabian Lander

Surfaces with Symmetry arise everywhere


I was raytracing a black hole spacetime...

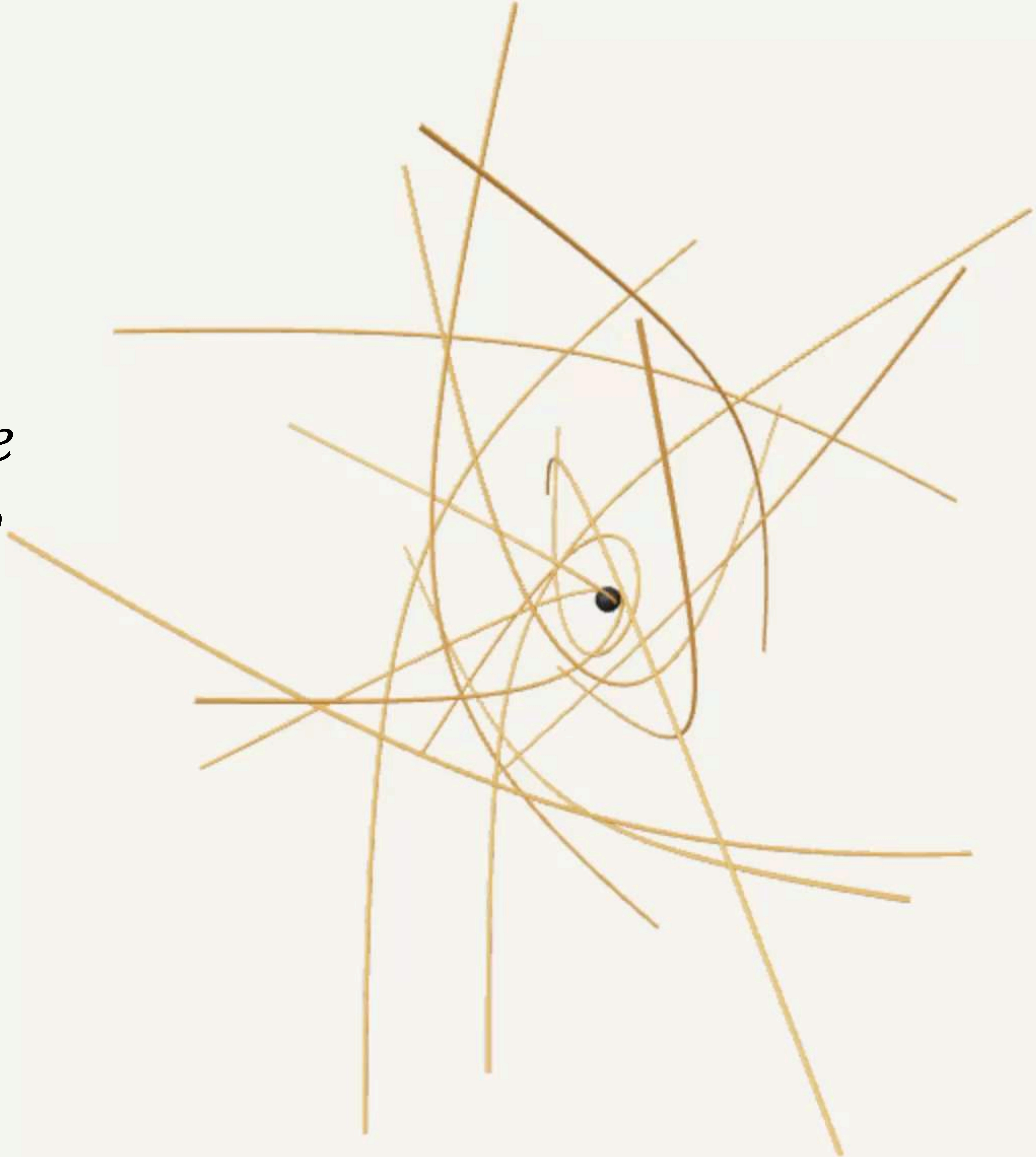
Technique: *fire out light rays along a light cone, see what they hit.*

The metric:

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 g_{\mathbb{S}^2}$$

Some nice properties of Null geodesics let you 'project off the time direction'

This leads us to studying a collection of curves in 3D, which happen themselves to be the geodesics of a Riemannian metric!

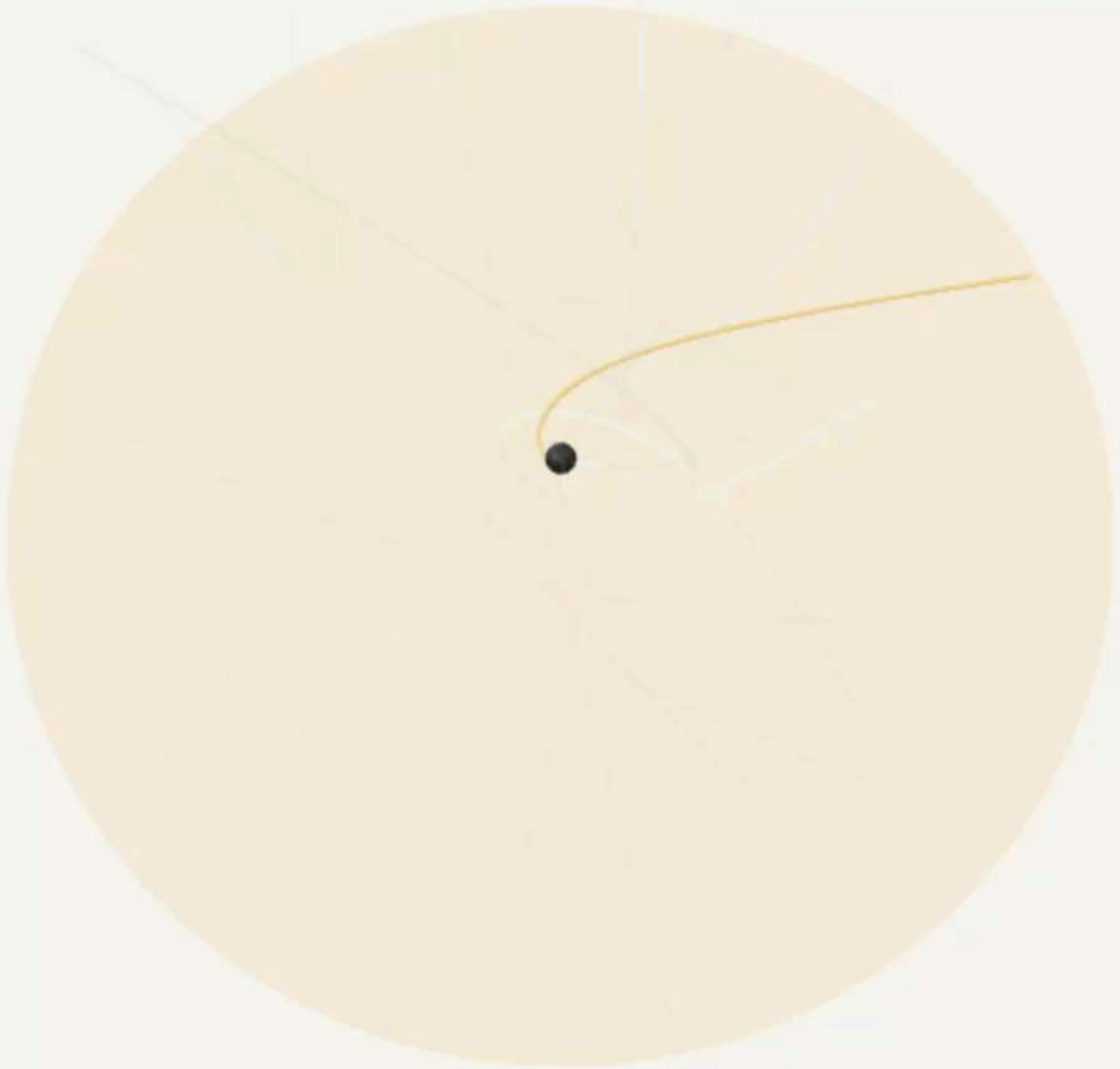


The metric:

$$g_{\text{opt}} = \frac{\left(1 + \frac{M}{2(x^2+y^2+z^2)}\right)^6}{\left(1 - \frac{M}{2(x^2+y^2+z^2)}\right)^2} \left(dx^2 + dy^2 + dz^2\right)$$

Spherical symmetry implies that all these curves actually lie in planes…and all planes are isometric!

Its enough to study a single slice of this metric, to understand all the optical properties of the black hole spacetime



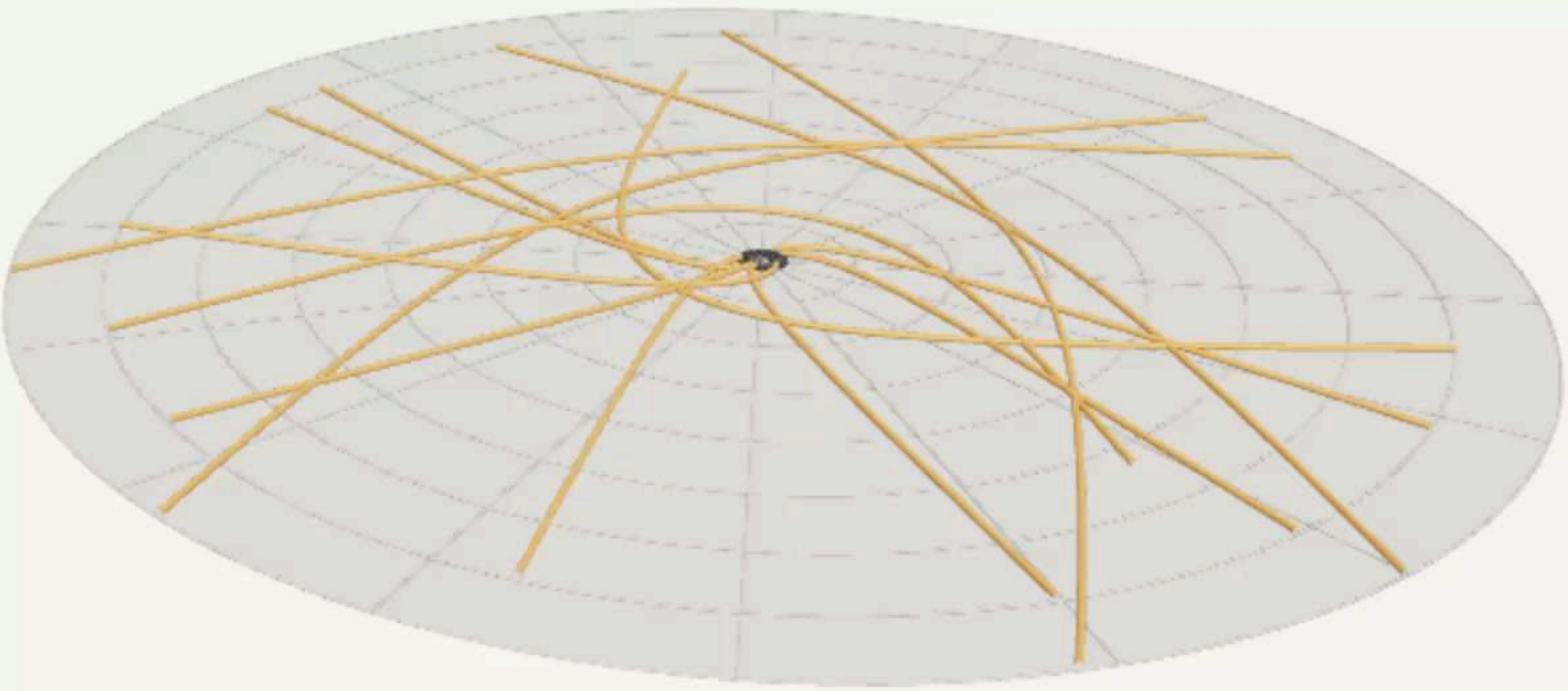
But spherical symmetry also implies this surface is rotationally symmetric!

The 'optical geometry' of a black hole is captured by an 'abstract surface of revolution'

...what does it look like?

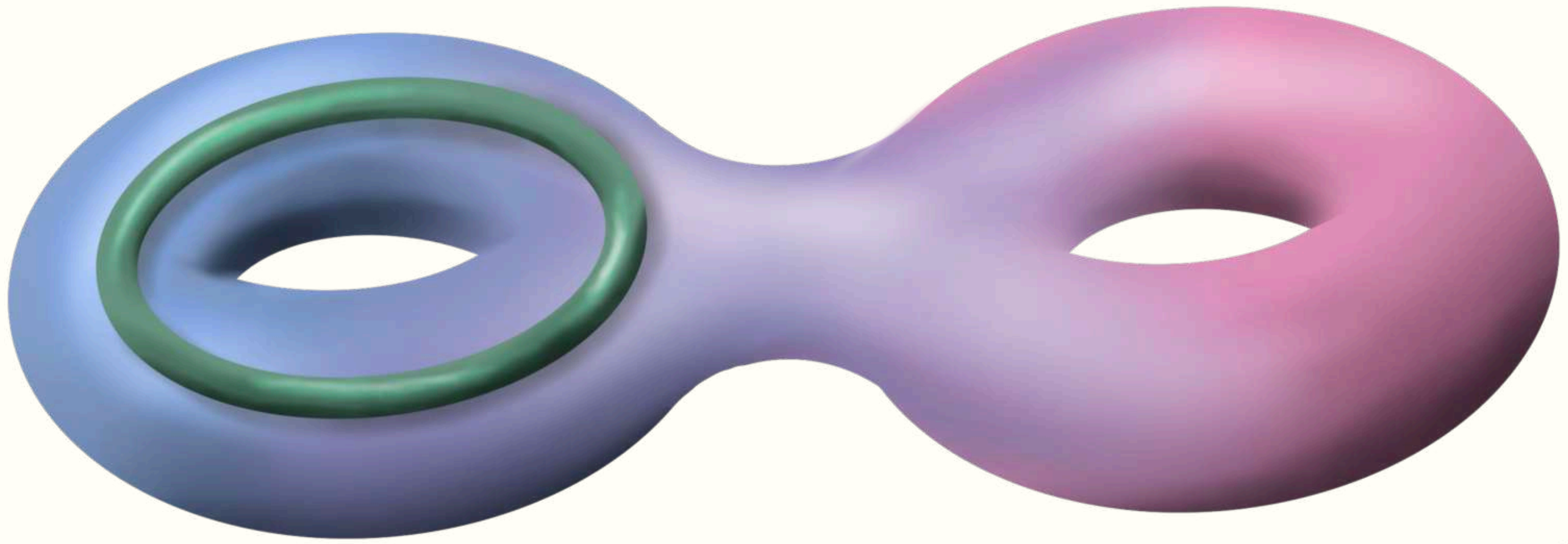
The metric:

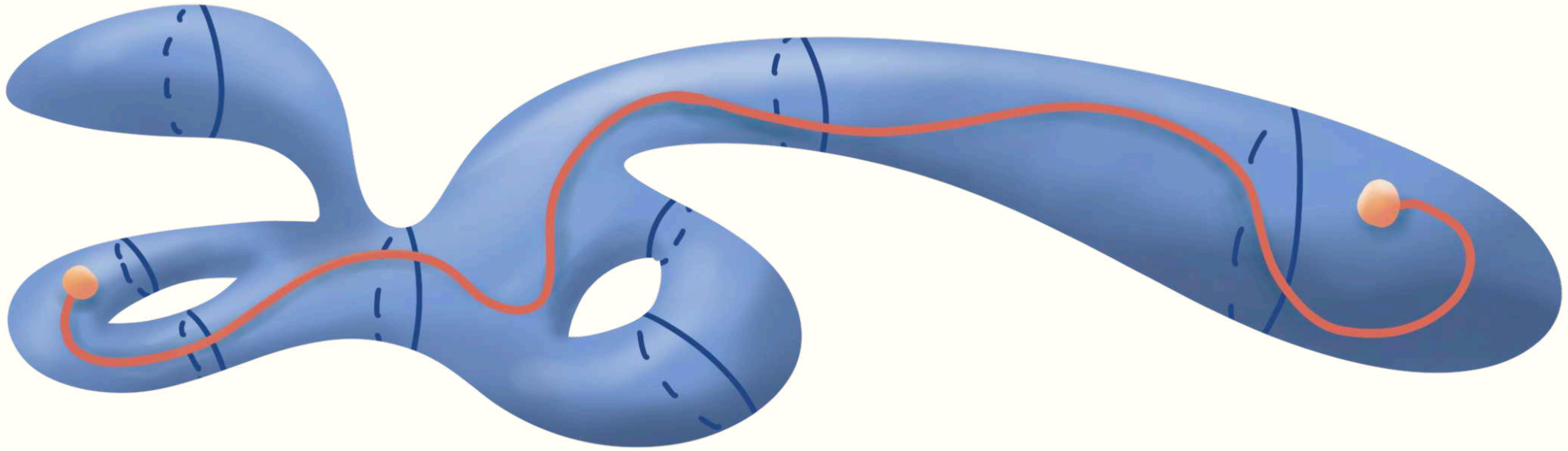
$$g = \frac{\left(1+\frac{M}{2\rho}\right)^6}{\left(1-\frac{M}{2\rho}\right)^2}\left(d\rho^2+\rho^2\,d\theta^2\right).$$



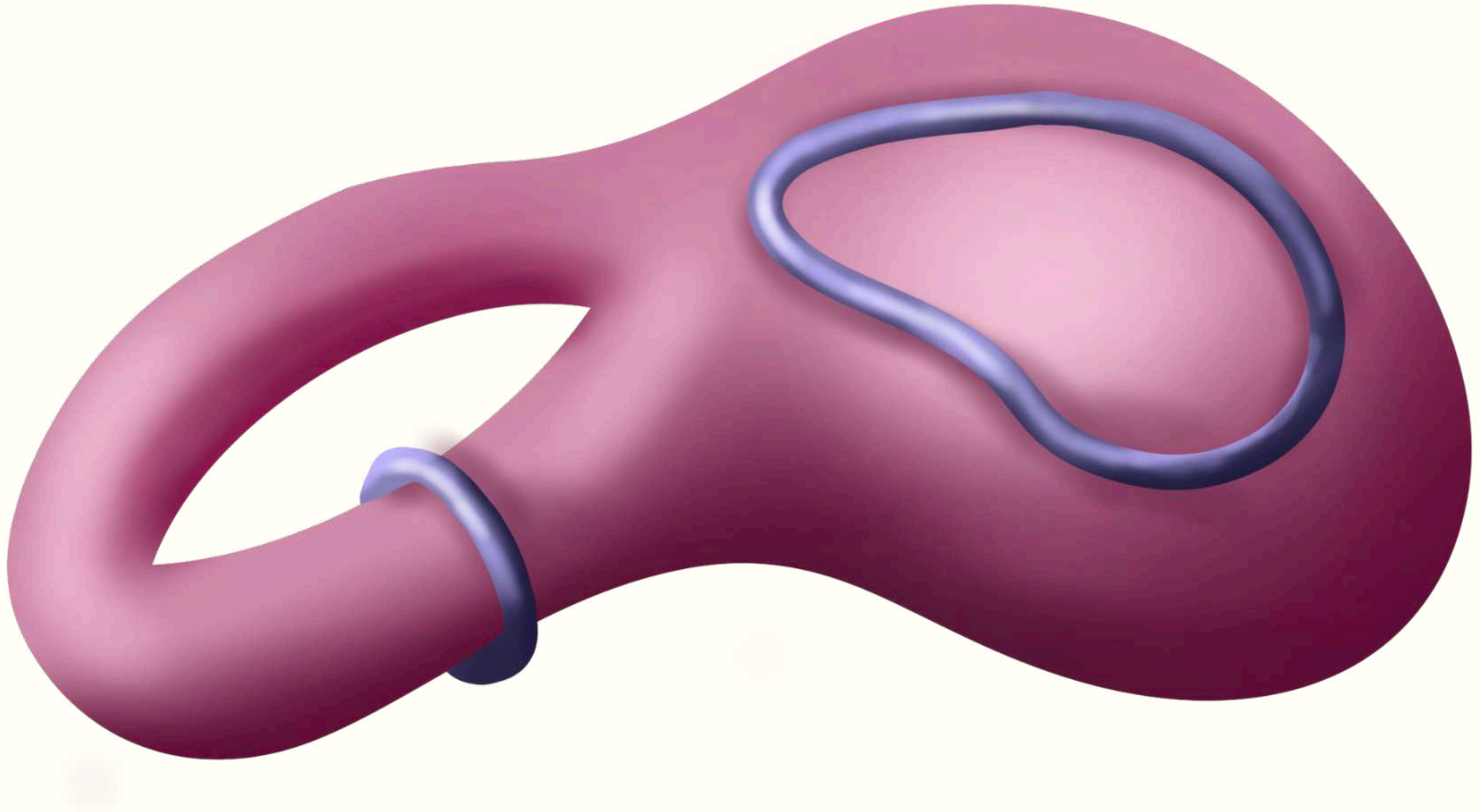
Example II: Neighborhoods of Curves

Theres a high dimensional space of surfaces of genus g. None of them are 'rotationally symmetric'!





Example II: Neighborhoods of Curves

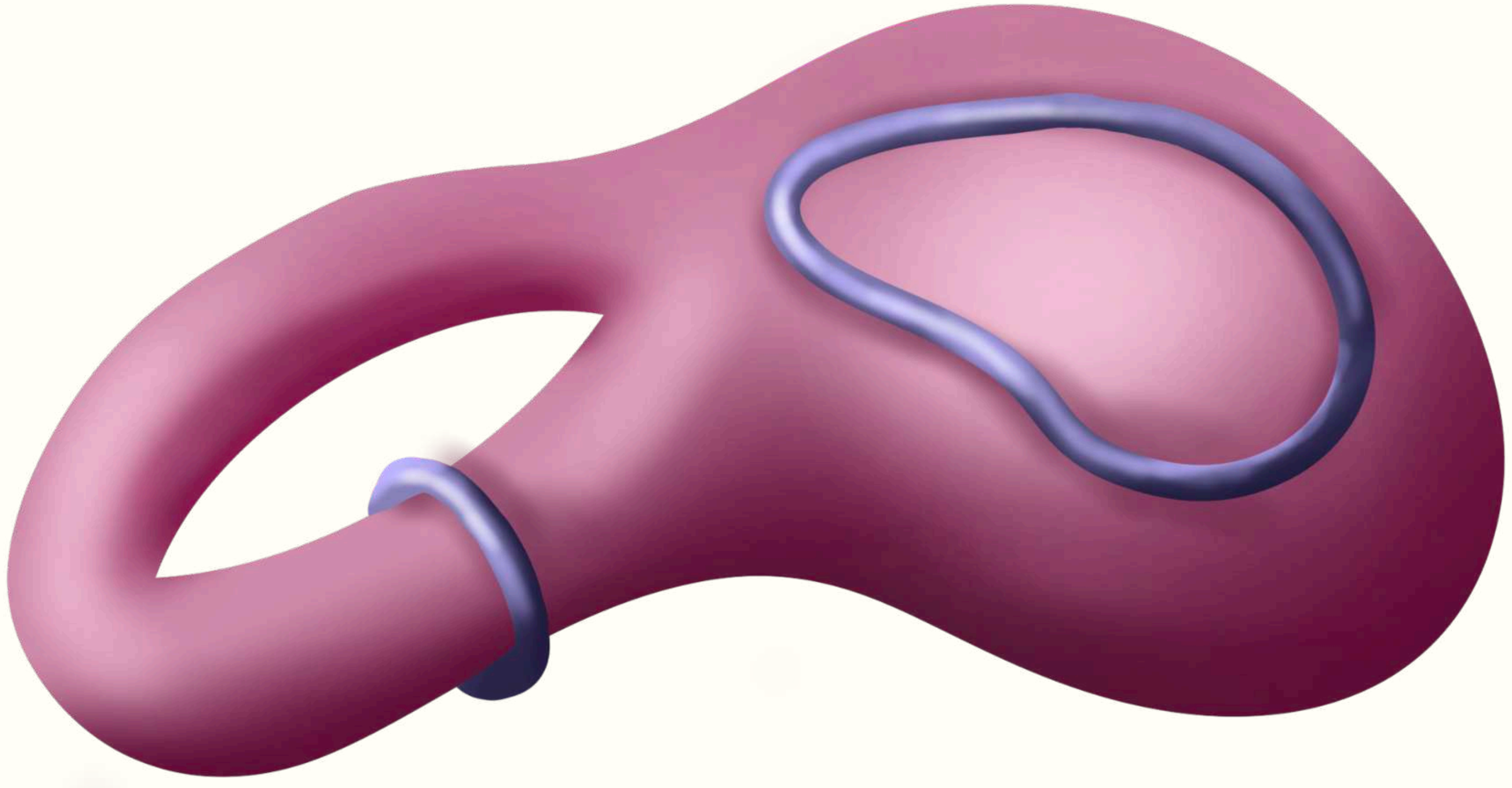


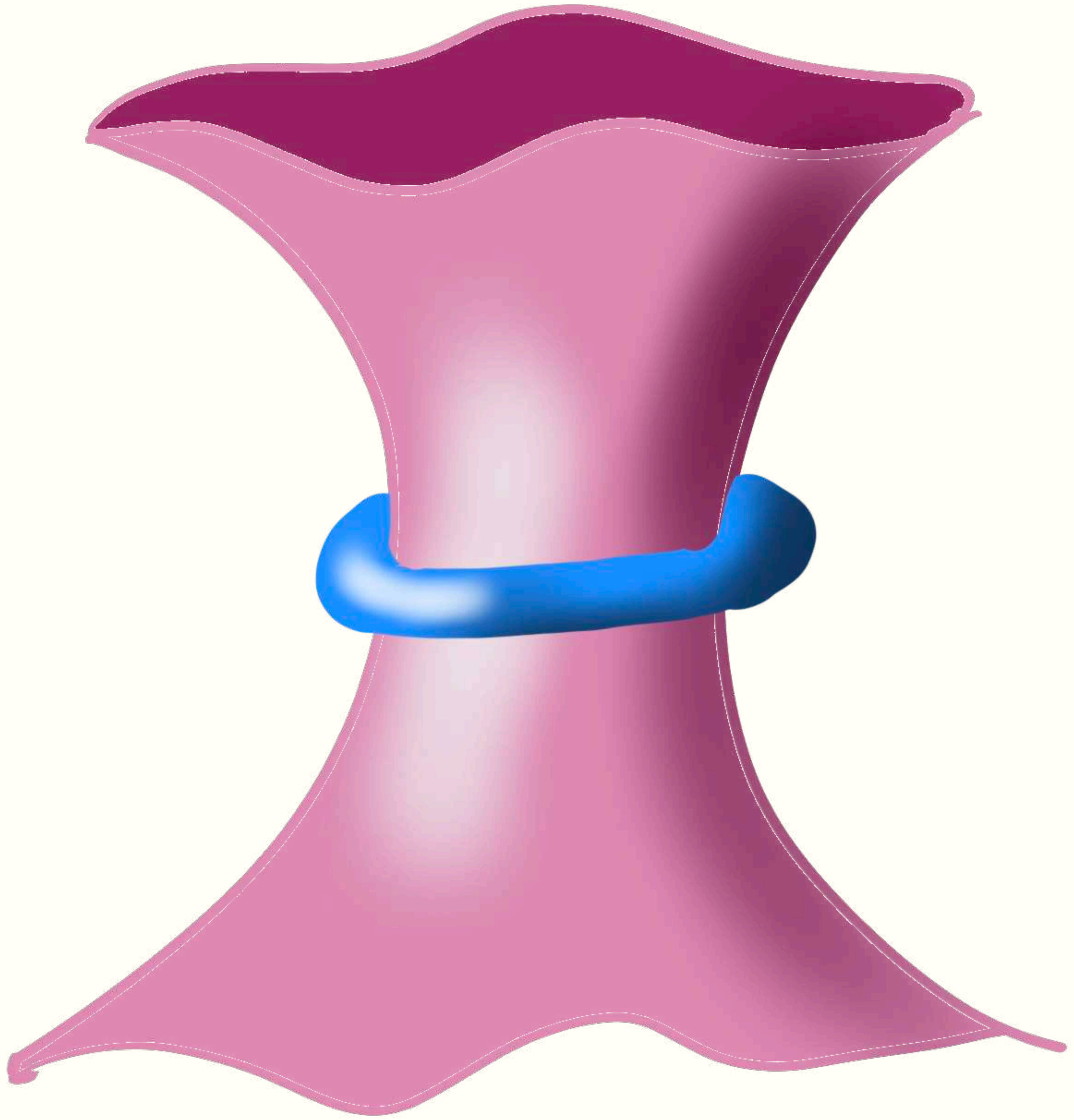
The space of such structures can be studied by measuring the length of curves

But, any nontrivial closed curve can be pulled tight to a geodesic....

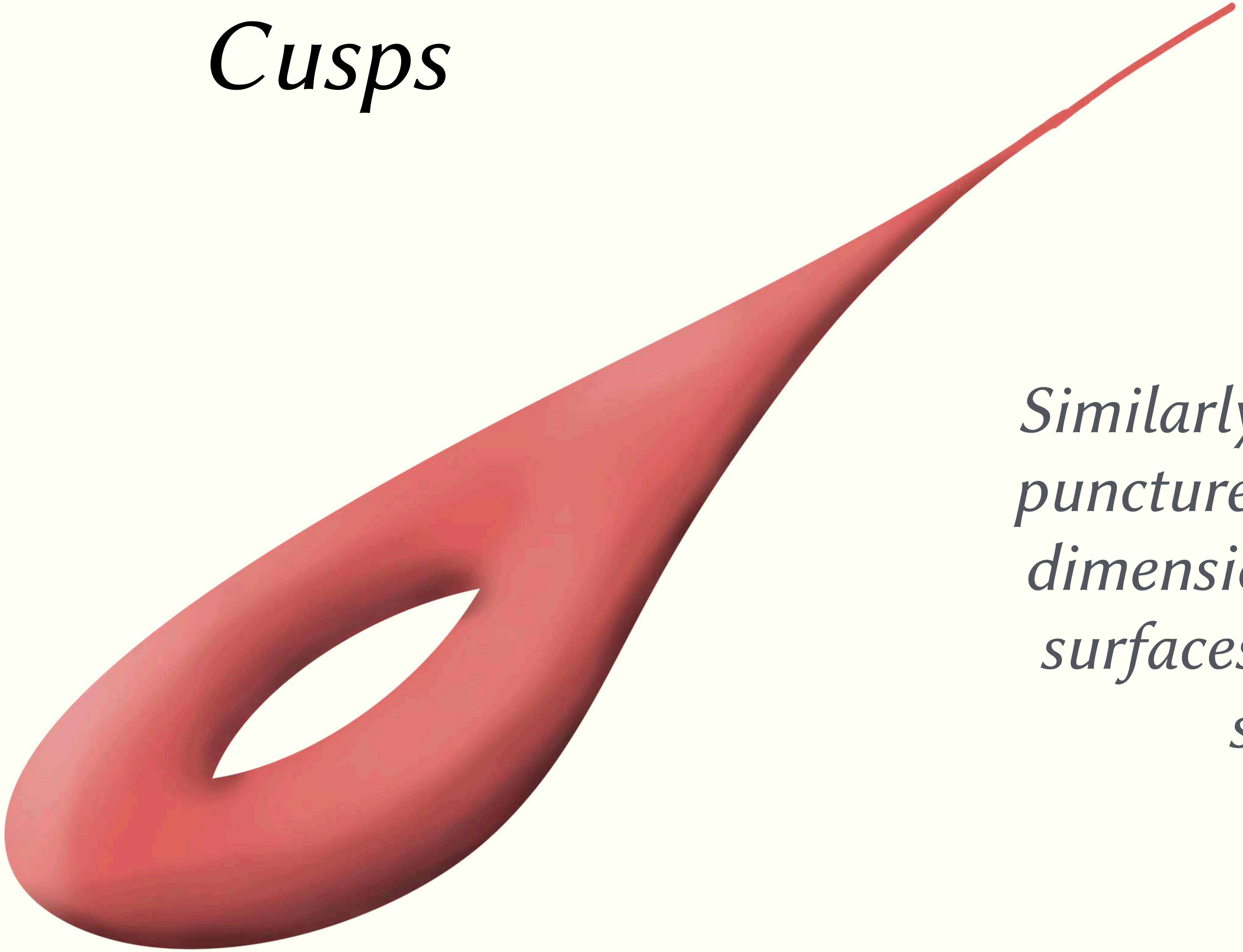
Example II: Neighborhoods of Curves

*And the neighborhood of such a curve *is* rotationally symmetric!*





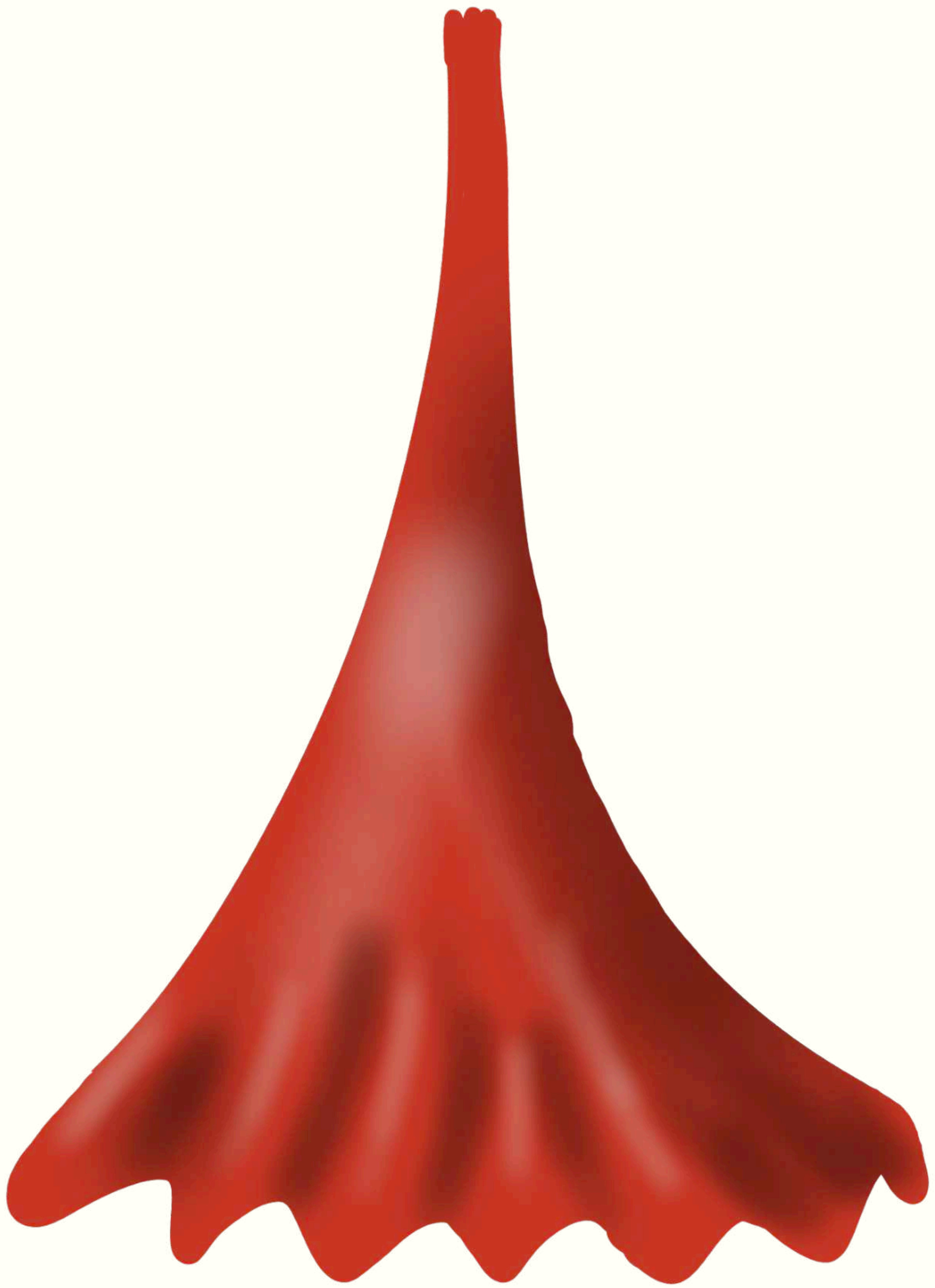
Example III: Exploring Cusps



Similarly, the space of the punctured surfaces is high dimensional, and no such surfaces are rotationally symmetric

Example III: Exploring Cusps

But every puncture has a **neighborhood which is rotationally symmetric!**

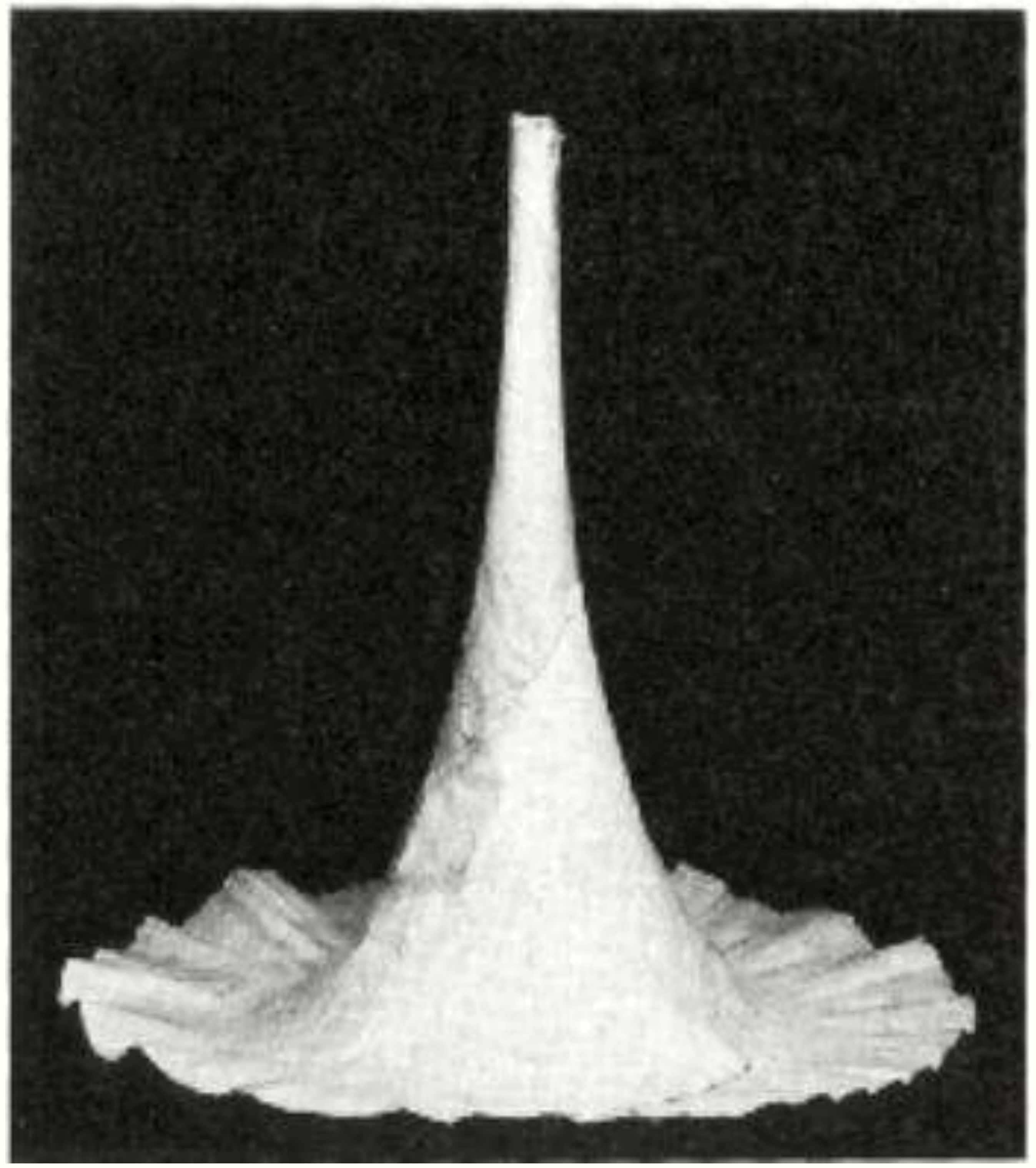


II

How to See a Surface

We're 3D creatures with brains evolved for 3D reasoning. So putting something abstract into 3D space can be a big help!

Eugenio Beltrami, 1860s



Physical models of a neighborhood of a cusp, and neighborhood of a hyperbolic geodesic, embedded in 3-dimensions

Embedded Surfaces

A surface (Σ, g) *isometrically embeds in 3-space by* $\Phi\colon \Sigma \to \mathbb{R}^3$ *if*

$$\Phi^* ds^2_{\text{Euc}} = g$$

Symmetric Surfaces

A surface (Σ, g) *is rotationally symmetric if* $\mathbb{S}^1 \hookrightarrow \text{Isom}(g)$

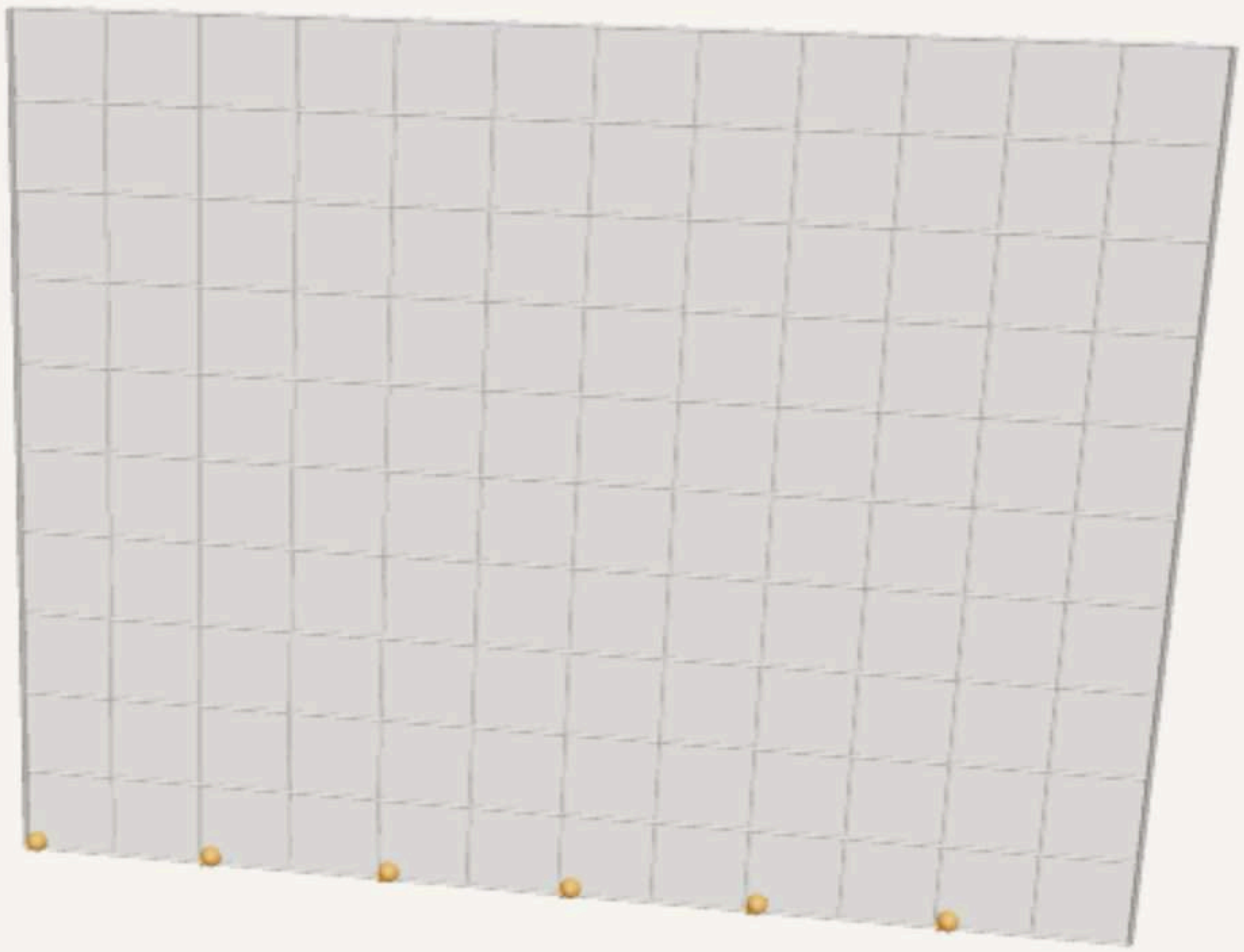
Surfaces of Revolution

An surface of revolution is an isometrically embedded rotationally symmetric surface, where the $\mathbb{S}^1$ *action comes from* $\mathbb{S}^1 \hookrightarrow \text{Isom}(ds^2_{\text{Euc}})$

Example:

The abstract surface $\mathbb{R}^2/\mathbb{Z}$ embeds in 3-space as a round cylinder

Geodesics on the abstract surface are geodesics on the embeddd surface

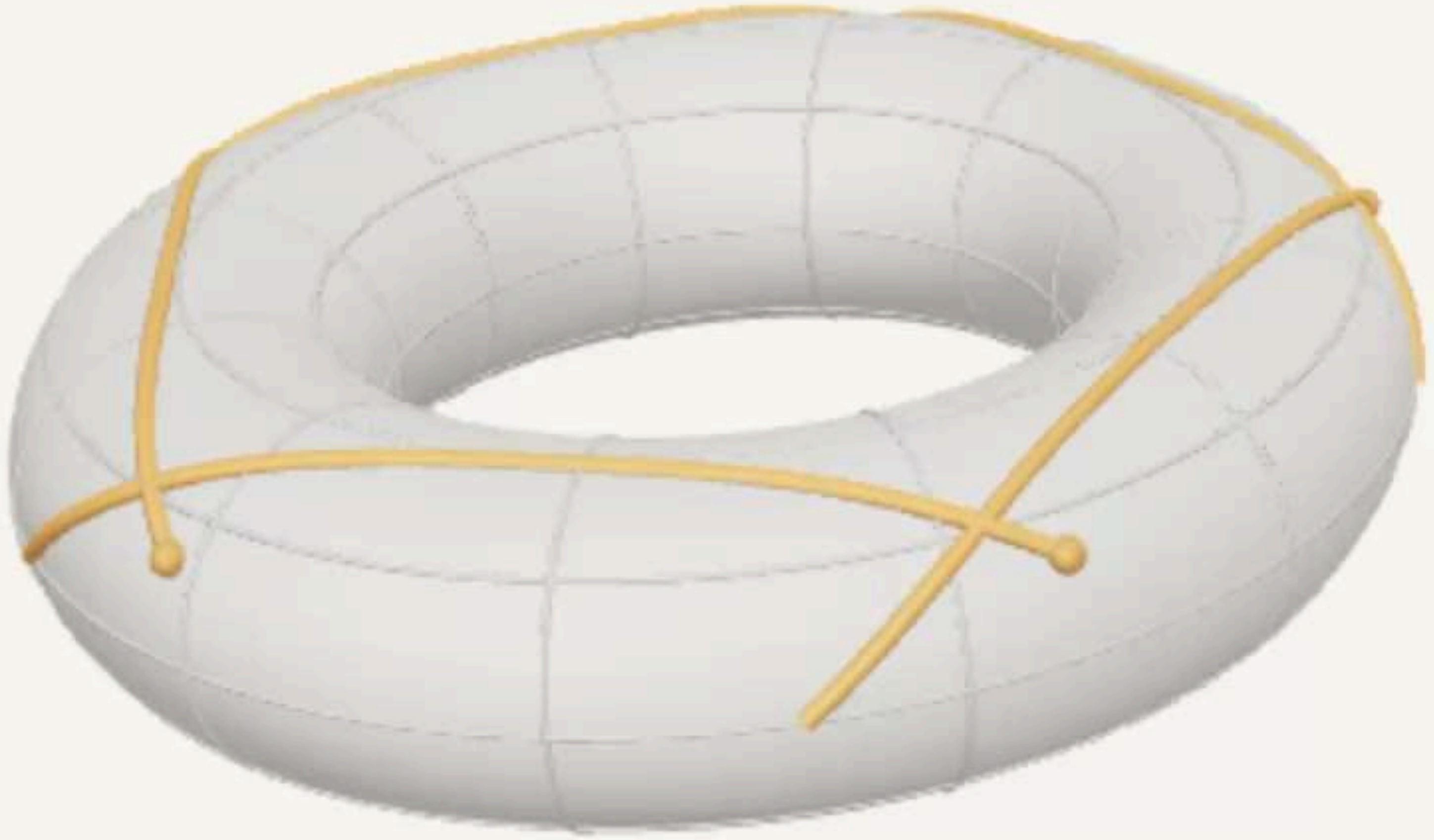


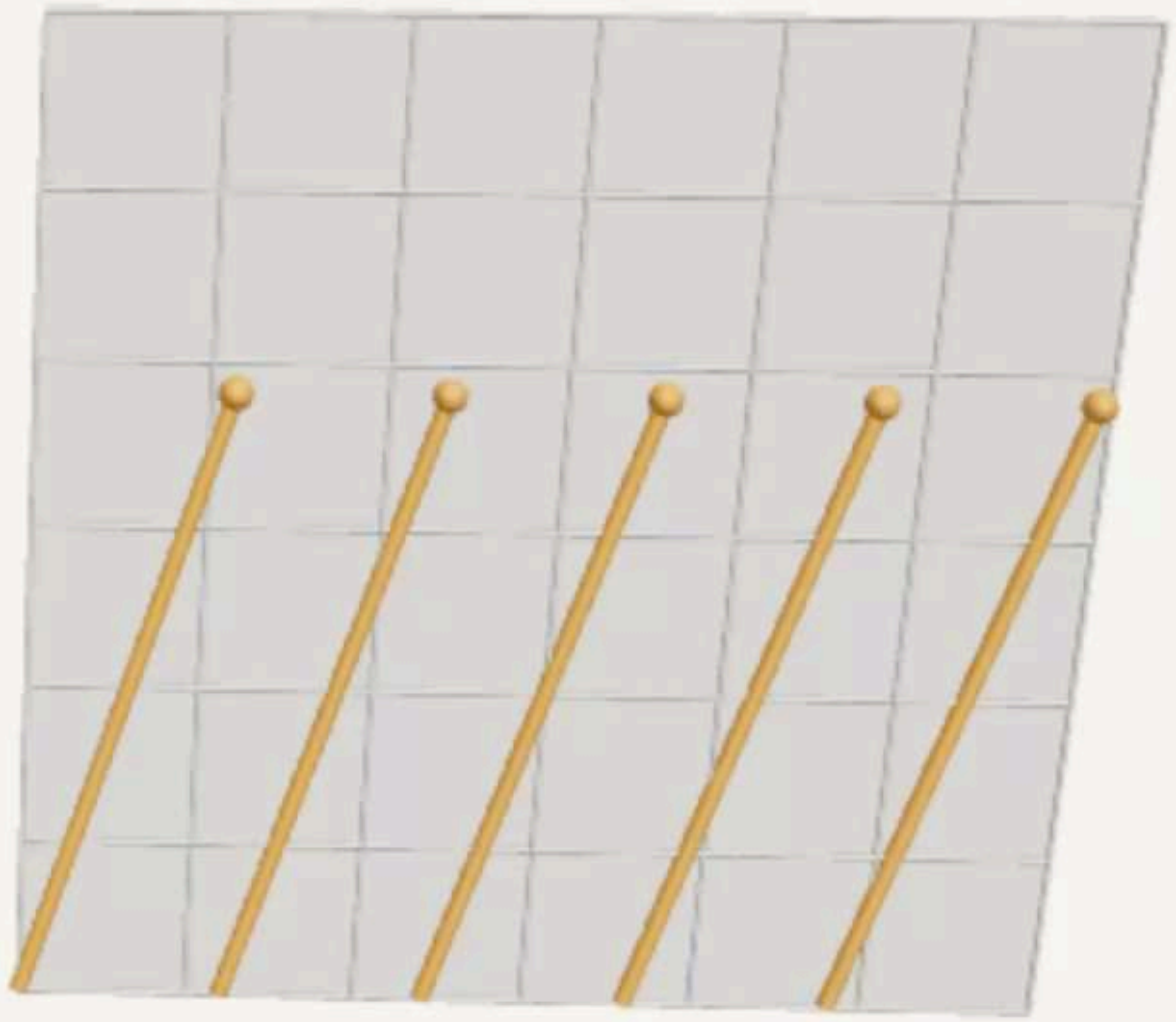
Non-Example:

The abstract surface $\mathbb{R}^2/\mathbb{Z}^2$ does not embed in 3-space as a surface of revolution

Geodesics of the flat metric

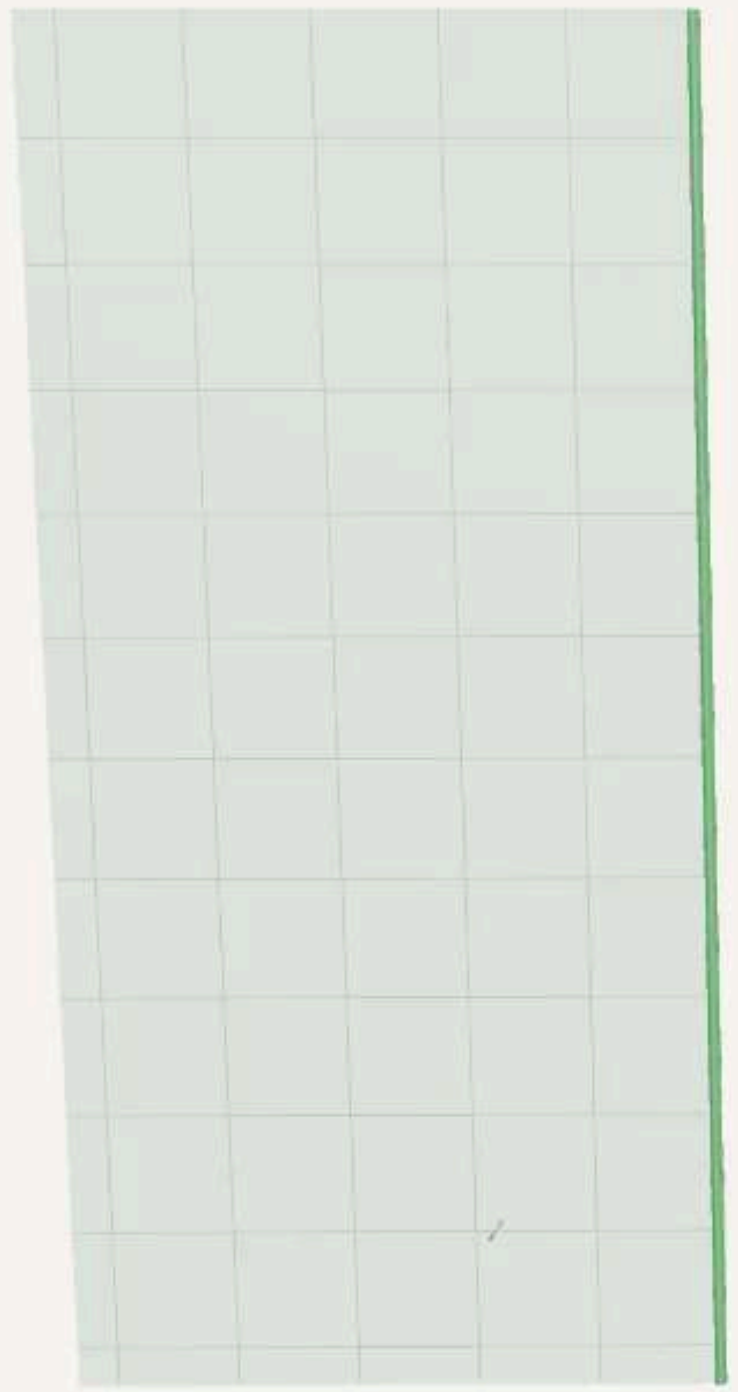
Geodesics of the embedded surface





Cooking Up a Surface:

$$\big(r(u), h(u)\big)$$



$$\left(\dot{r}^2 + \dot{h}^2\right) du^2 + r^2 d\theta^2$$

Given a curve in the plane, we can easily work out the induced metric.

We just need to reverse the process!

Cooking Up a Cusp:

The pseudosphere $\mathbb{H}^2/\langle P \rangle$ *has metric* $\frac{1}{y^2}(dx^2 + dy^2)$

$$\left(\dot{r}^2 + \dot{h}^2\right) du^2 + r^2 d\theta^2$$

$$\begin{array}{ccc} \| & & \| \\ \dfrac{1}{y^2} & & \dfrac{1}{y^2} \end{array}$$

Cooking Up a Cusp:

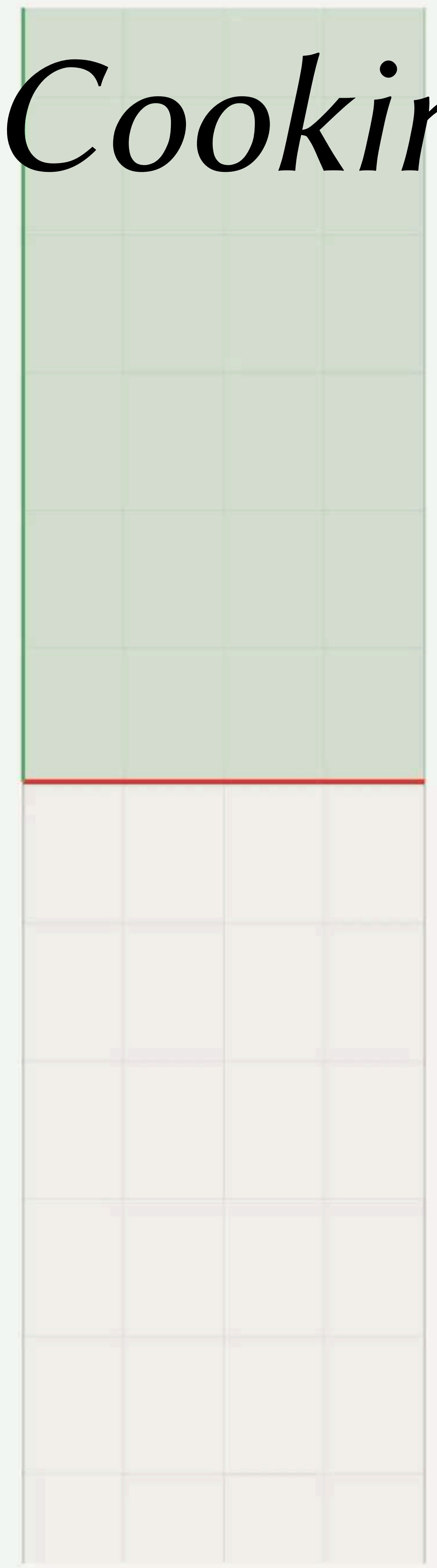
The pseudosphere $\mathbb{H}^2/\langle P \rangle$ *has metric* $\frac{1}{y^2}(dx^2 + dy^2)$

$$\left\{ \begin{array}{c} \dot{r}^2 + \dot{h}^2 = \dfrac{1}{y^2} \\ r = \dfrac{1}{y} \end{array} \right\} \implies \dot{h}^2 = \frac{y^2 - 1}{y^4}$$

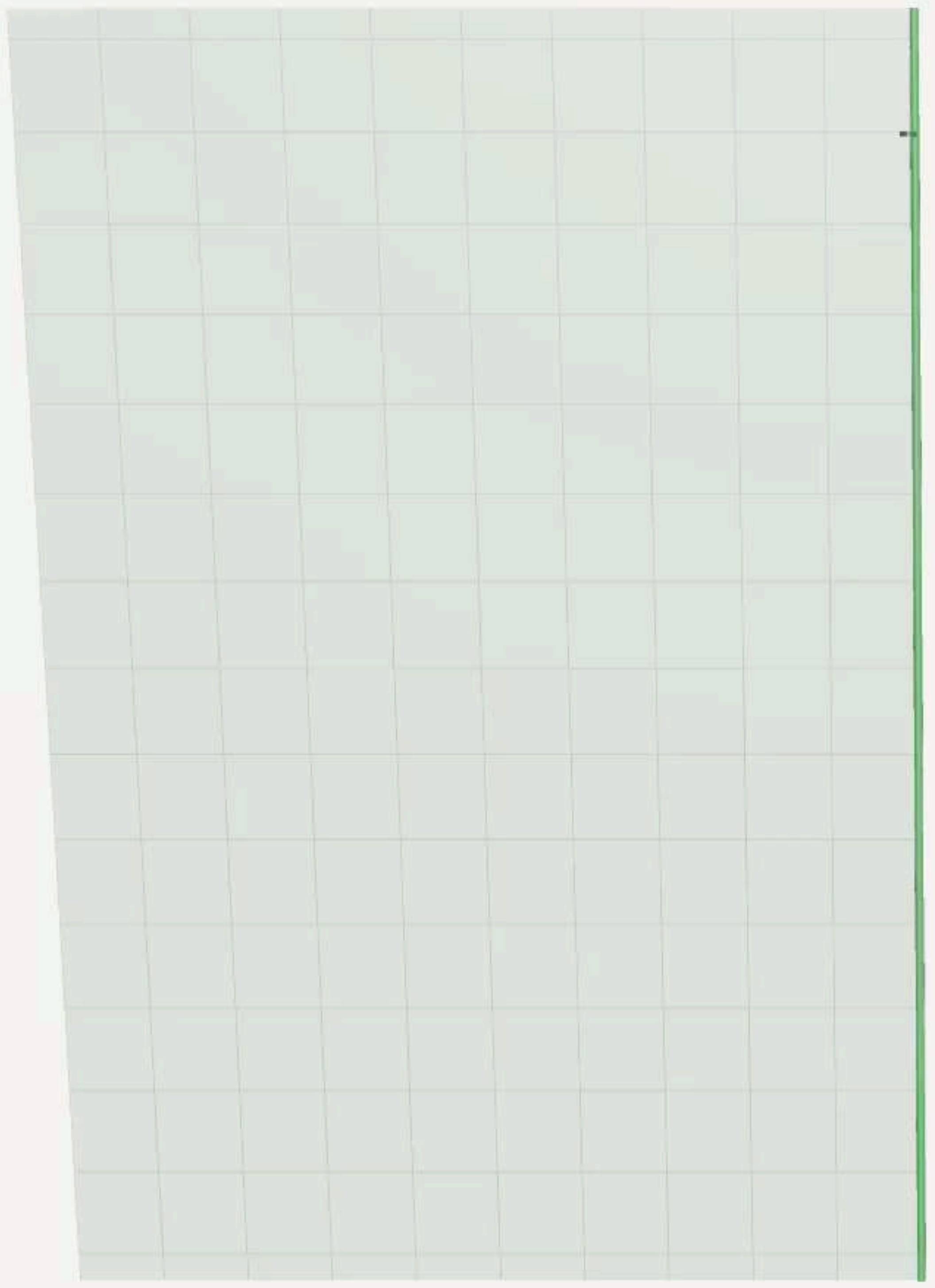
$$\implies y \geq 1$$

And then just integrate!

Cooking Up a Cusp:



Domain in $\mathbb{H}^2 : y \geq 1$

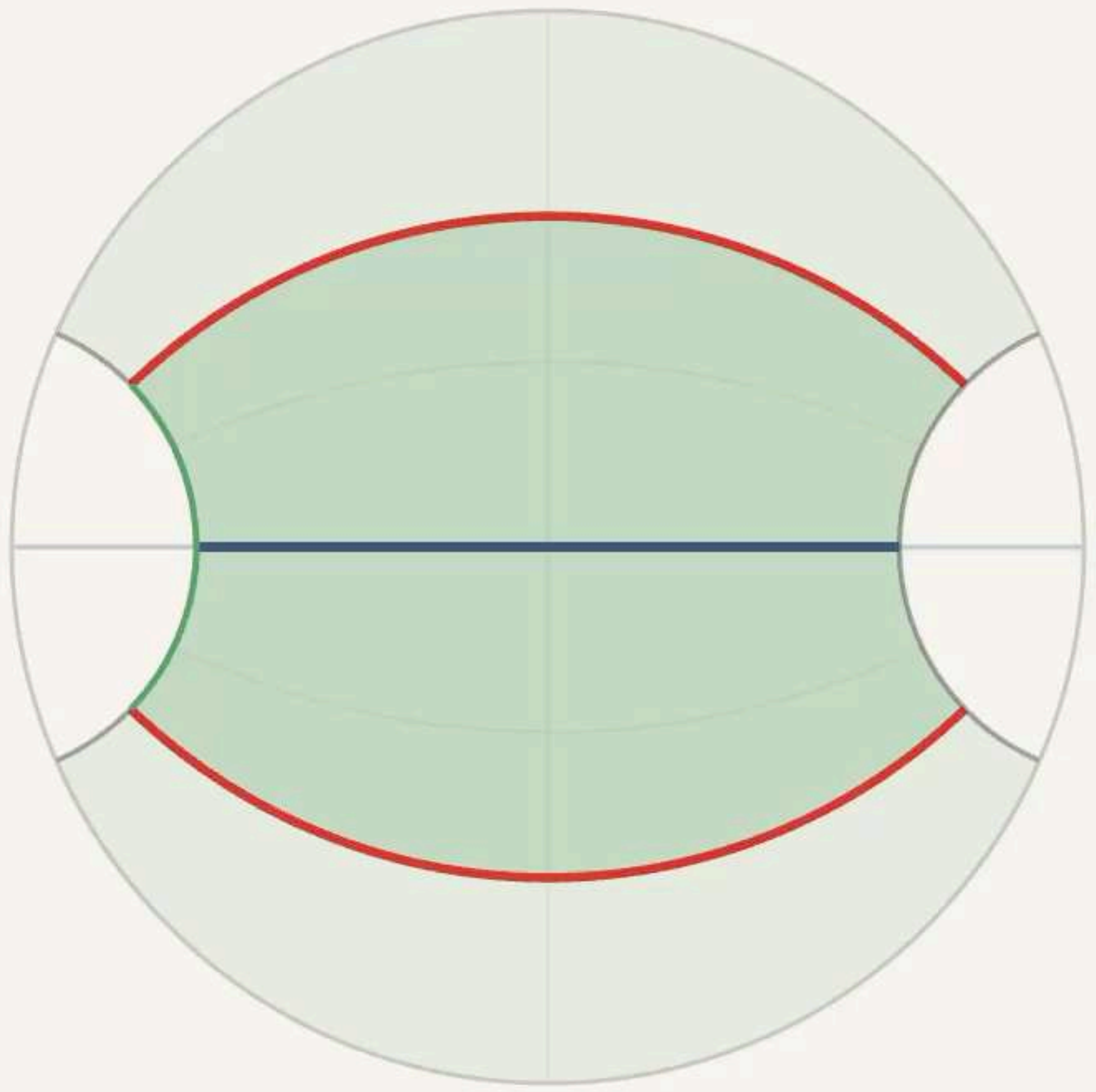


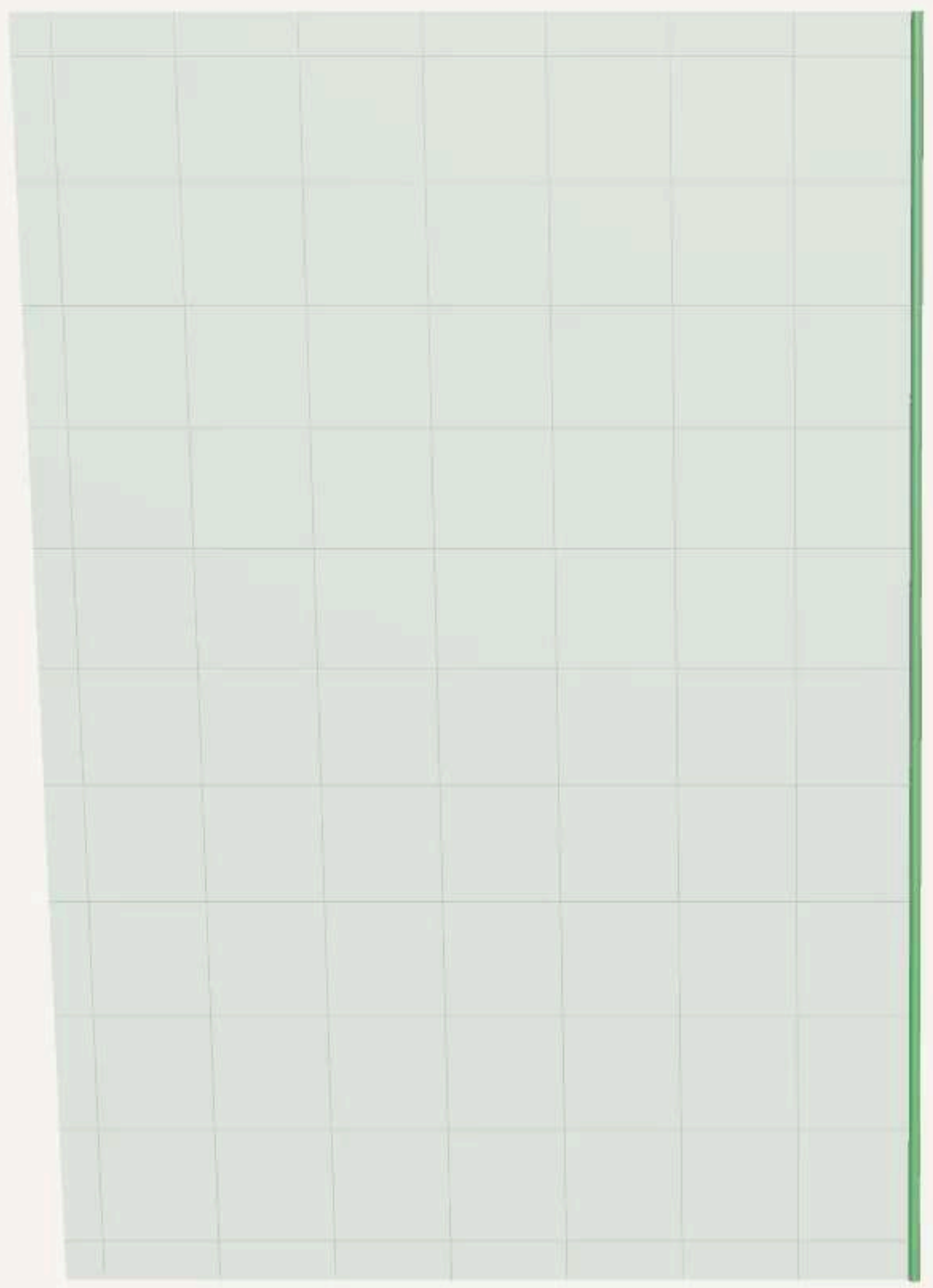
Curve:

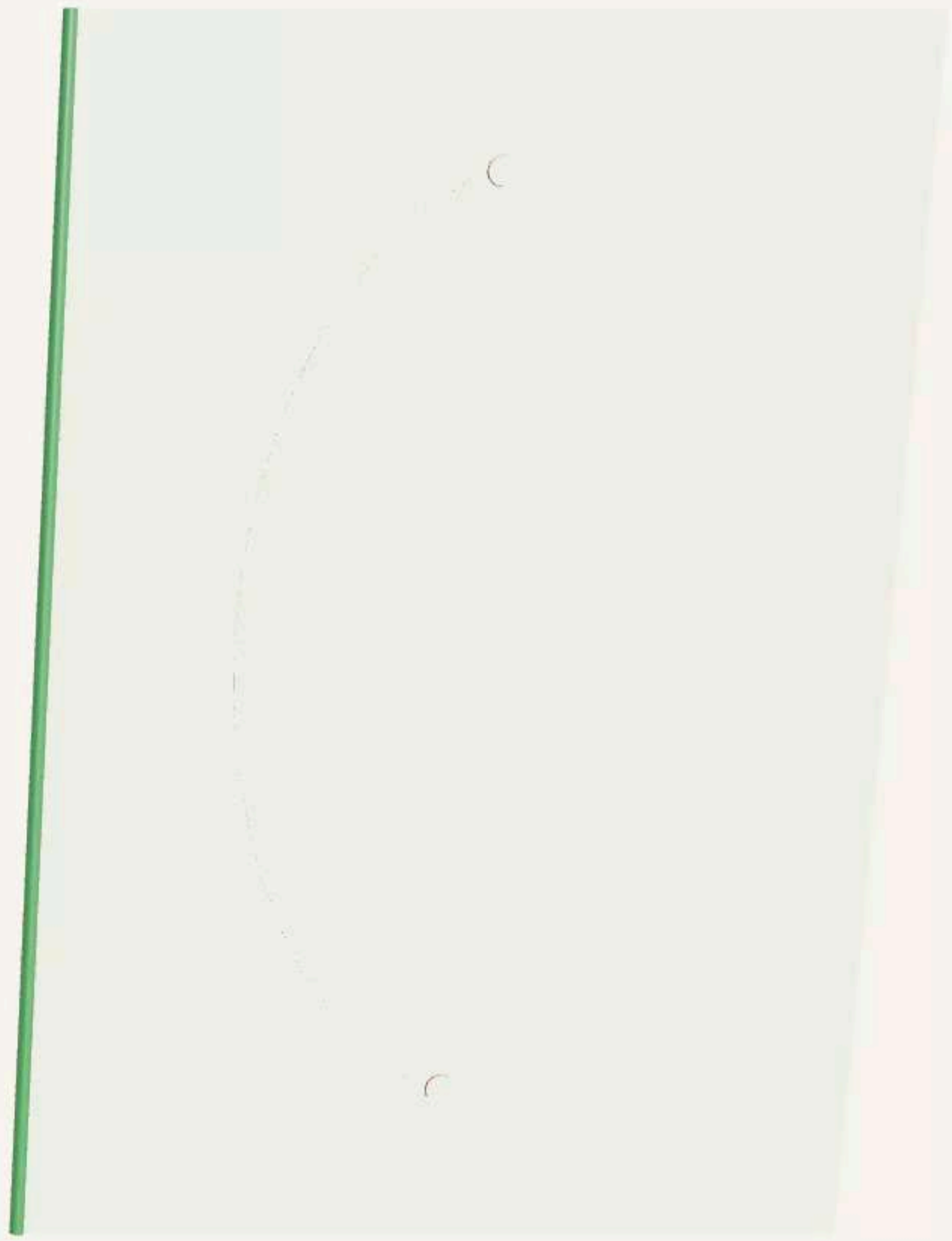
$$y \mapsto \left(\frac{1}{y}, \int \frac{\sqrt{y^2 - 1}}{y^2} \, dy \right)$$

Surface isometric to $\mathbb{H}^2 / \langle P \rangle$ for $y \geq 1$

Cooking Up a Collar:

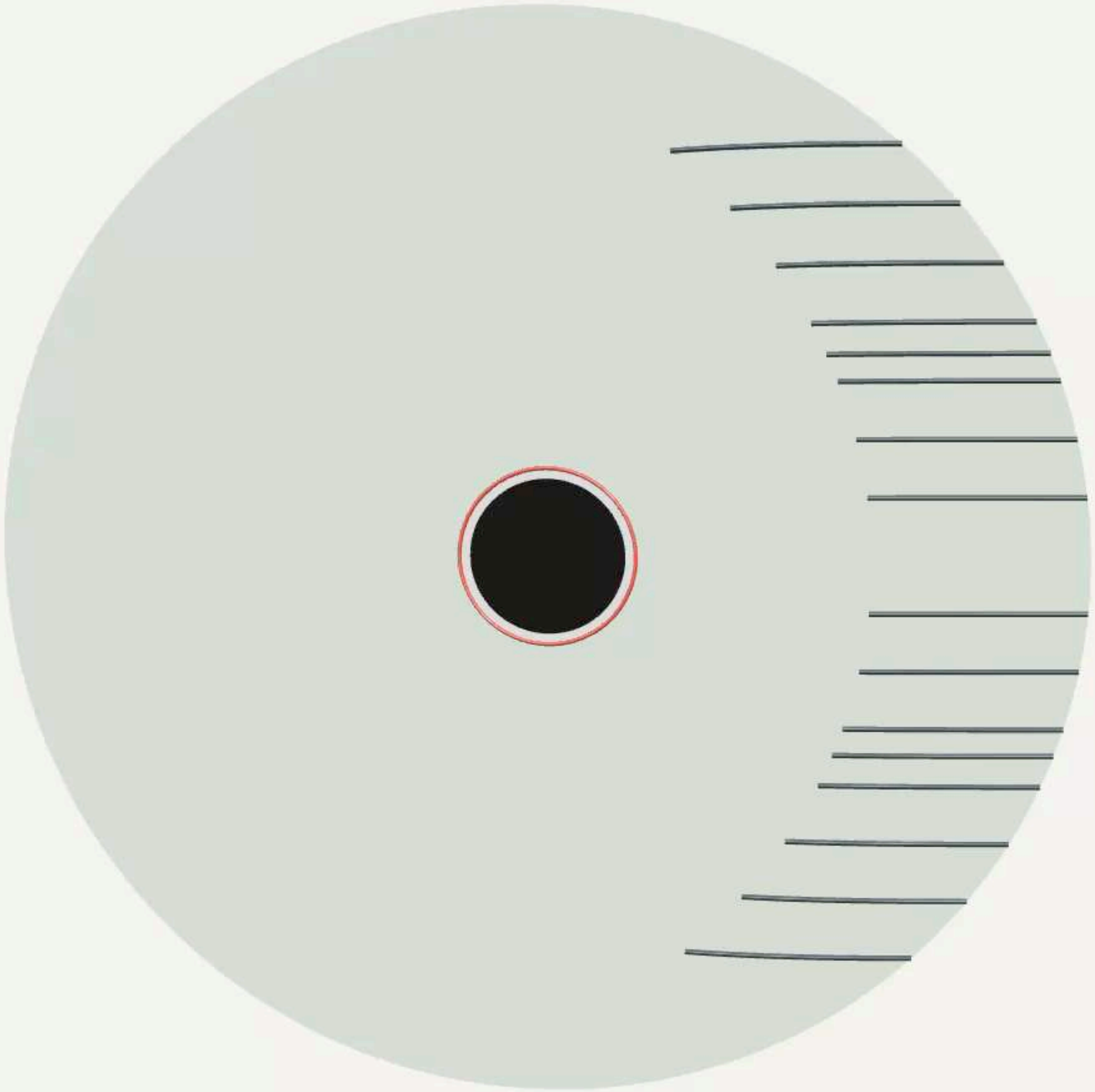




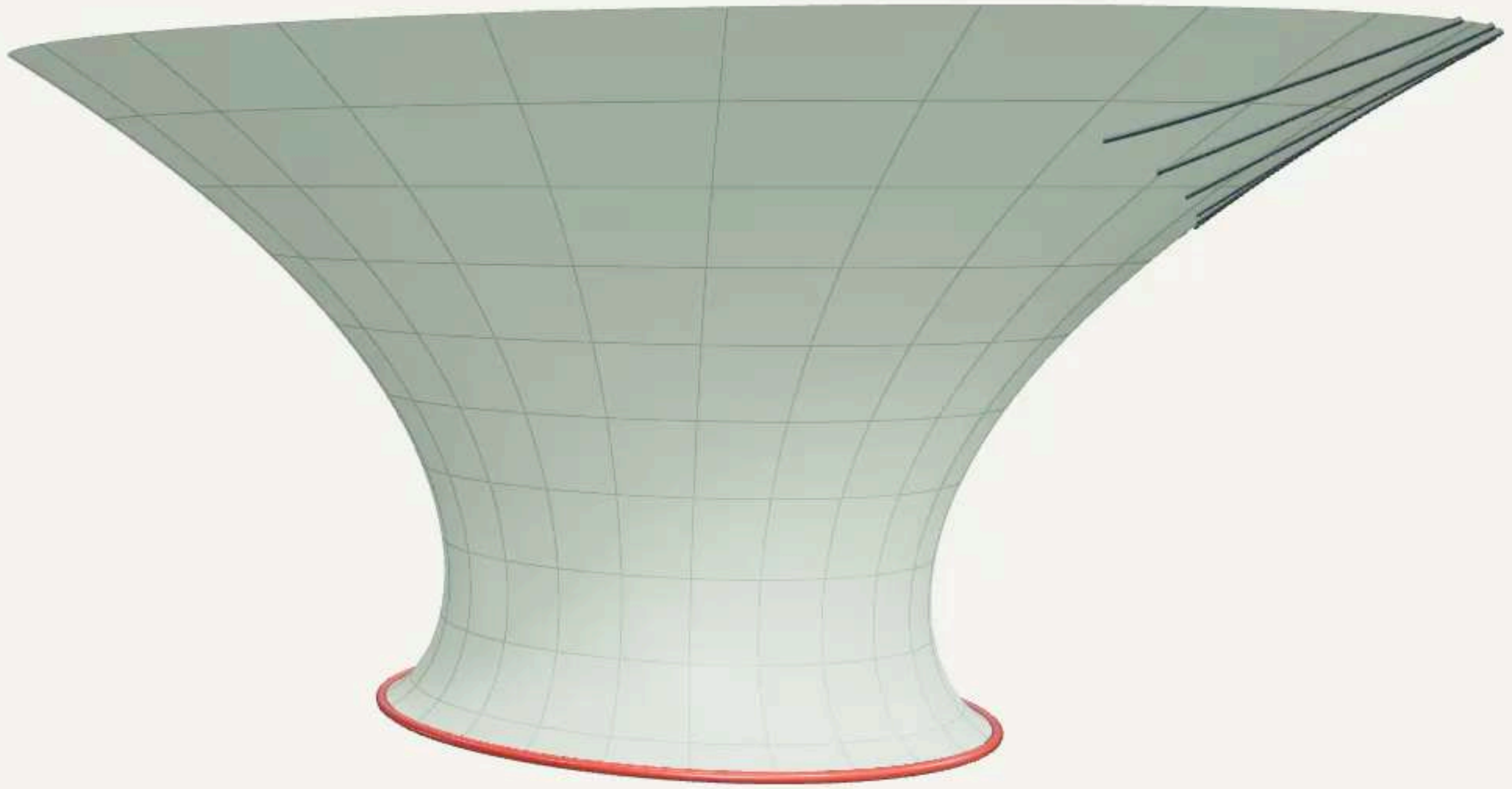


Embeds as a surface of revolution up to width $w = \operatorname{asinh}\left(\frac{2\pi}{\ell}\right)$

Cooking Up a Black Hole:



We already can learn a ton by *just looking at* the 3D embedding!

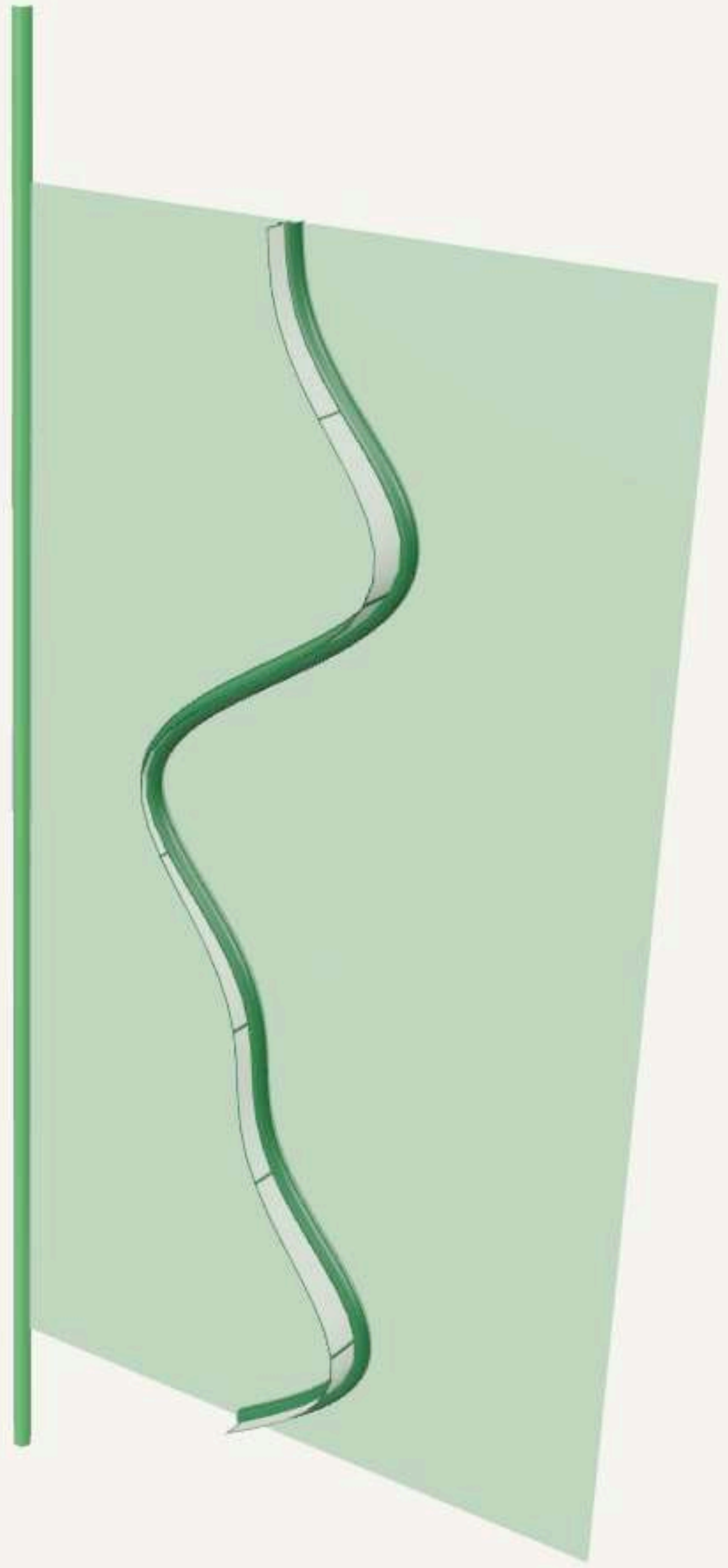


The surface also *does not embed all the way:* it cuts off at $9/8 \times$ the event horizon!

When a surface embeds

We know the
induced metric, so
reverse-engineer!

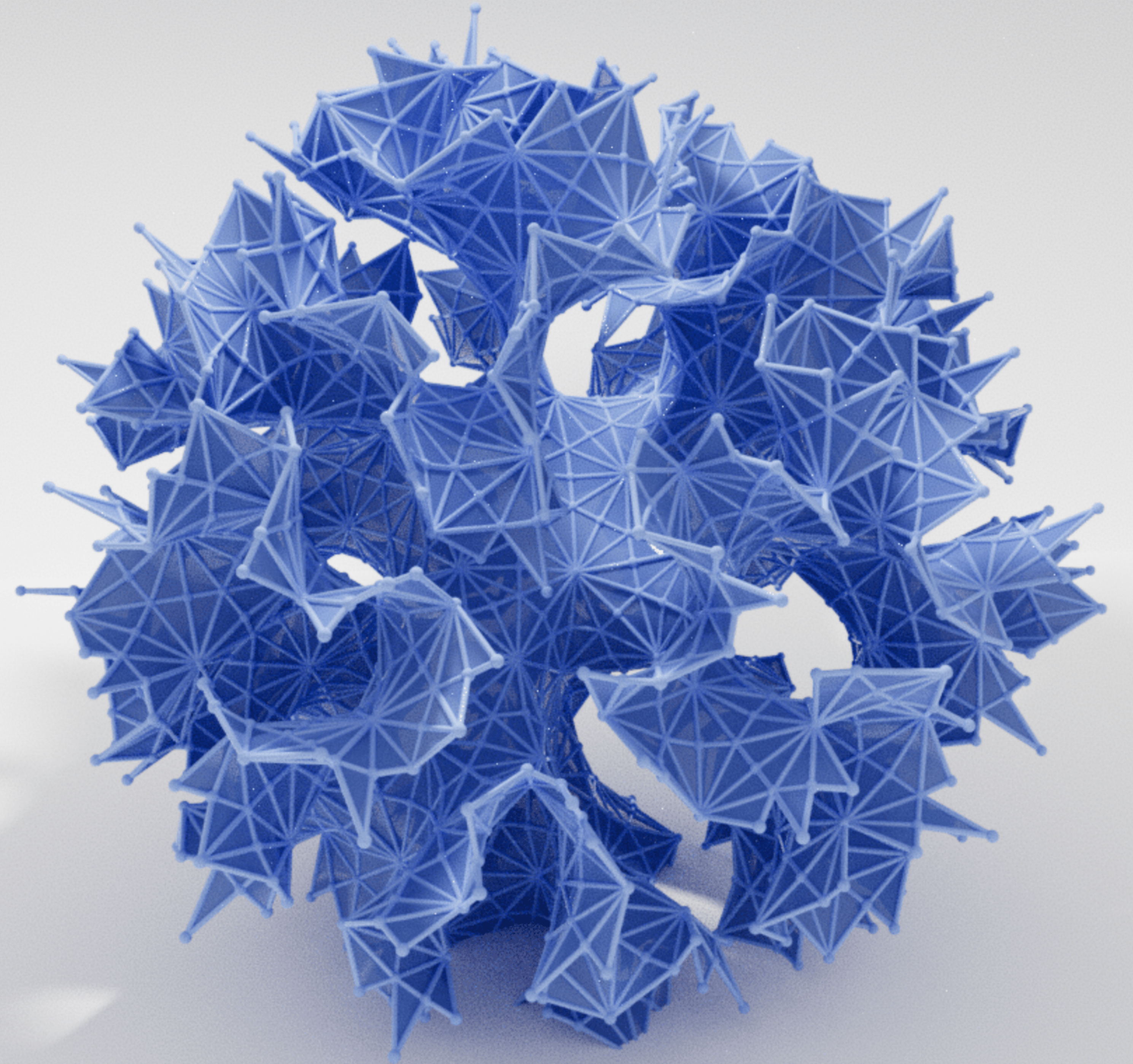
A surface with metric
$$ds^2 + f^2(s)\, d\theta^2$$
embeds in $\mathbb{E}^3$ *if and only if*
$$\dot{f}^2 \leq 1$$



III

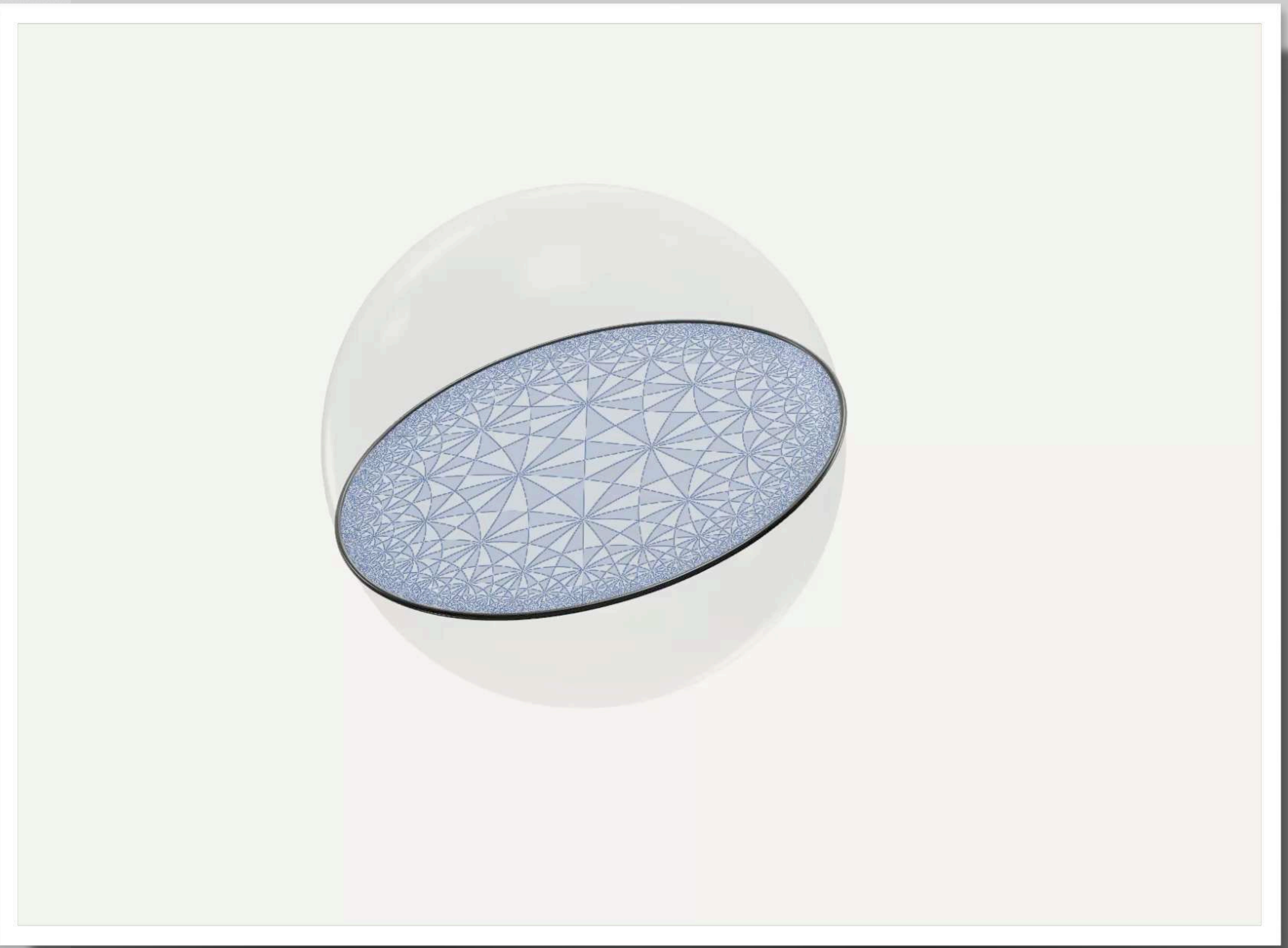
Where to See a Surface

We should probably broaden our view

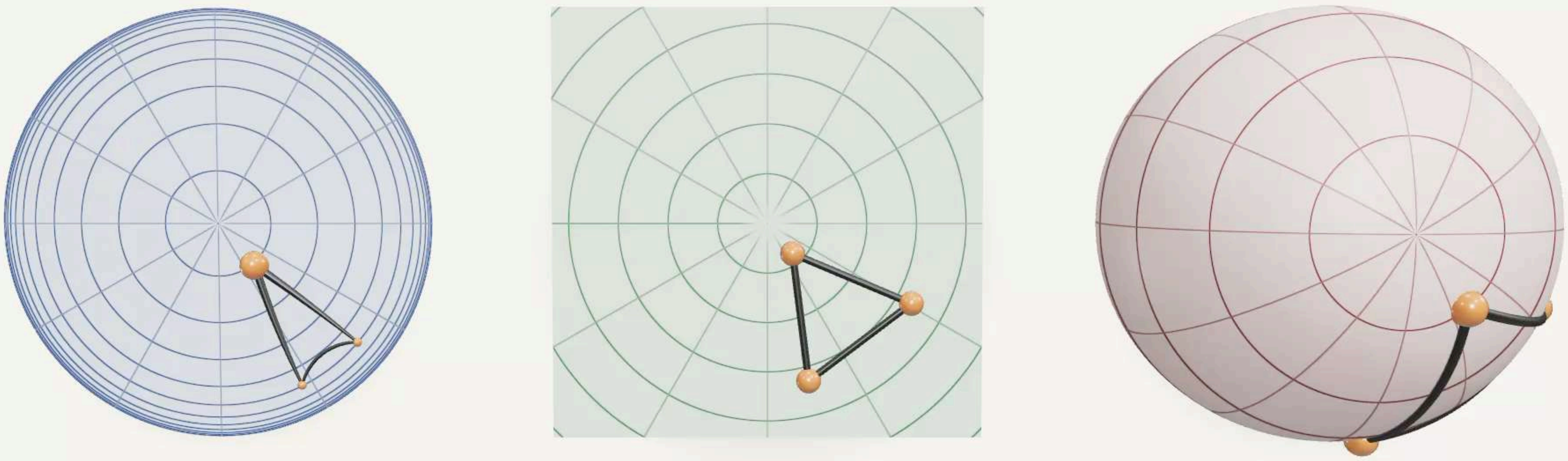


Attempts to embed a hyperbolic disk in Euclidean space go poorly

But the entire hyperbolic plane embeds quite nicely in $\mathbb{H}^3$ of course!



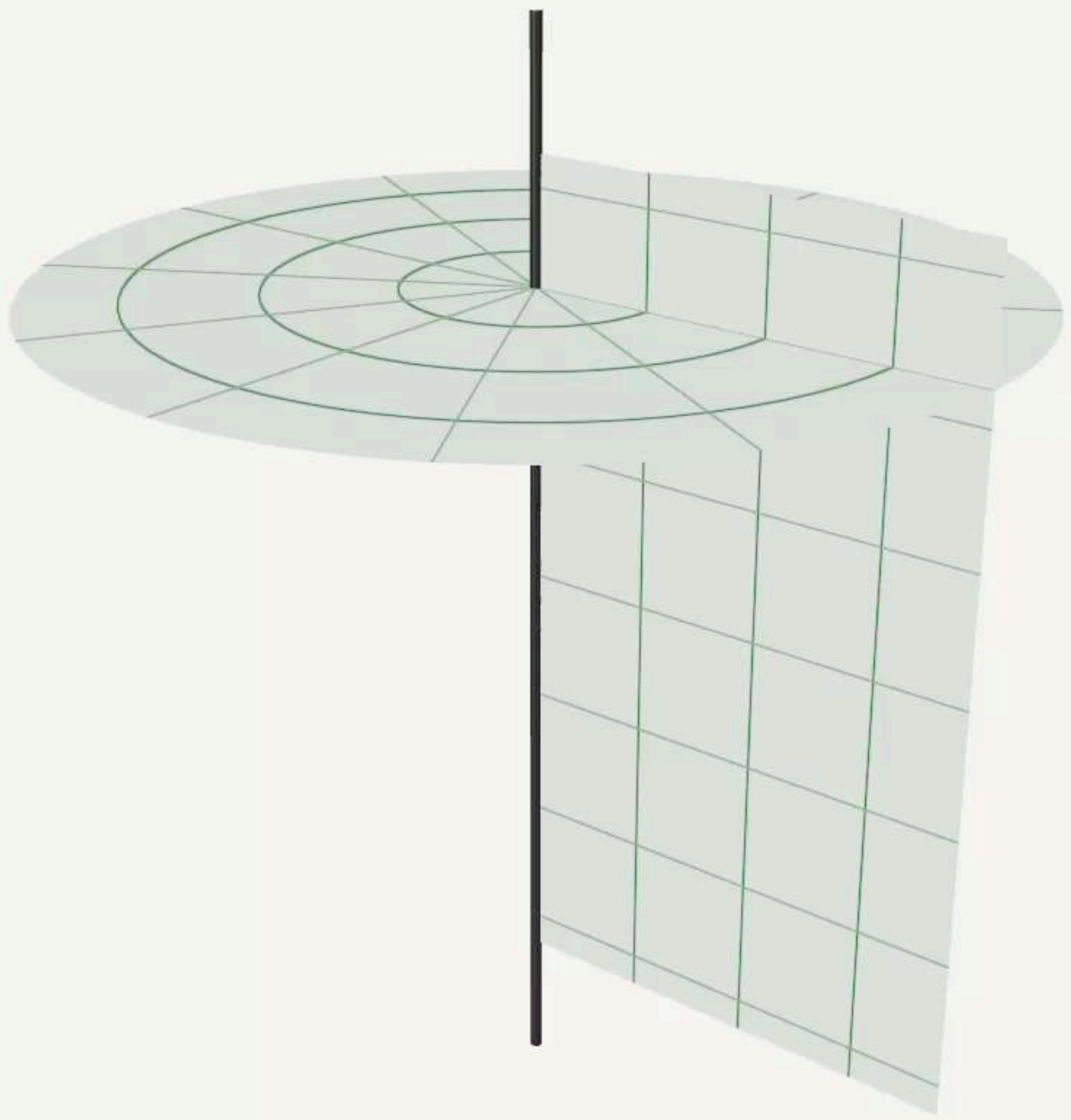
Other 3-Manifolds we know well: Constant Curvature!

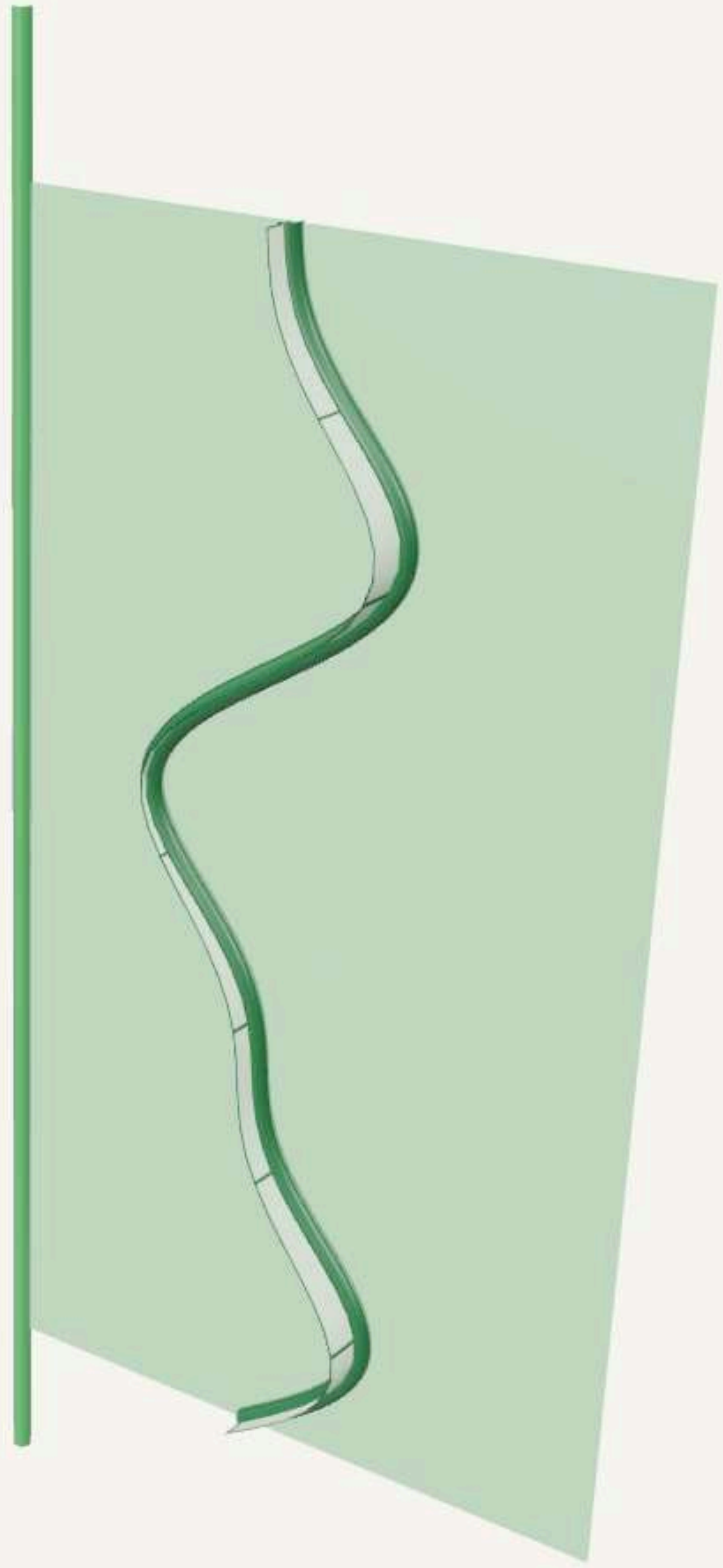


We have intuition for these spaces from doing mathematics, so they are also useful playgrounds for embedding surfaces.

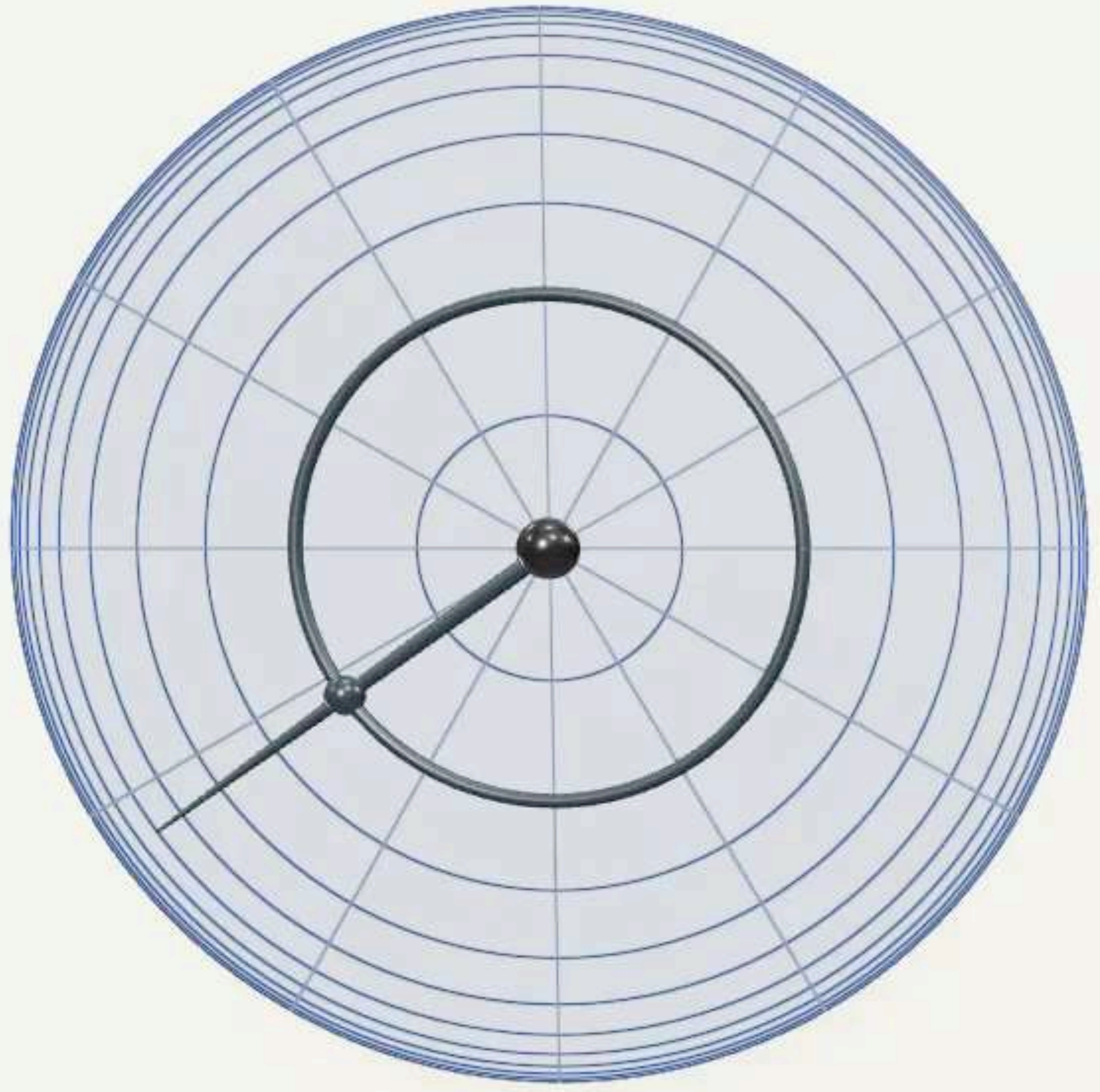
Generalizing the technique

Our euclidean calculation relied on us understanding circles, and also parallel curves from the axis

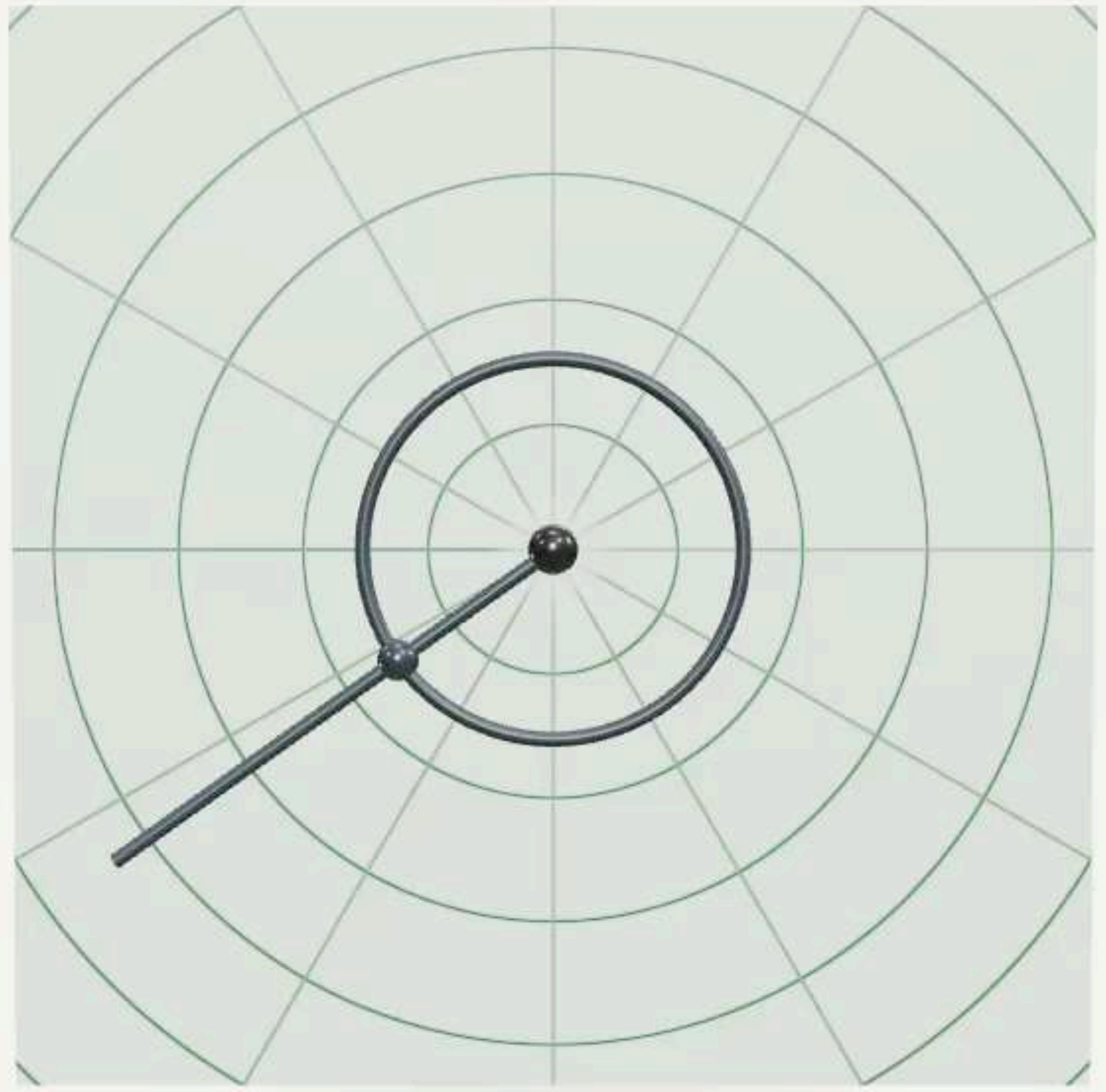




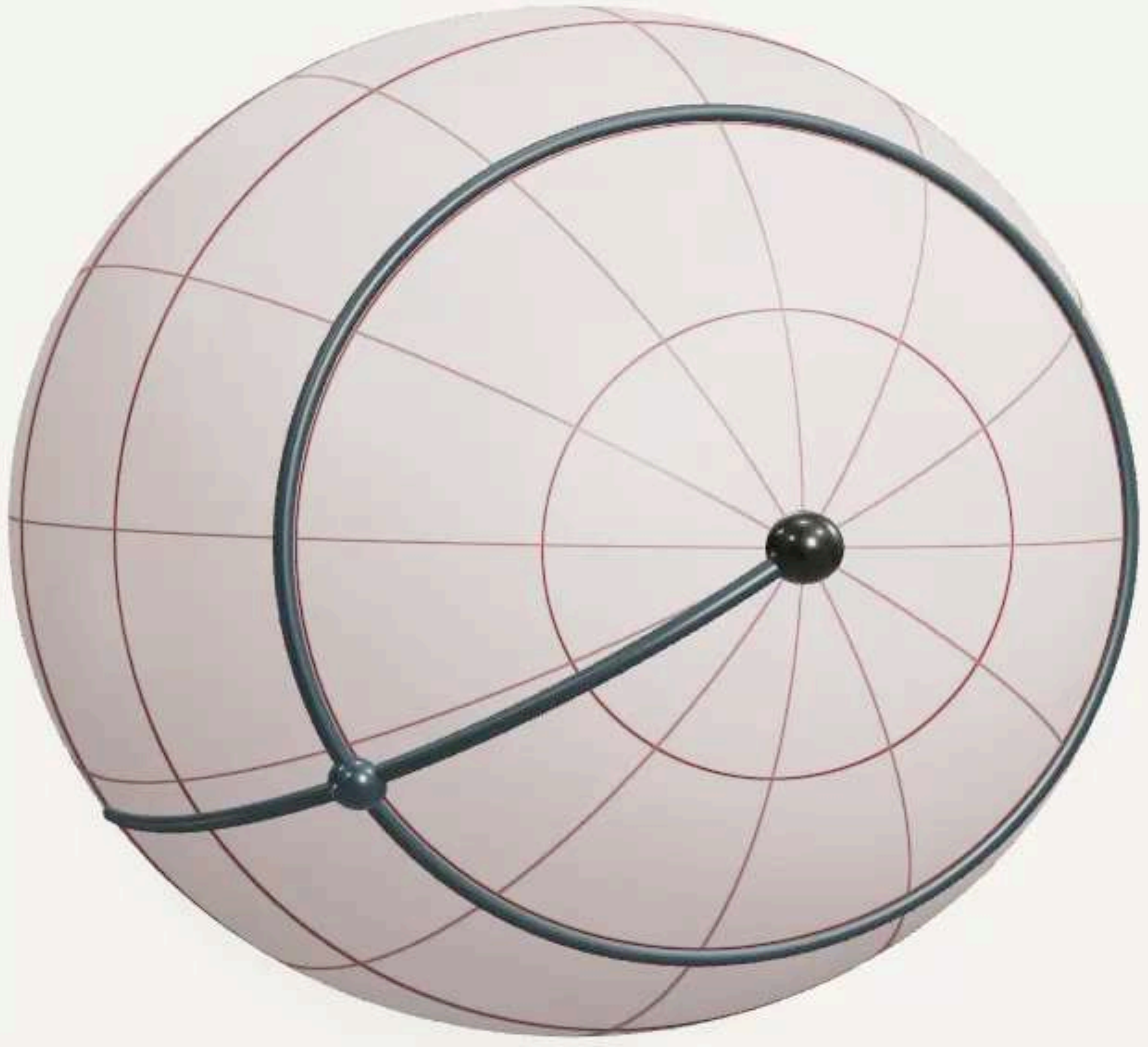
Understanding Circles



$C = \sinh(r)$



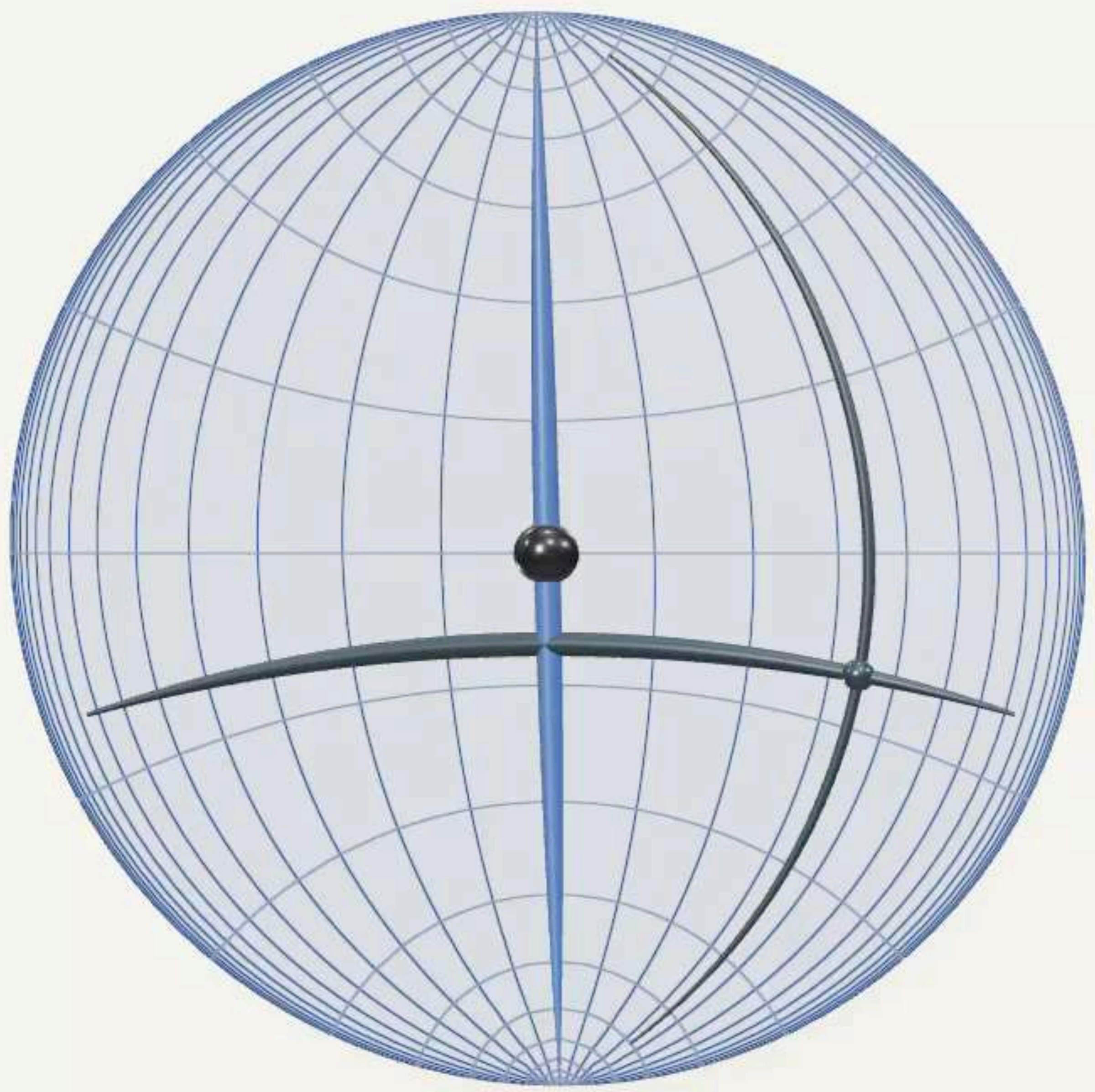
$C = r$

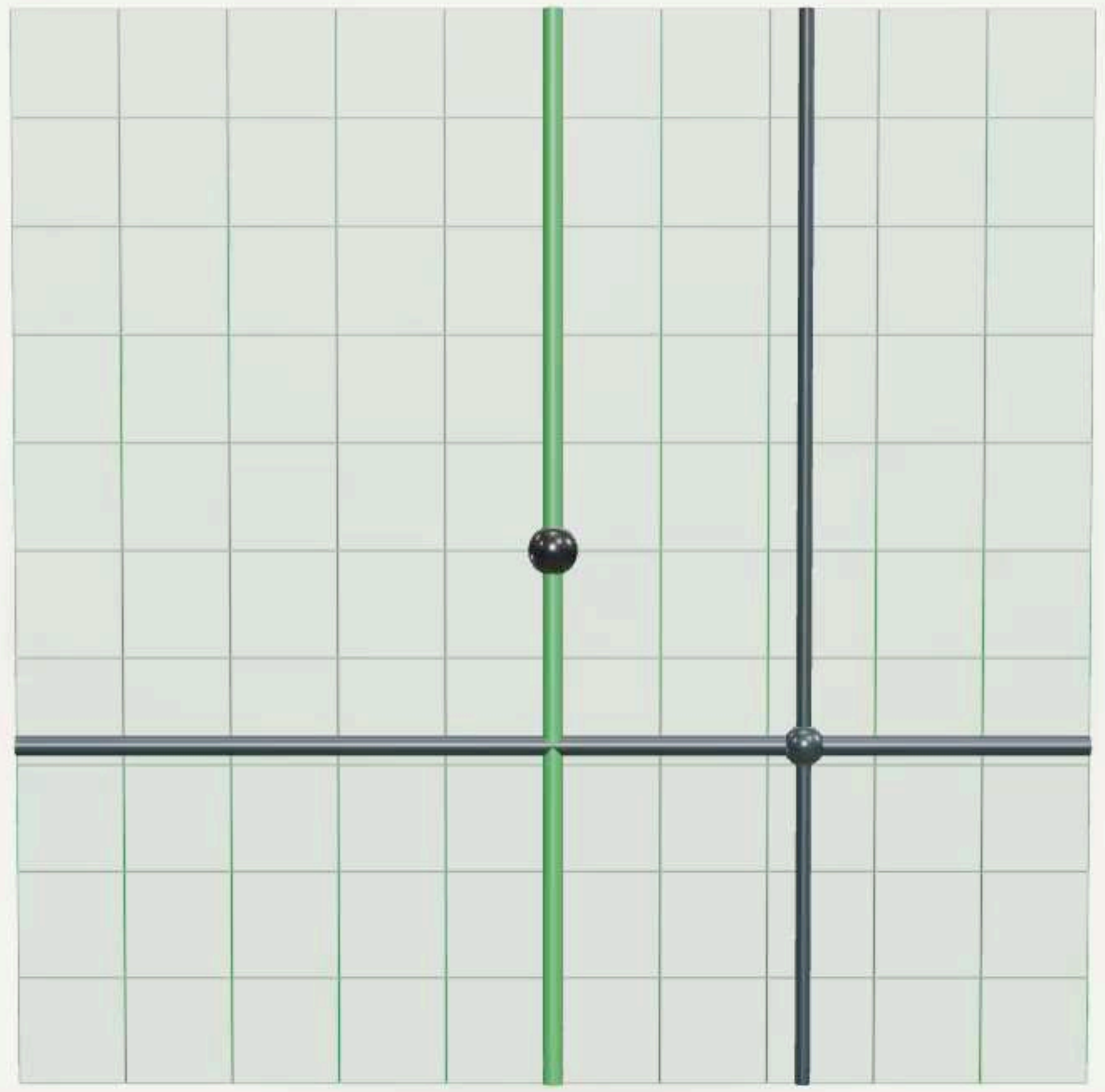


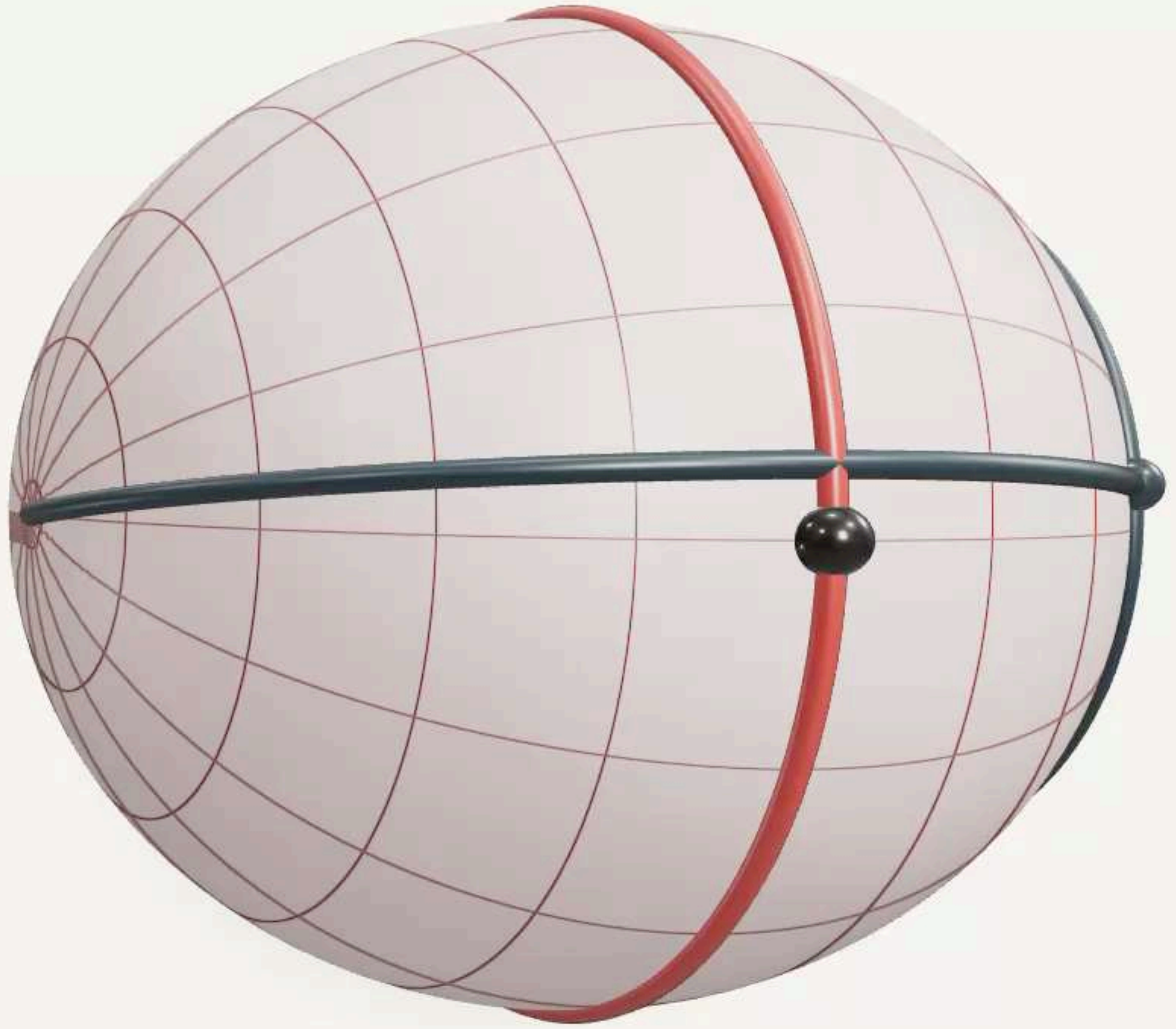
$C = \sin(r)$

Understanding Divergence

How much do things spread out as you move away from the geodesic axis?







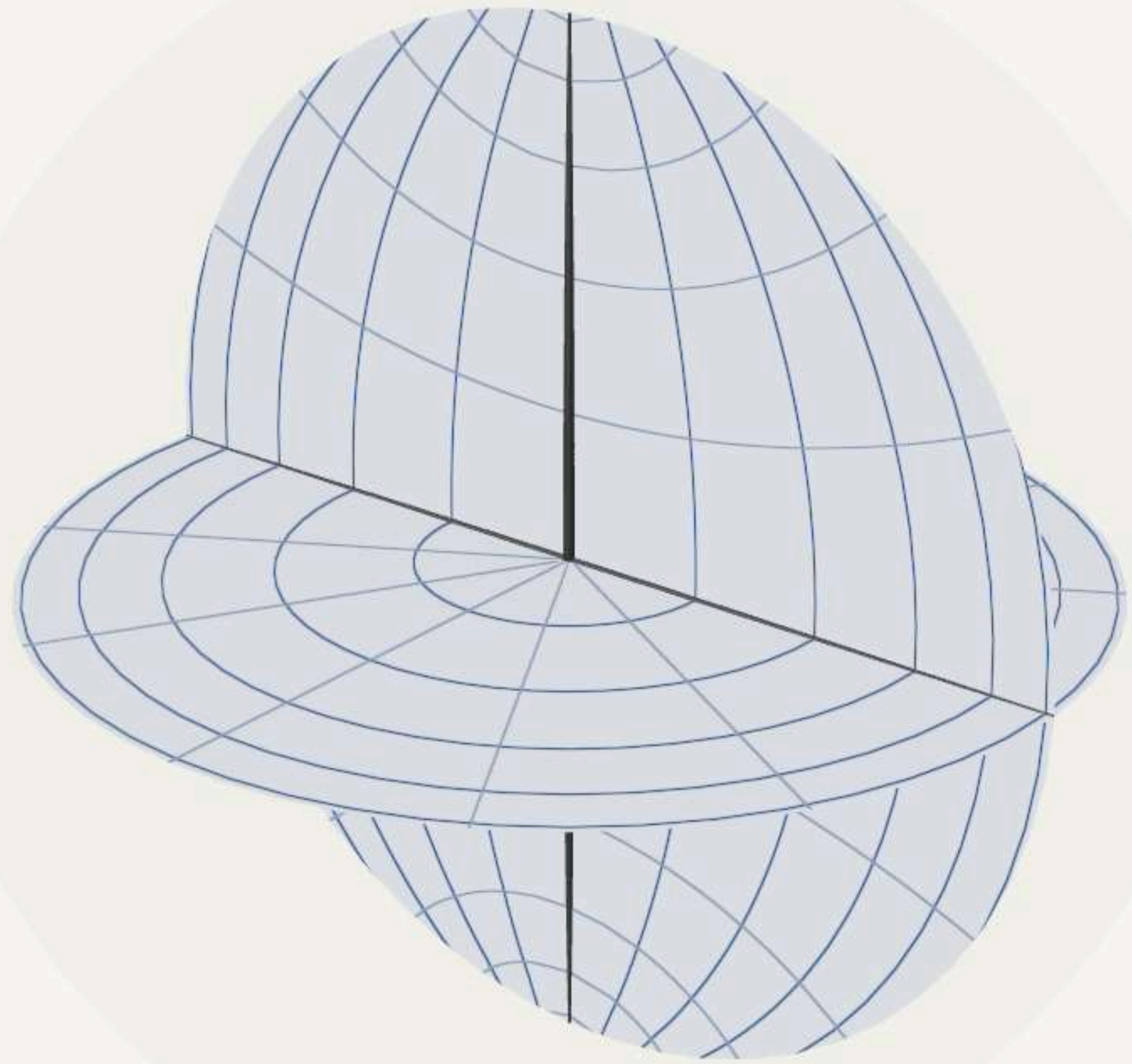
$D = \cosh(r)$

$D = 1$

$D = \cos(r)$

Building Coordinates

The distance r from the central geodesic, height h along it, and angle θ around it assemble into a cylindrical coordinate system



$$ds^2 = dr^2 + C^2\, d\theta^2 + D^2\, dh^2$$

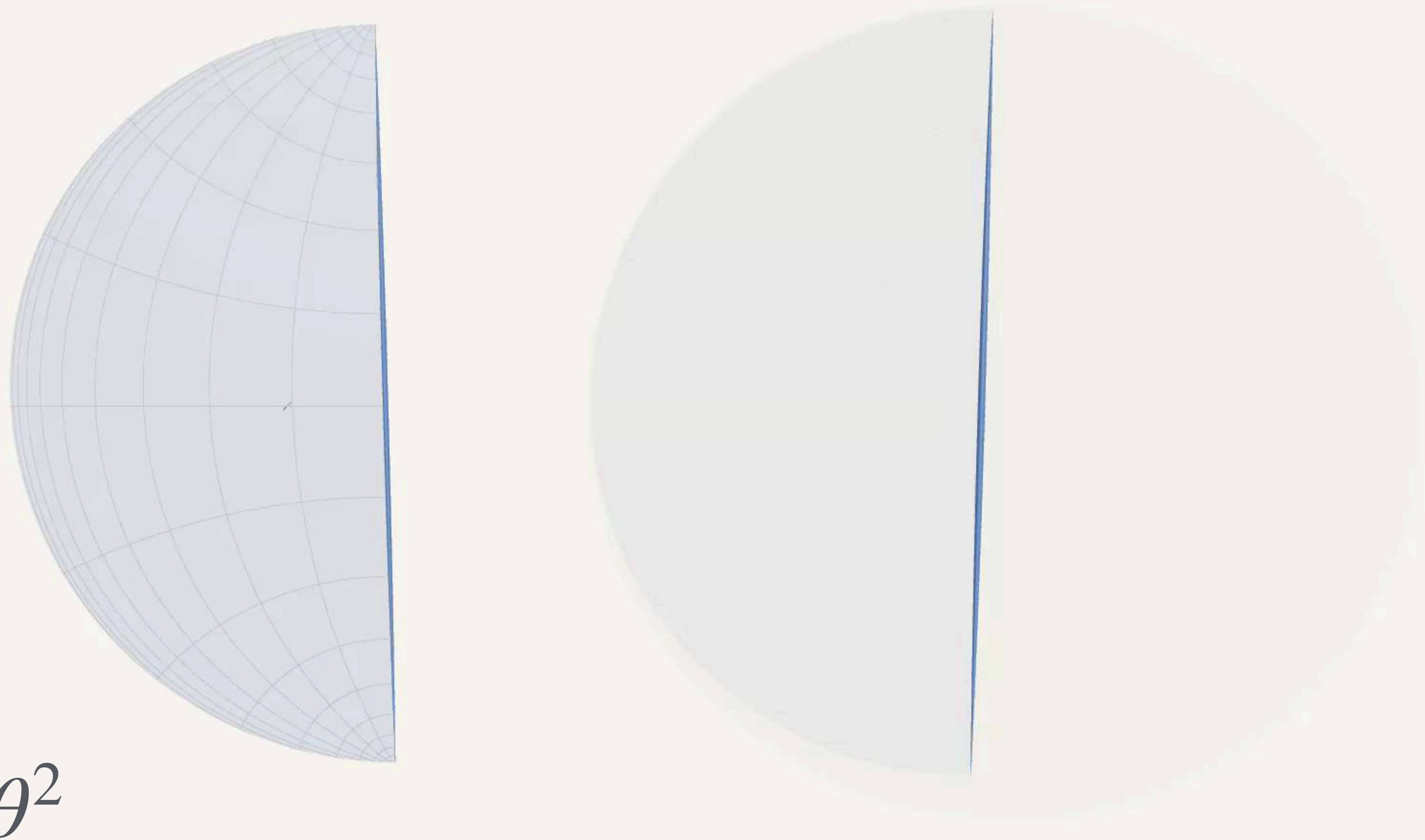
Revolving in Hyperbolic Space

Curve:

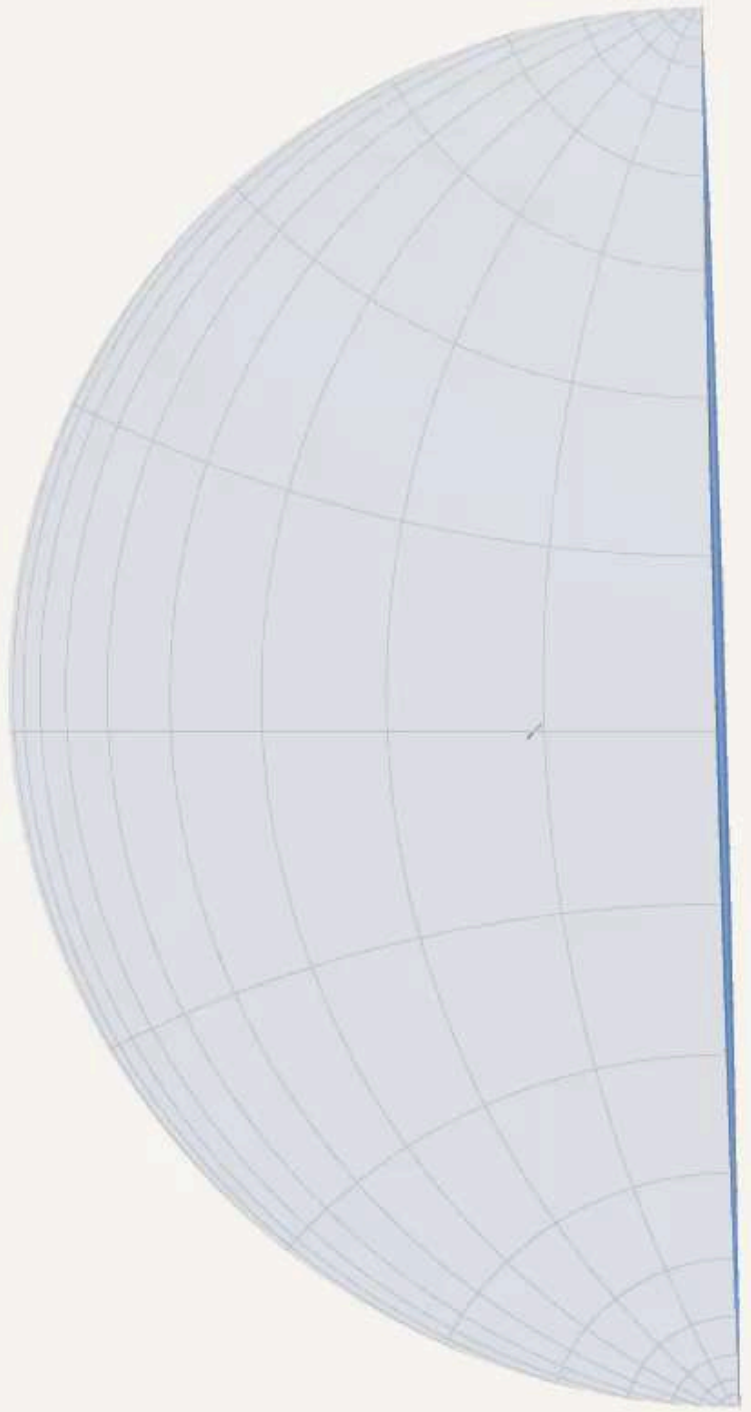
$$u \mapsto (r(u), h(h))$$

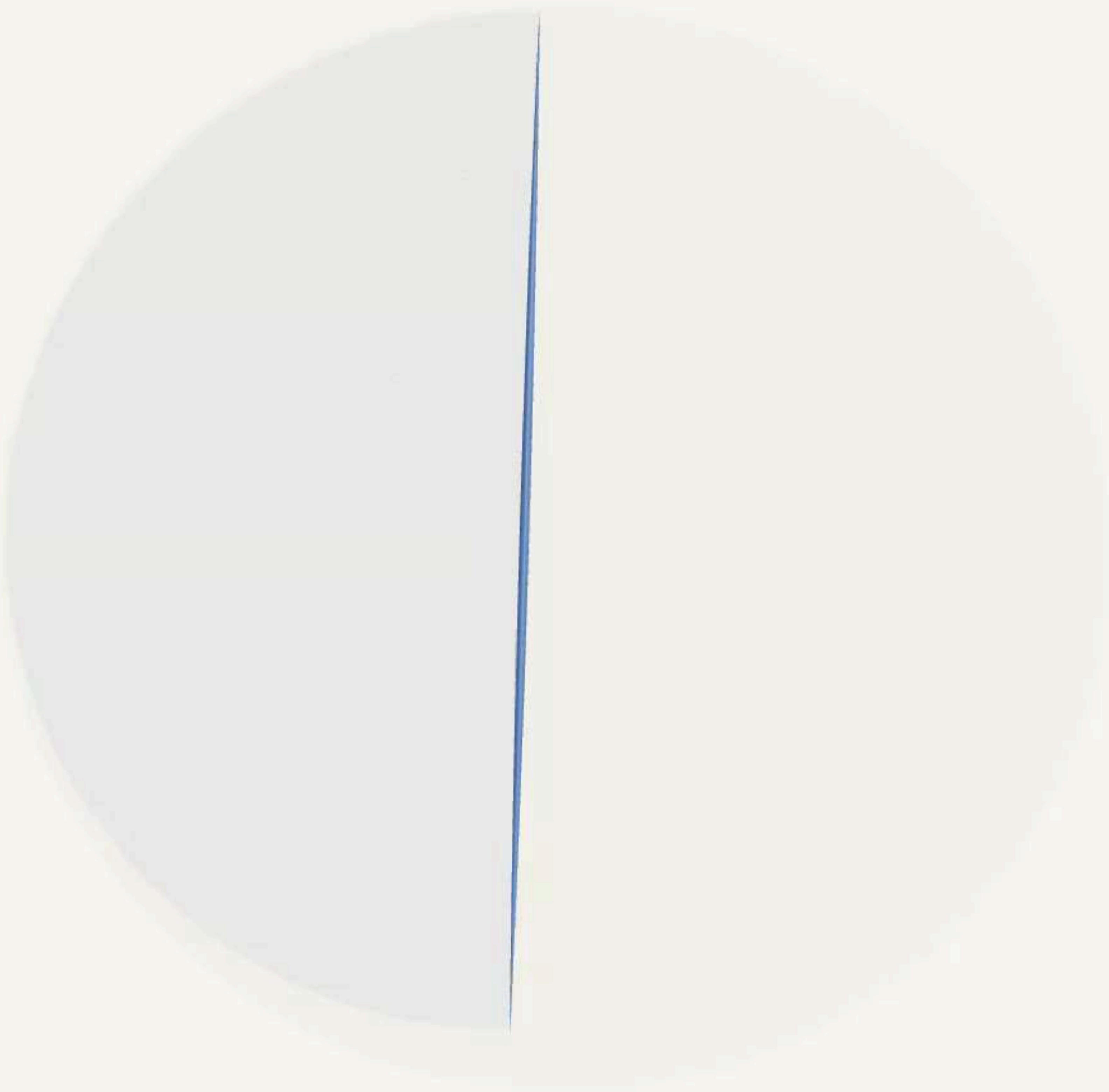
Surface:

$$ds^2 = (\dot{r} + D^2\,\dot{h}^2)\,du^2 + C^2\,d\theta^2$$



Revolving in Hyperbolic Space





Reverse this!

A surface with metric

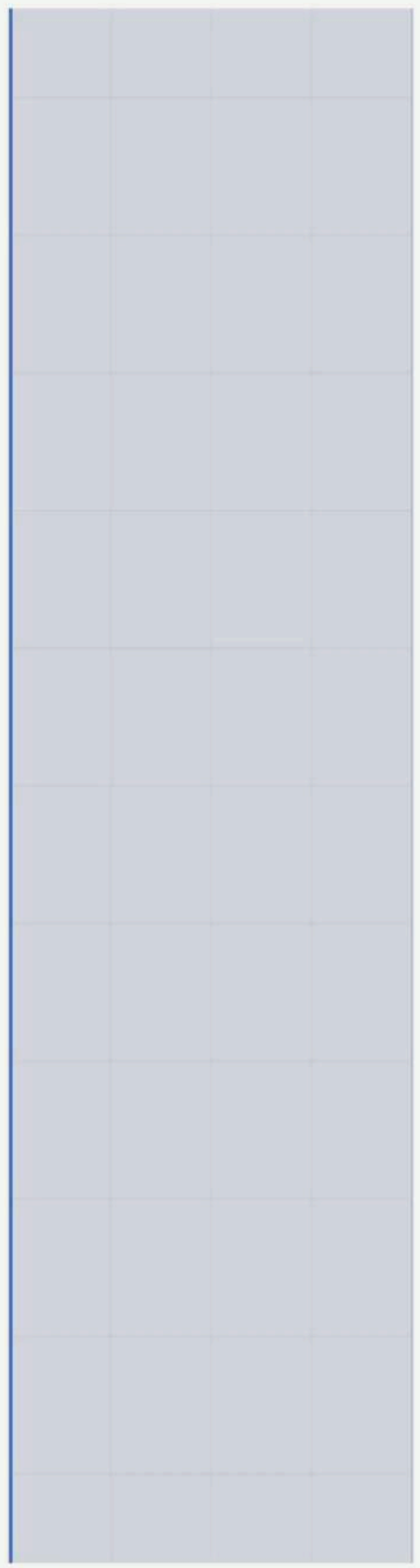
$$ds^2 + f^2(s)\,d\theta^2$$

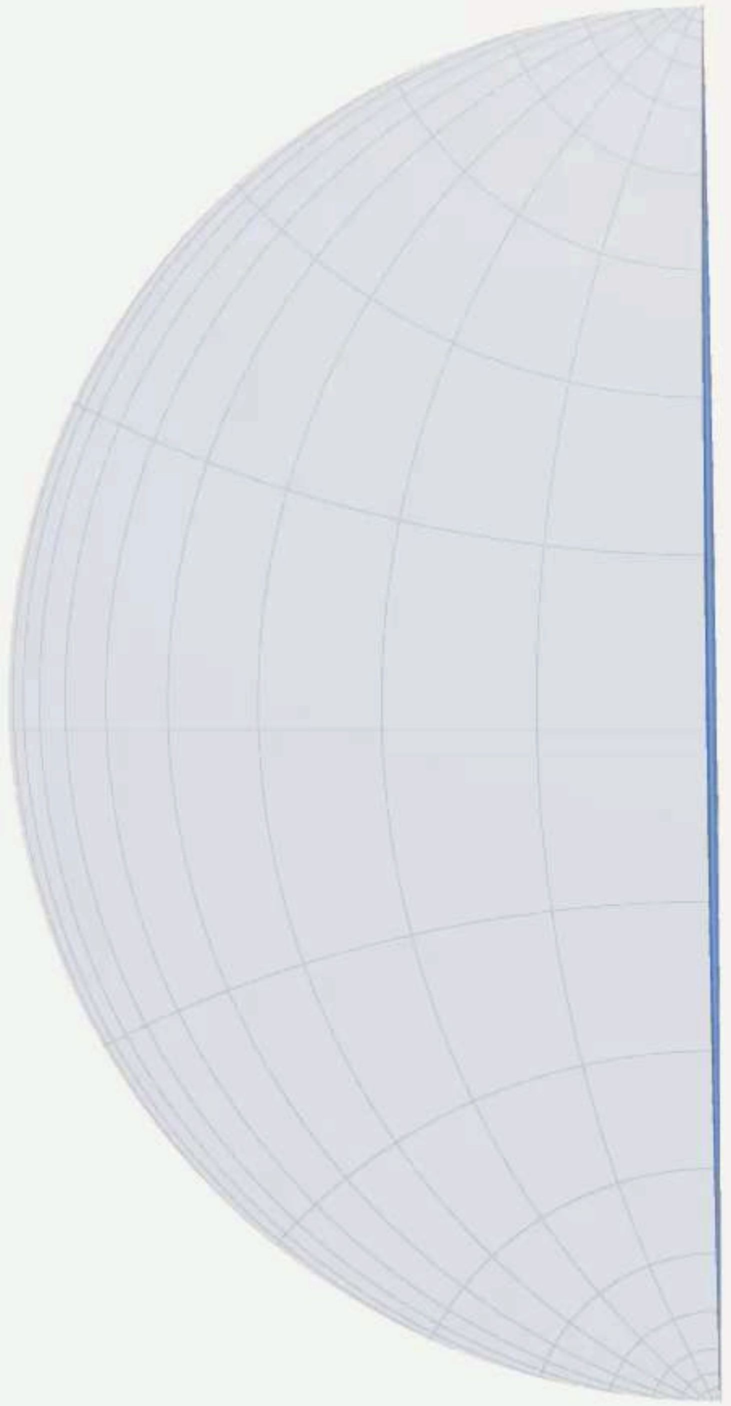
embeds in $\mathbb{H}^3$ *if and only if*

$$\dot{f}^2 \leq 1 + f^2$$

Cooking Up a Cusp, II:

Now the entire surface $\mathbb{H}^2/\langle P \rangle$ embeds!





Revolving in the 3-Sphere

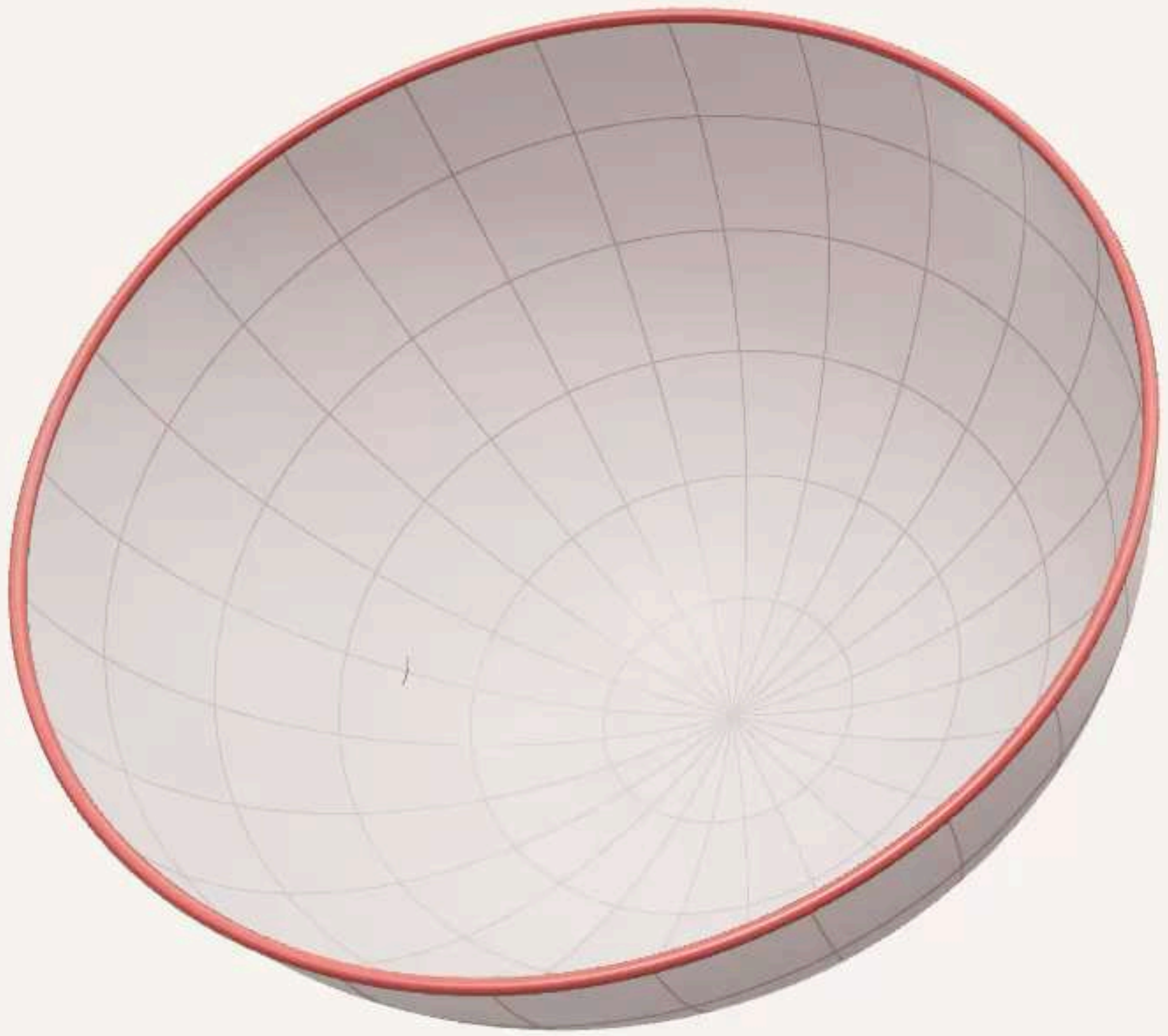
The same formula holds, phrased geometrically in terms of circumference (C) and divergence (D)

$\Longrightarrow$ *A surface with metric*

$$ds^2 + f^2(s)\,d\theta^2$$

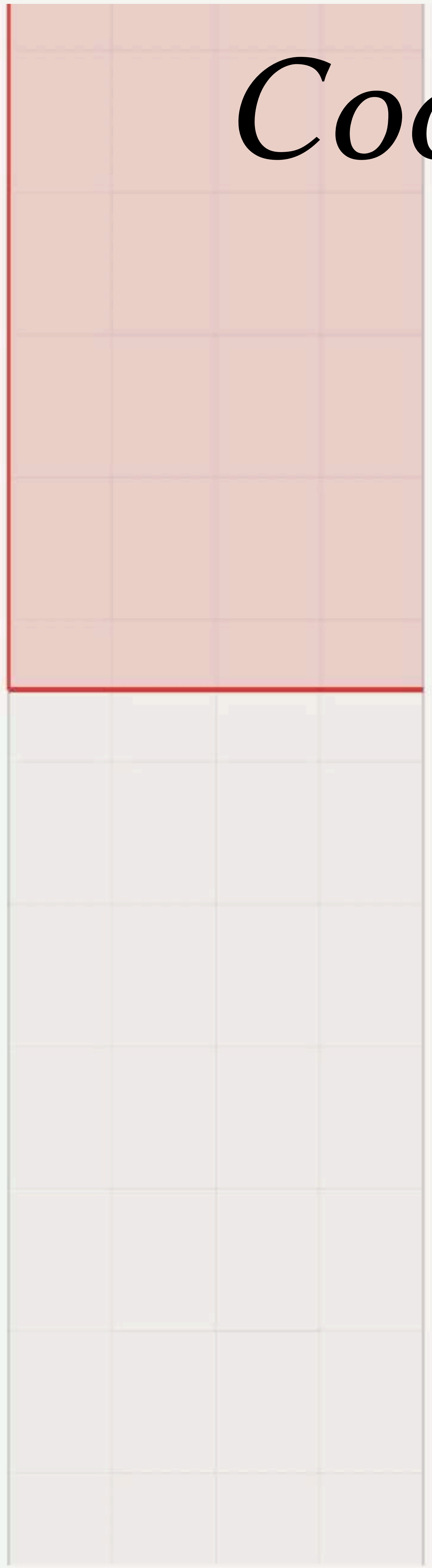
embeds in $\mathbb{S}^3$ *if and only if*

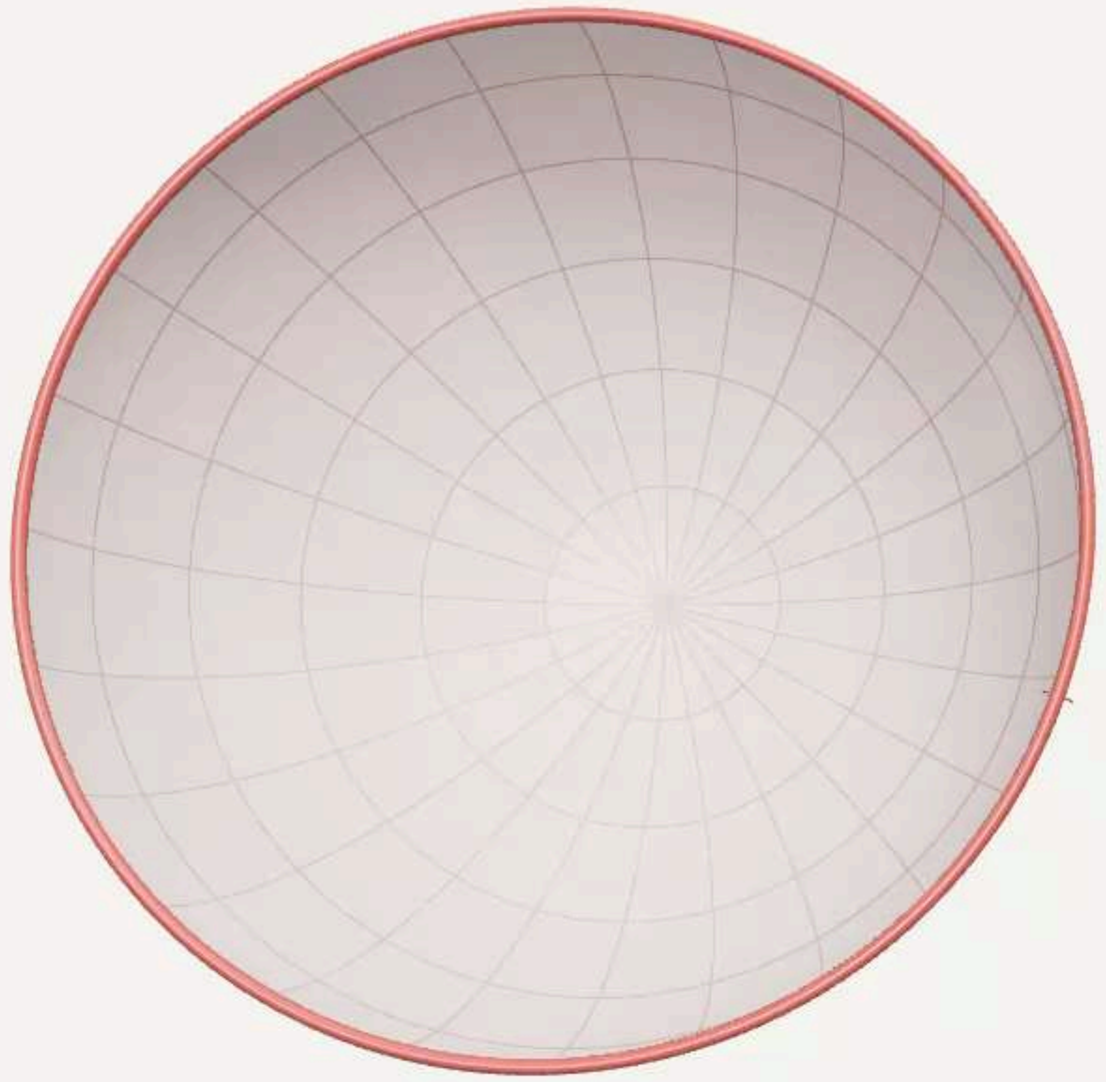
$$\dot{f}^2 \leq 1 - f^2$$



Cooking Up a Cusp, III:

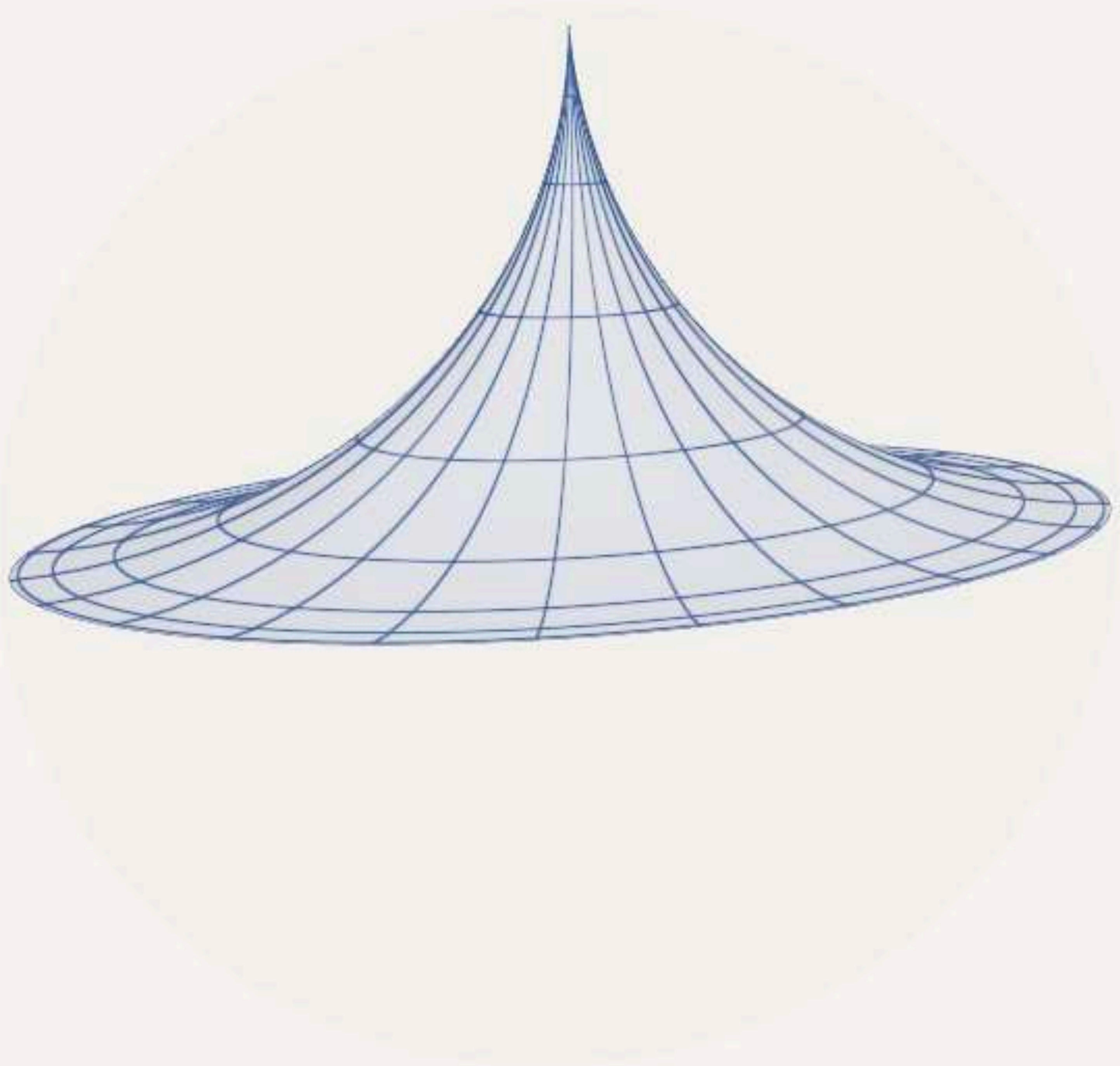
And you thought Euclidean space did a bad job!



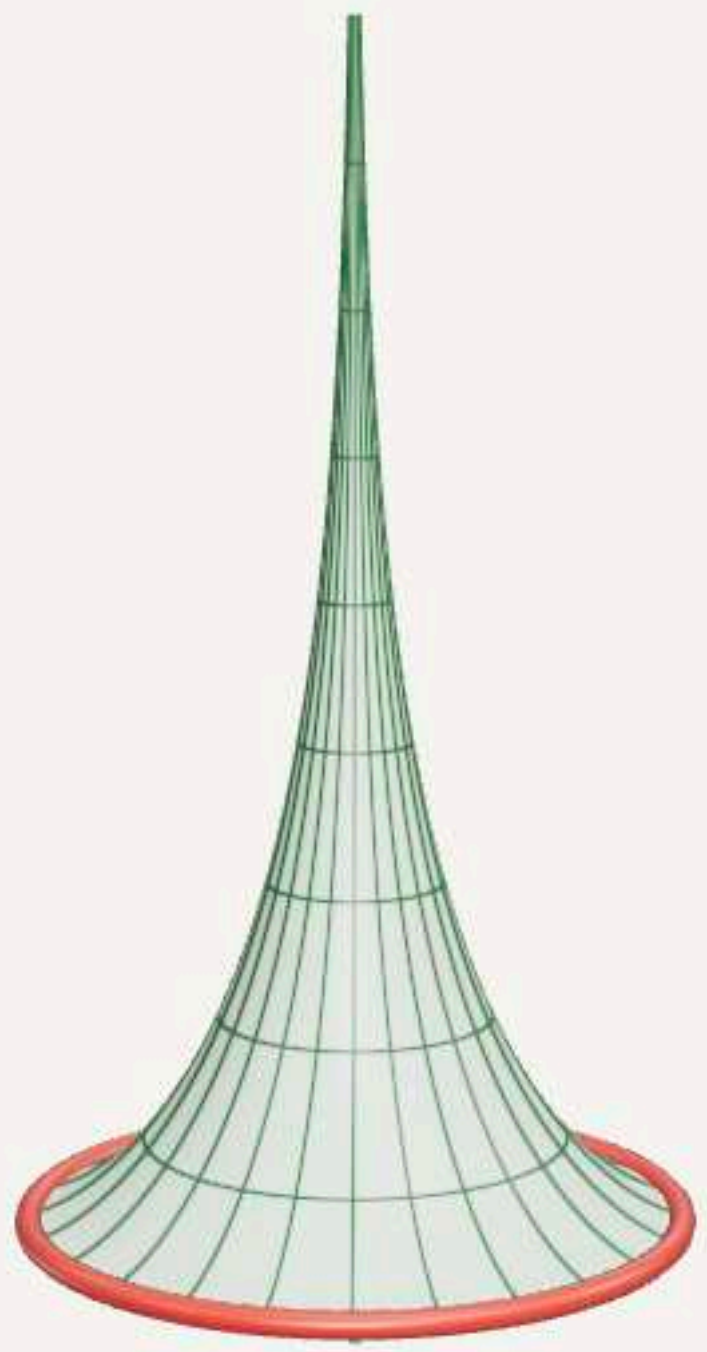


The pseudosphere only embeds for $u \geq \sqrt{2}$

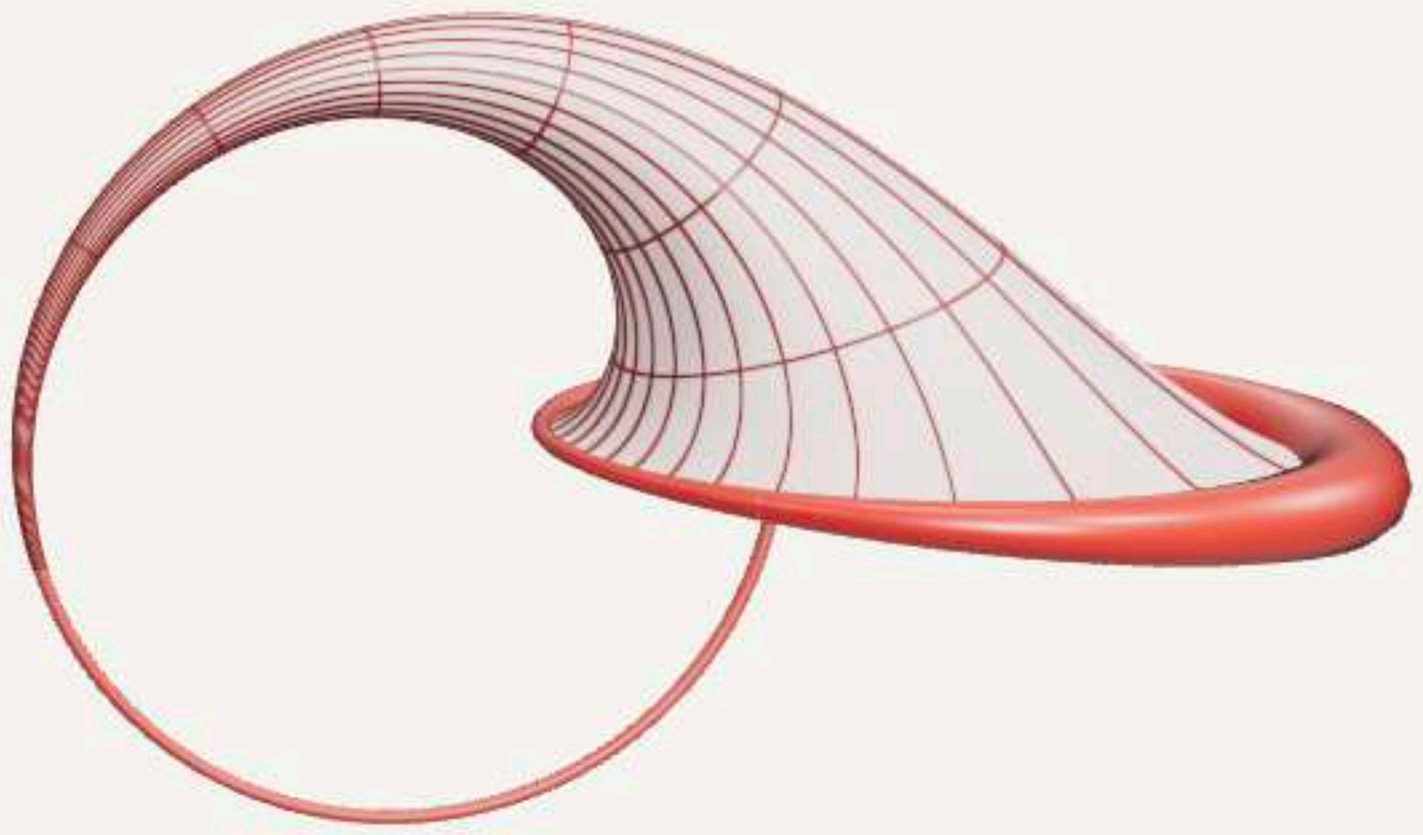
Our friend across all three worlds



k=-1, embeds all the way



k=0, embeds for $u \geq 1$

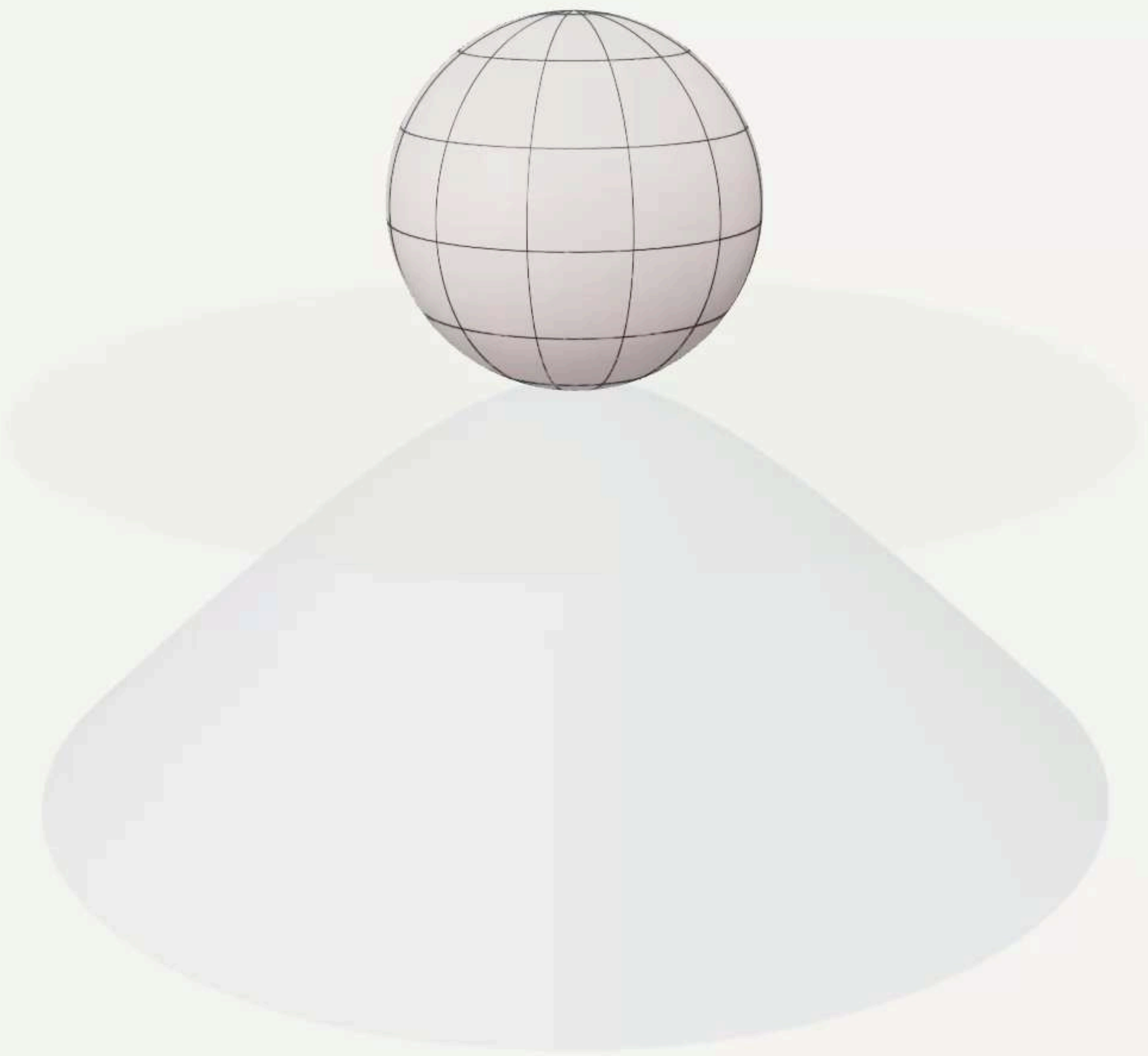


k=1, embeds for $u \geq \sqrt{2}$

But there's more than 3 worlds!

The geometries of constant curvature *k* form a *continuous family* as *k* varies

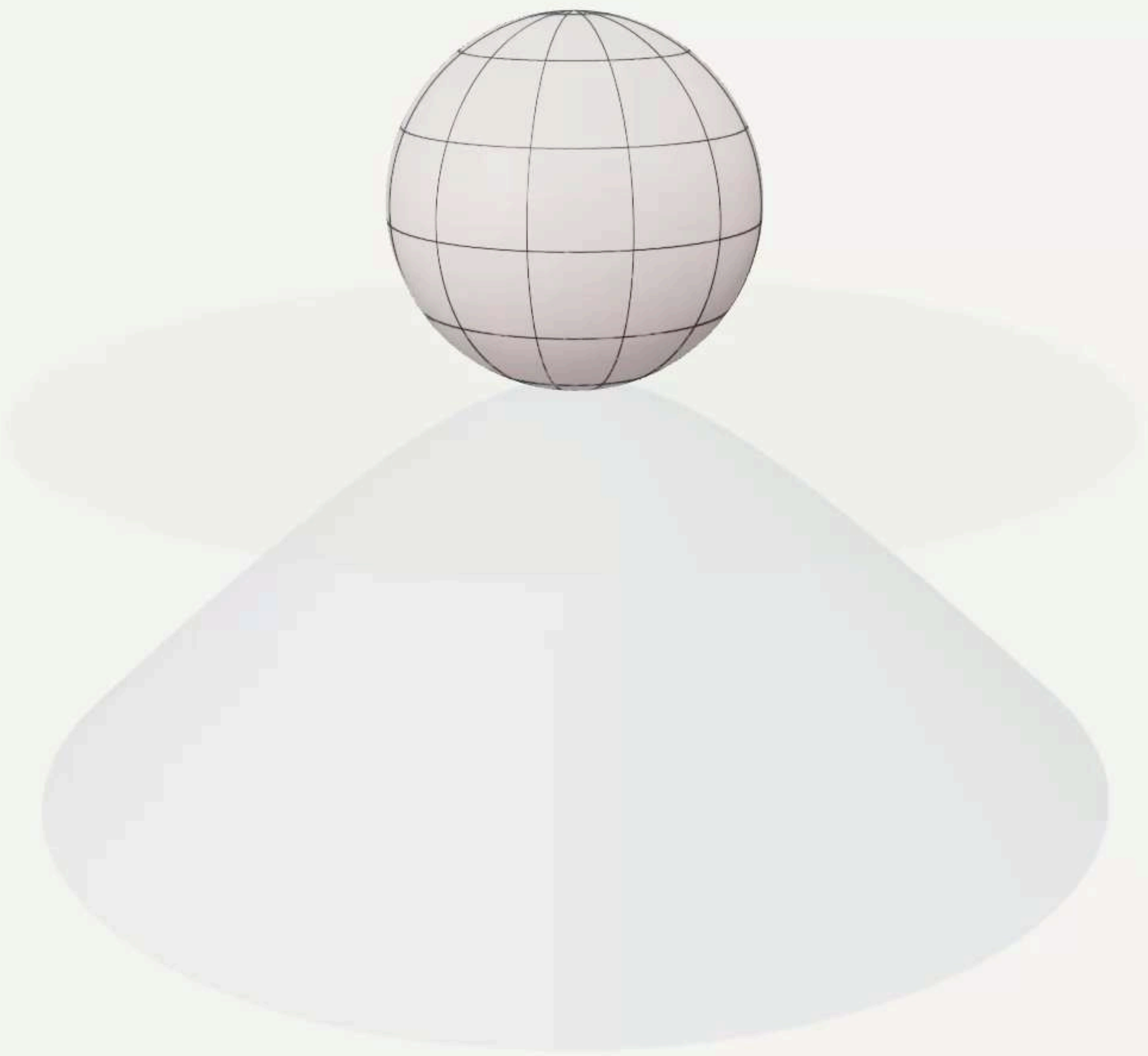
This is the natural home to talk about how embeddings depend on curvature!



But there's more than 3 worlds!

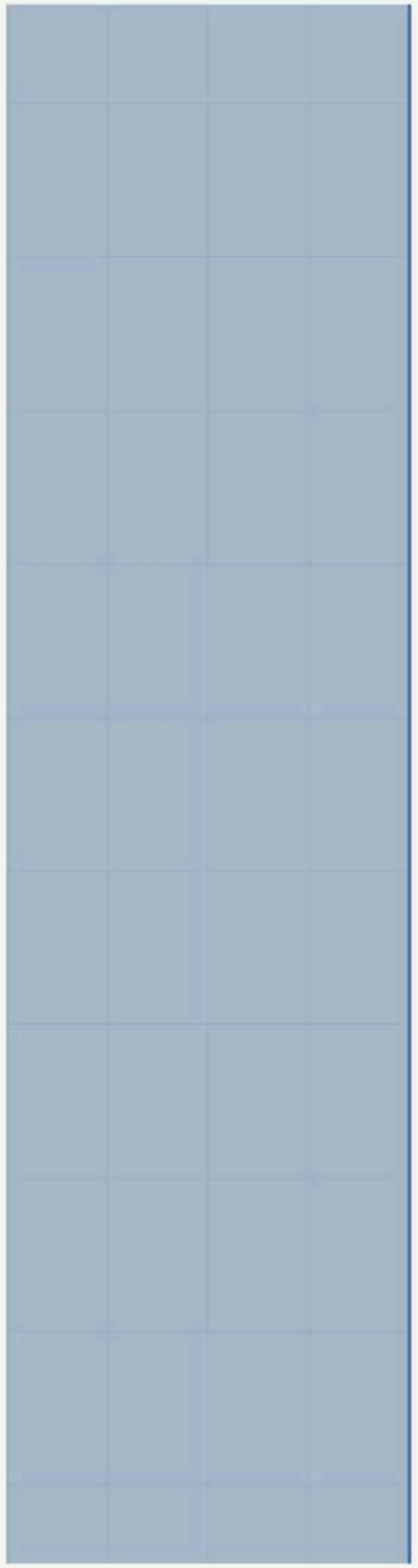
$$C_k(r) = \begin{cases} \dfrac{\sin(\sqrt{k}\, r)}{\sqrt{k}}, & k > 0, \\ r, & k = 0, \\ \dfrac{\sinh(\sqrt{-k}\, r)}{\sqrt{-k}}, & k < 0. \end{cases}$$

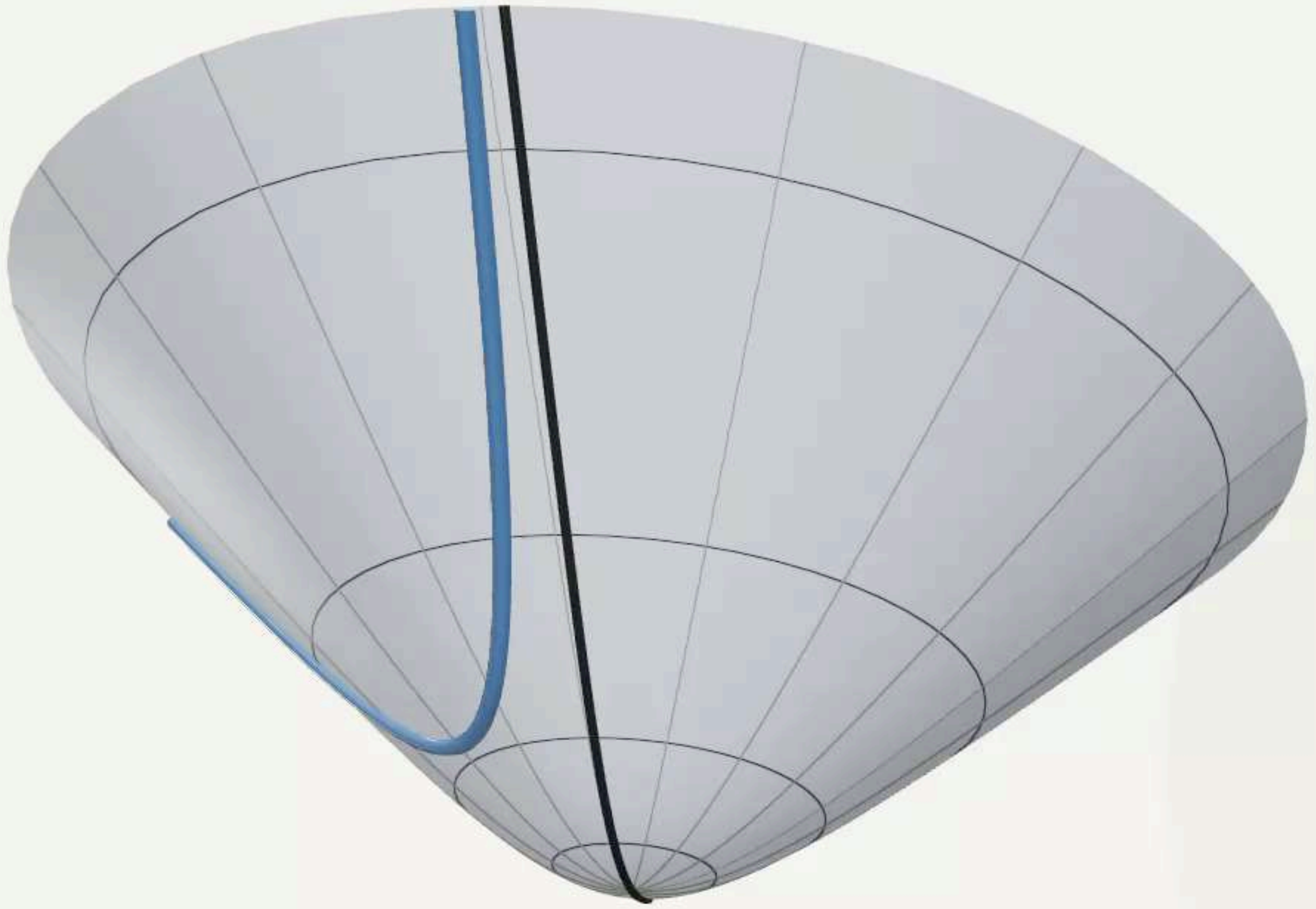
$$D_k(r) = \begin{cases} \cos(\sqrt{k}\, r), & k > 0, \\ 1, & k = 0, \\ [2mm]\cosh(\sqrt{-k}\, r), & k < 0. \end{cases}$$

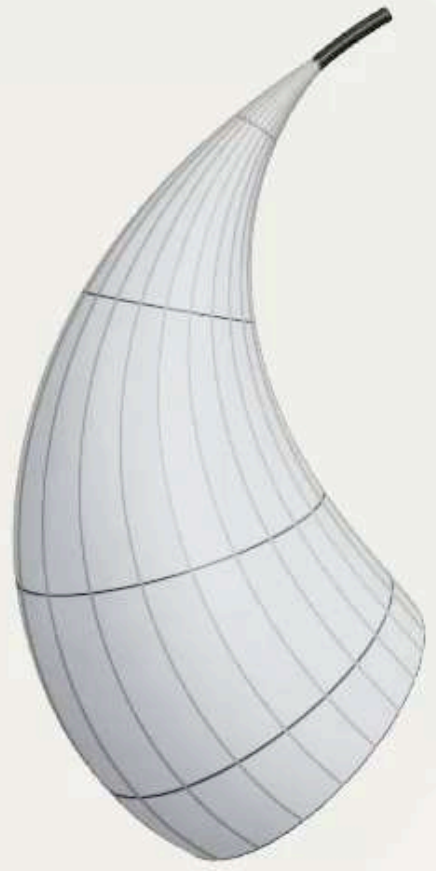


k = −1.75

The Pseudosphere in Curvature k



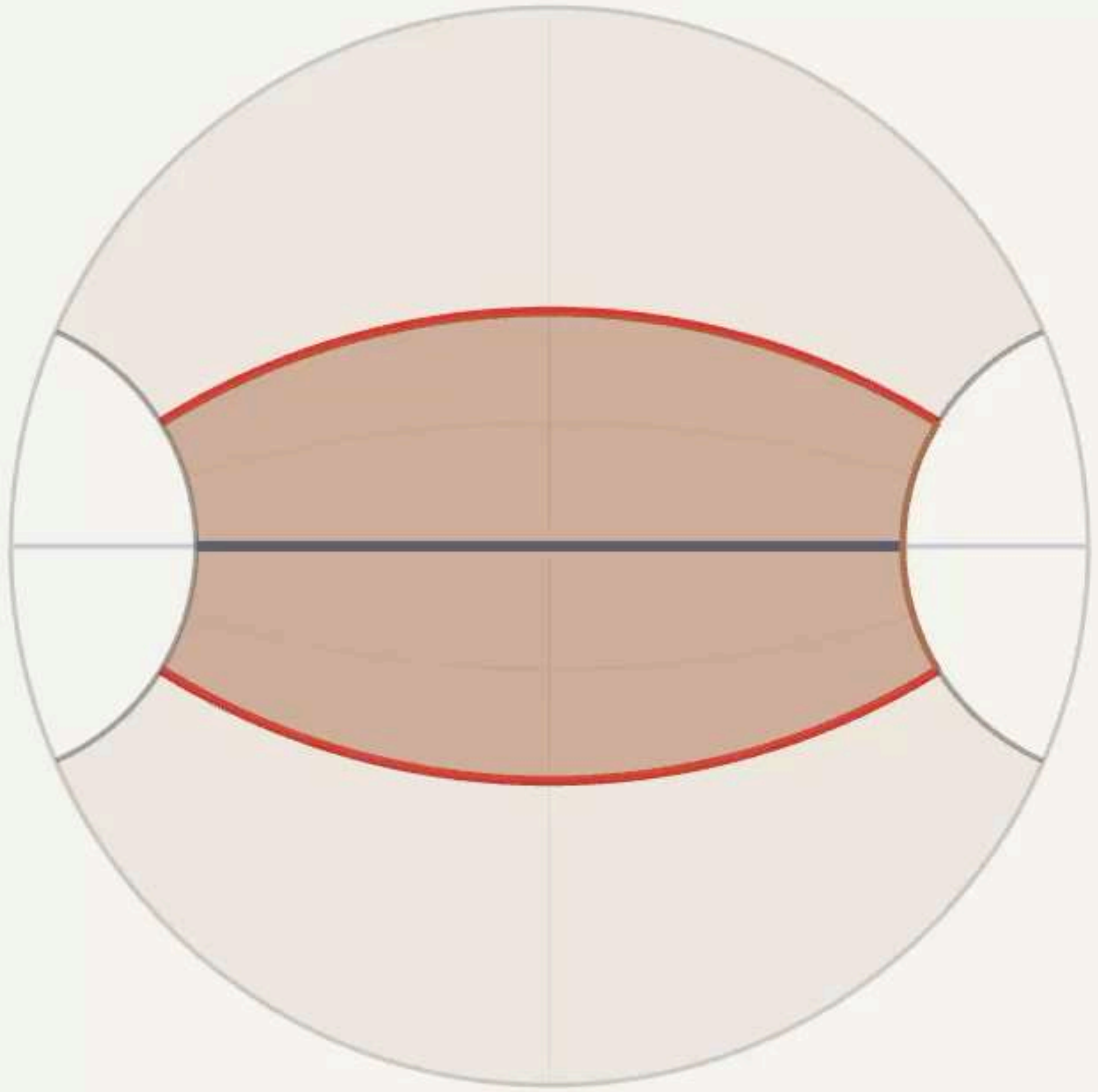


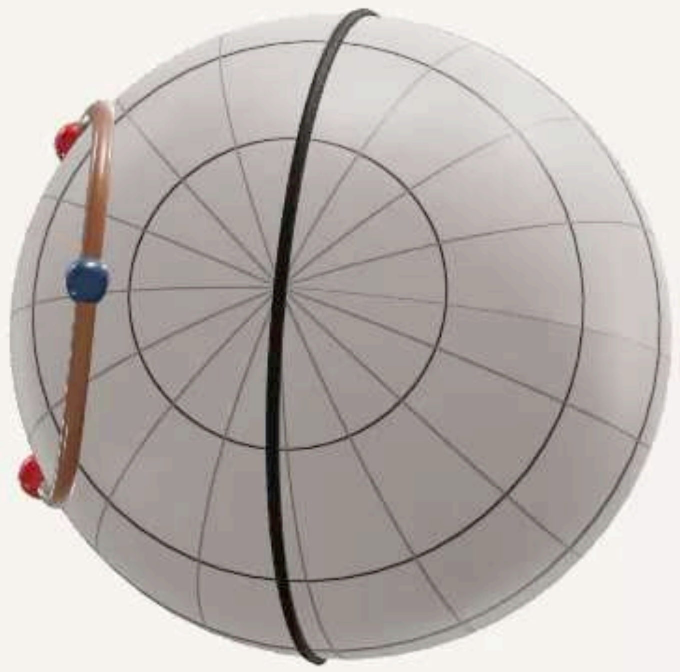


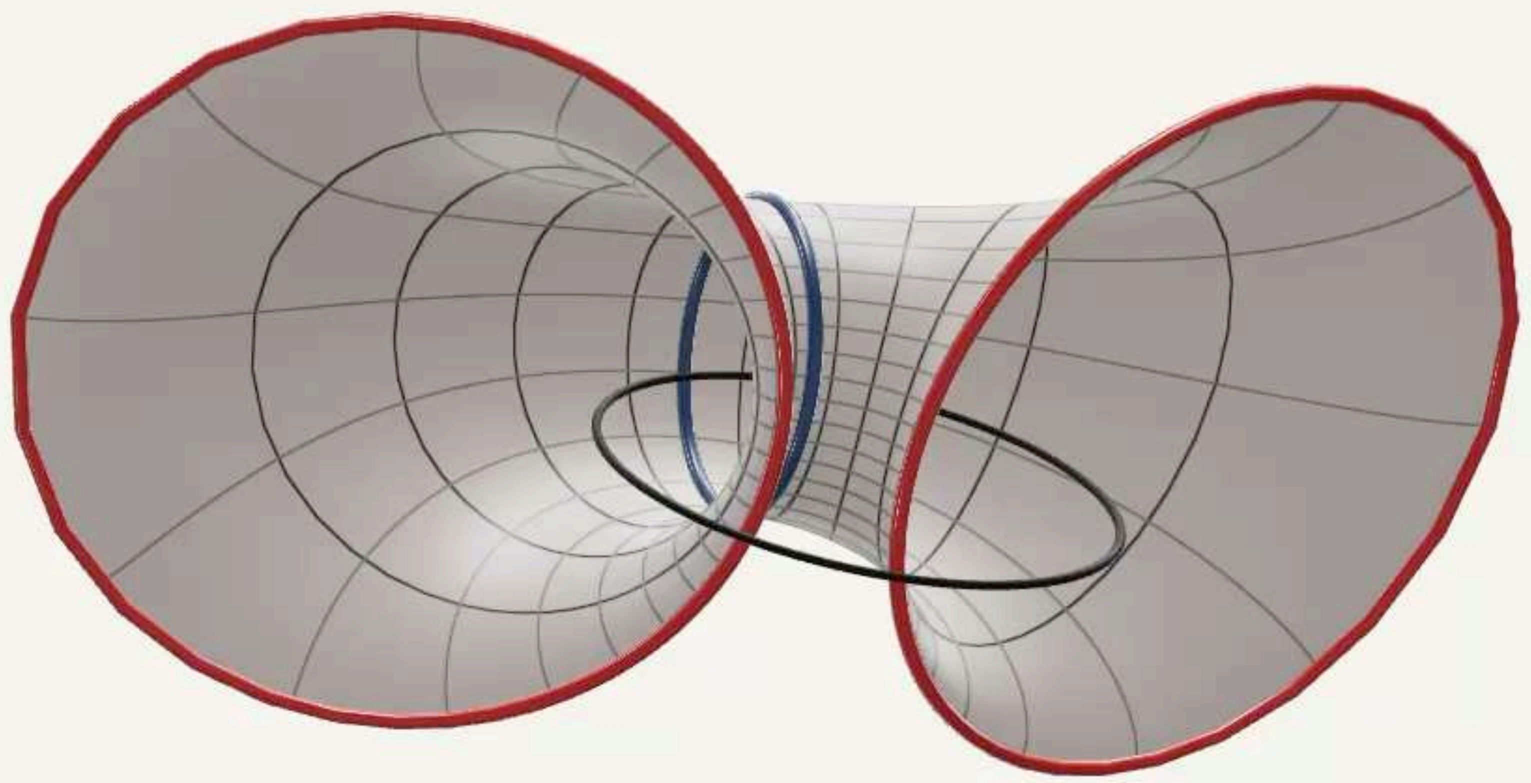
Embeds for
$u \geq \sqrt{k+1}$

k = +1.32

The Collar in Curvature k





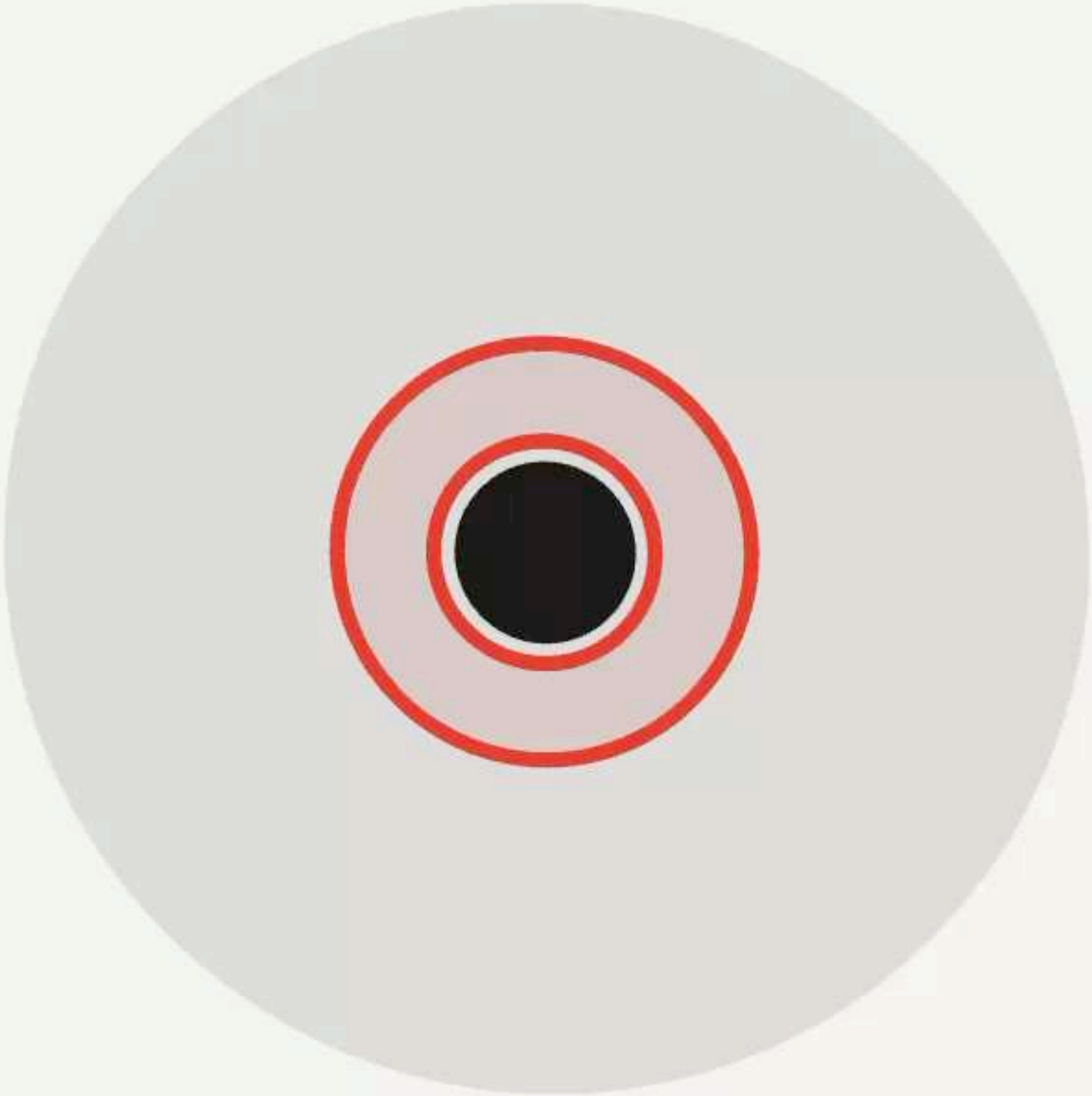


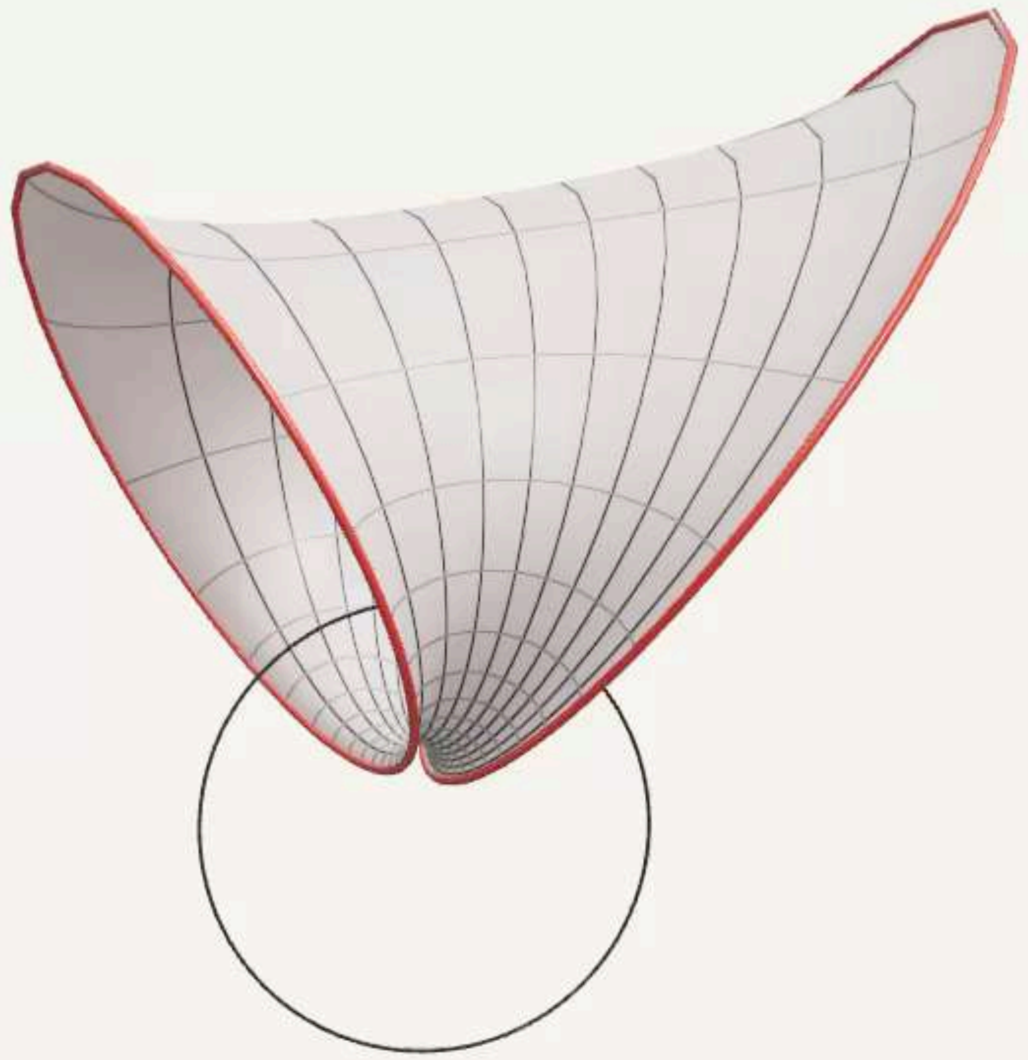
Embeds to width

$$w_k = \begin{cases} \infty, & k \leq -1, \\ \operatorname{arsinh}\sqrt{\dfrac{\left(\frac{2\pi}{\ell}\right)^2 - k}{1+k}}, & -1 < k < \left(\frac{2\pi}{\ell}\right)^2, \\ 0, & k = \left(\frac{2\pi}{\ell}\right)^2. \end{cases}$$

Black Hole Geometry in Curvature k

`k = +0.34 · lip at r = 2.44M`





The full surface embeds once

$$k < \frac{-1}{16M^2}$$

Theorem

(Lander, T-)

A surface with metric

$$ds^2 + f^2(s)\, d\theta^2$$

embeds in constant curvature k if and only if

$$\dot{f}^2 \leq 1 - kf^2$$

$$\Longrightarrow$$

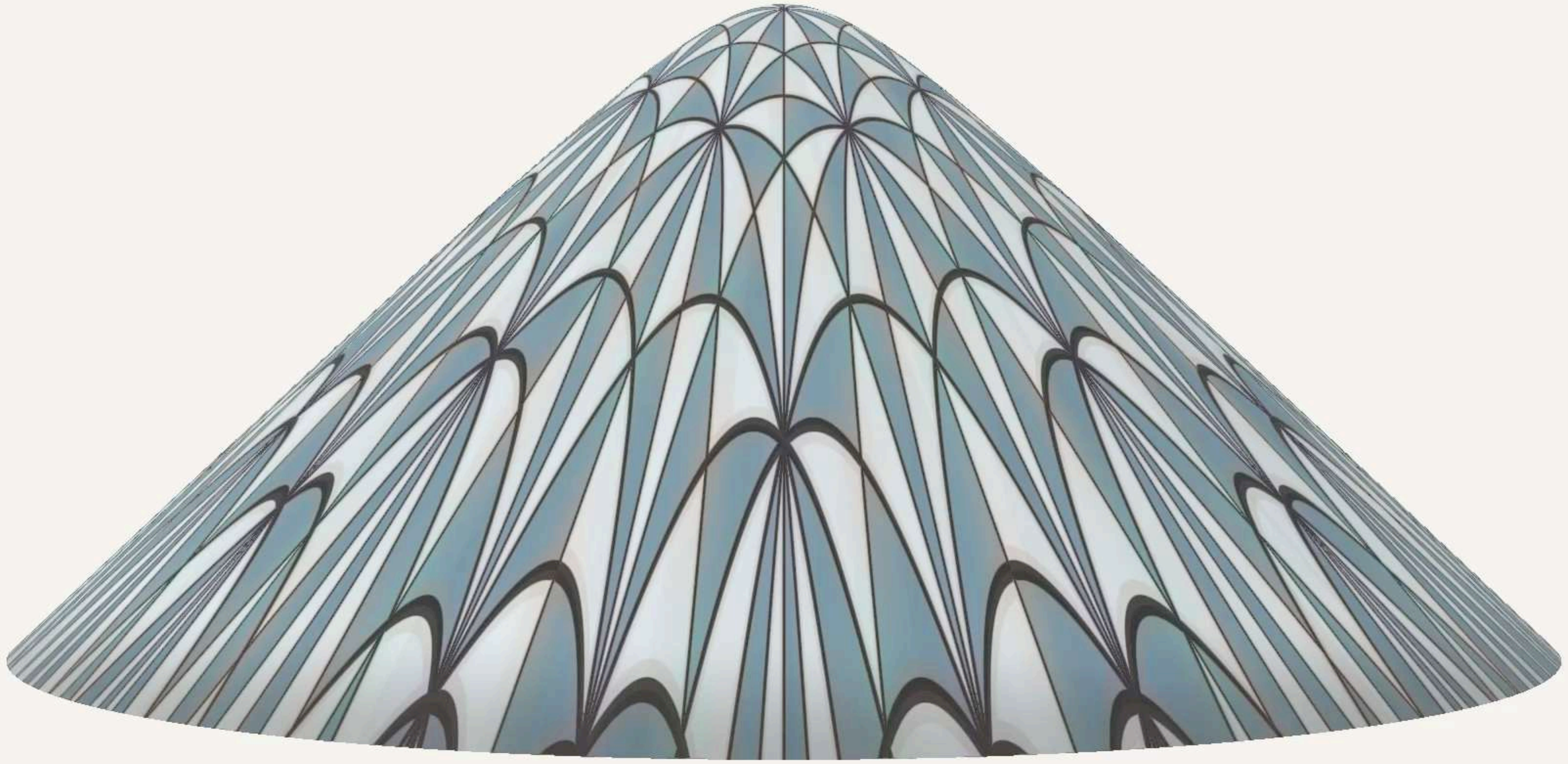
If a surface embeds as a surface of revolution in the 3 sphere (for example), then it also does in hyperbolic space.

IV

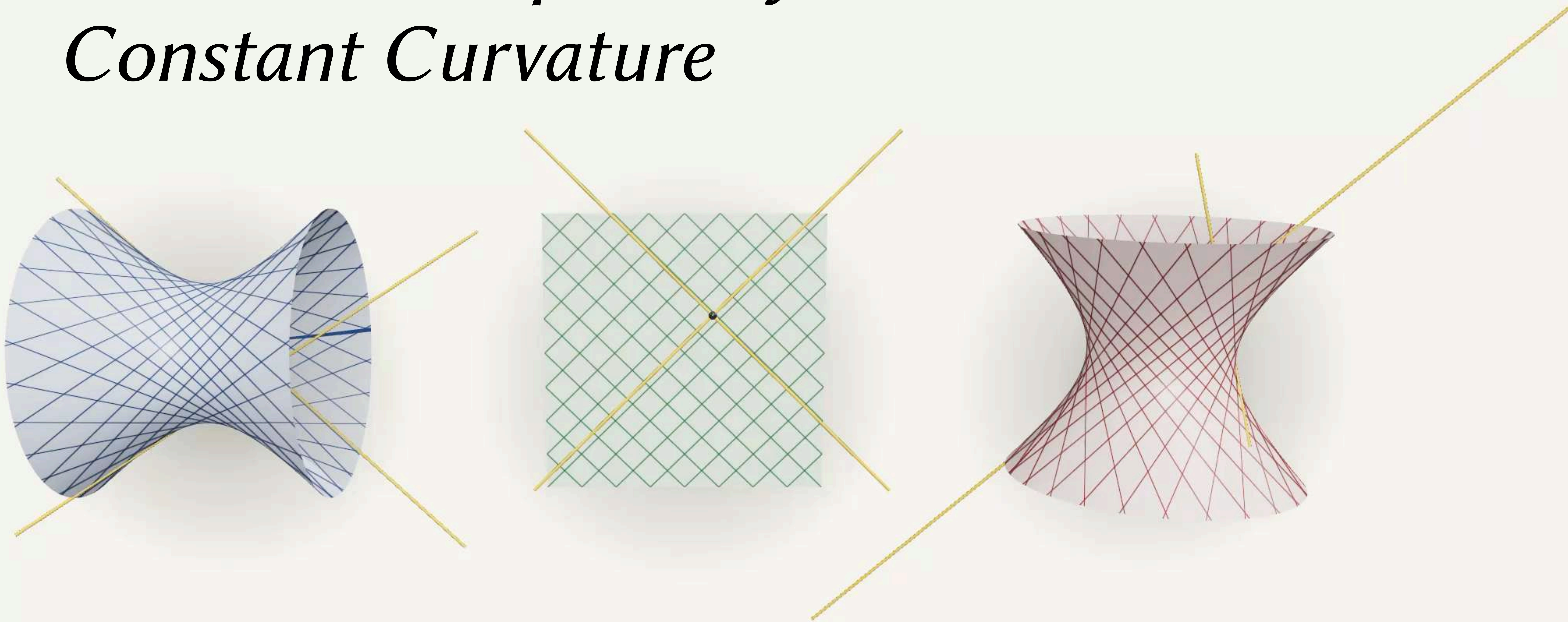
The Lorentzian World

While the hyperbolic plane does not embed in flat Euclidean space, it **does embed in flat Lorentzian space!**

So there's a whole new world out there..



The Lorentzian Spaces of Constant Curvature

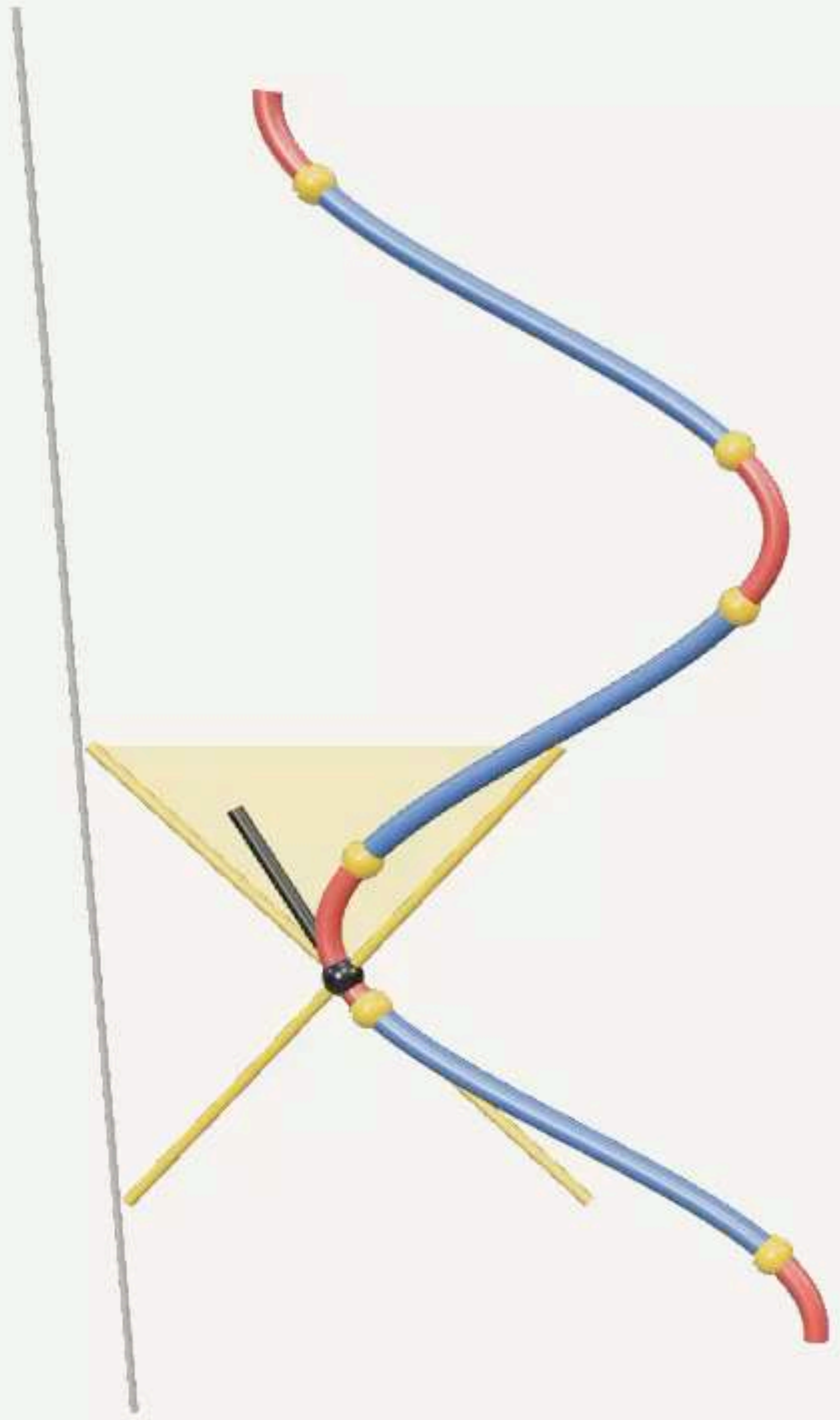


Anti de Sitter,
negatively curved

Minkowski,
flat

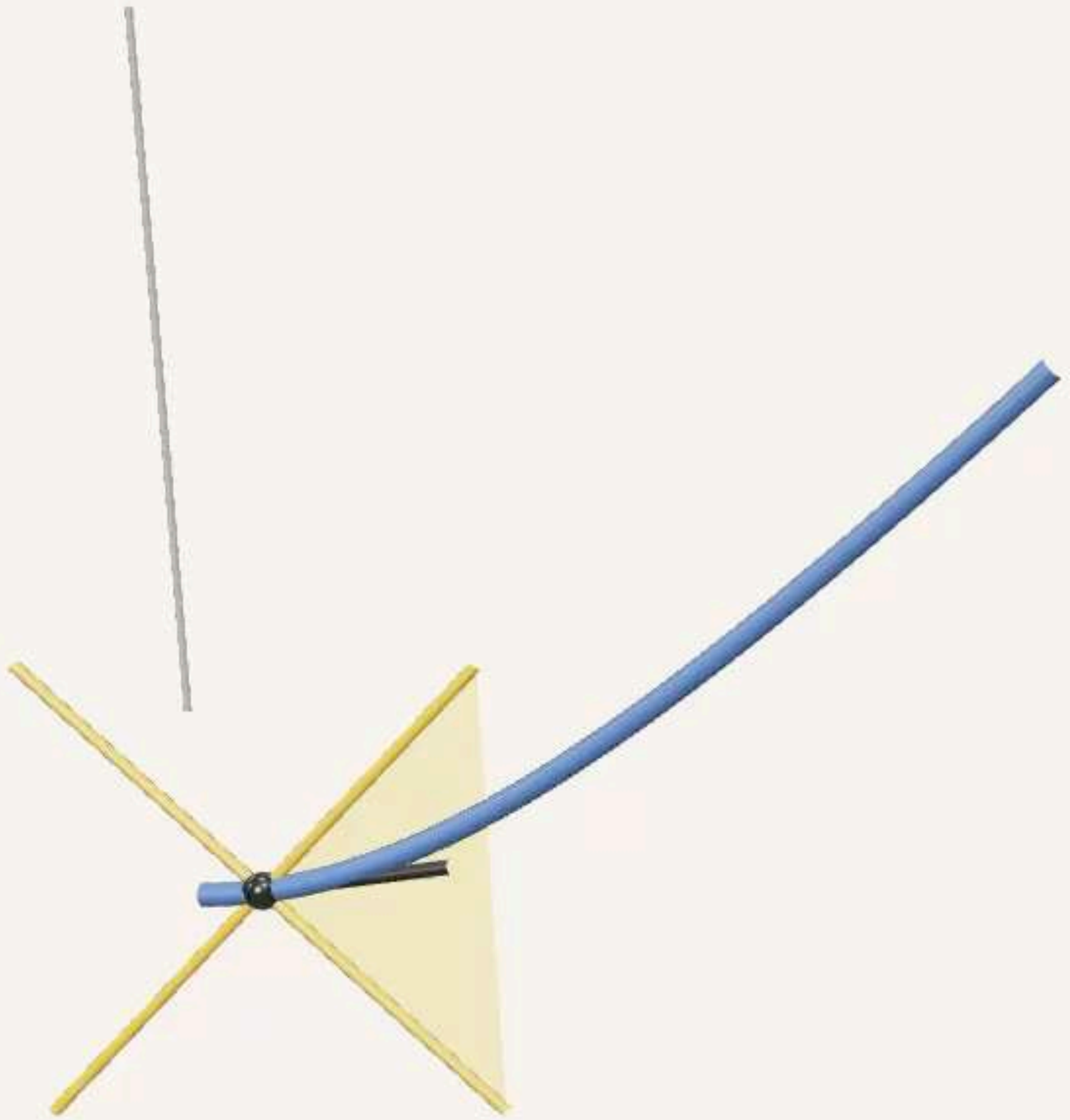
de Sitter,
positively curved

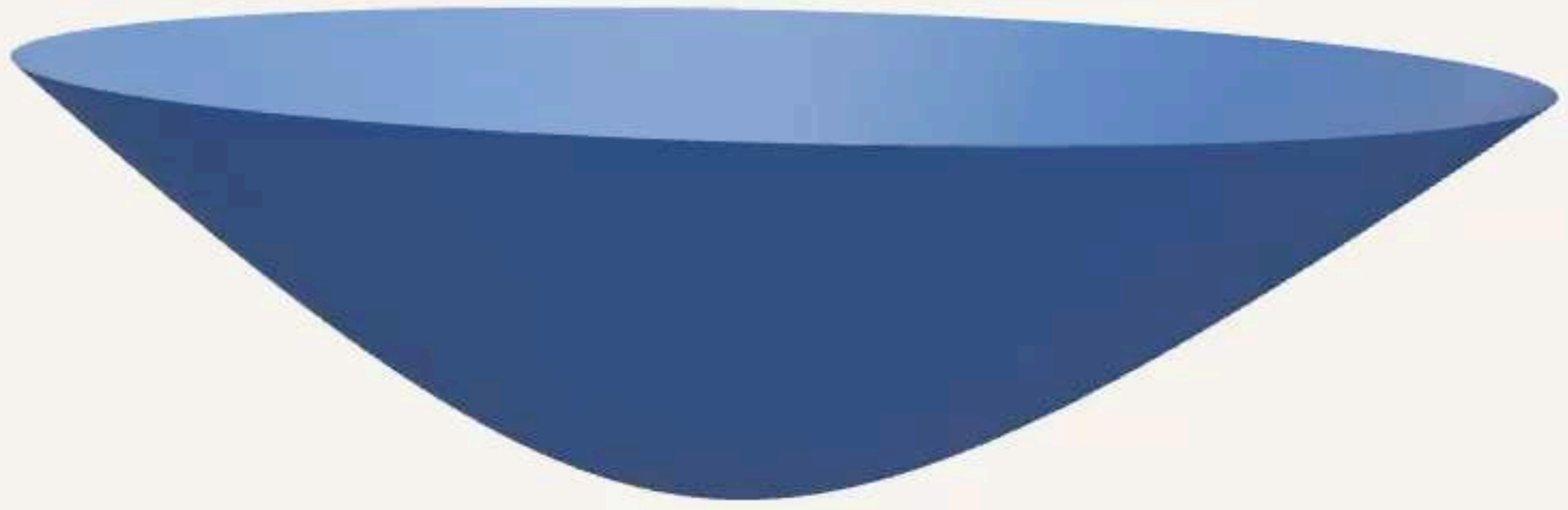
Curves in Lorentzian Geometry



A curve's direction can be either space like or time like.

Curves in Lorentzian Geometry



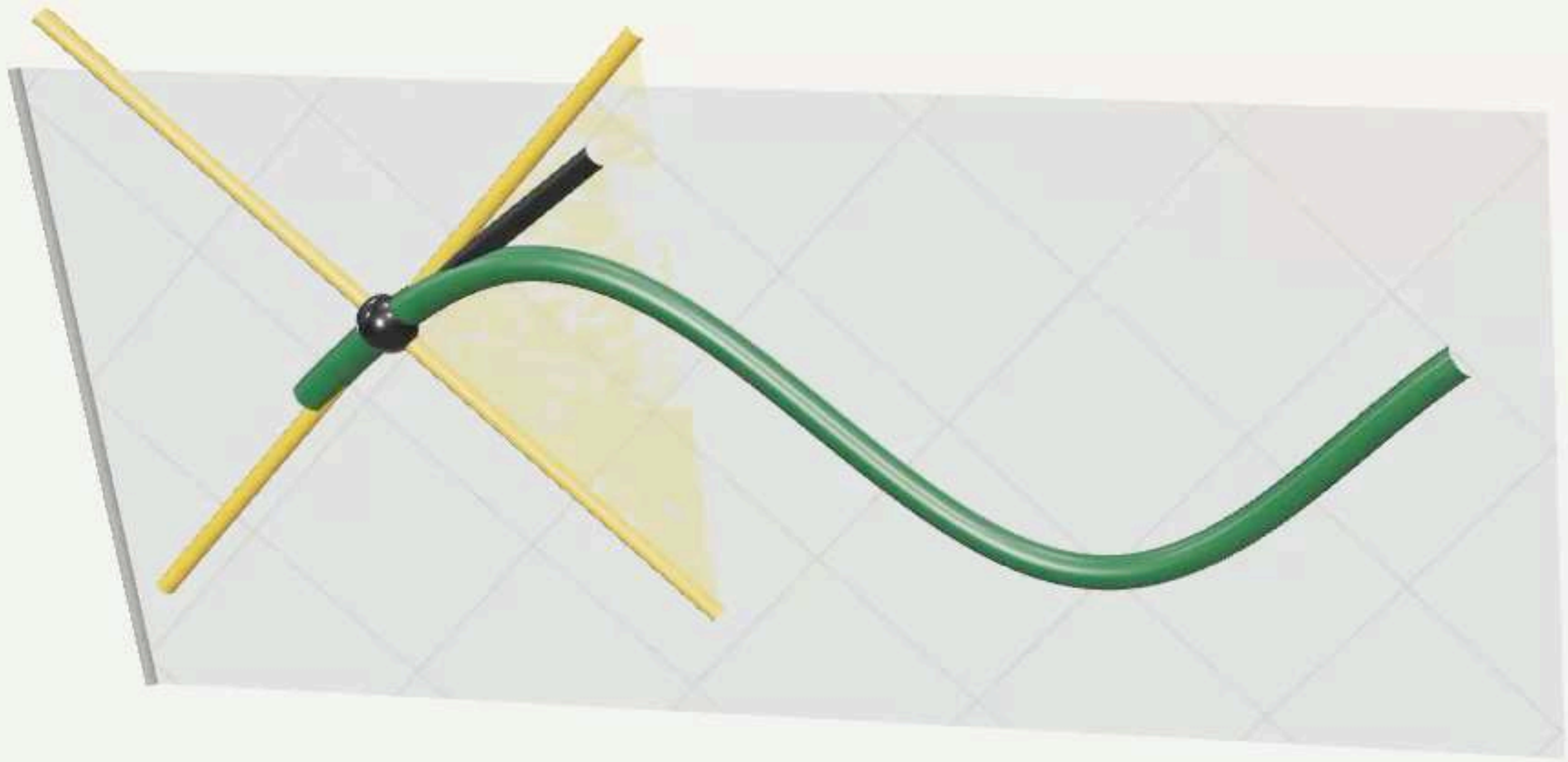


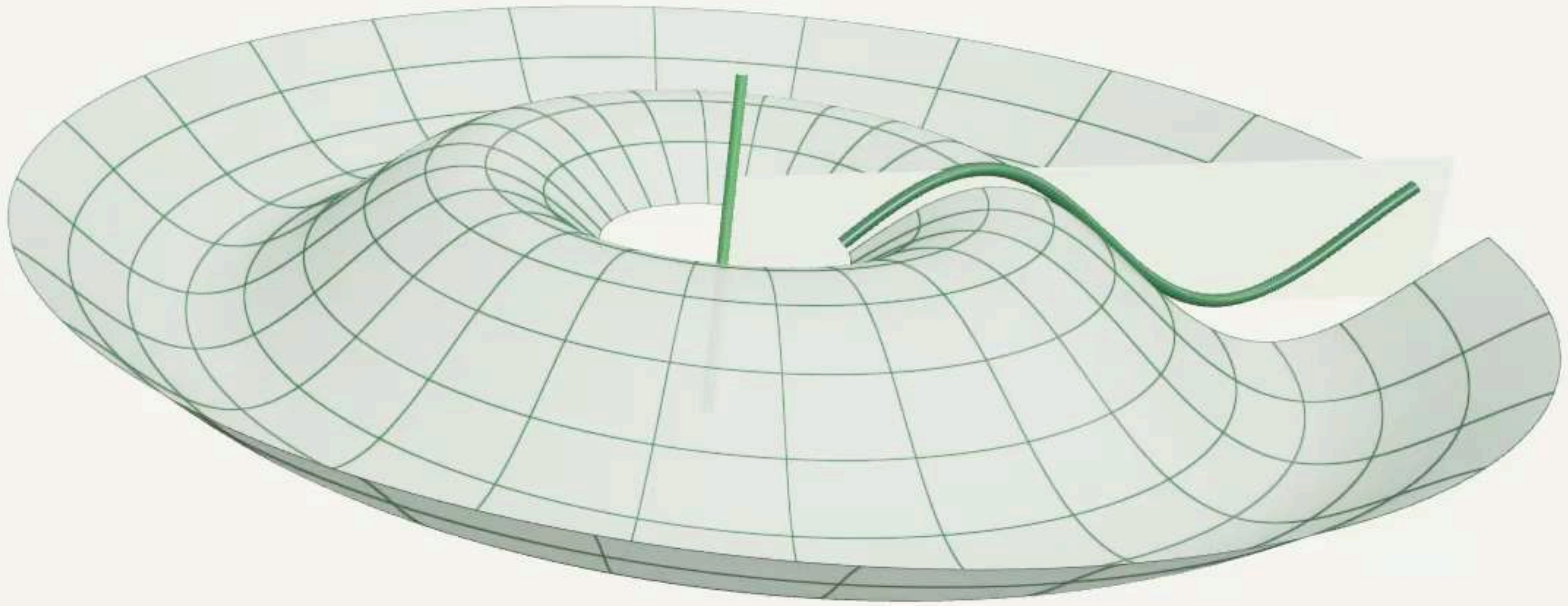
If the curve is always space like, the rotated surface is Riemannian.

Curves in Lorentzian Geometry

If the curve is always time like, the rotated surface is Lorentzian.

Revolution in Minkowski Geometry

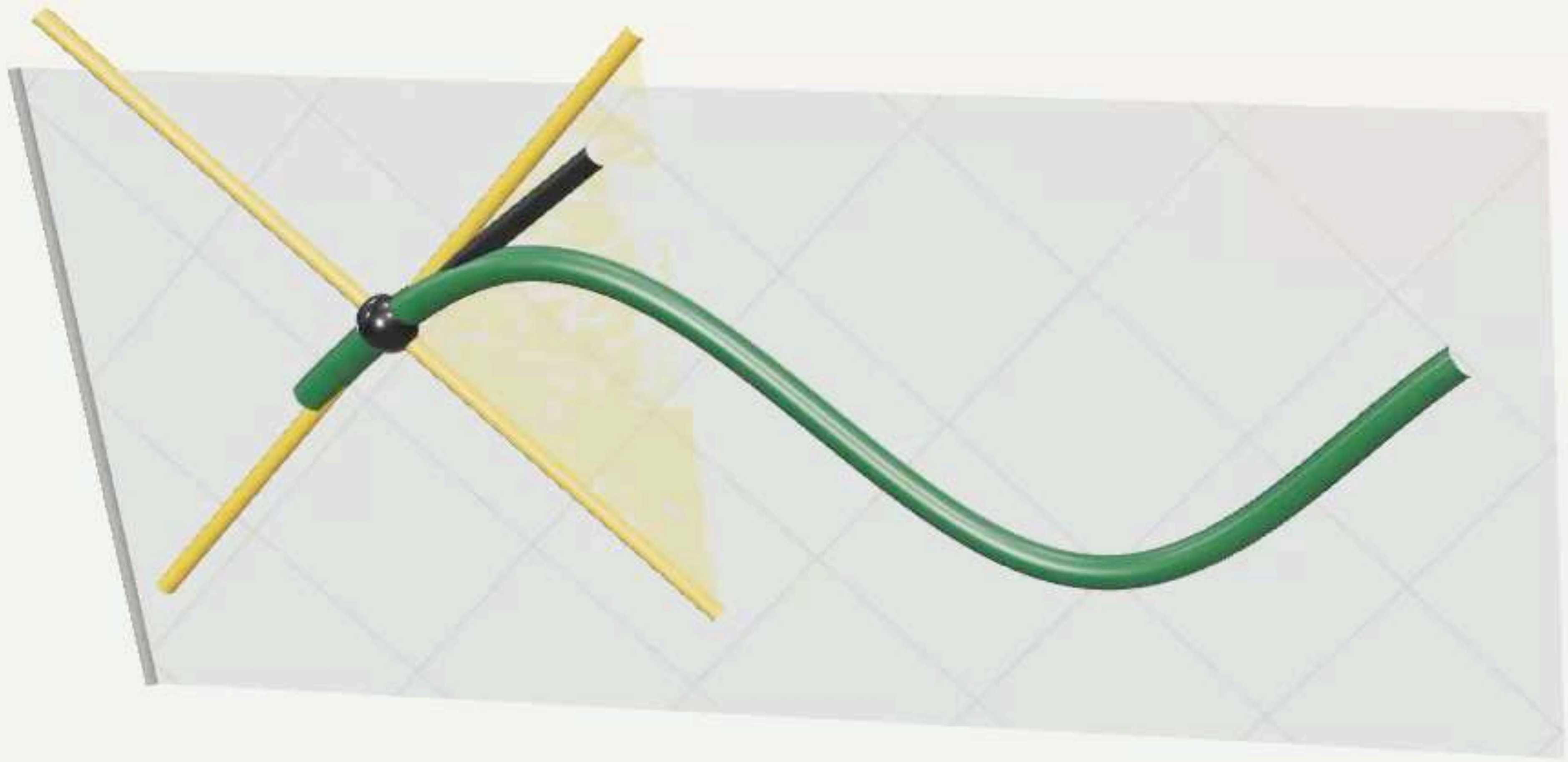


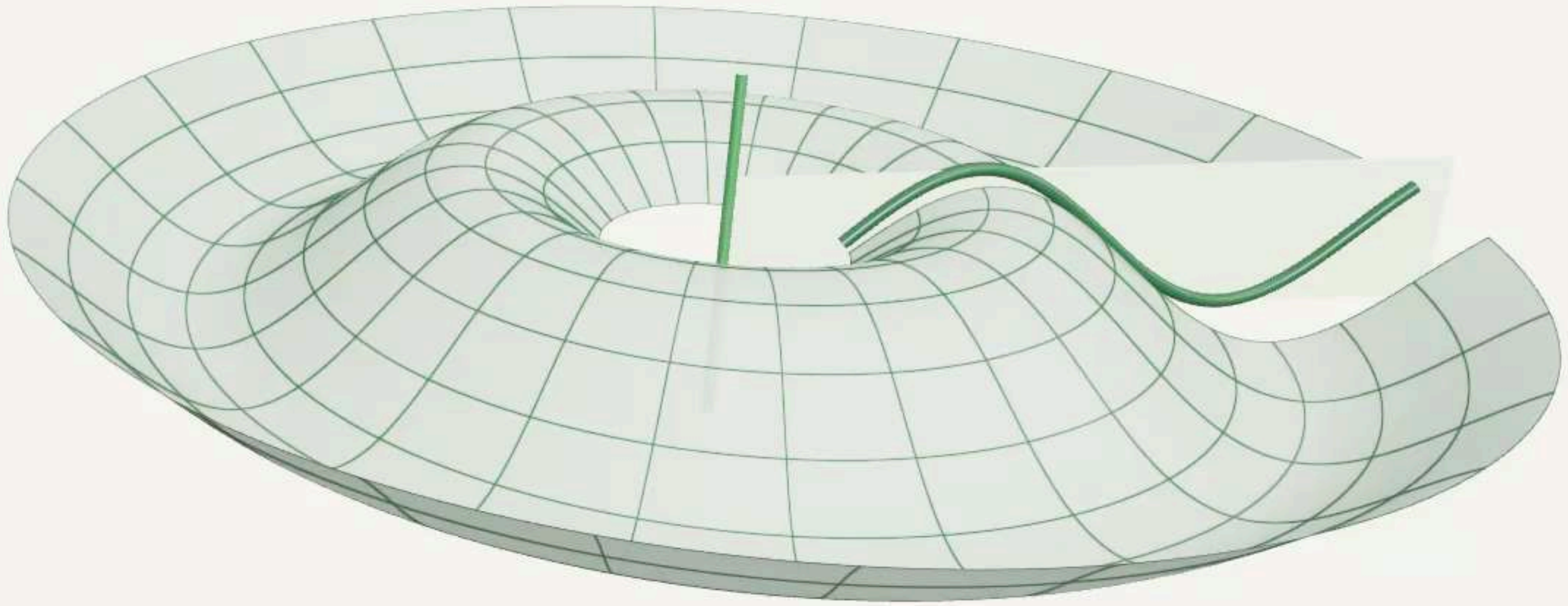


Like Euclidean, but there's a minus sign

$$ds^2 = (\dot{r} \boxed{-} \dot{\tau}^2)\, du^2 + C^2\, d\theta^2$$

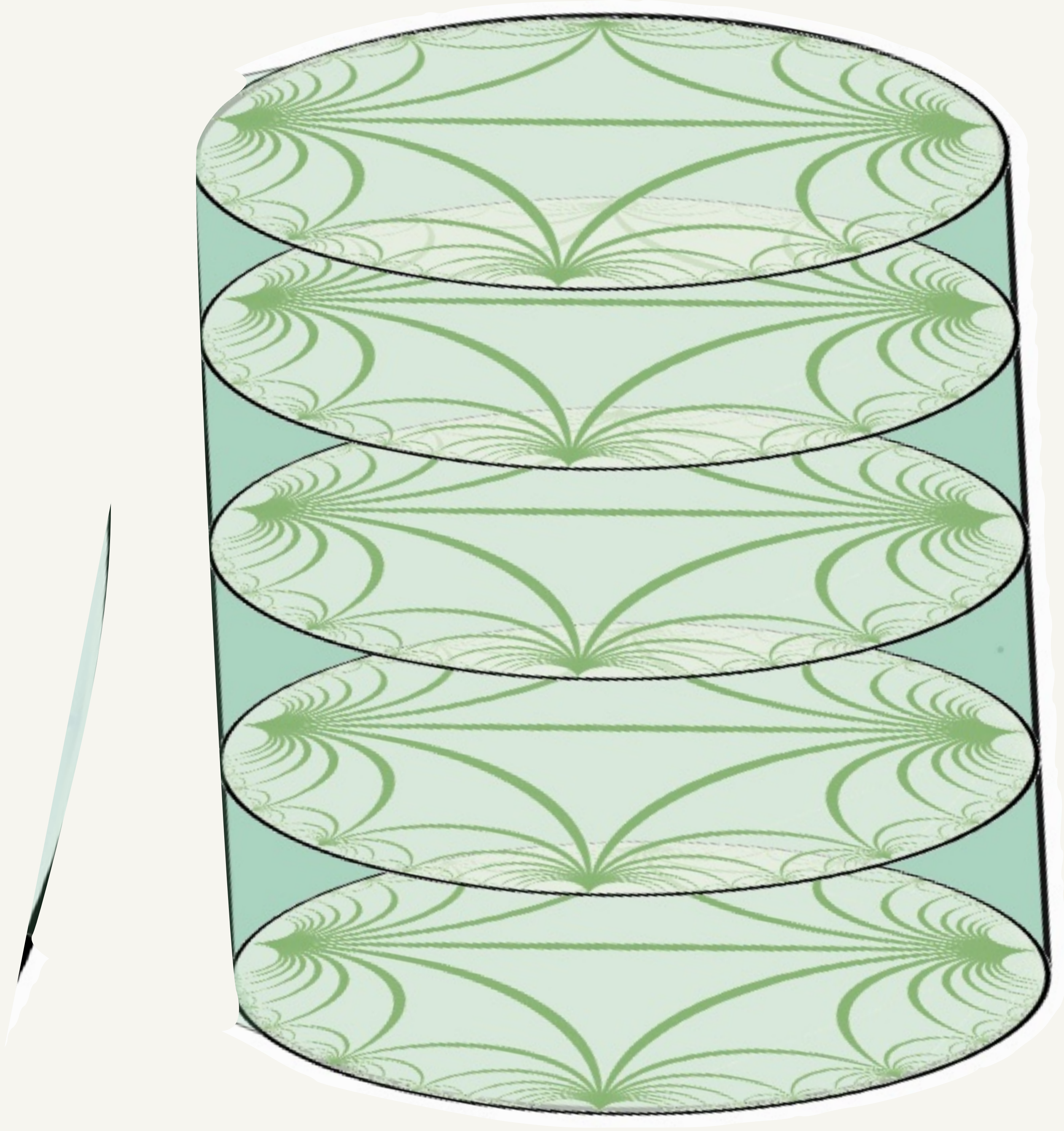
Revolution in Minkowski Geometry



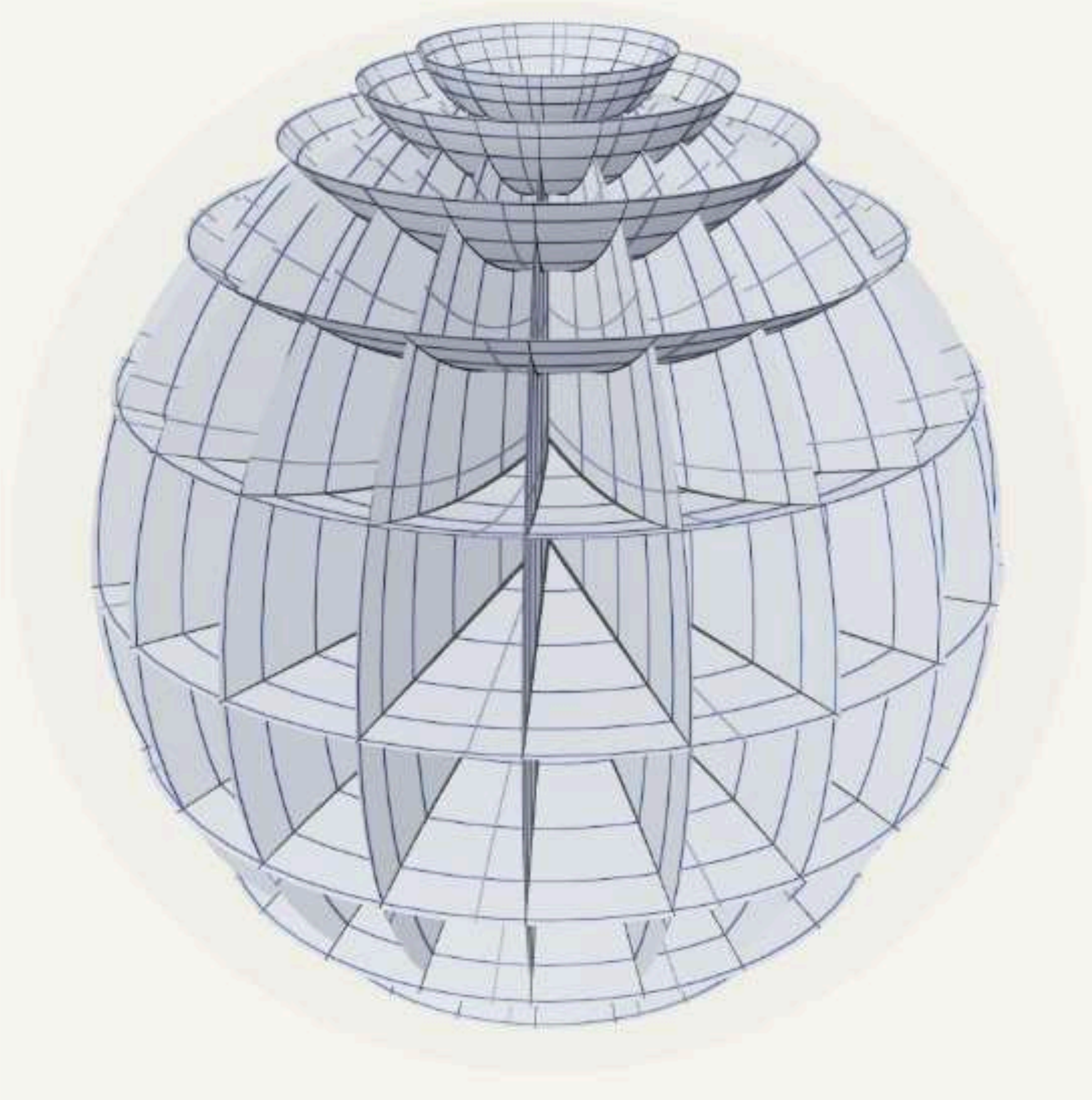


A surface with metric $ds^2 + f^2(s)\, d\theta^2$
embeds in $\mathbb{M}^3$ *if and only if*
$$\dot{f}^2 \boxed{\geq} 1$$

Anti de Sitter Space



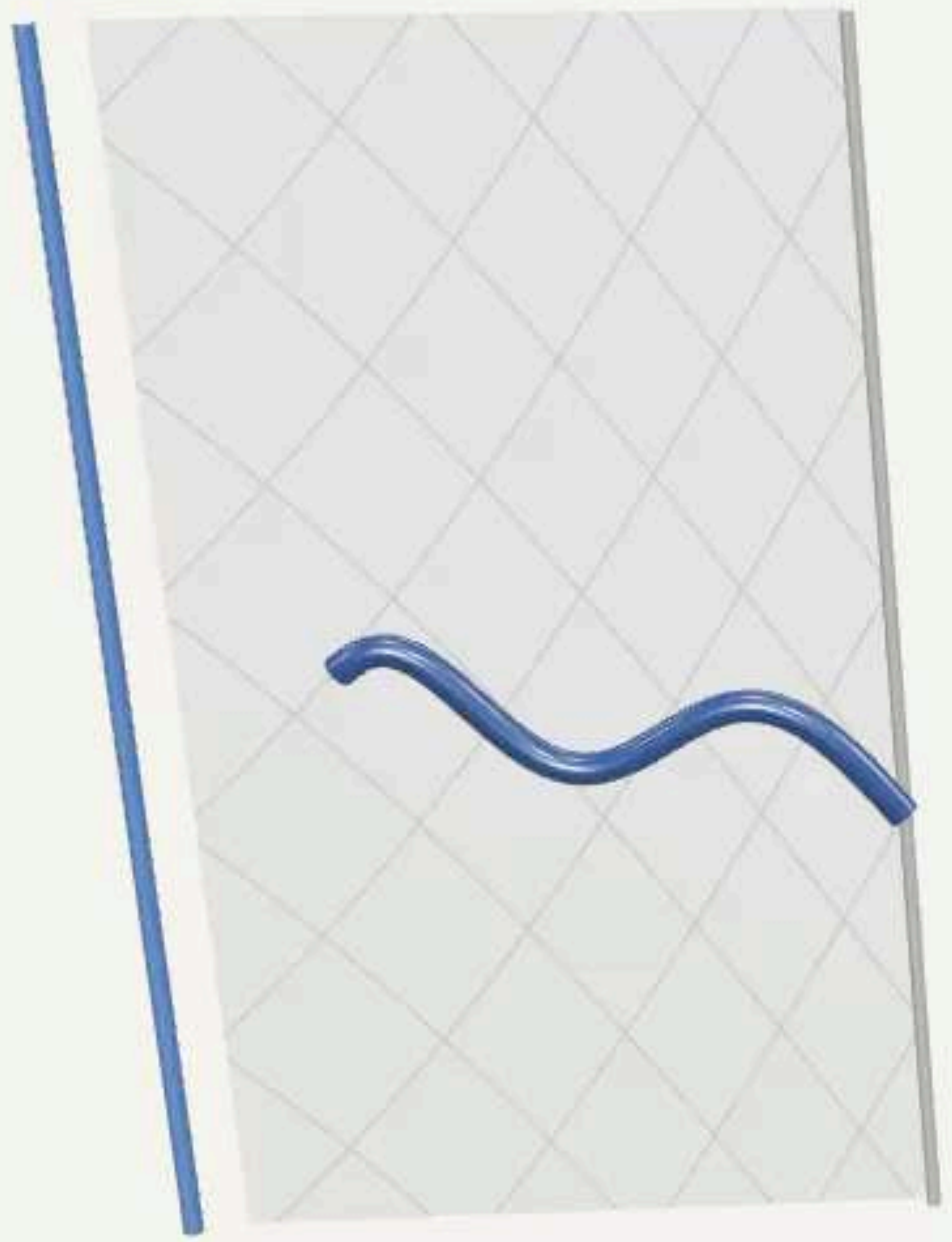
There's a way to build global coordinates, that look very much like $\mathbb{H}^3$

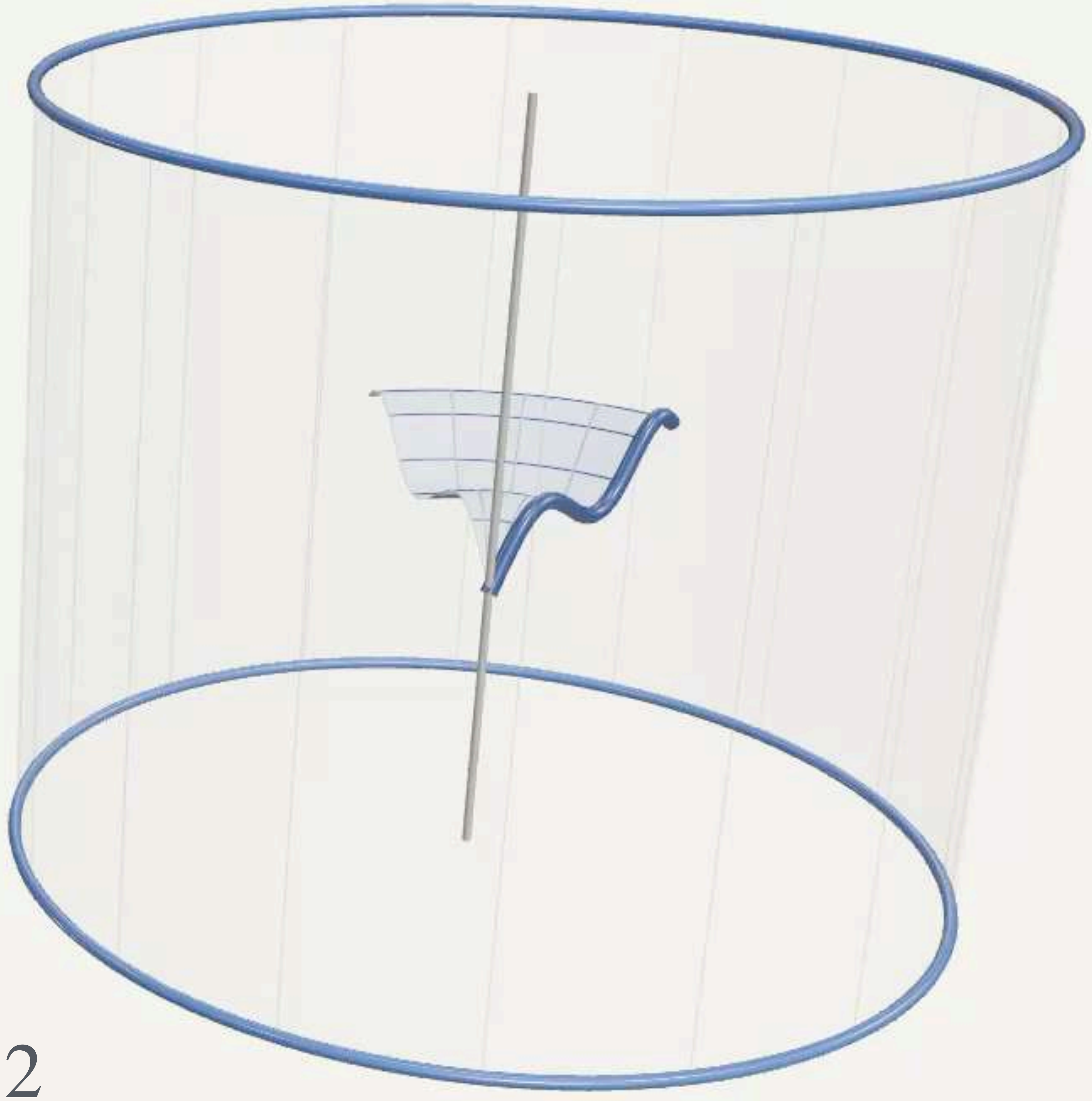


Anti de Sitter,
spacelike slices are hyperbolic planes

$$ds^2 = dr^2 + \sinh(r)^2\, d\theta^2 \boxed{-} \cosh(r)^2\, d\tau^2$$

Anti de Sitter Space





A surface with metric $ds^2 + f^2(s)\,d\theta^2$
embeds in AdS^3 *if and only if*

$$\dot{f}^2 \boxed{\geq} 1 + f^2$$

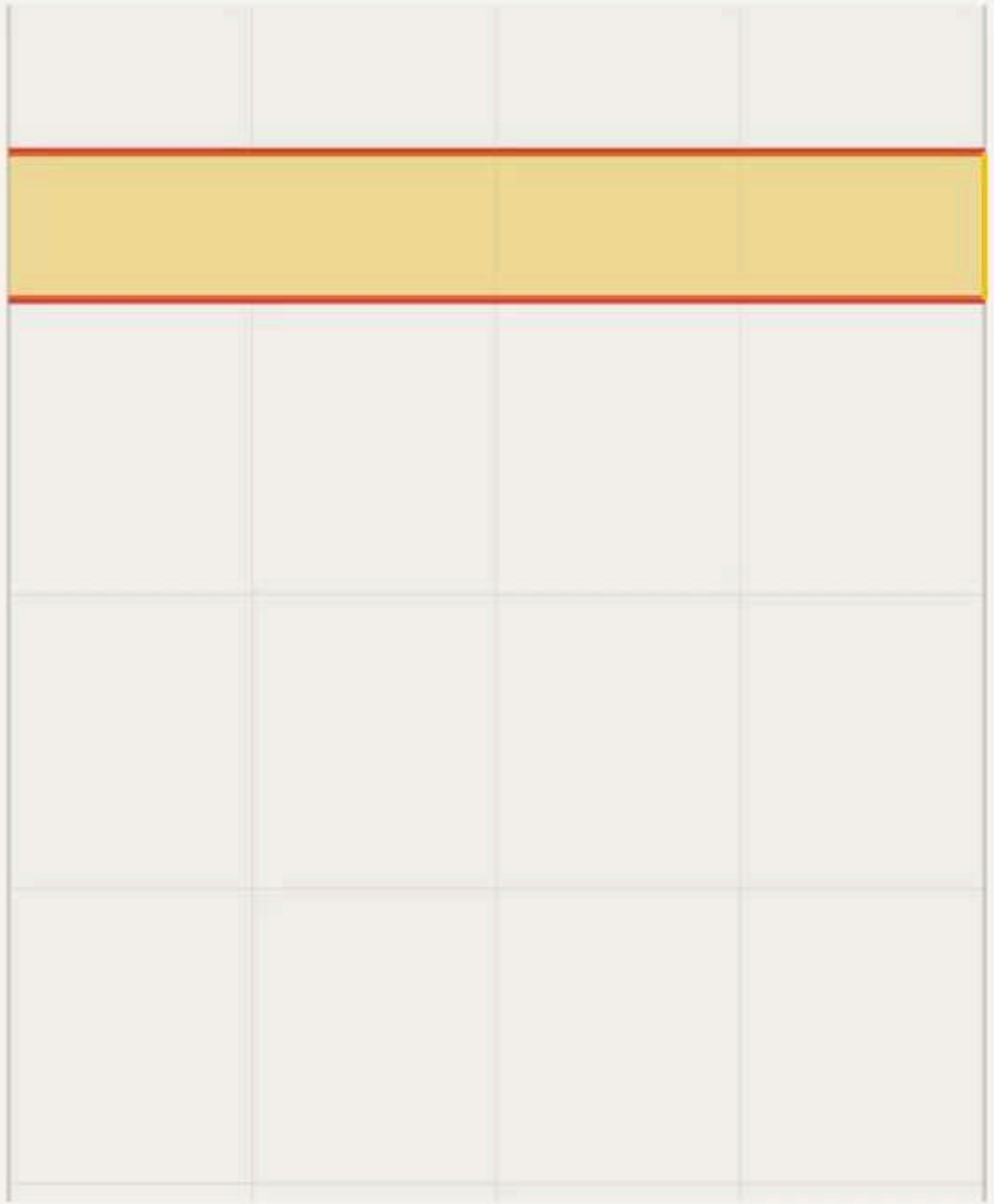
de Sitter Space

There's a way to build coordinates, that look very much like $\mathbb{S}^3$ *(but not exactly globally!)*

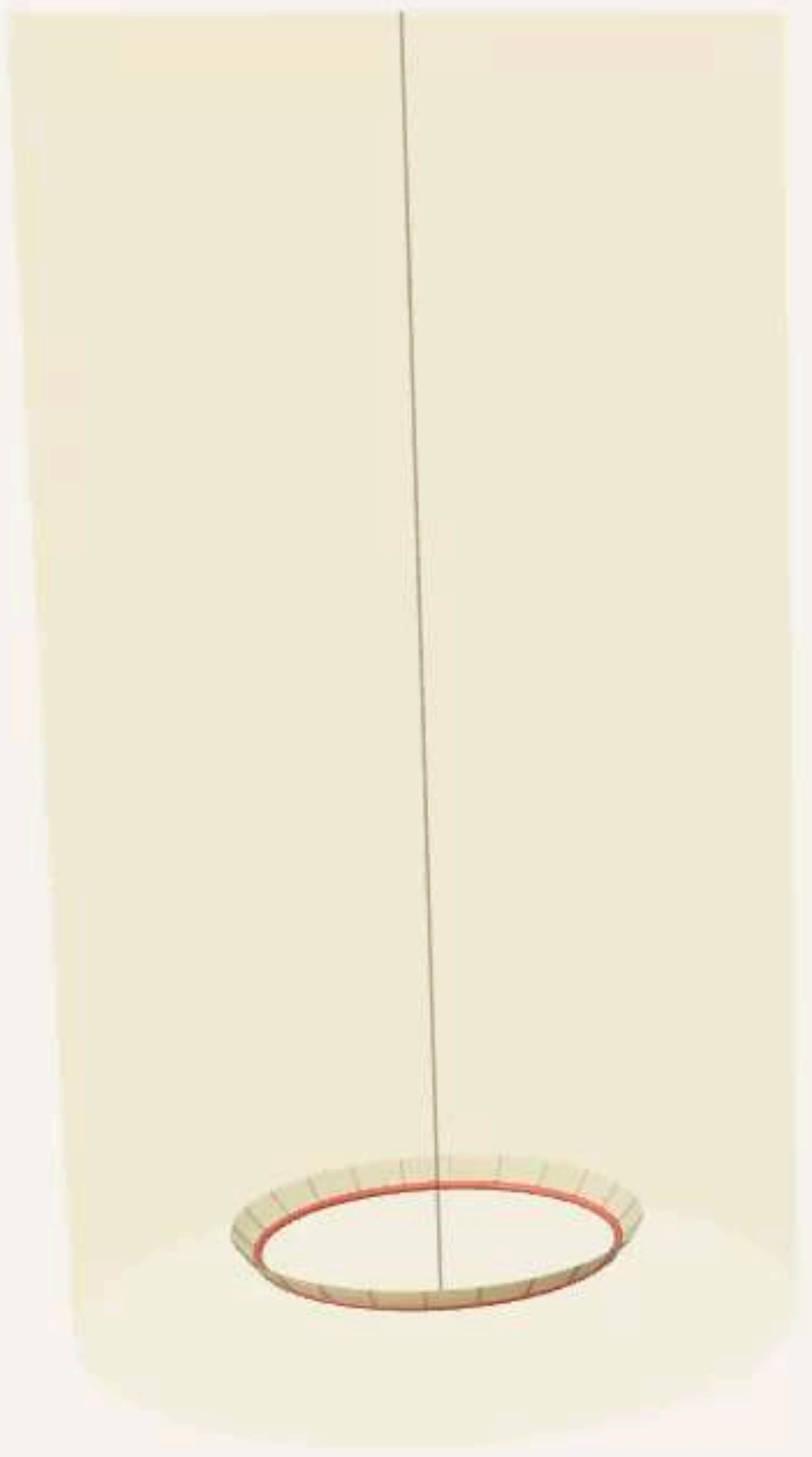
$$ds^2 = dr^2 + \sin(r)^2\, d\theta^2 \boxed{-} \cos(r)^2\, d\tau^2$$

de Sitter Space

In these coordinates, only a strip of the pseudosphere embeds.



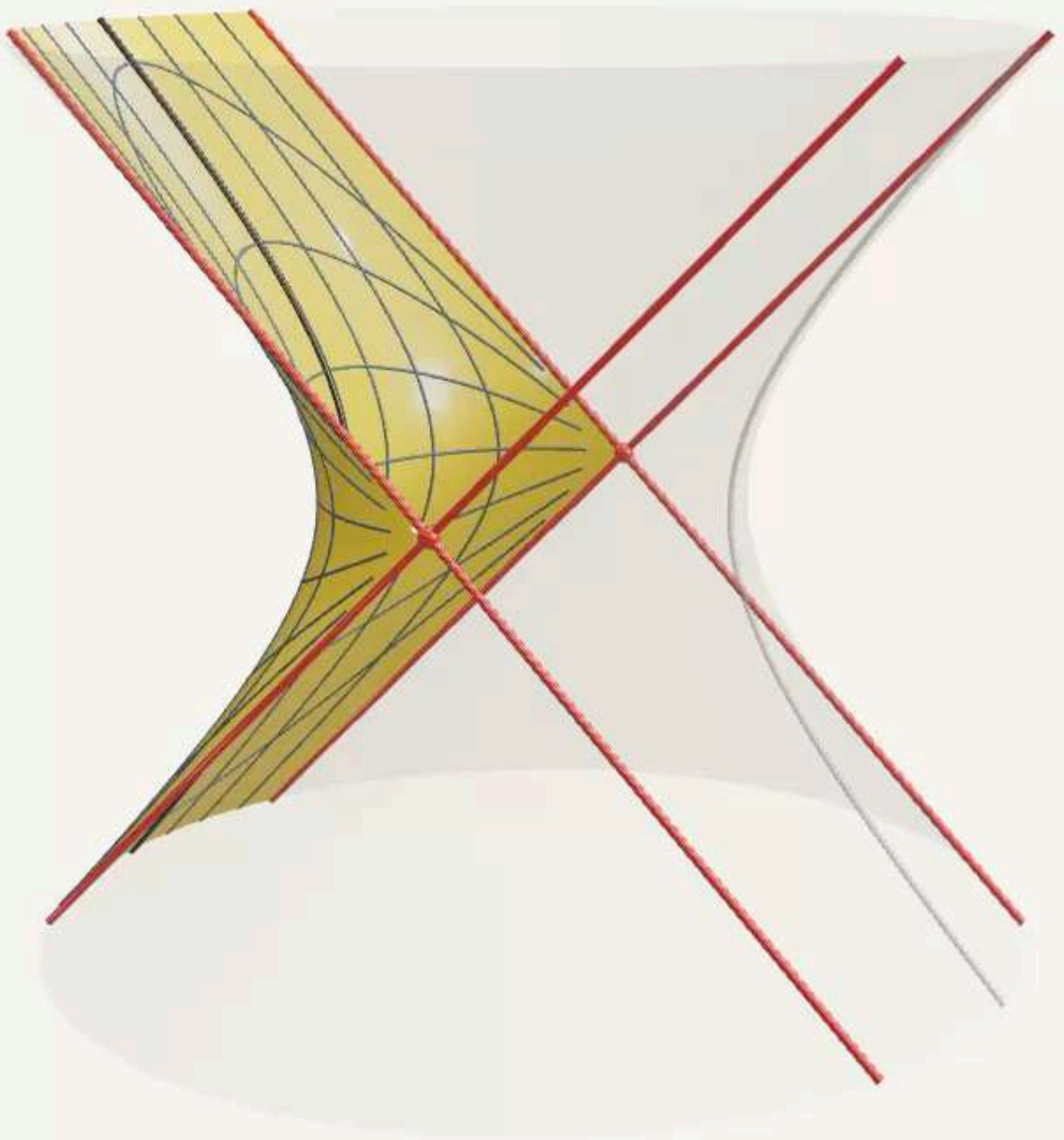
But…a hint!
It reaches infinity in finite time



The de Sitter Horizon

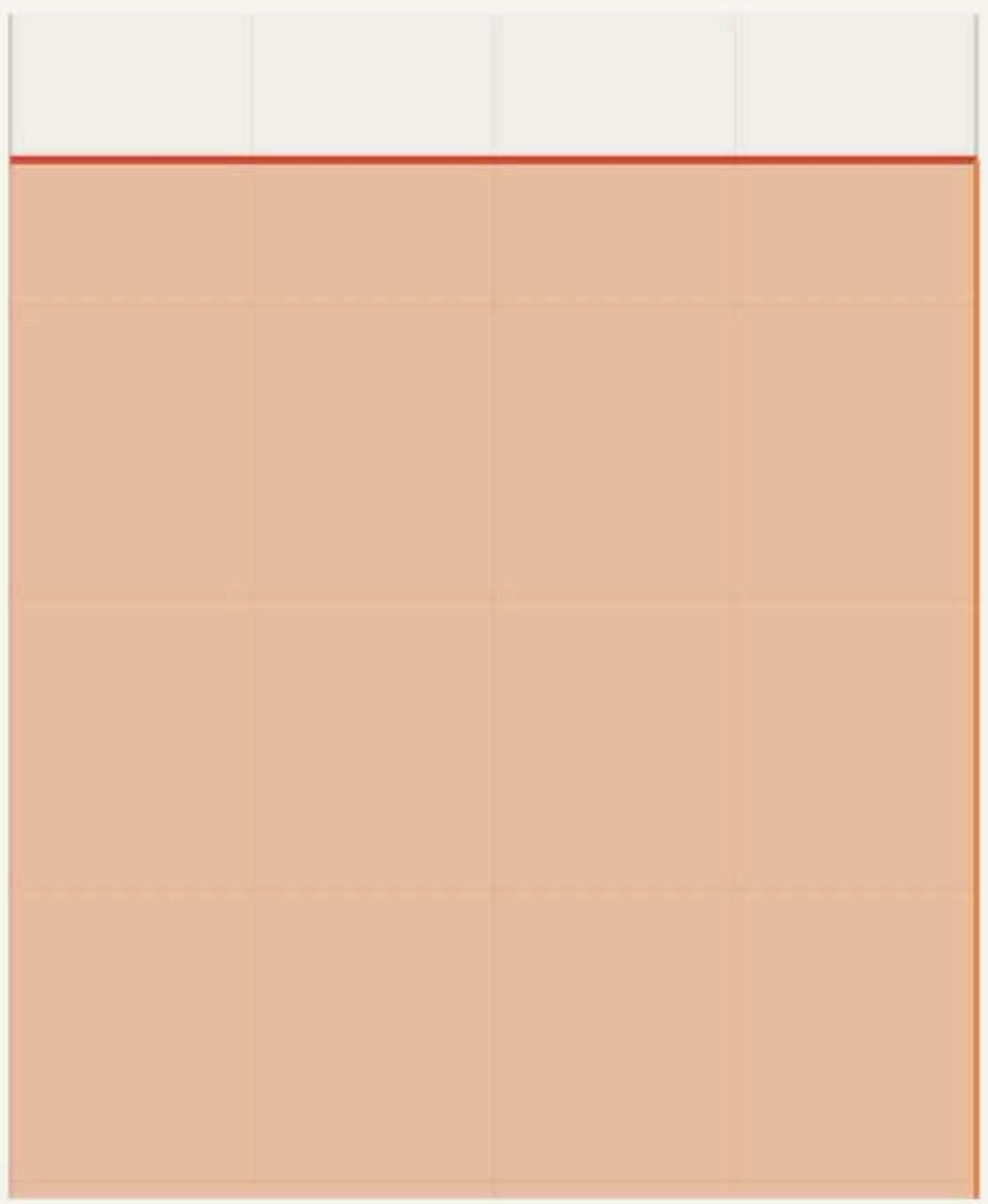
The sphere-like coordinates trace out only a small portion of the entire spacetime, the "static patch"

Infinity in these coordinates is like the horizon of a black hole

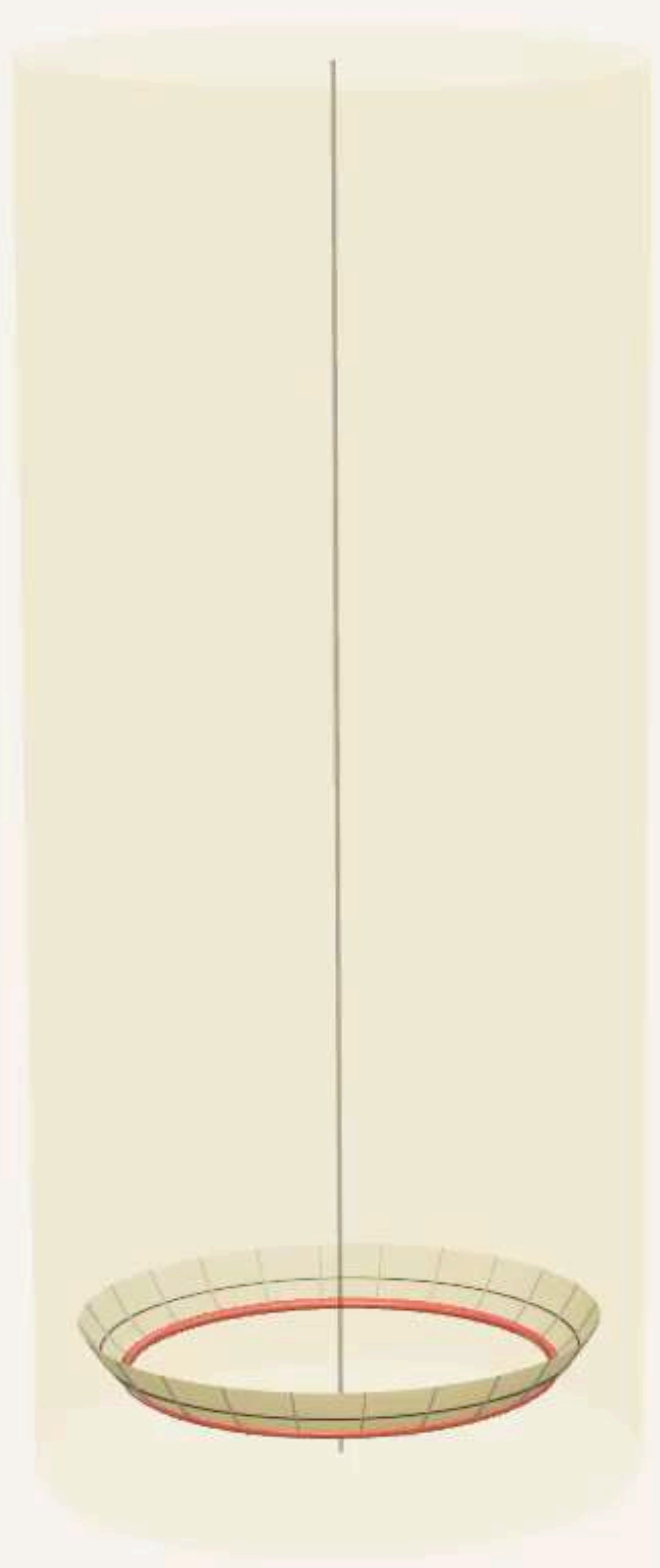


Beyond the Horizon

Globally, the pseudosphere continues through the horizon without issue!

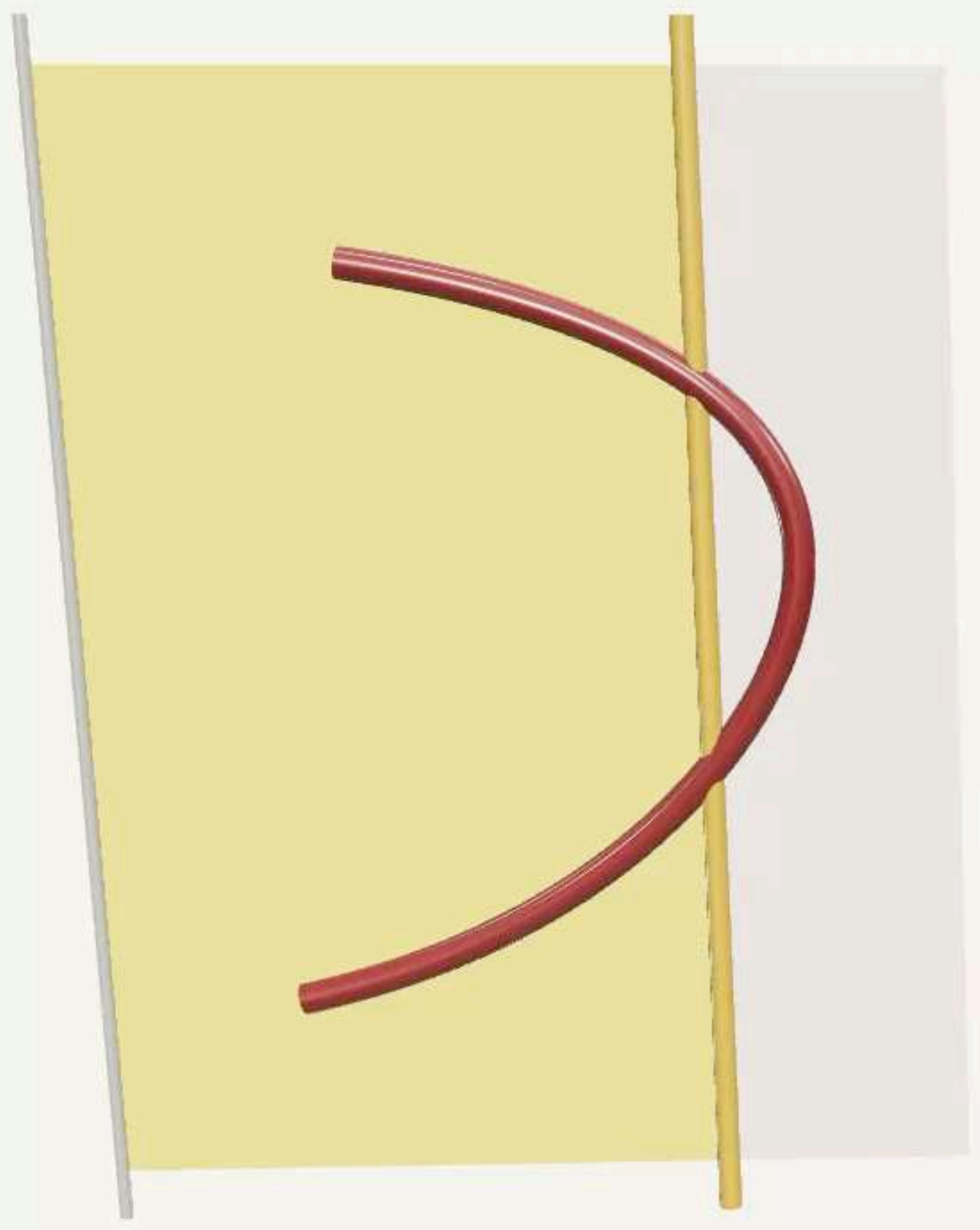


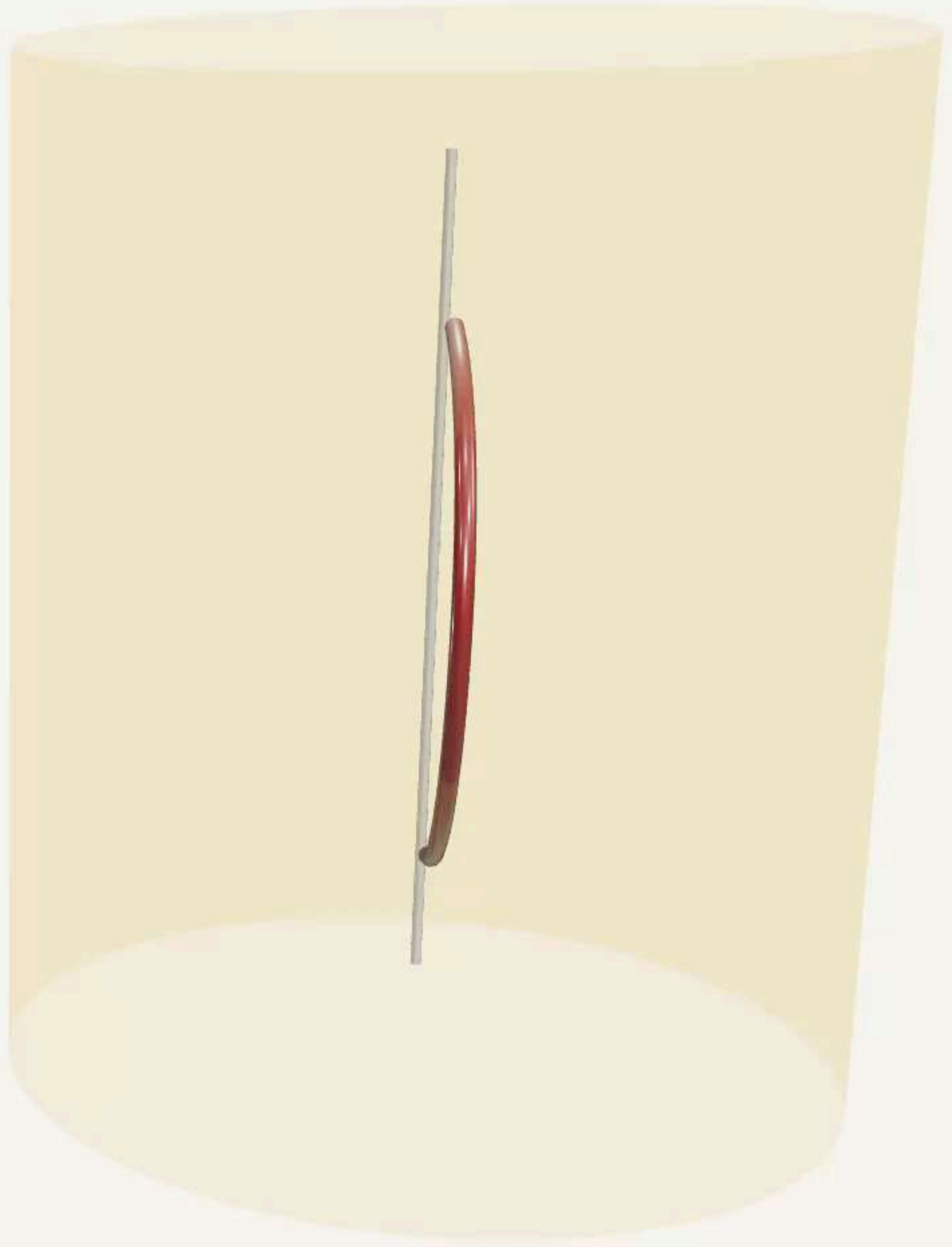
Indeed, everything *below* $u = \sqrt{2}$ embeds: the exact opposite of $\mathbb{S}^3$



Embeddability in de Sitter

A surface with metric $ds^2 + f^2(s)\, d\theta^2$ *embeds in* dS^3 *if and only if* $\dot{f}^2 \boxed{\geq} 1 - f^2$





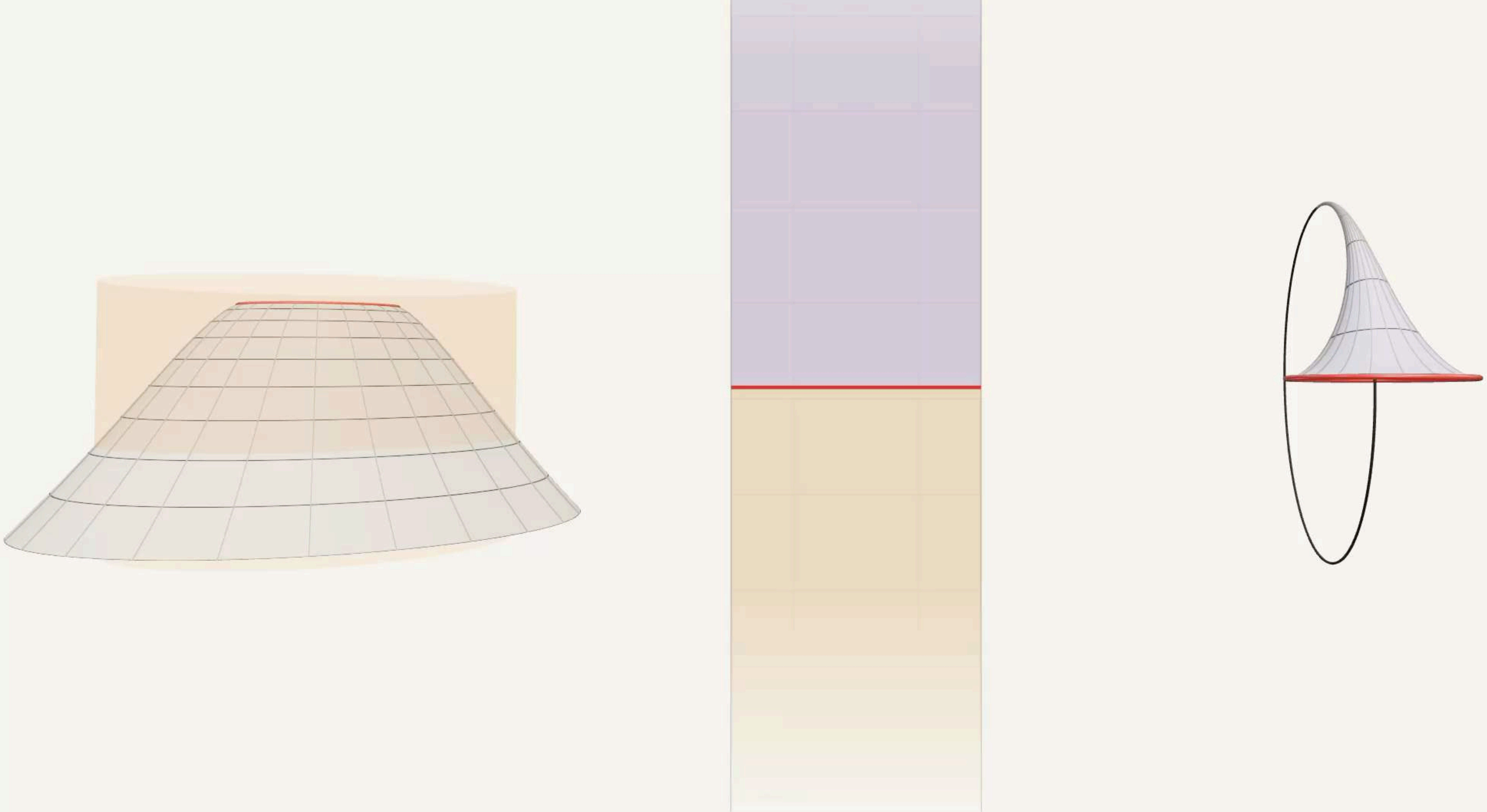
Theorem
(Lander, T-)

Whatever portions of a surface with rotational symmetry does not *embed in Riemannian curvature k, does embed in Lorentzian curvature k, and vice versa.*

$\Longrightarrow$

*(For example) Since a hyperbolic disk is not a surface of revolution in Euclidean space, it *must be* in Minkowski*

The pseudosphere, split between worlds



$k = +0.18$ · $u_\star = \sqrt{(1+k)}$ · spherical | de Sitter

Thanks!