What do 3-Manifolds Look Like?

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Joint work with Remi Coulon, Sabetta Matsumoto, and Henry Segerman

3 Manifolds are Interesting

We live in a 3-Manifold

Evolution endowed us with a high powered geometric computation device: (the visual cortex), to survive in the 3-manifold we live in.

Our 3-Manifold is boring

No interesting topology: its contractible. No interesting geometry: its flat.

What are interesting 3 manifolds like?

I Raytracing in Manifolds

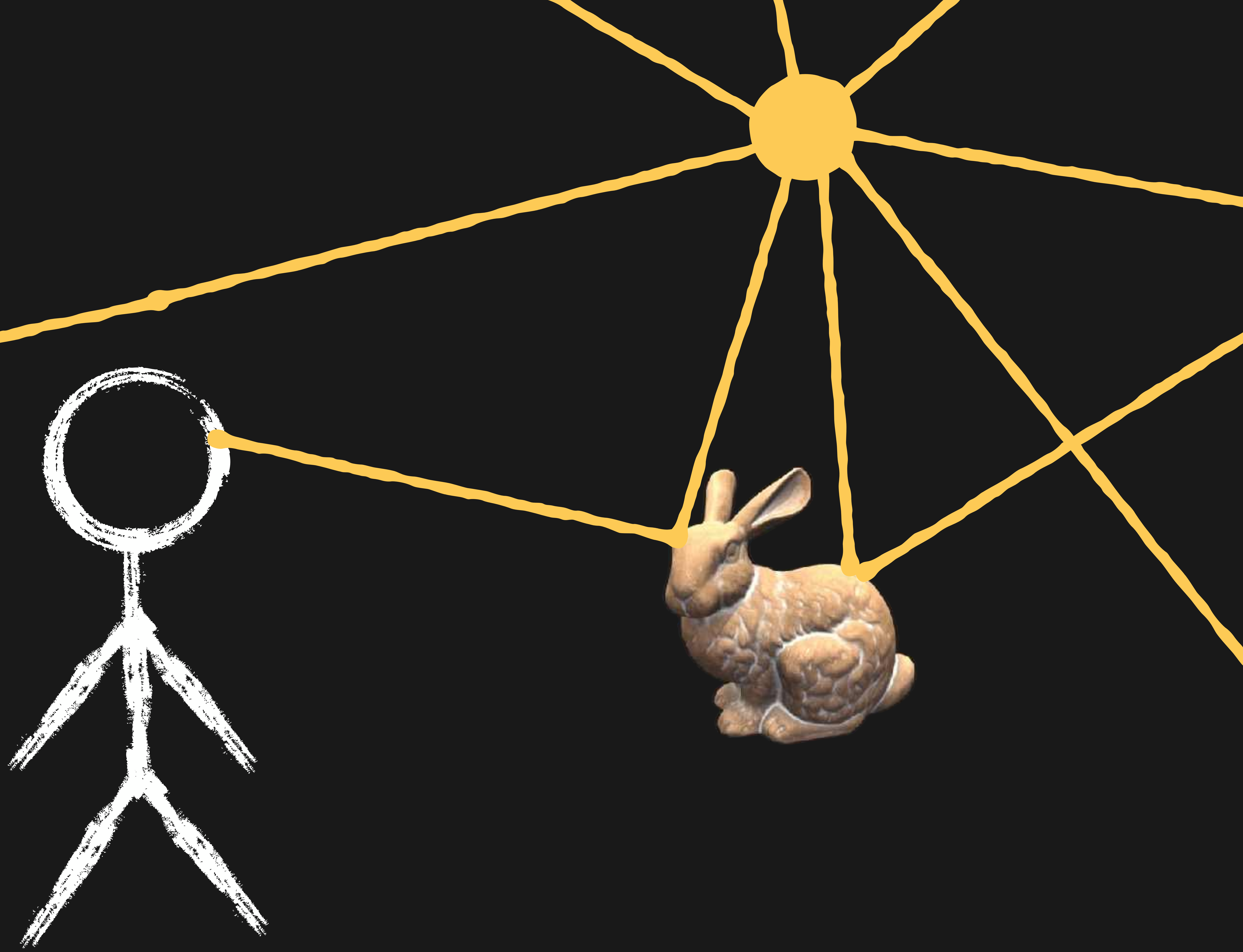
II Understanding What we See

III Seeing general 3-Manifolds

Raytracing in Manifolds

What does it mean to 'see'?

We see an object
when light
bounces off it
into our eye.

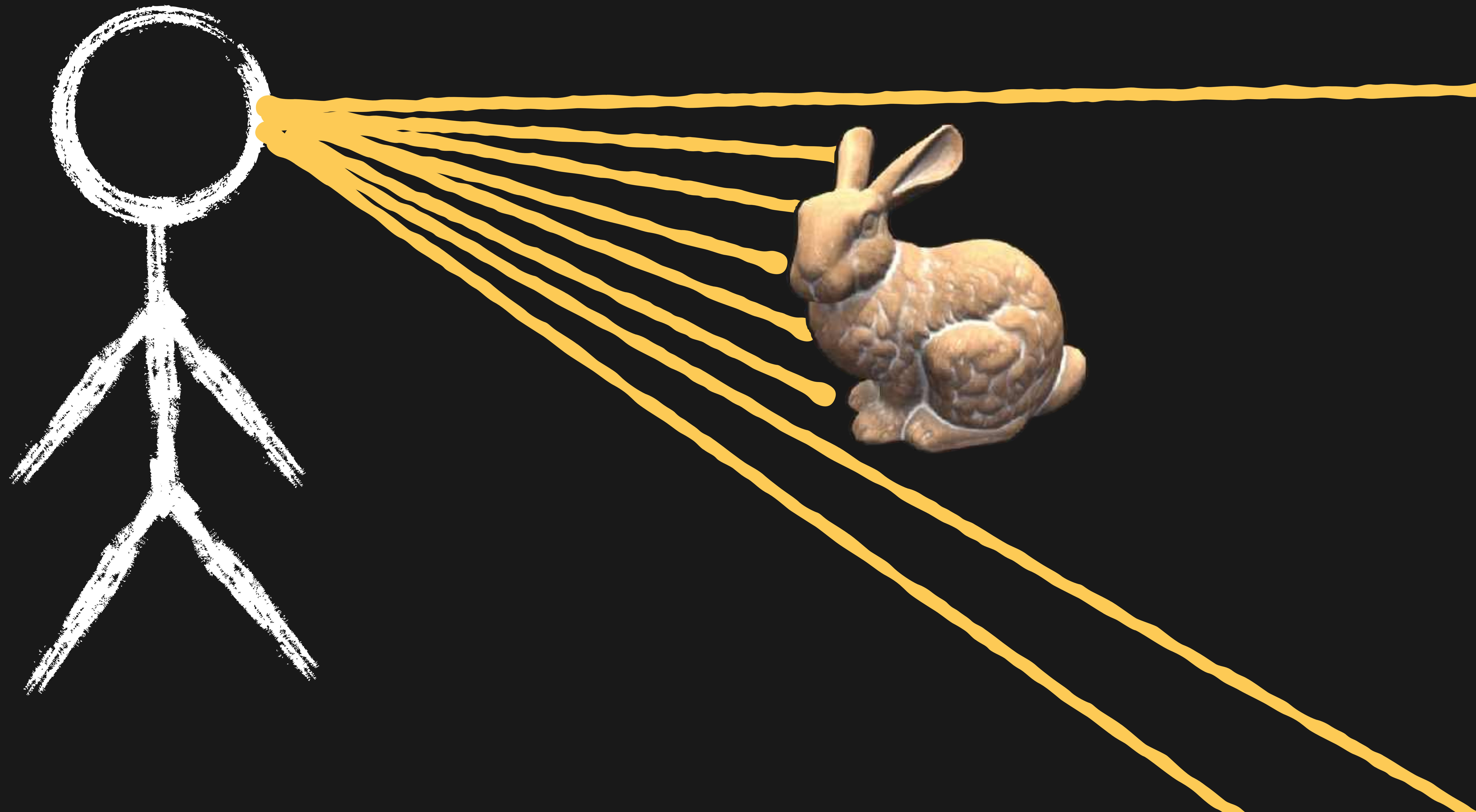


$\mathrm{viz}\big($  $\big) = \mathbb{P}\,\exp^{-1}$  $\big($  $\big) \in \mathbb{P}T$  Space

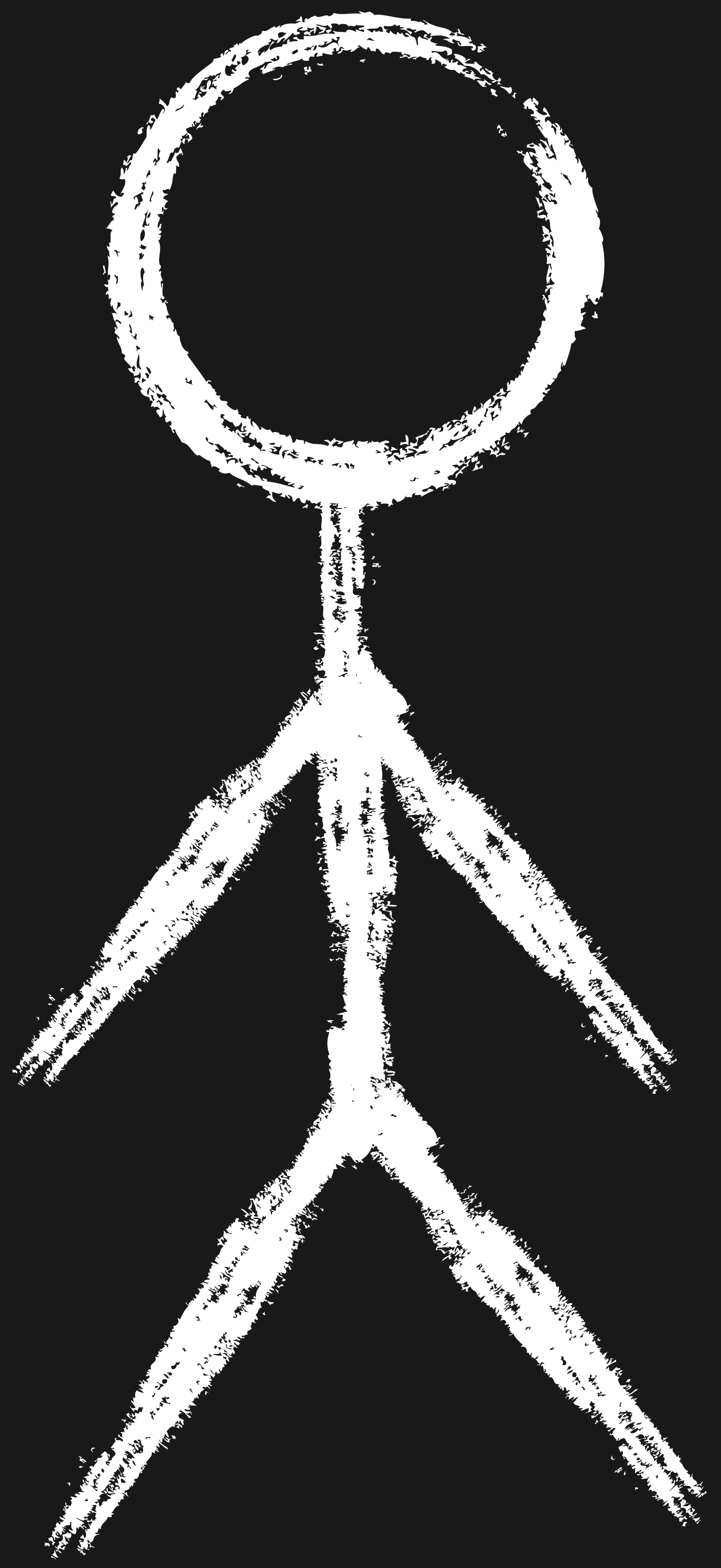
This is the inverse of the exponential map in Riemannian geometry

On the computer....reverse this!

Calculate for each initial direction in the tangent space at your eye a geodesic out into the world, and see what it hits



Computing Intersections



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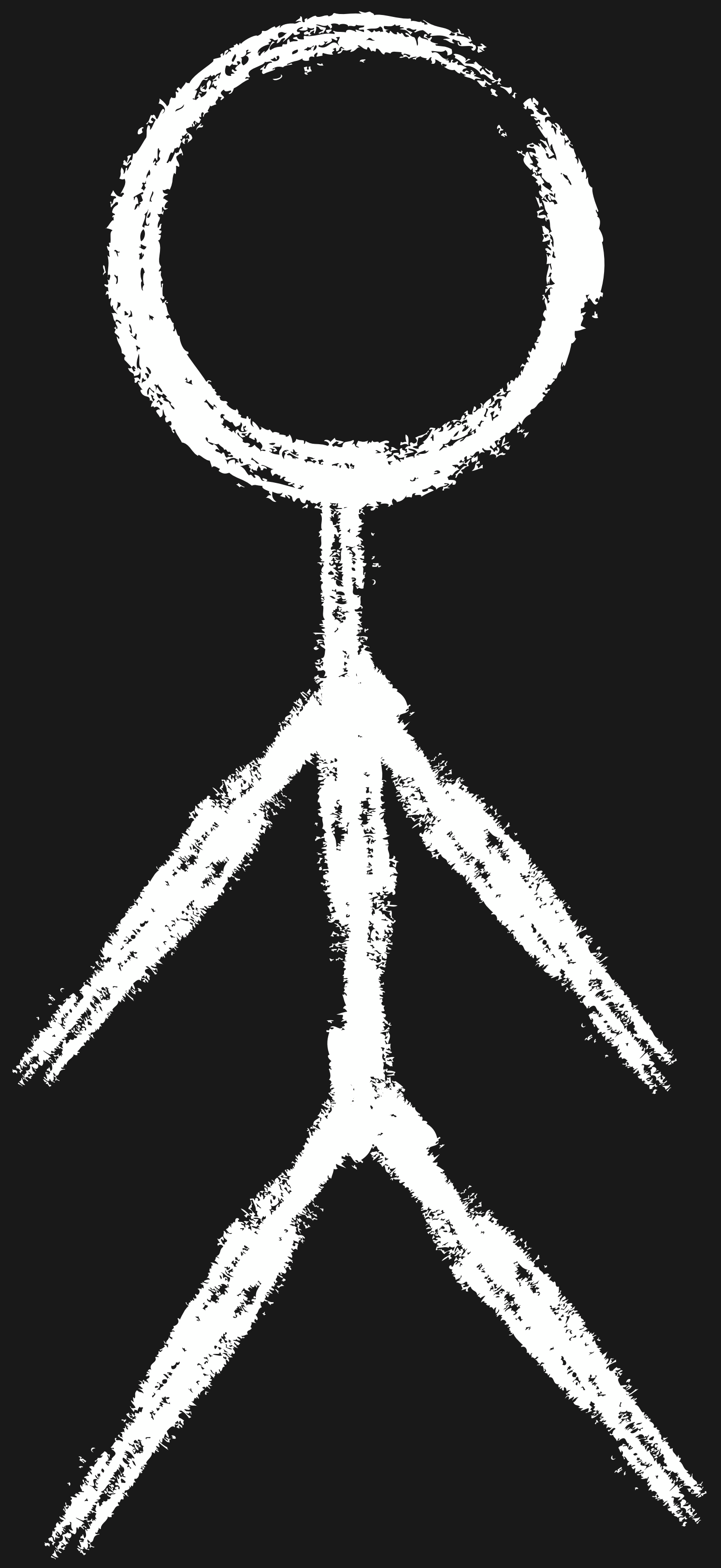
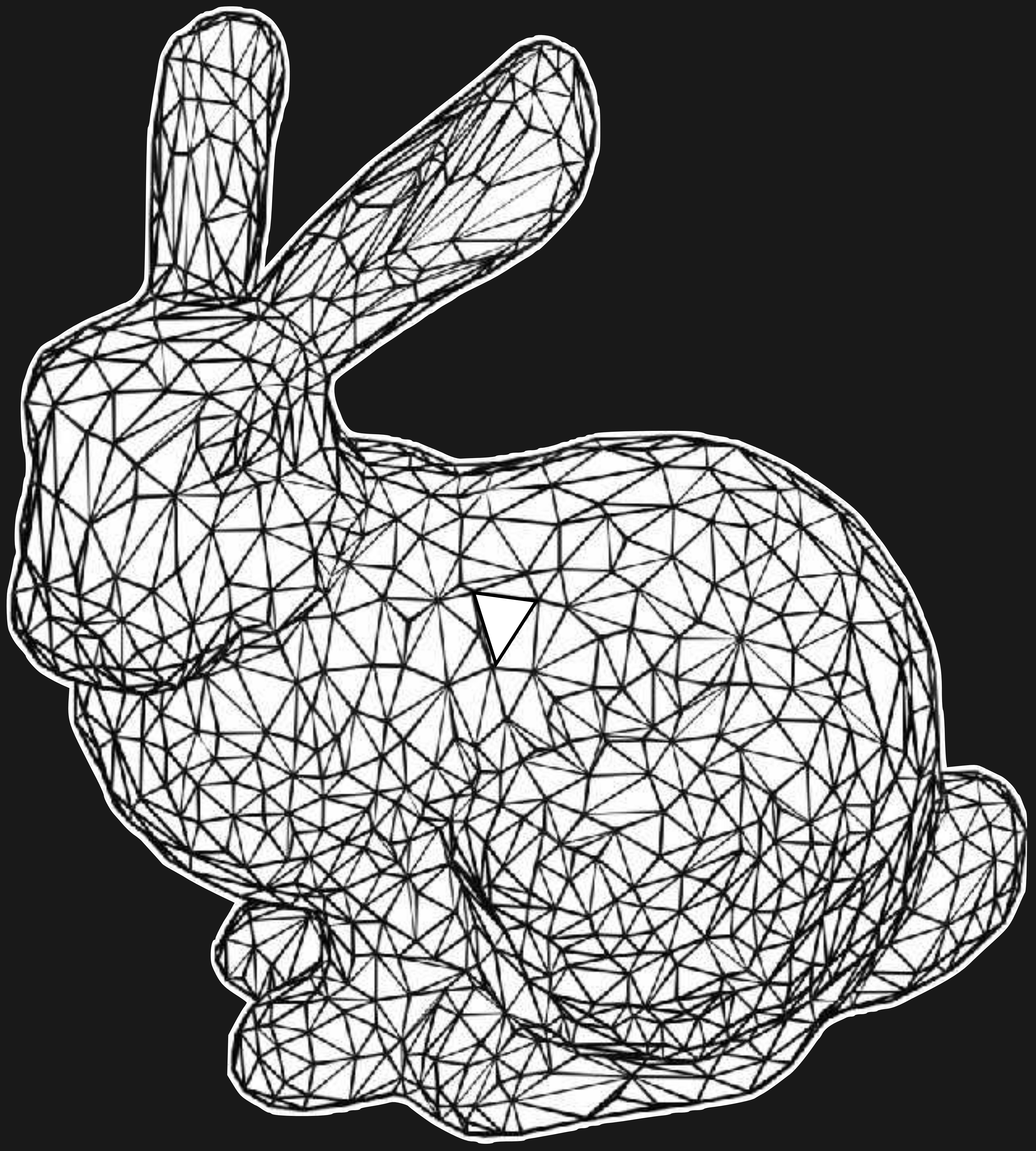
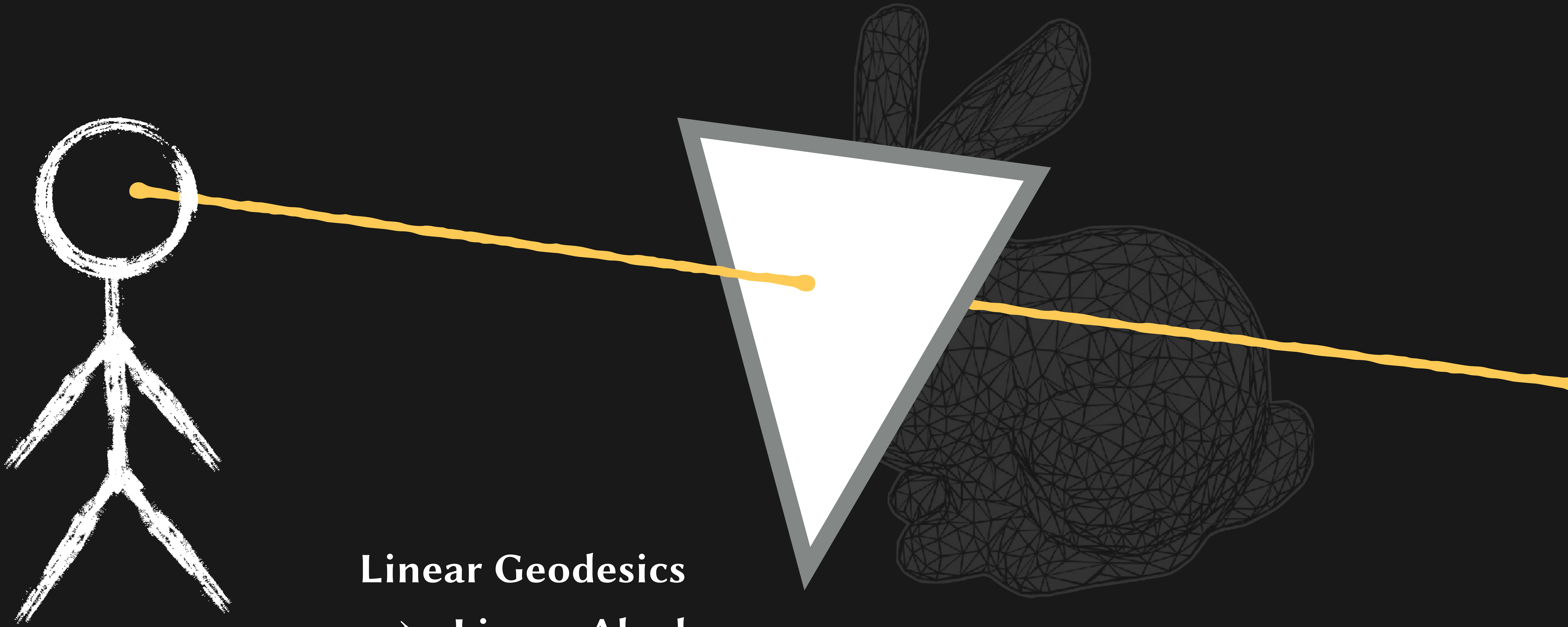




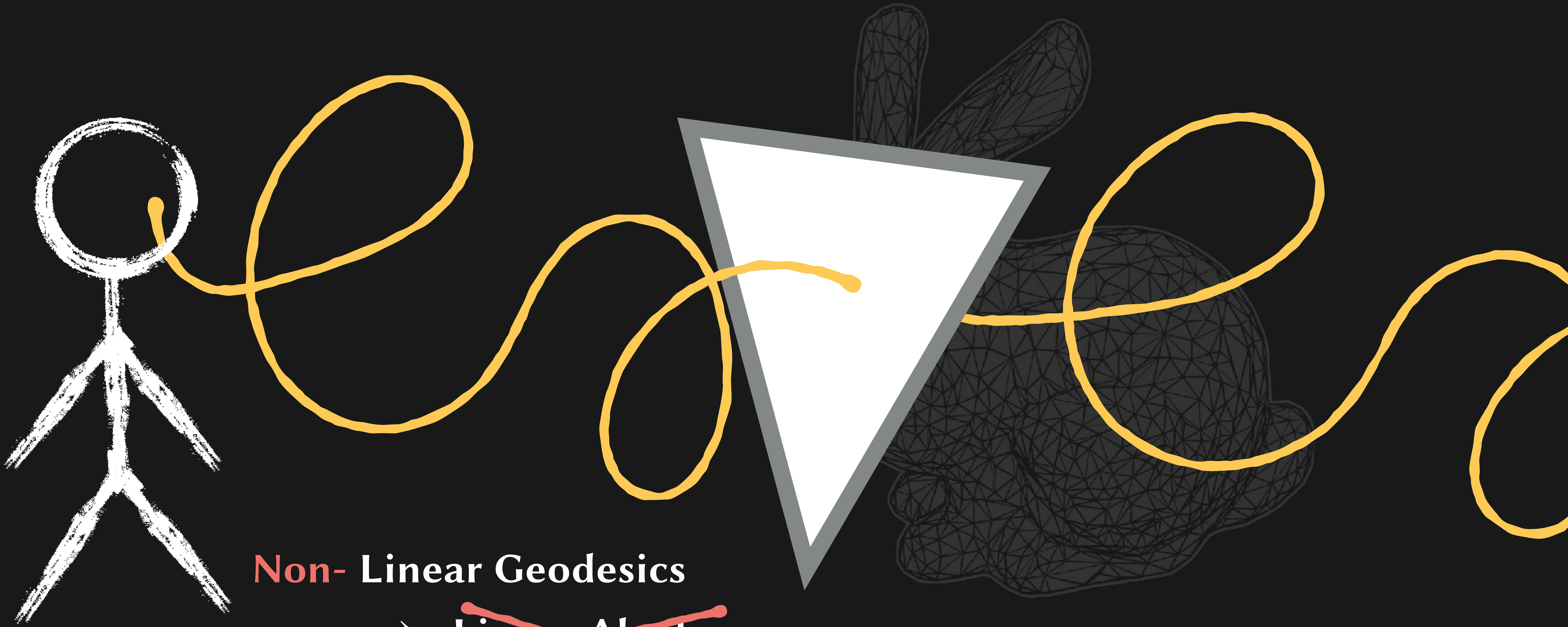
Computing Intersections



Linear Geodesics

$\Longrightarrow$ Linear Algebra

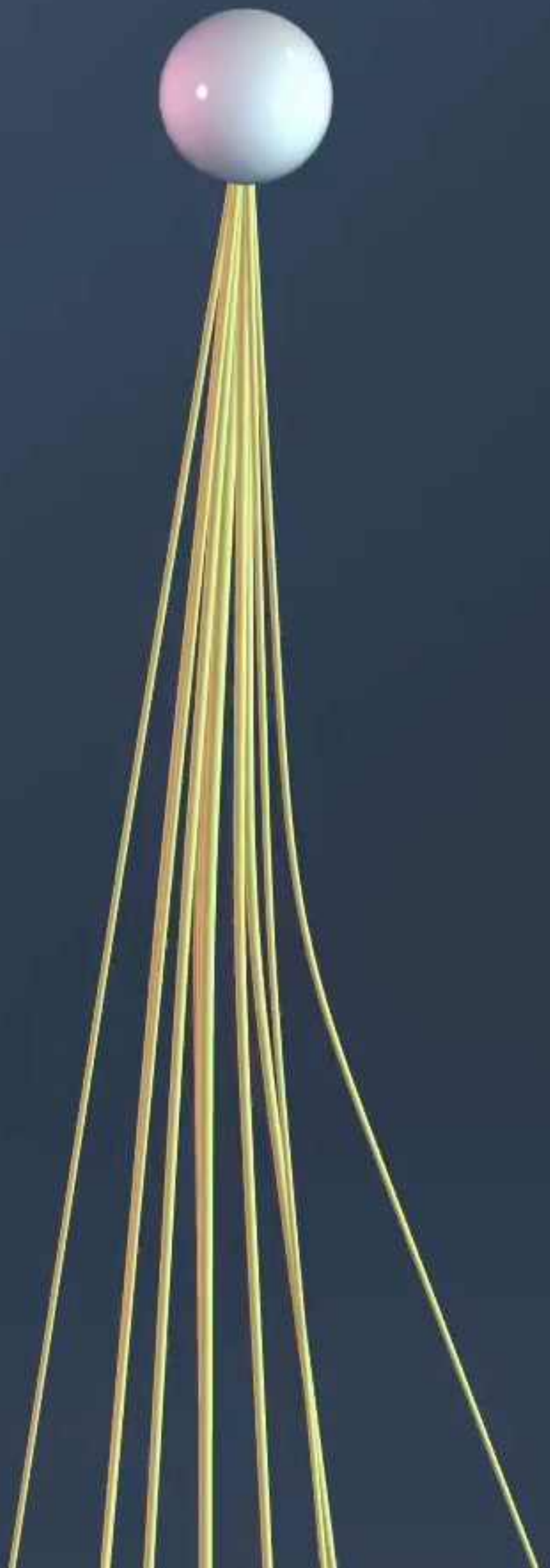
Computing Intersections



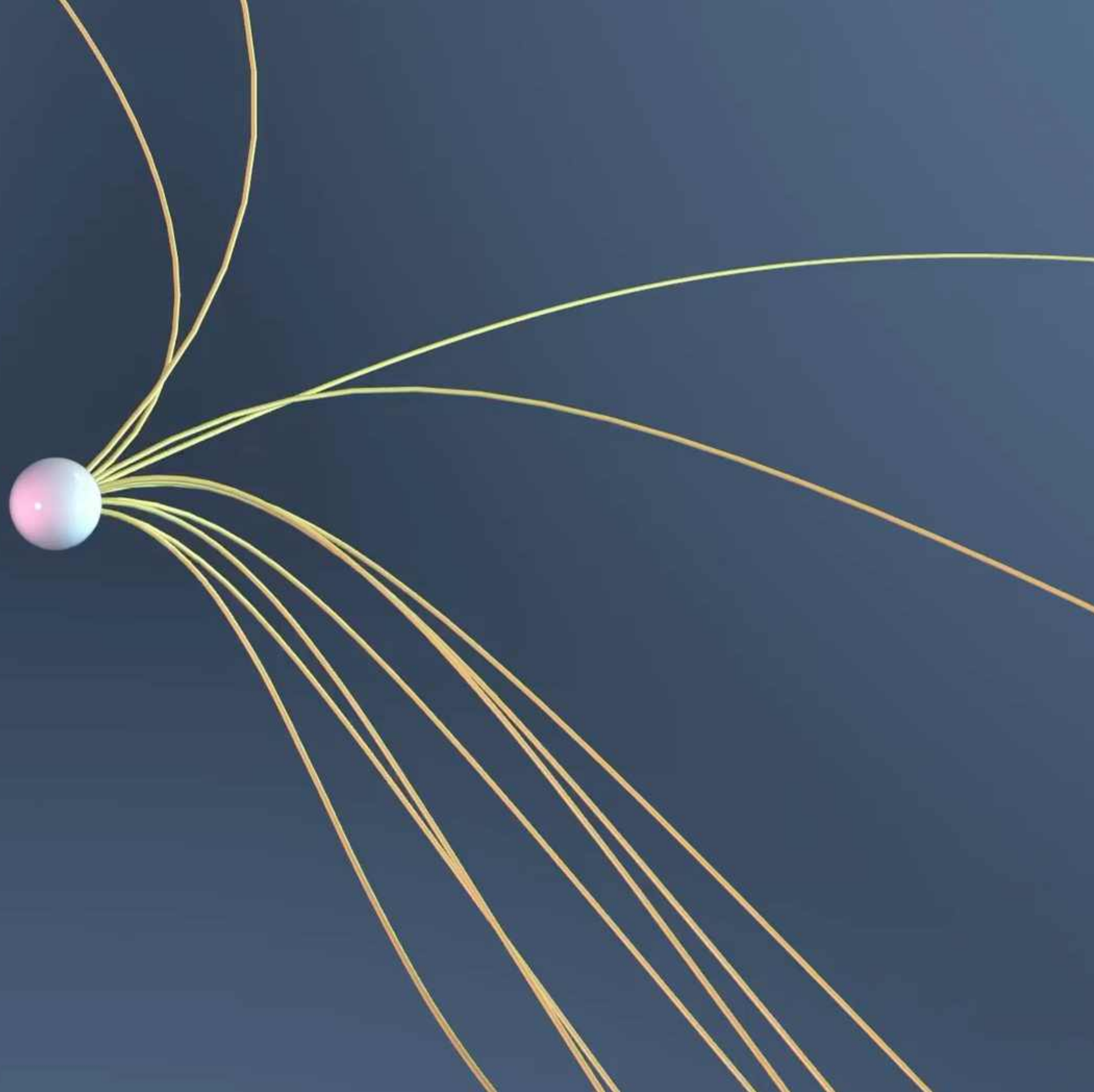
Non- Linear Geodesics

⟹ ~~Linear Algebra~~

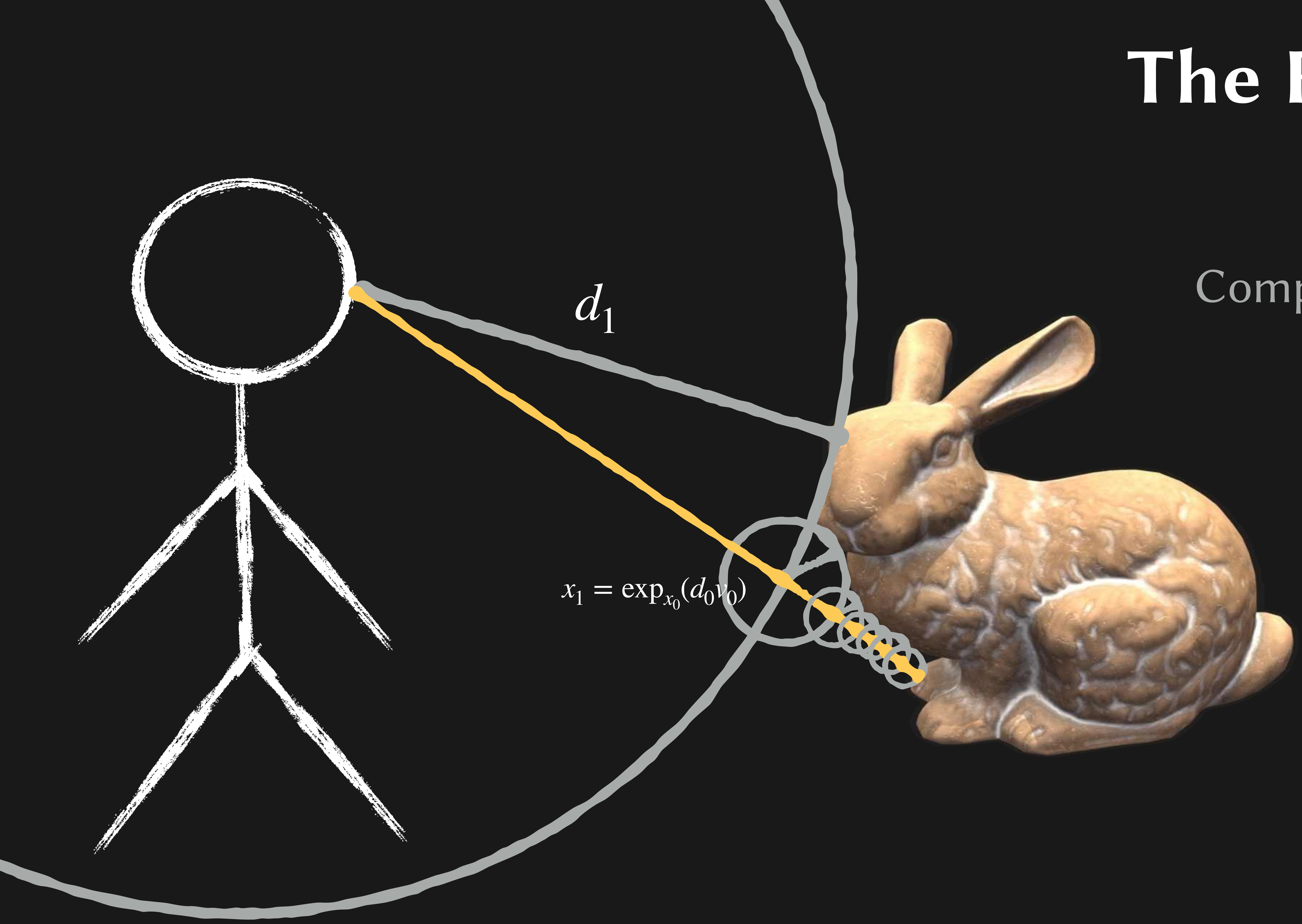
$$dx^2 + dy^2 + dz^2 + e^{-2r^2}(x\,dx + y\,dy + z\,dz)^2$$



$$dx^2 + dy^2 + (dz - x\,dy)^2$$



The Raymarching Algorithm



Compute the distance to object

$$d_n = \mathrm{dist}(x_n, \mathrm{Obj})$$

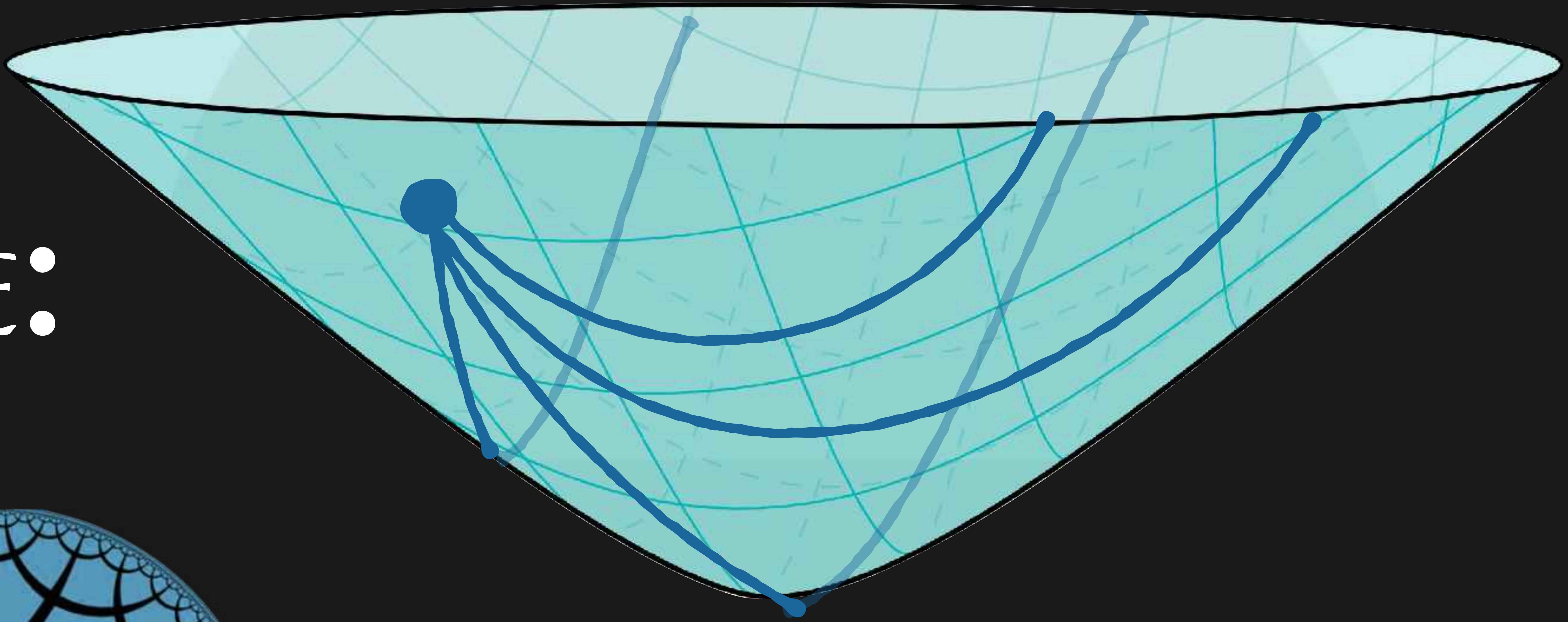
March this distance along geodesic

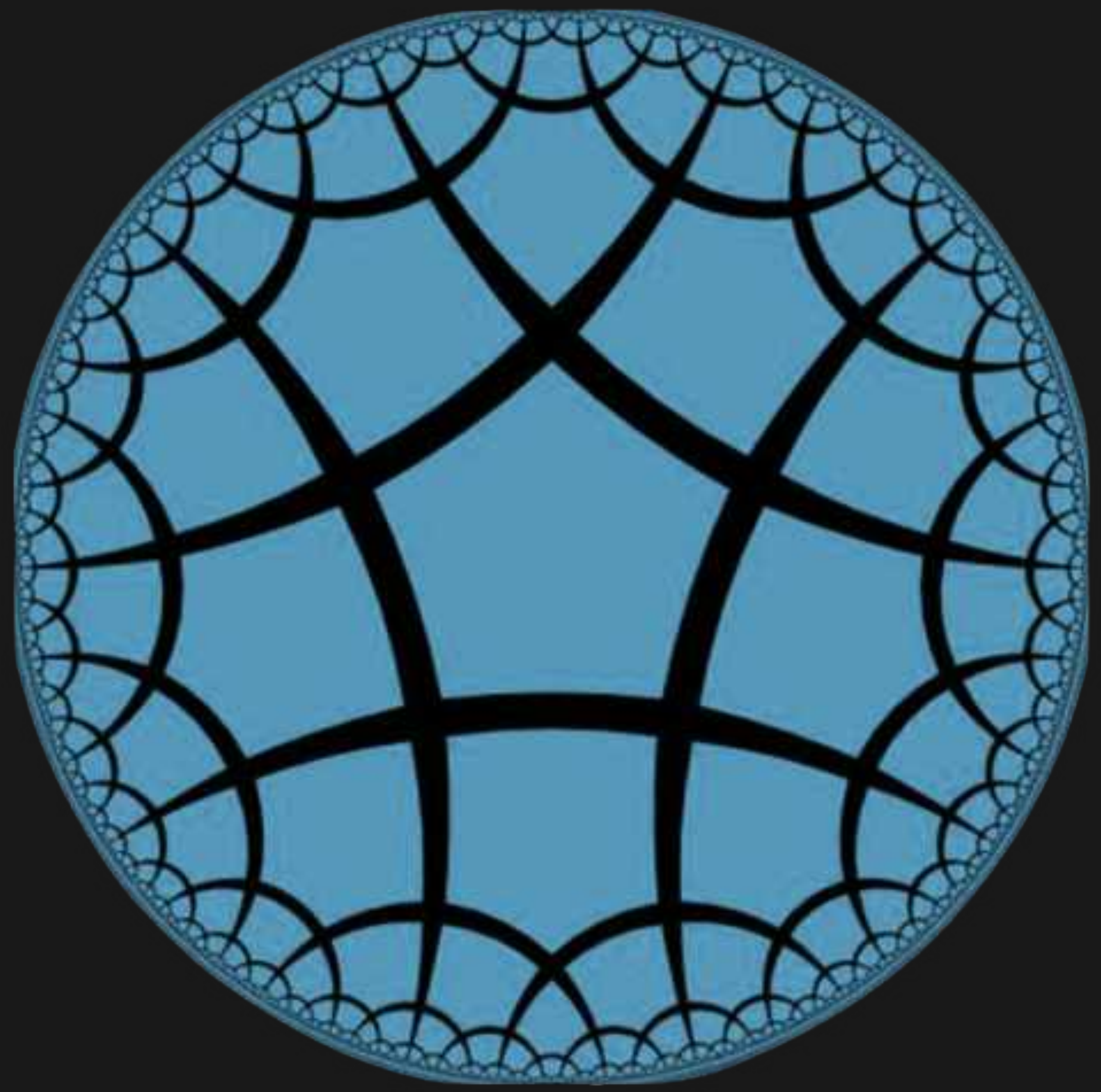
$$x_{n+1} = \exp_{x_n}(d_n v_n)$$

$$v_{n+1} = \frac{d}{dt}\bigg|_{t=d_n} \exp_{x_n}(t v_n)$$

Repeat until $d_N \approx 0$

Example:



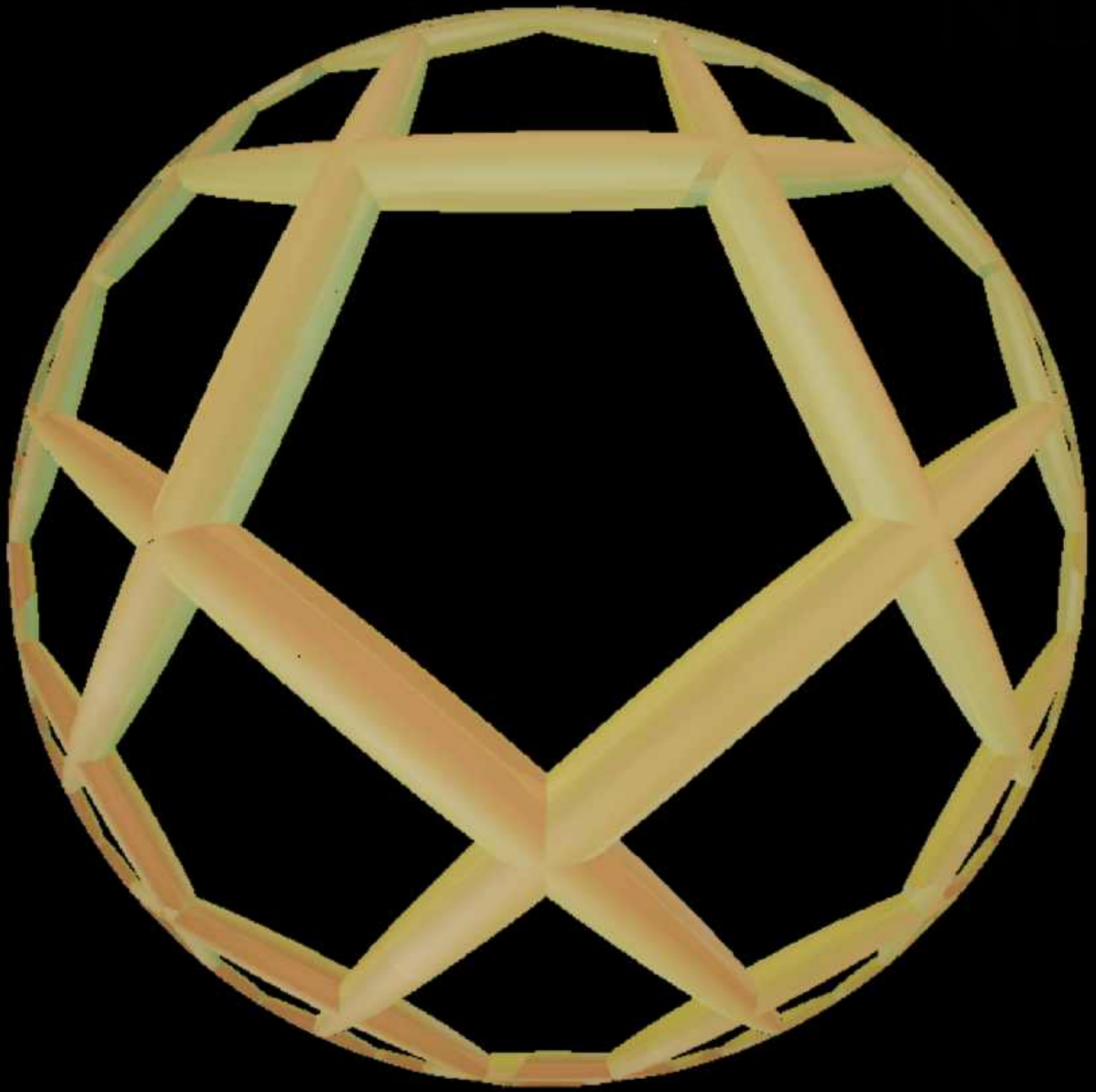


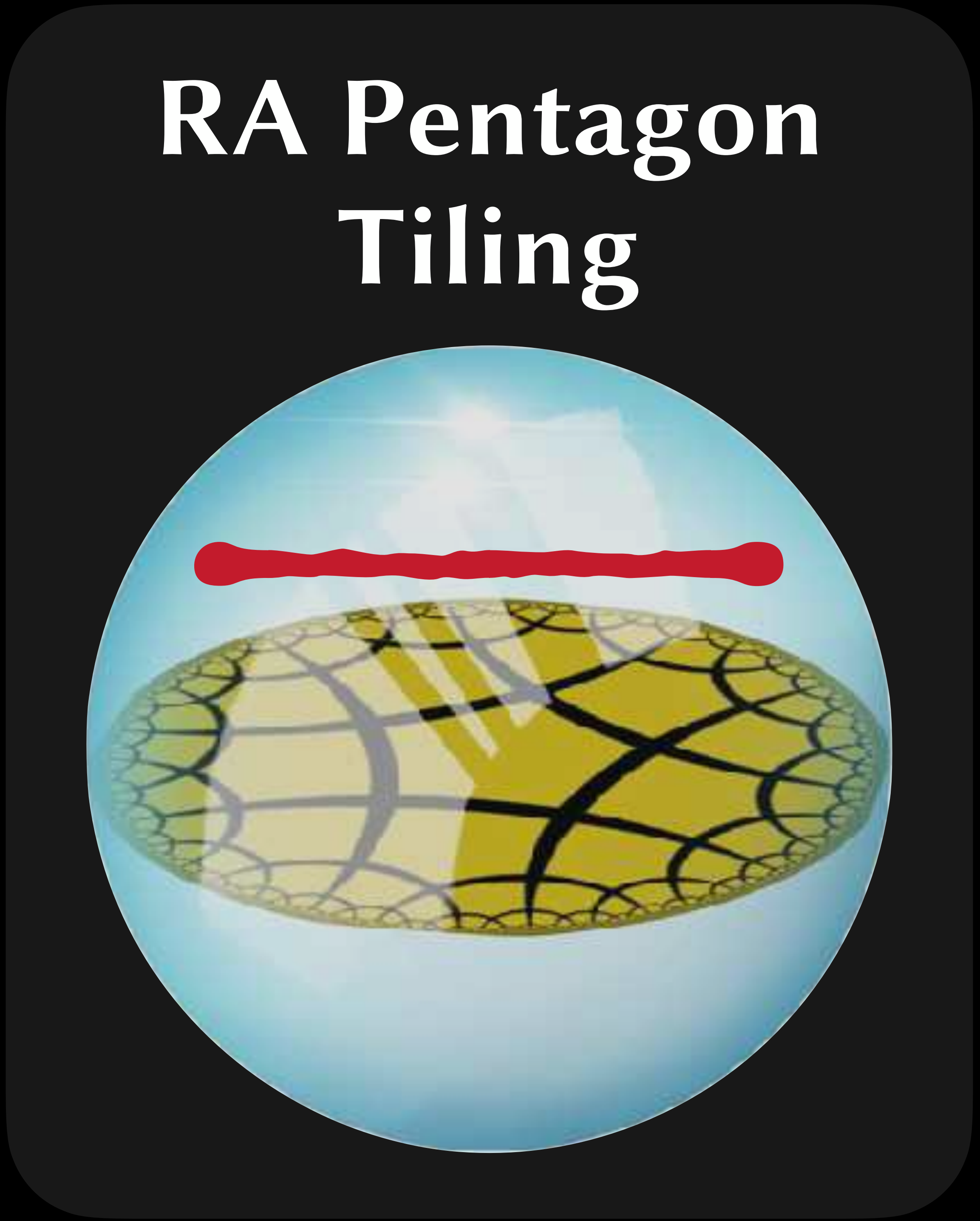
Hyperbolic Space

Geodesics are easy to compute: intersection with a linear space.

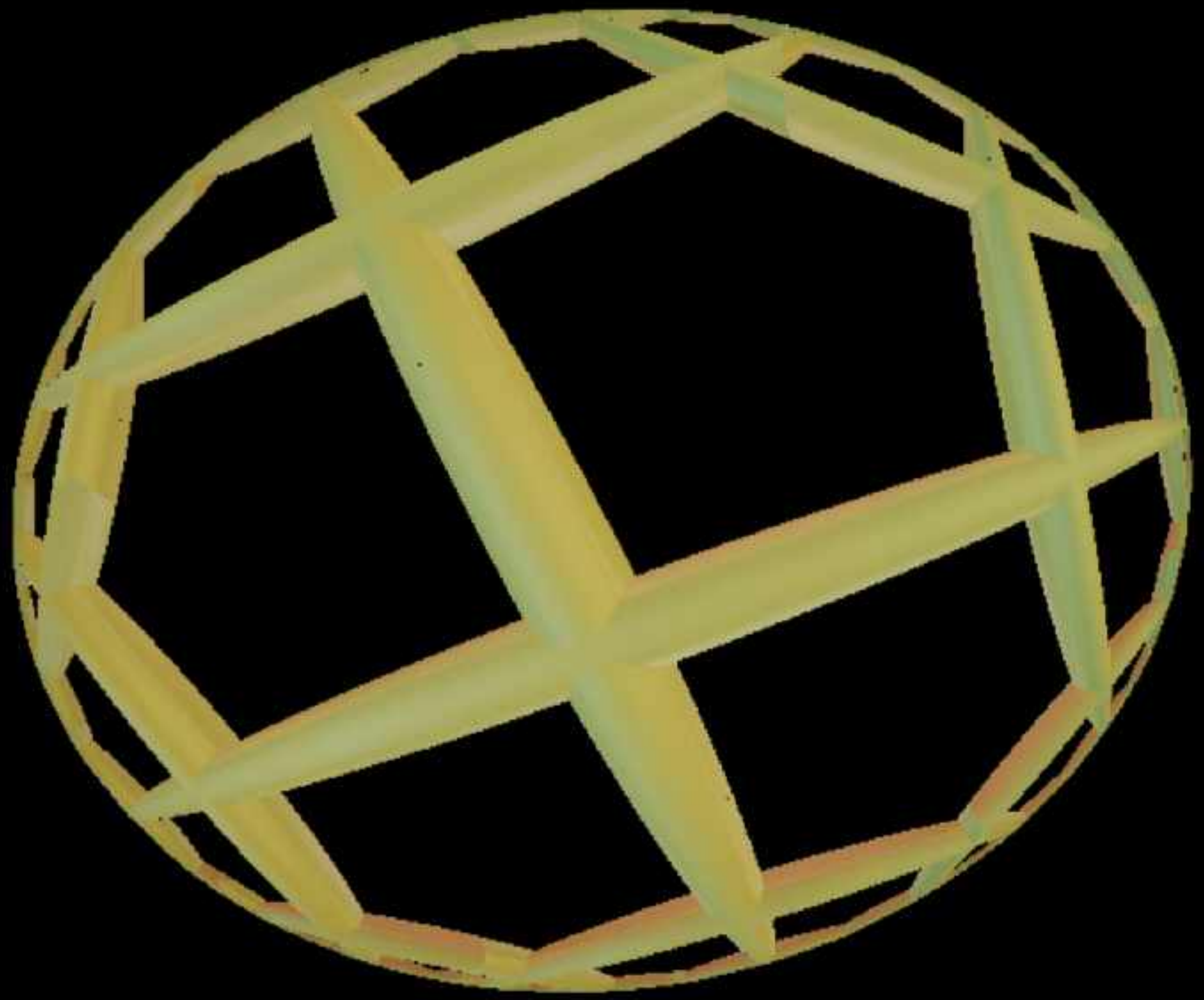
Distance is easy to compute: Minkowski dot product.

Hyperbolic Space

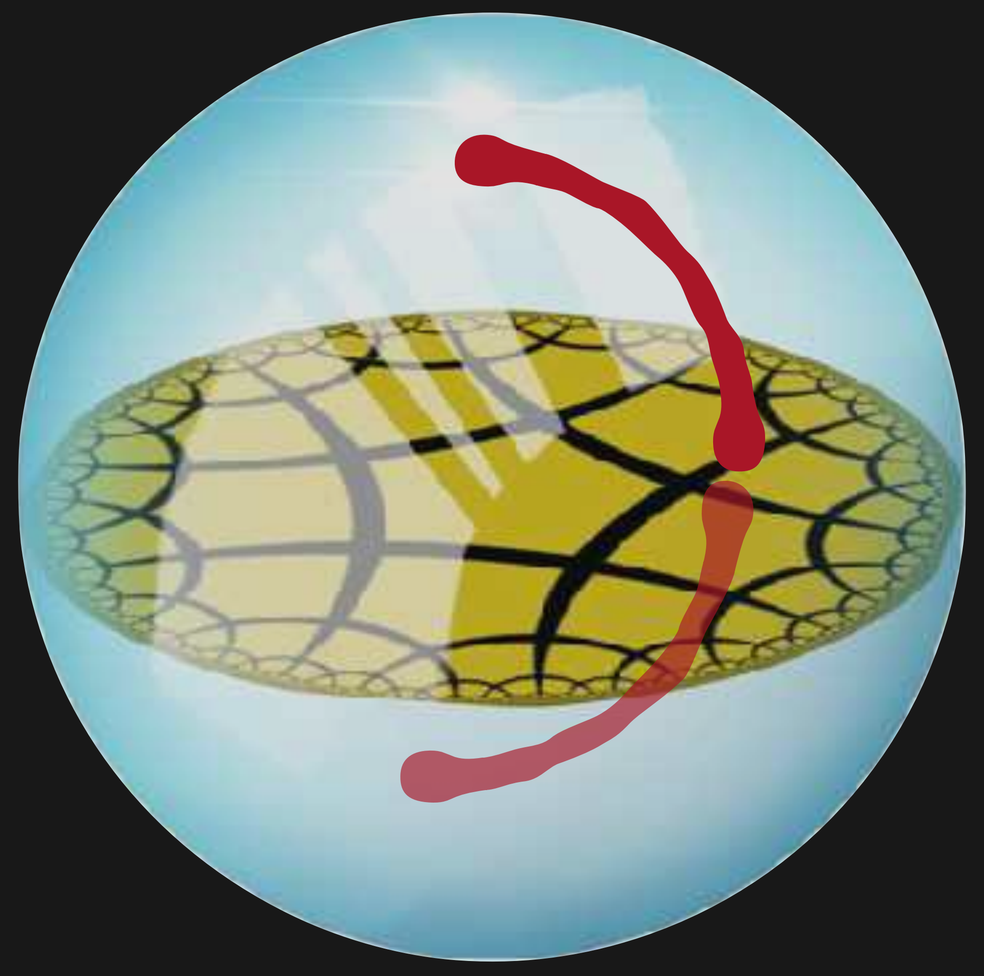




Hyperbolic Space

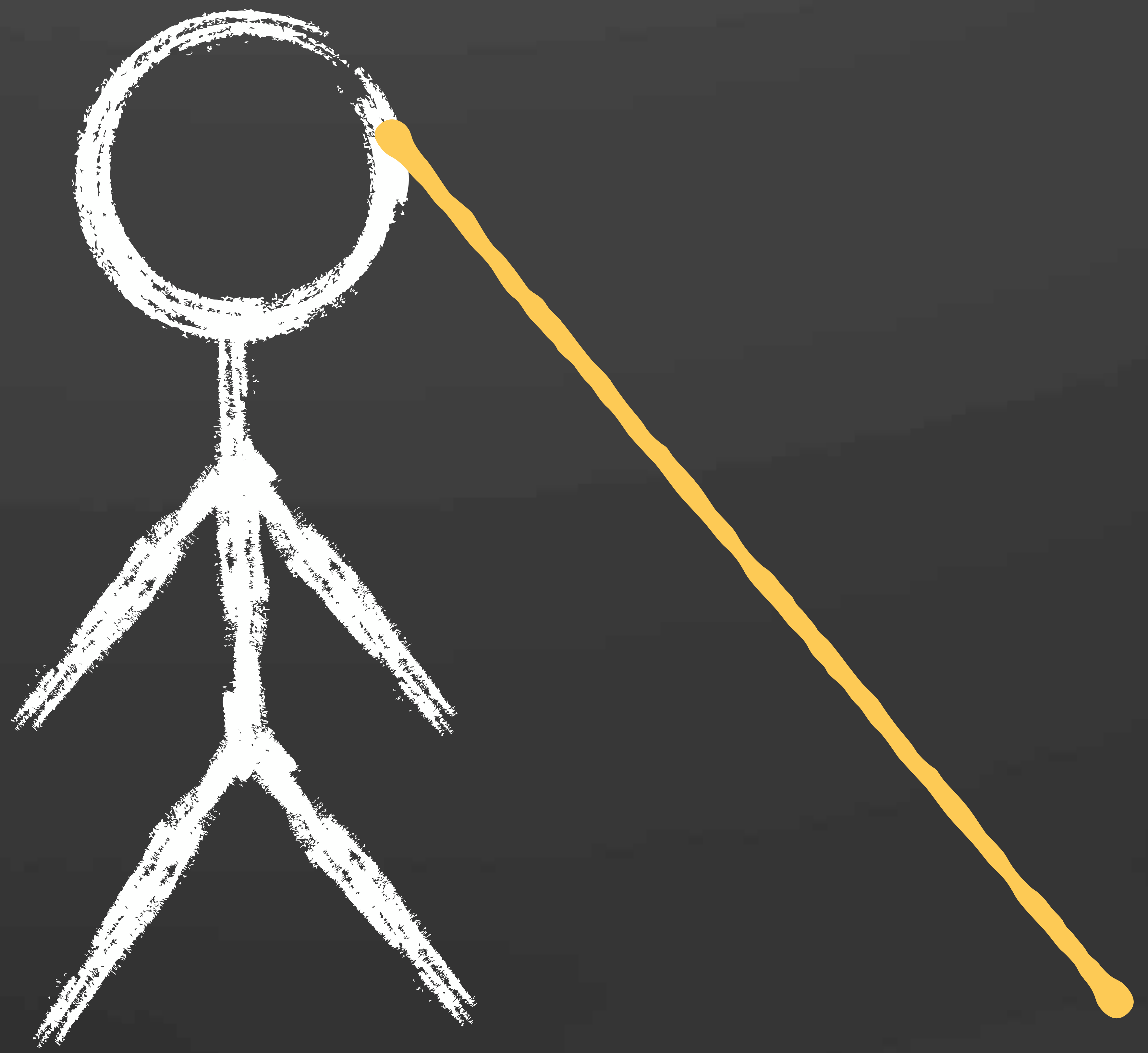


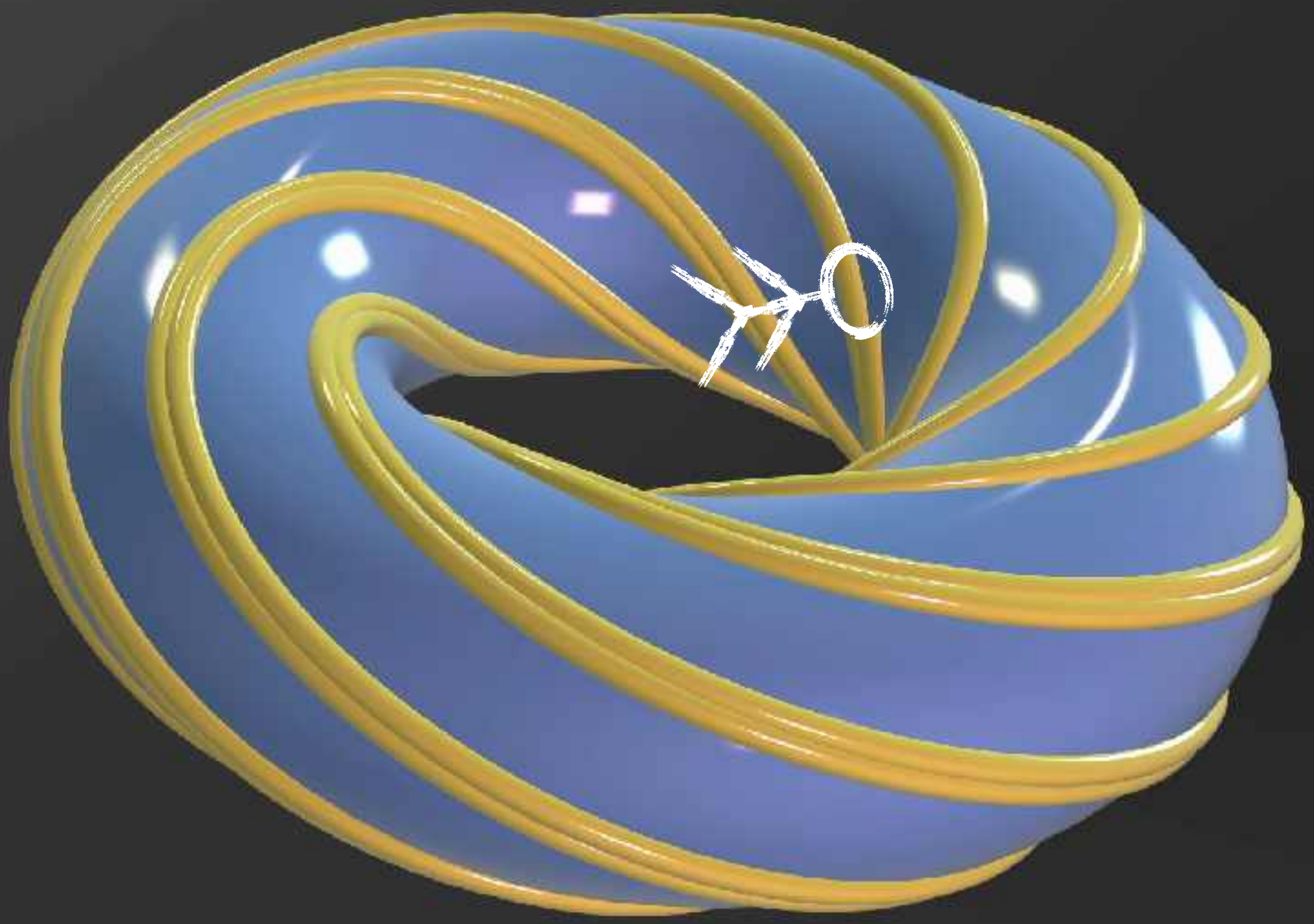
RA Pentagon Tiling

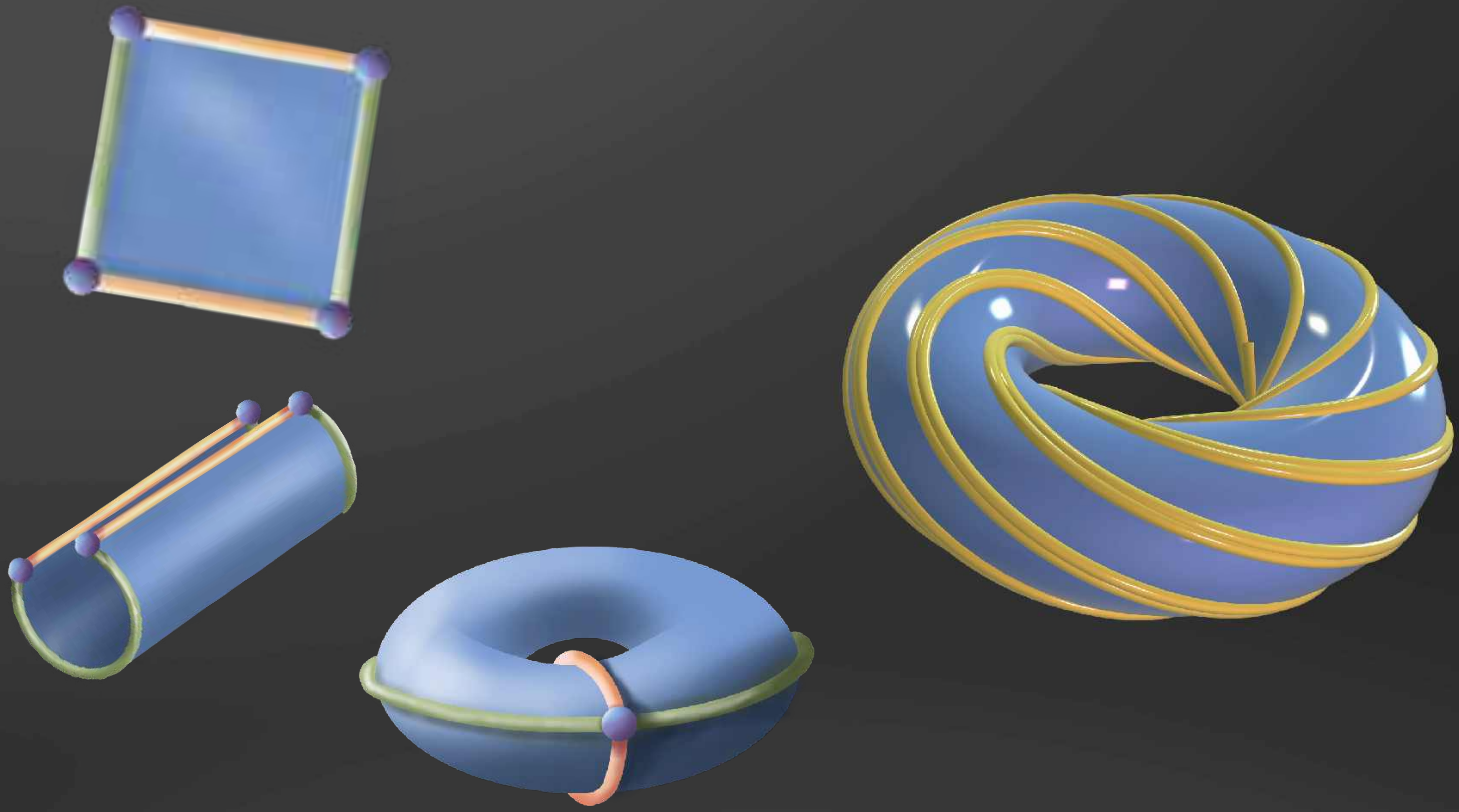


Raytracing in Multiply Connected Manifolds

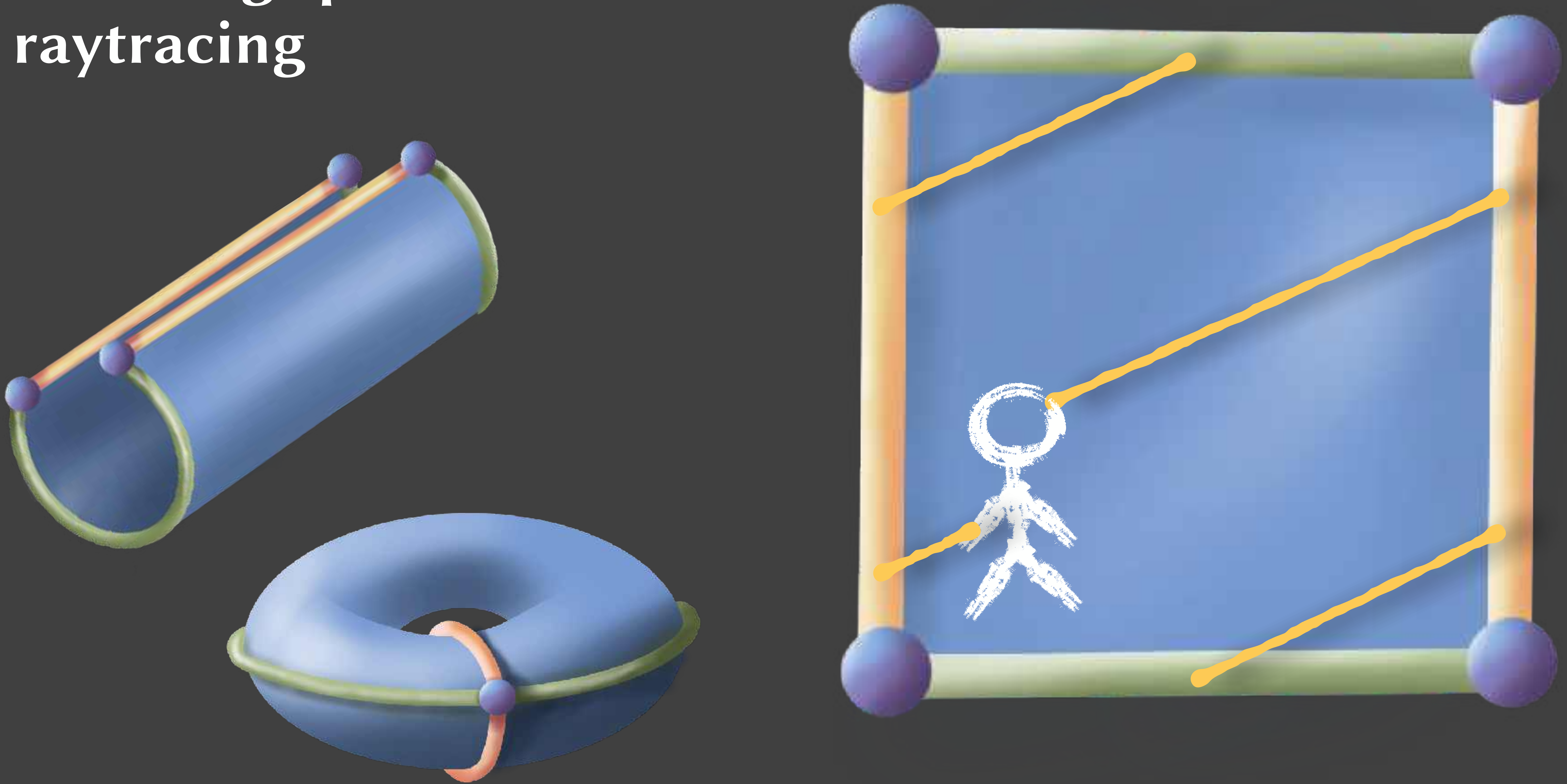
Light on a closed manifold spirals around and around.....





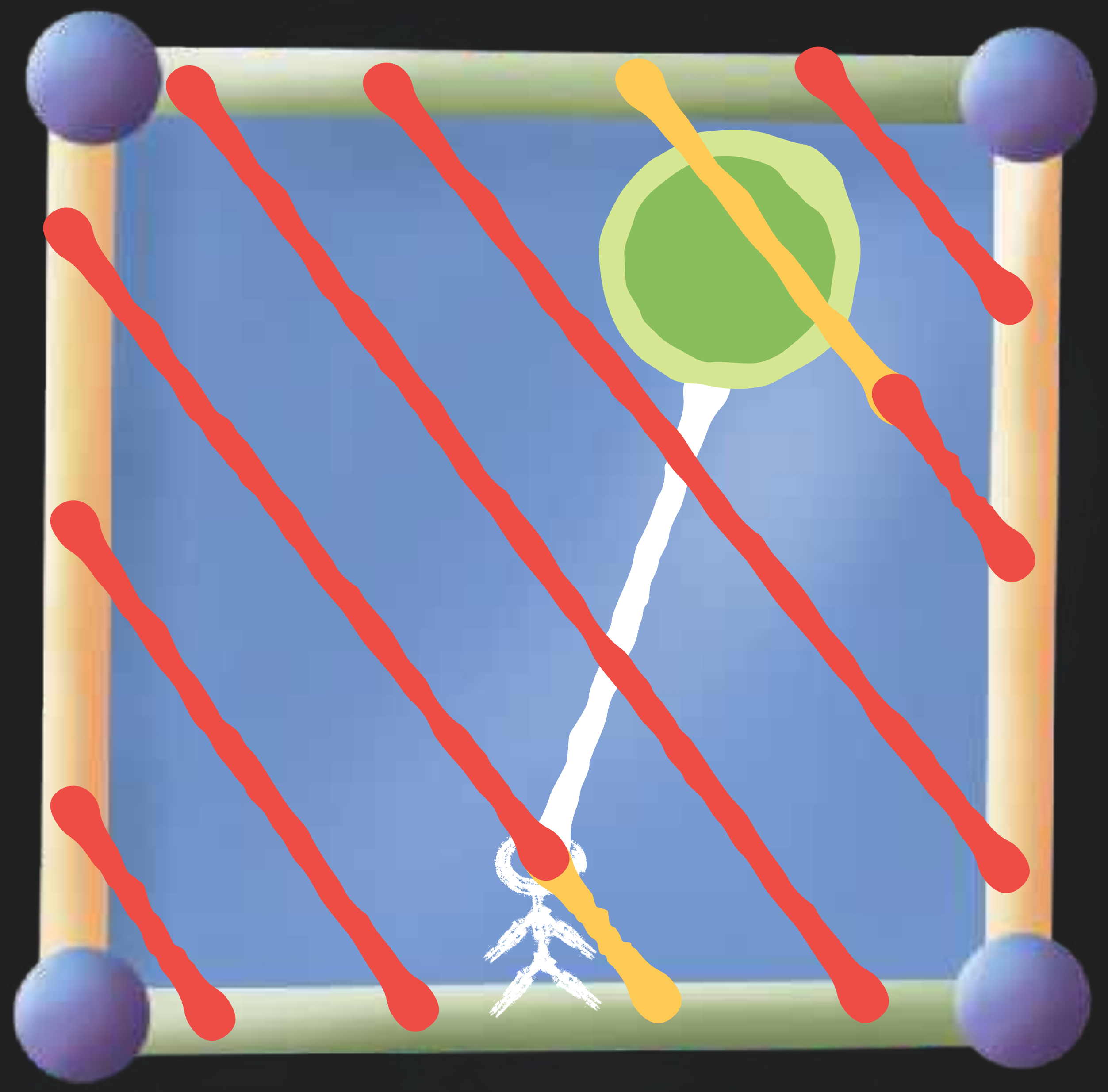


Covering spaces & raytracing



Covering spaces & raytracing

The original ray marching algorithm is fails when using distance function on a fundamental domain.



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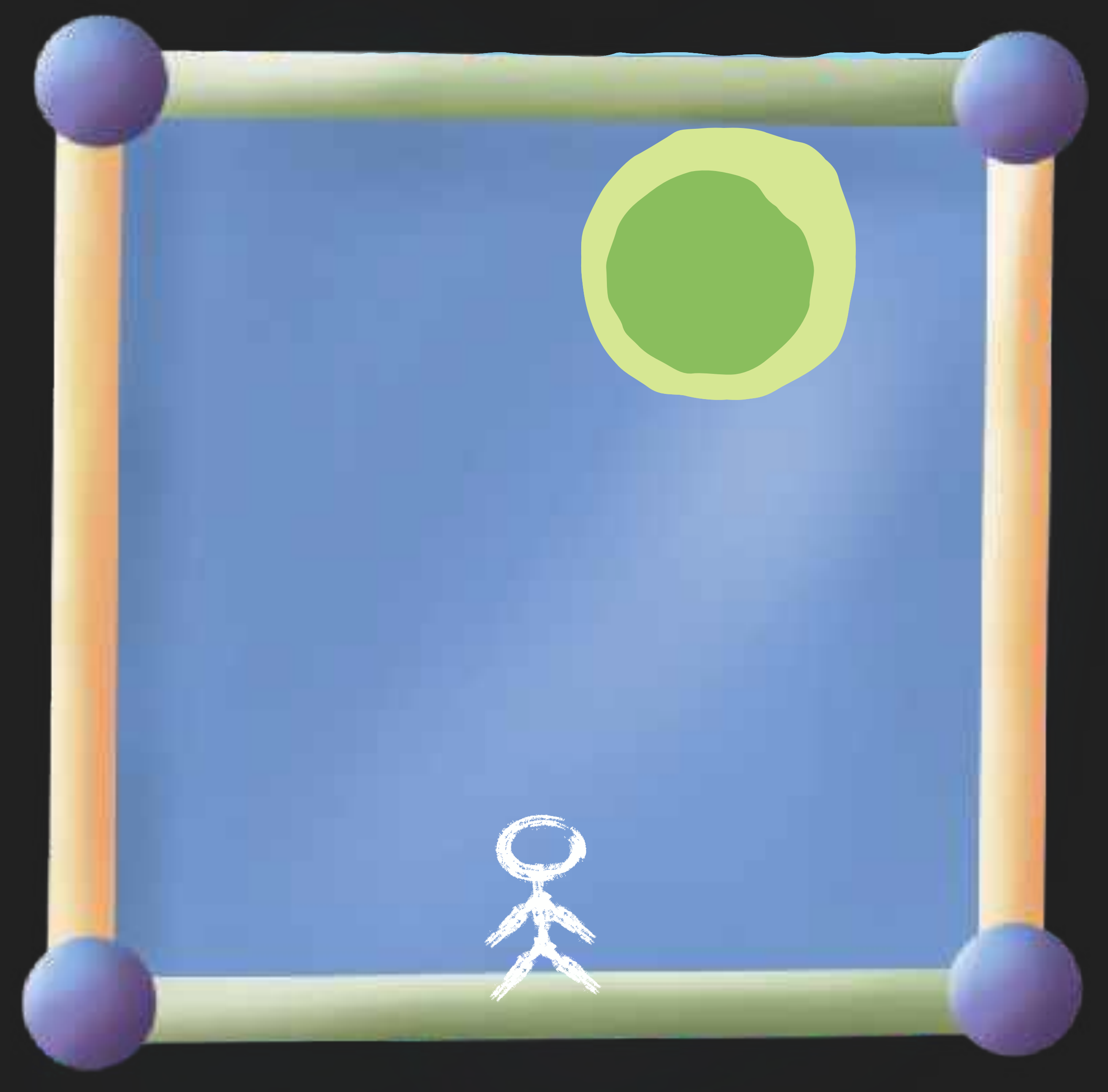
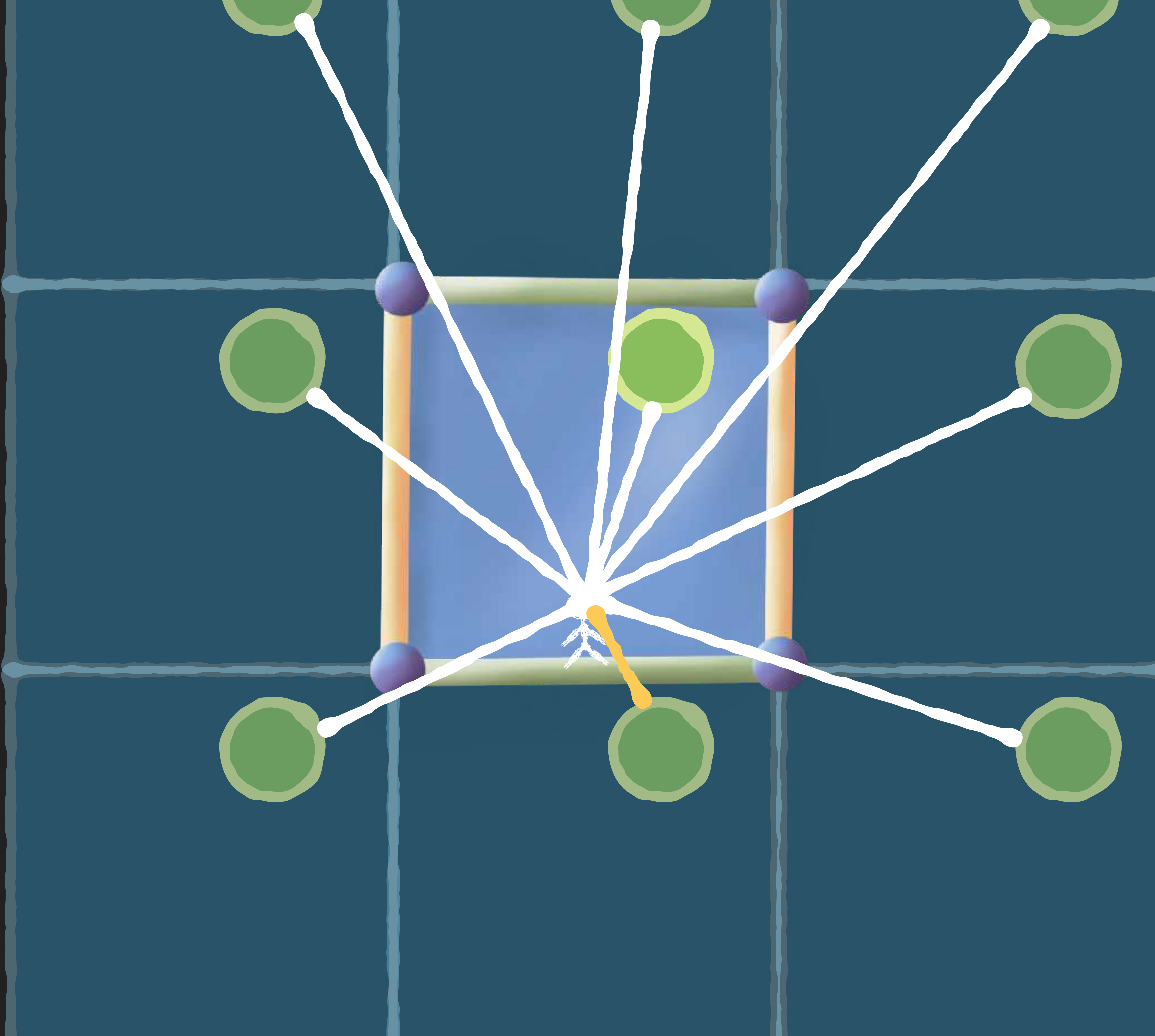


Covering spaces & raytracing

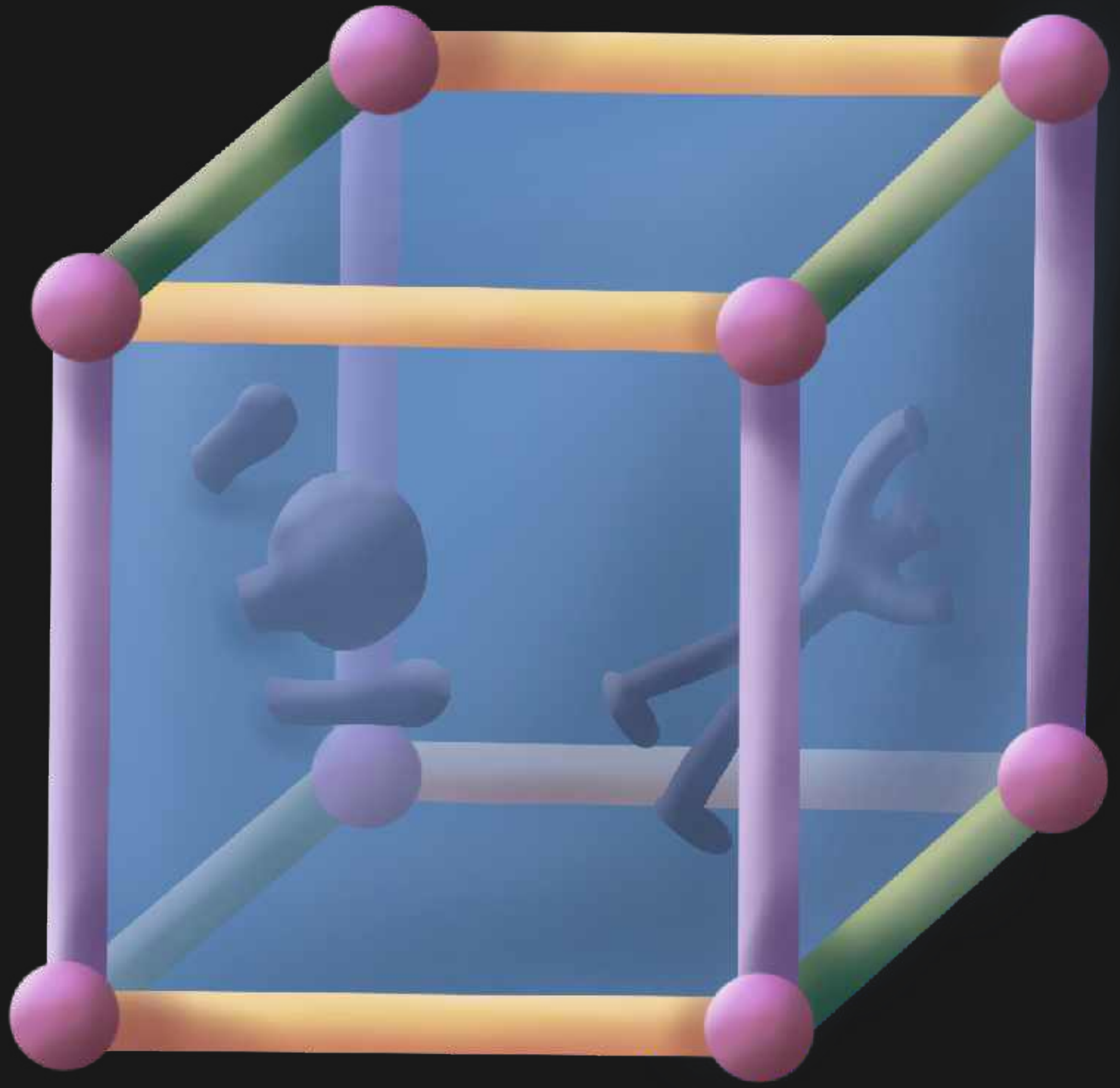
The original ray marching algorithm is fails when using distance function on a fundamental domain.

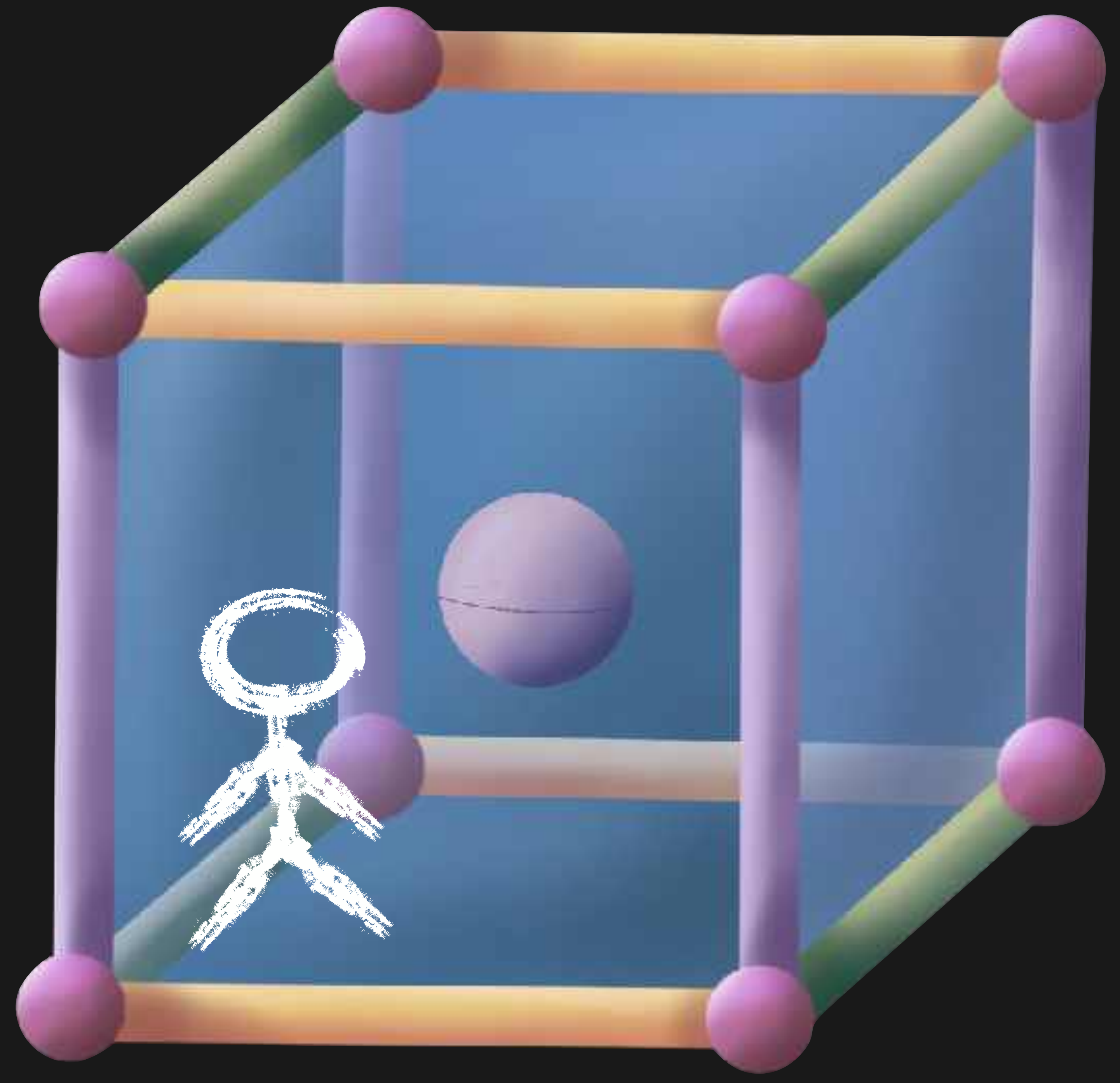
Need to take into account certain elements of π_1



The 3-Torus

Fundamental domain is a cube / parallelepiped with opposite sides identified via translation.





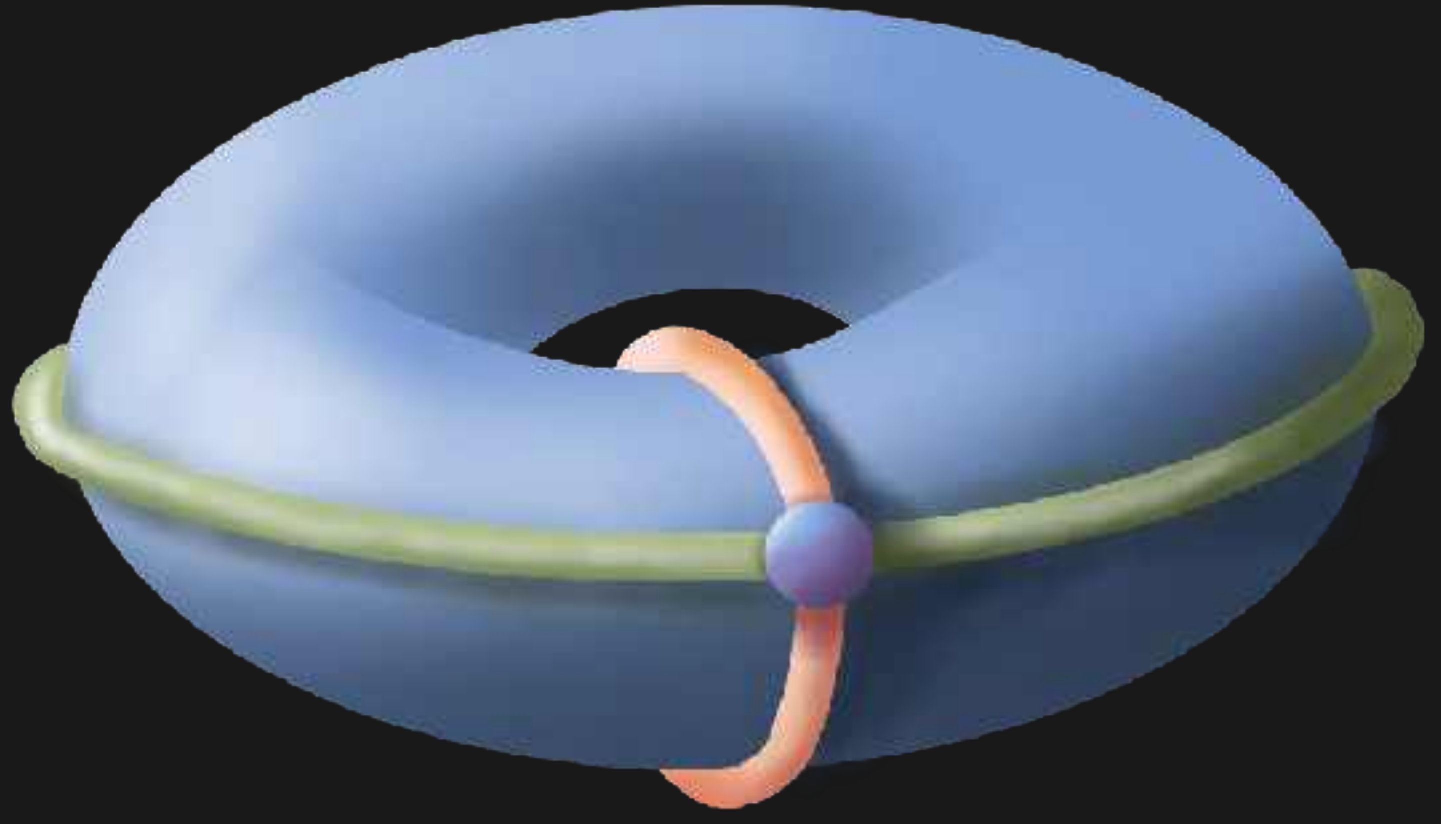
What do you expect to see, if the 3-torus is empty except for a single white sphere?

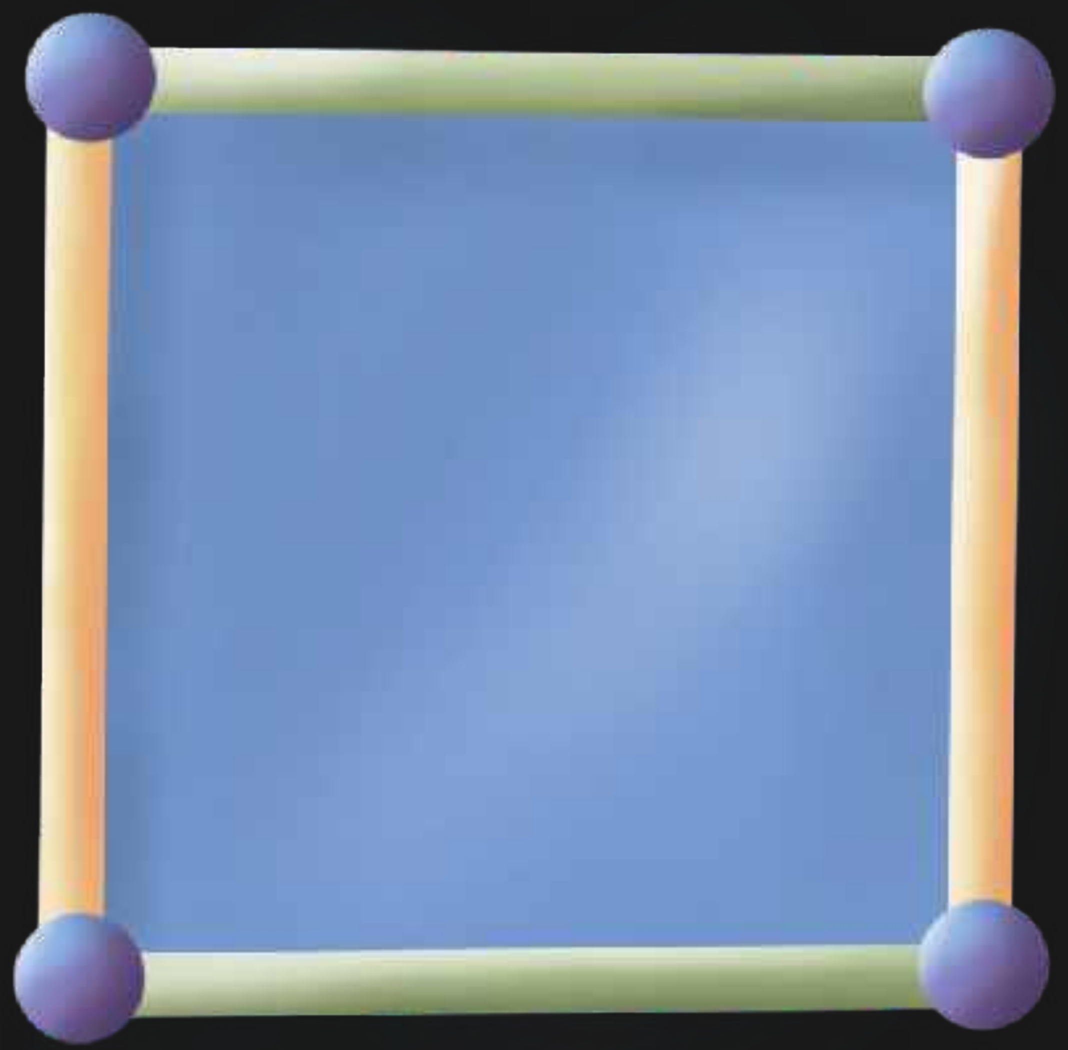

$\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$

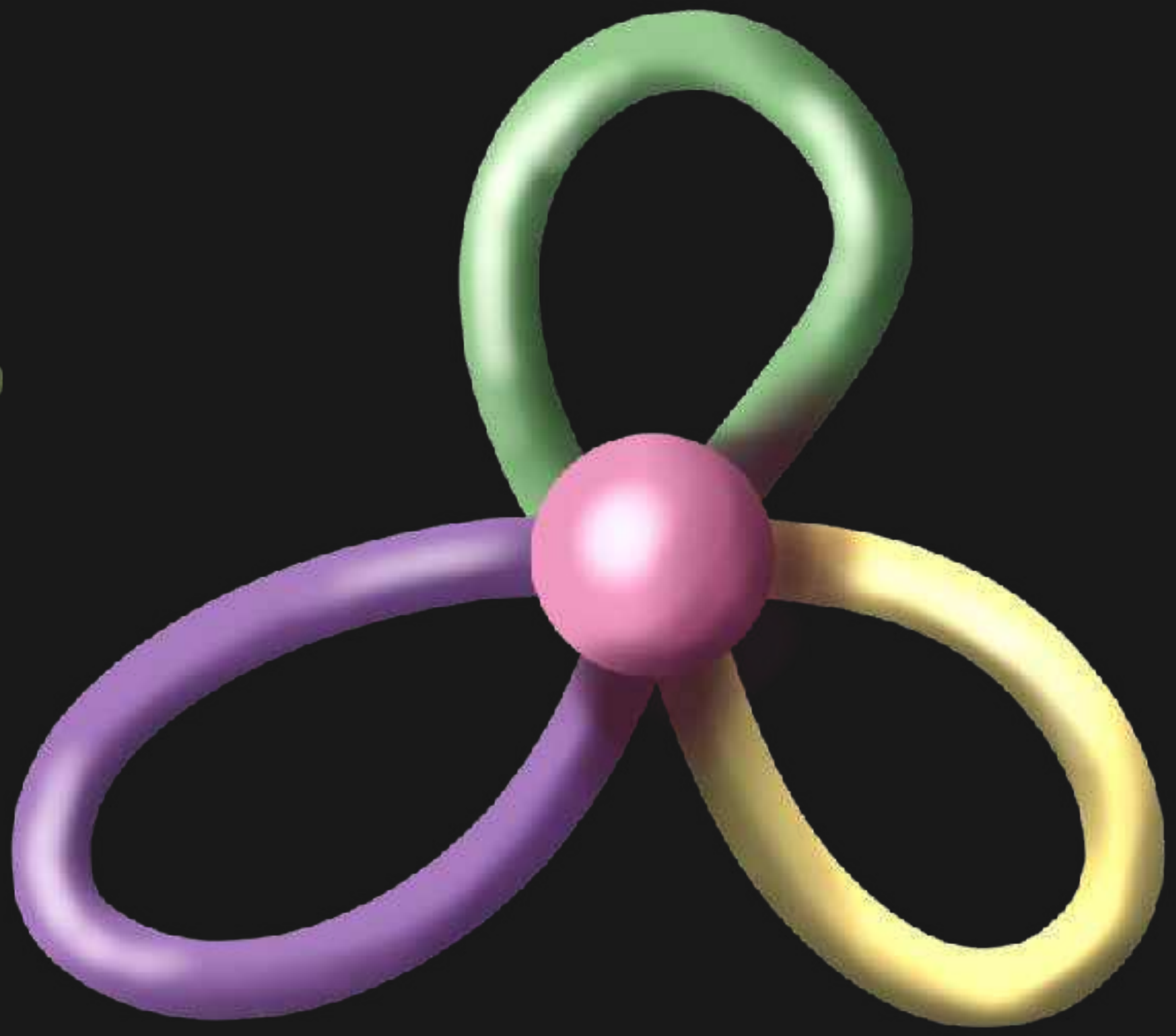

$\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$

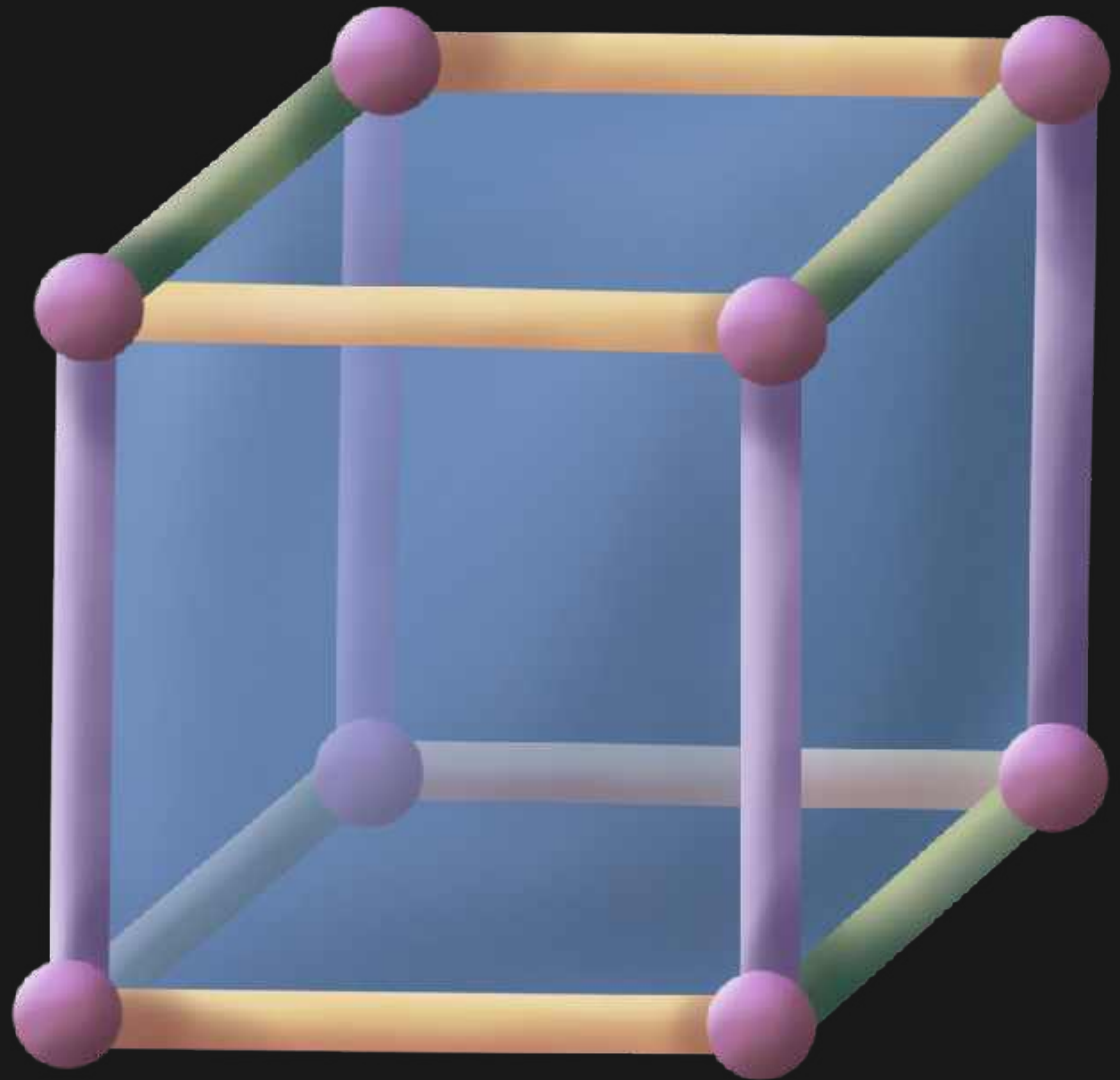
The 3-Torus

Generators of π_1










$\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$

Seifert-Weber Dodecahedral Space

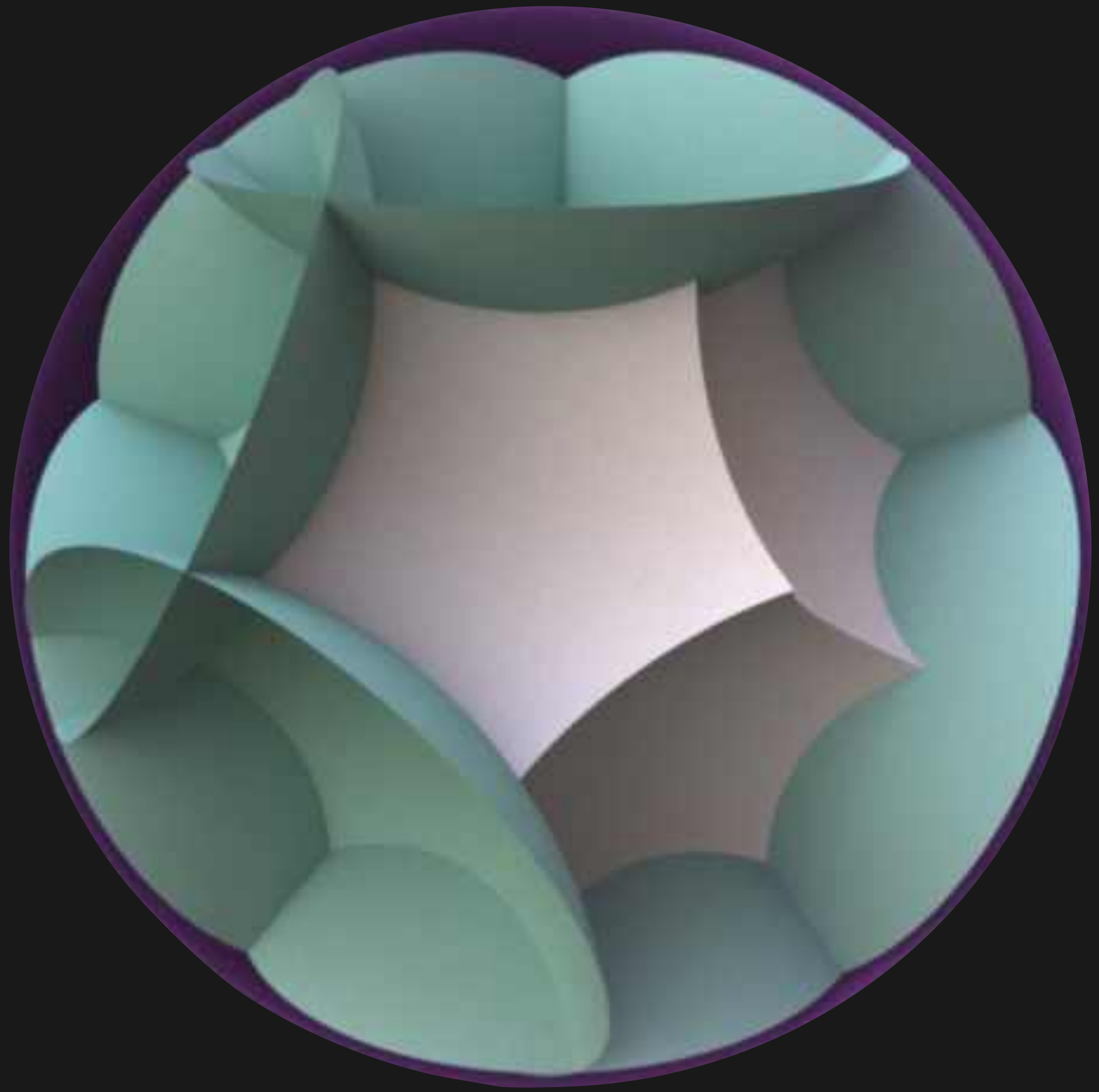
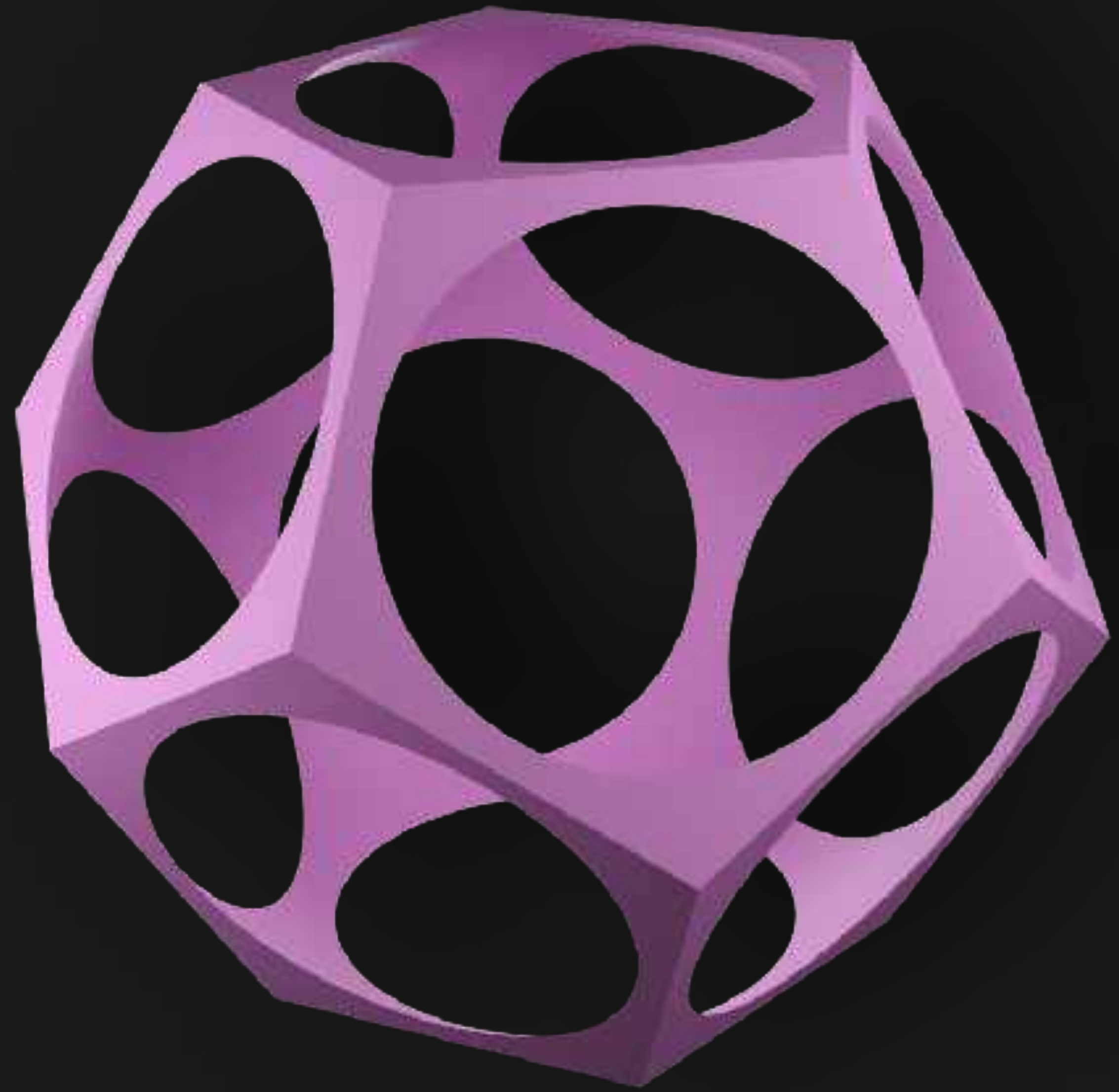


Image: Arnaud Cheritat

Fundamental domain is a hyperbolic dodecahedron, with opposite sides paired by a 3/10 twist.



To raytrace:

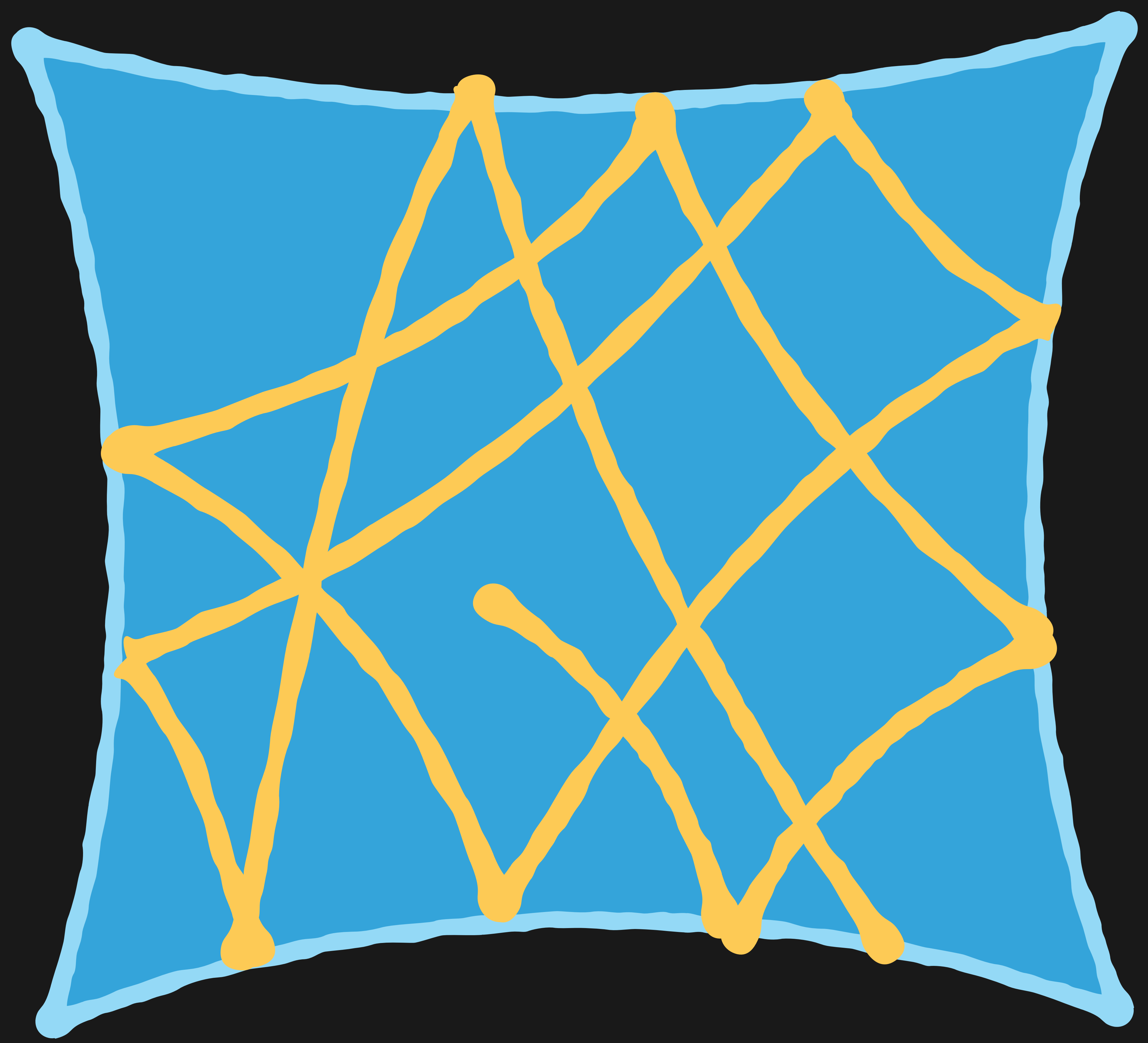
Domain walls built by deleting a hyperbolic sphere from its center


Seifert-Weber


Seifert-Weber

Hyperbolic Coxeter Groups

Set face pairings to be reflections!




Hyperbolic
Coxeter Group

Understanding What we See

The geometry and topology of the Riemannian exponential.

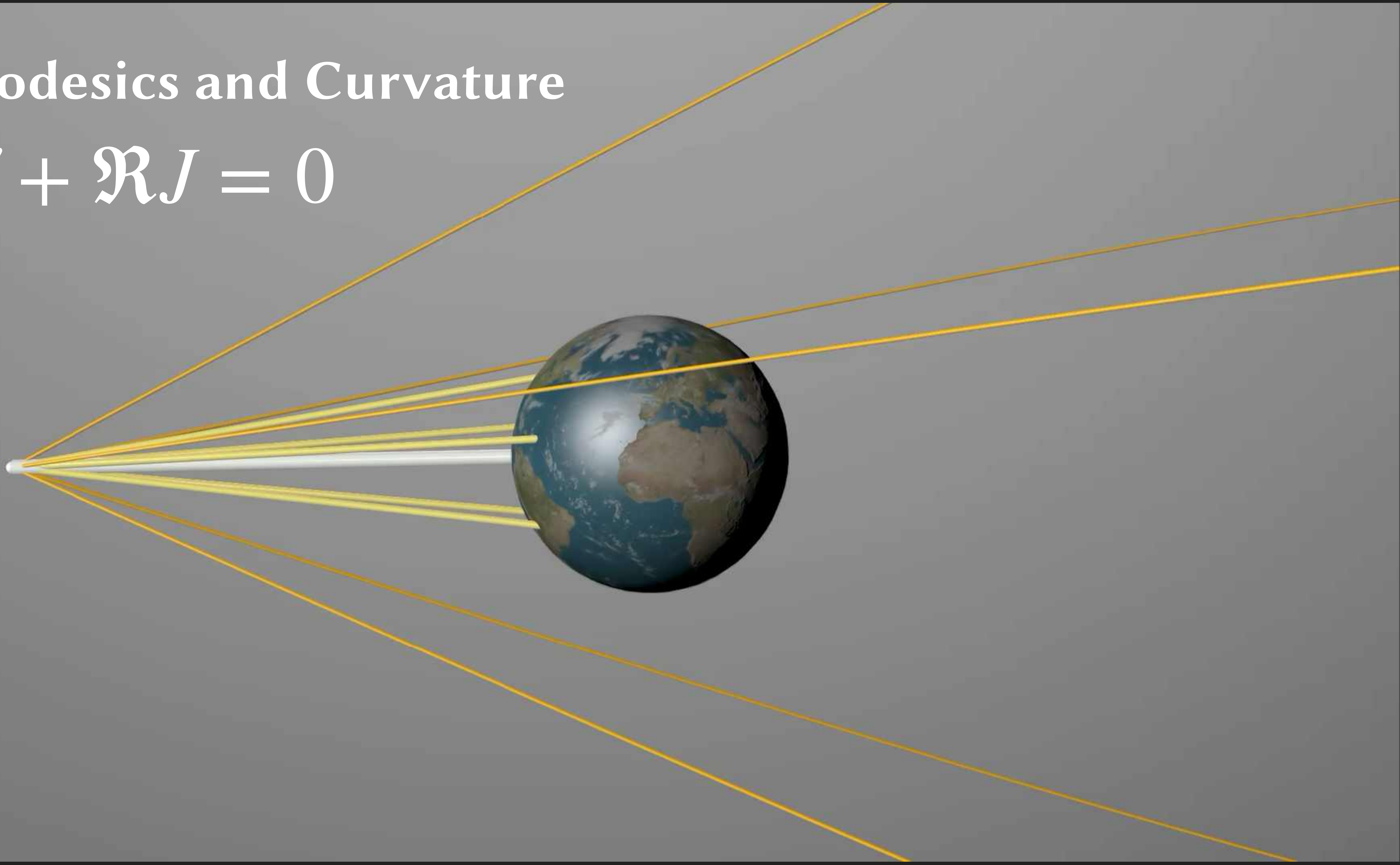
Topology & Vision

Looking around in a closed manifold is the same as looking around at a G invariant scene in the universal cover.

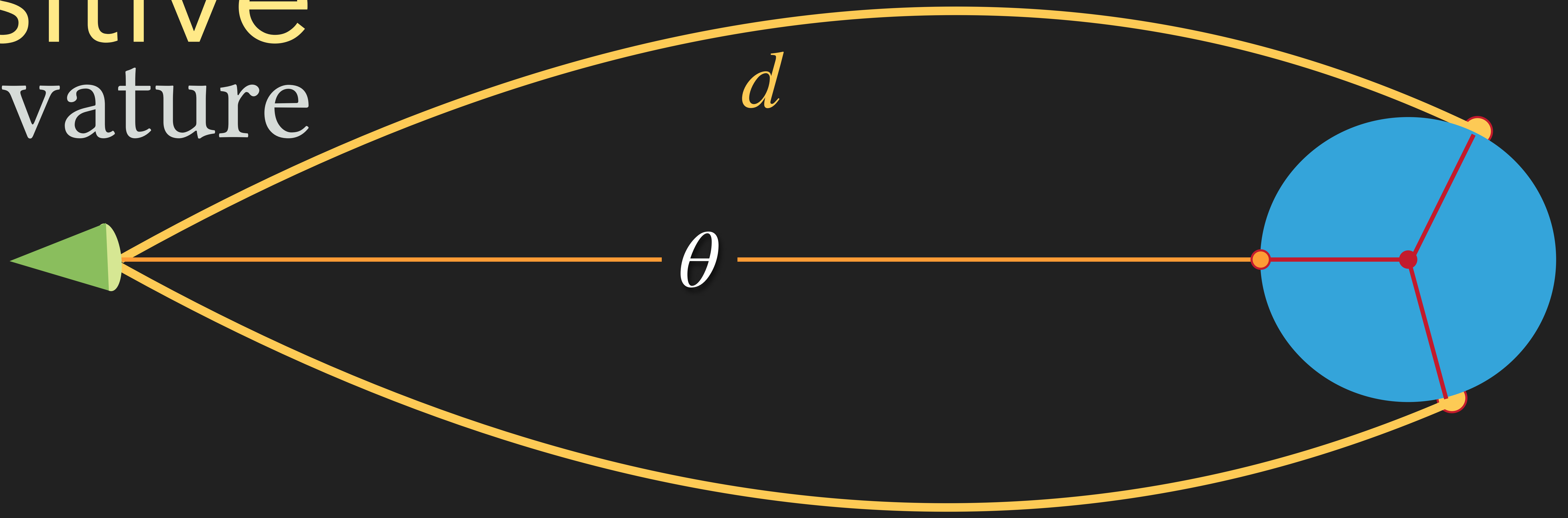
Curvature & Vision

Geodesics and Curvature

$$J'' + \mathfrak{R}J = 0$$

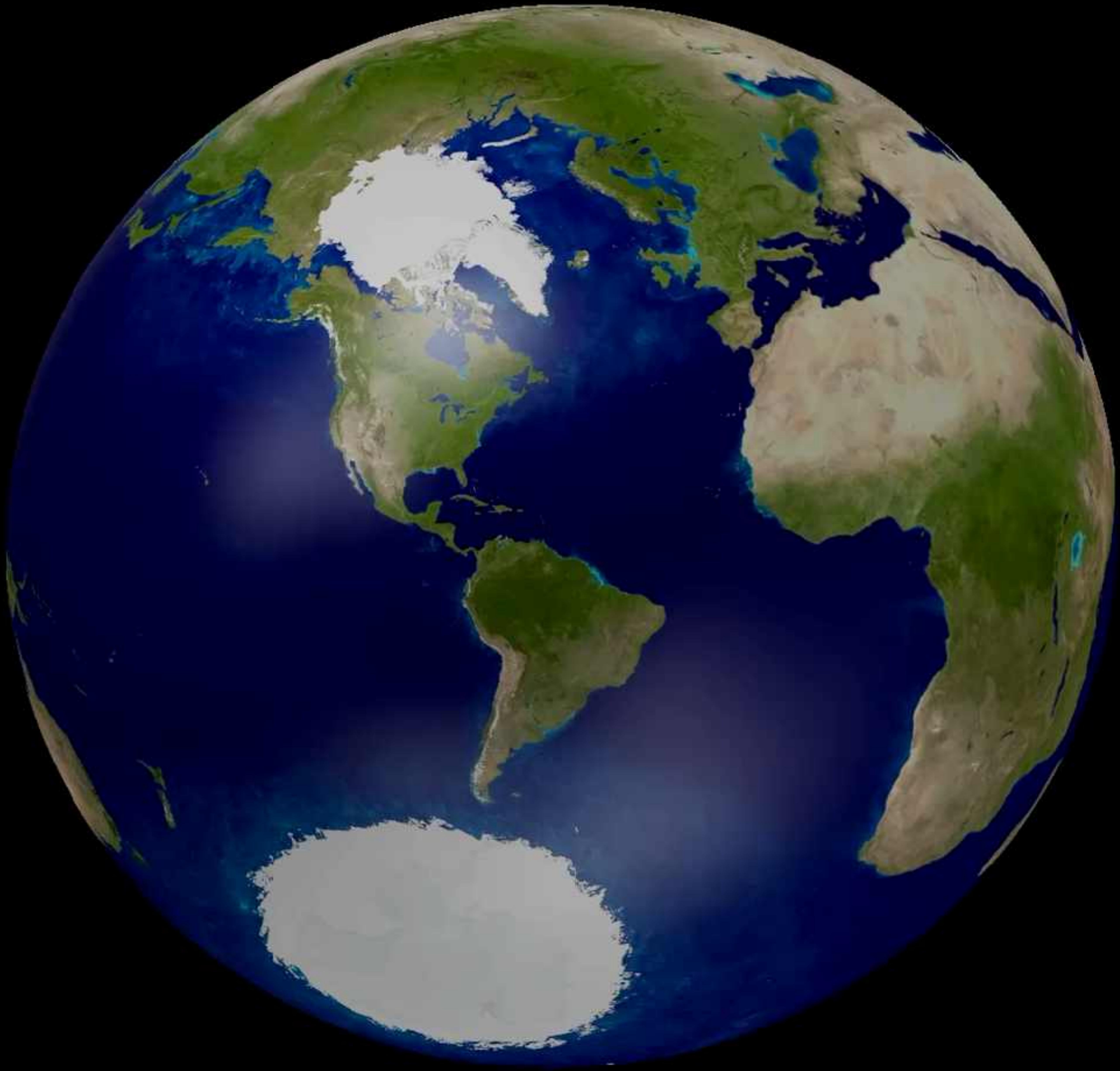


Vision in Positive curvature



$$\theta \approx 2\tan^{-1}\left(\frac{\tan r}{\sin d}\right) \cong \frac{2r}{\sin(d)}$$

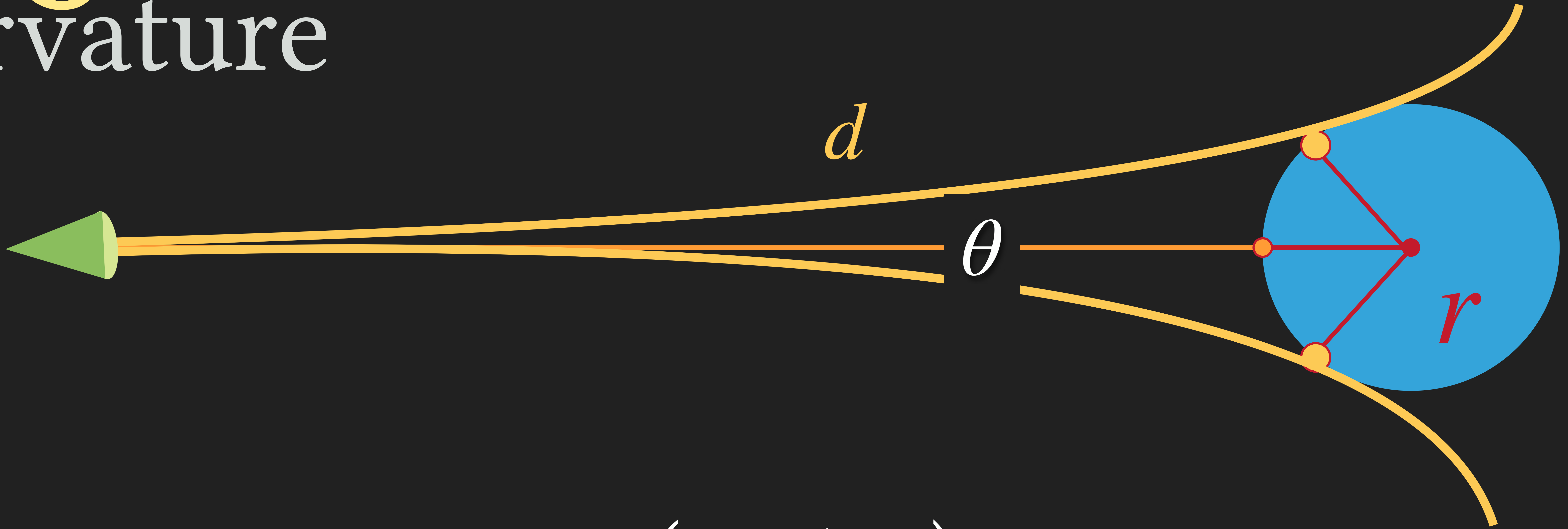
Vision in Positive curvature




Vision in
Positive
curvature


Vision in
Negative
curvature

Vision in Negative curvature

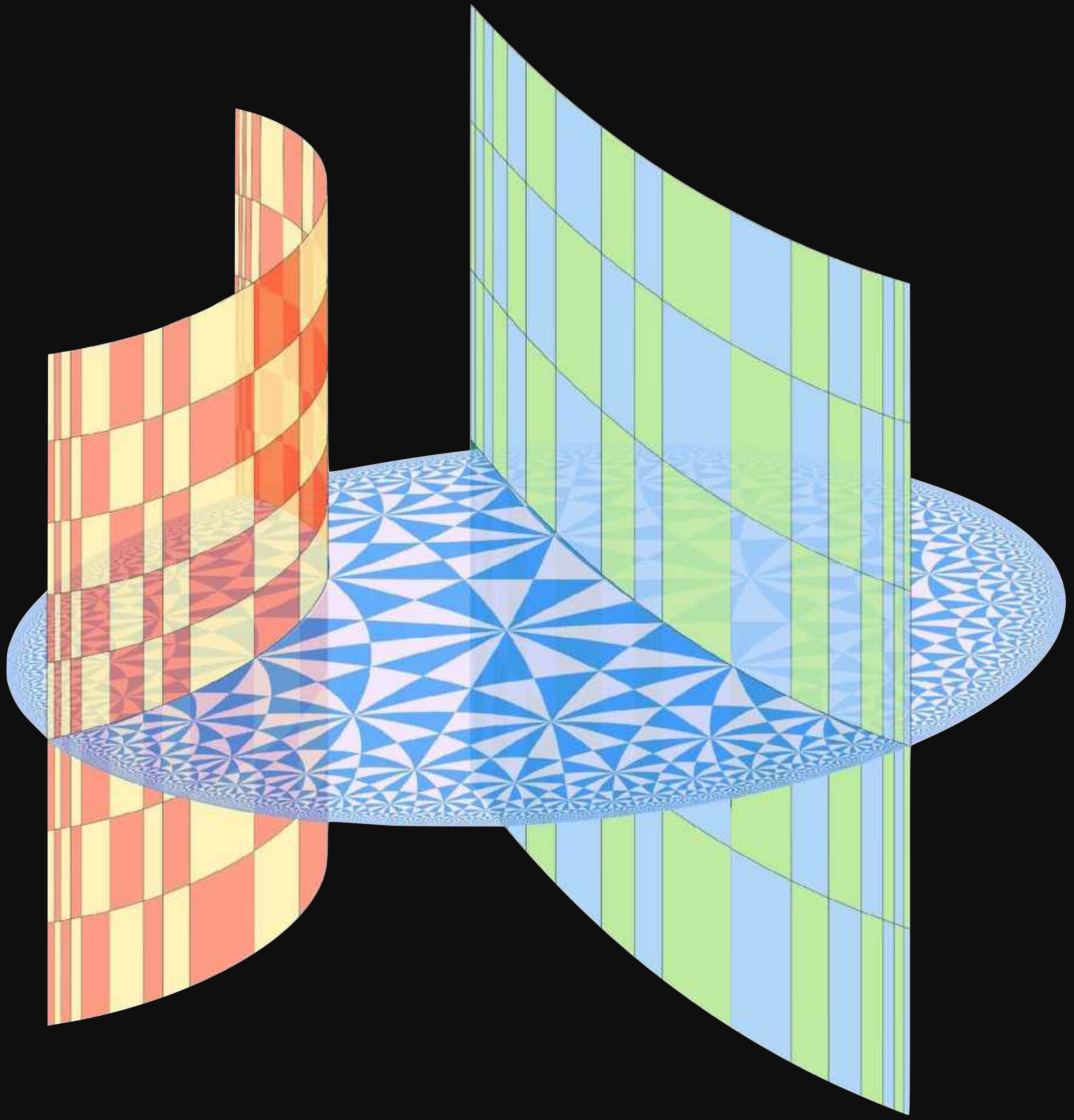


$$\theta \approx 2\tan^{-1}\left(\frac{\tanh r}{\sinh d}\right) \cong \frac{2r}{e^{d}}$$

Vision in Negative curvature

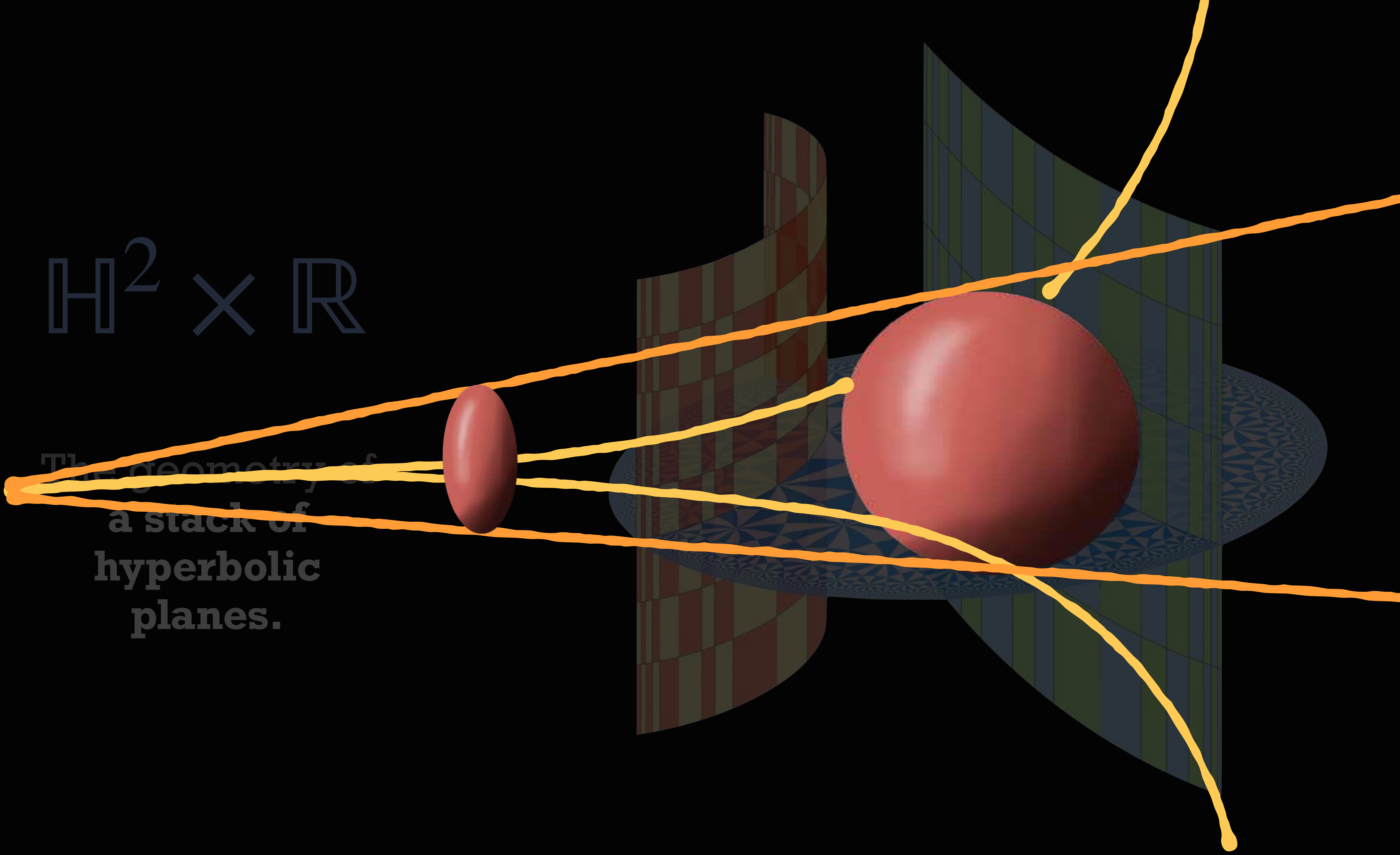
$\mathbb{H}^2 \times \mathbb{R}$

The geometry of
a stack of hyperbolic planes.

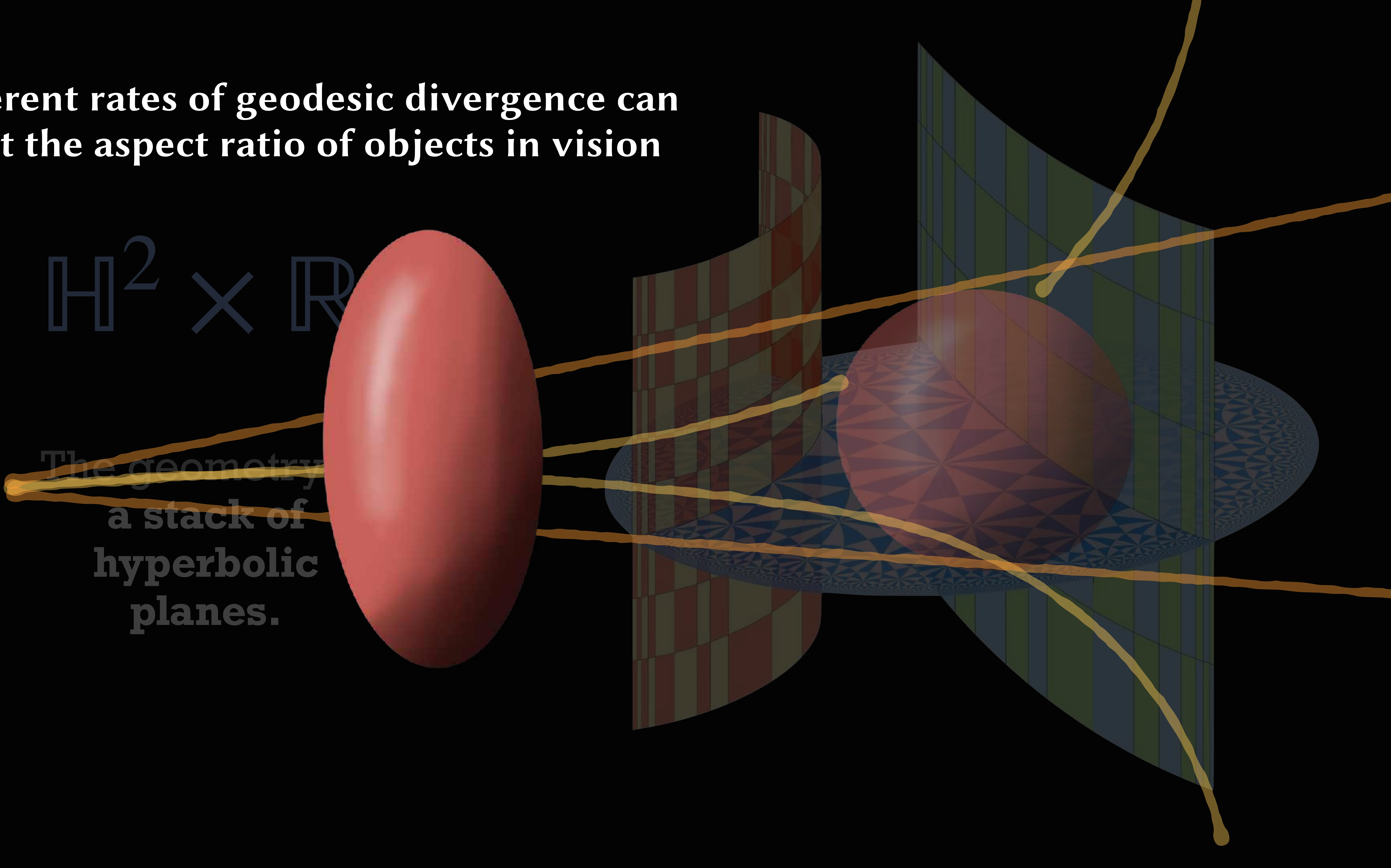


$\mathbb{H}^2 \times \mathbb{R}$

The geometry of a stack of hyperbolic planes.



Different rates of geodesic divergence can affect the aspect ratio of objects in vision

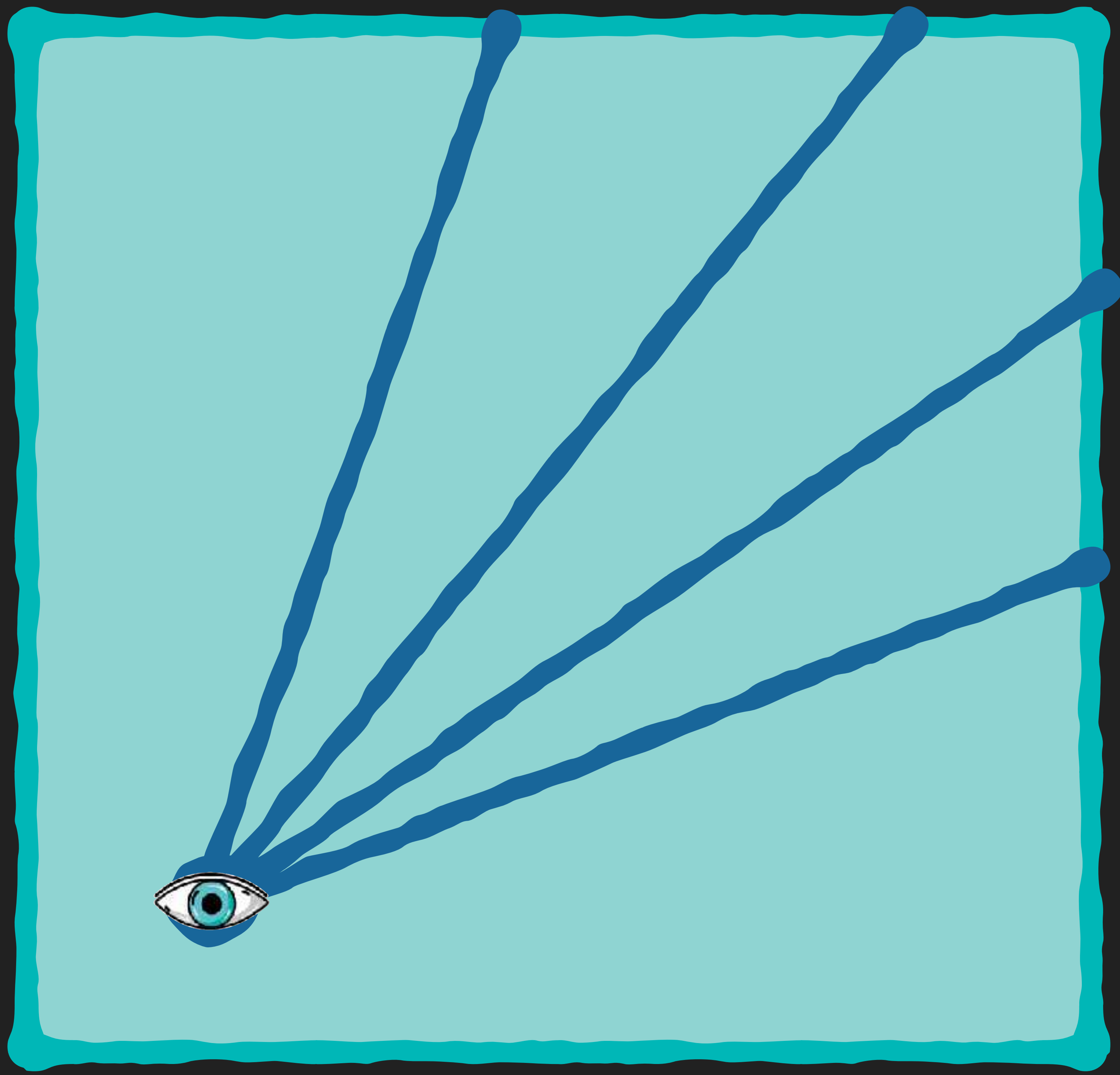


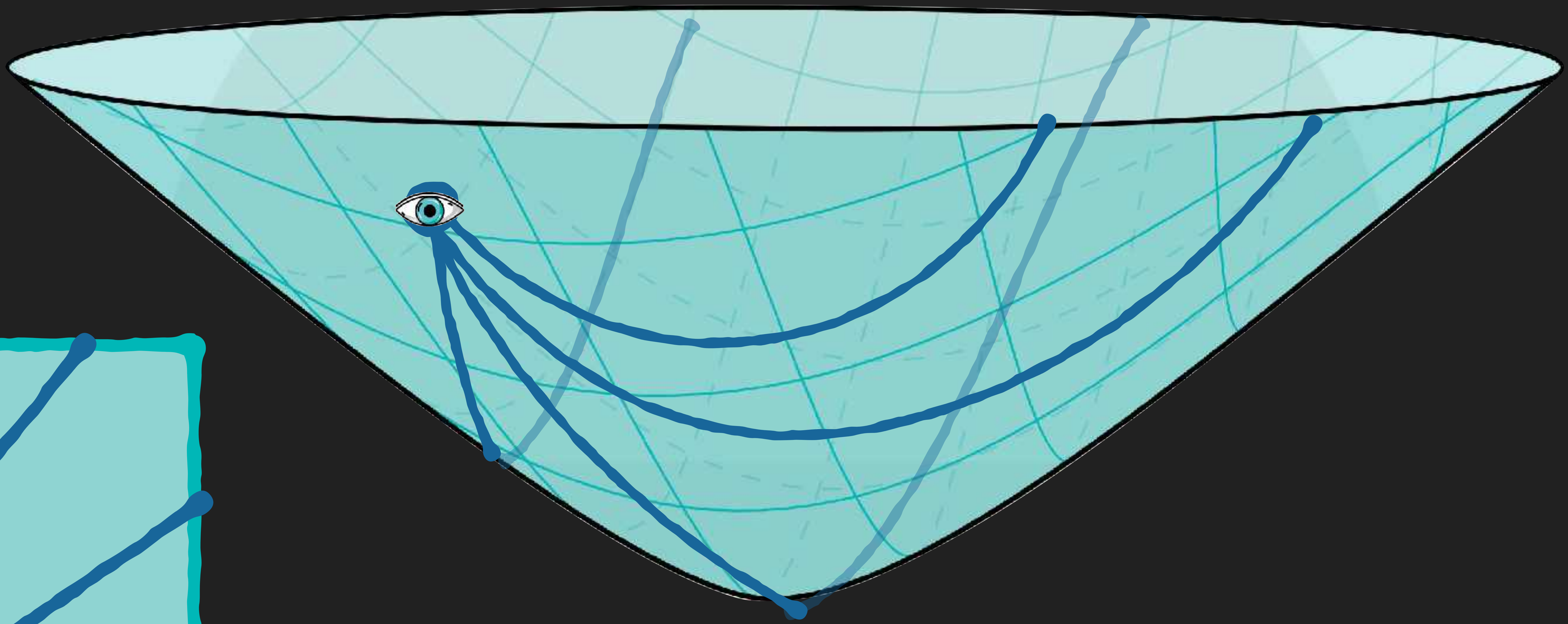
Vision in $\mathbb{H}^2 \times \mathbb{R}$

Geometric Lensing

So we may not be able to trust the size or aspect ratio of things in our vision, but how does the general 'topology' of what we see relate to reality?

exp



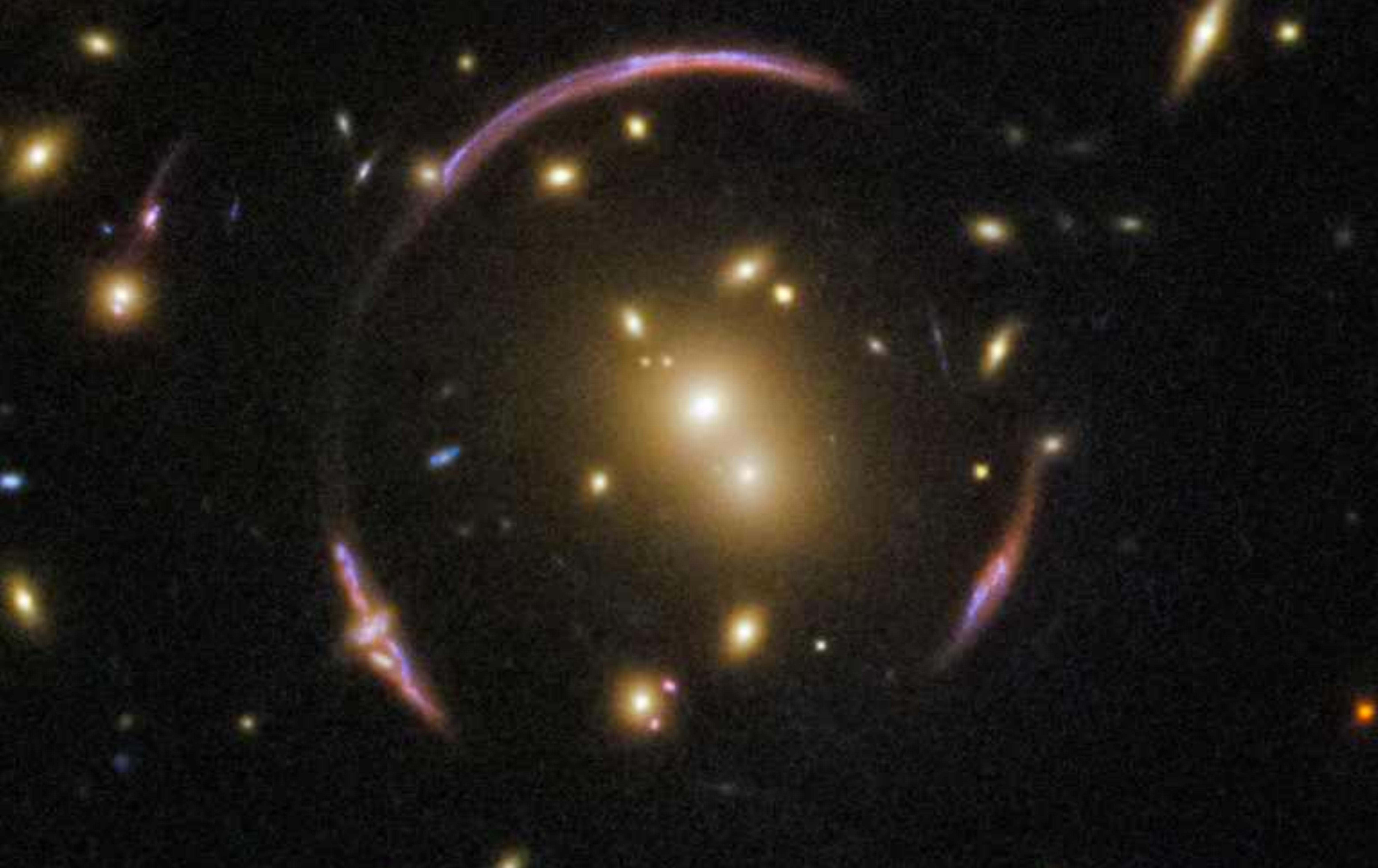


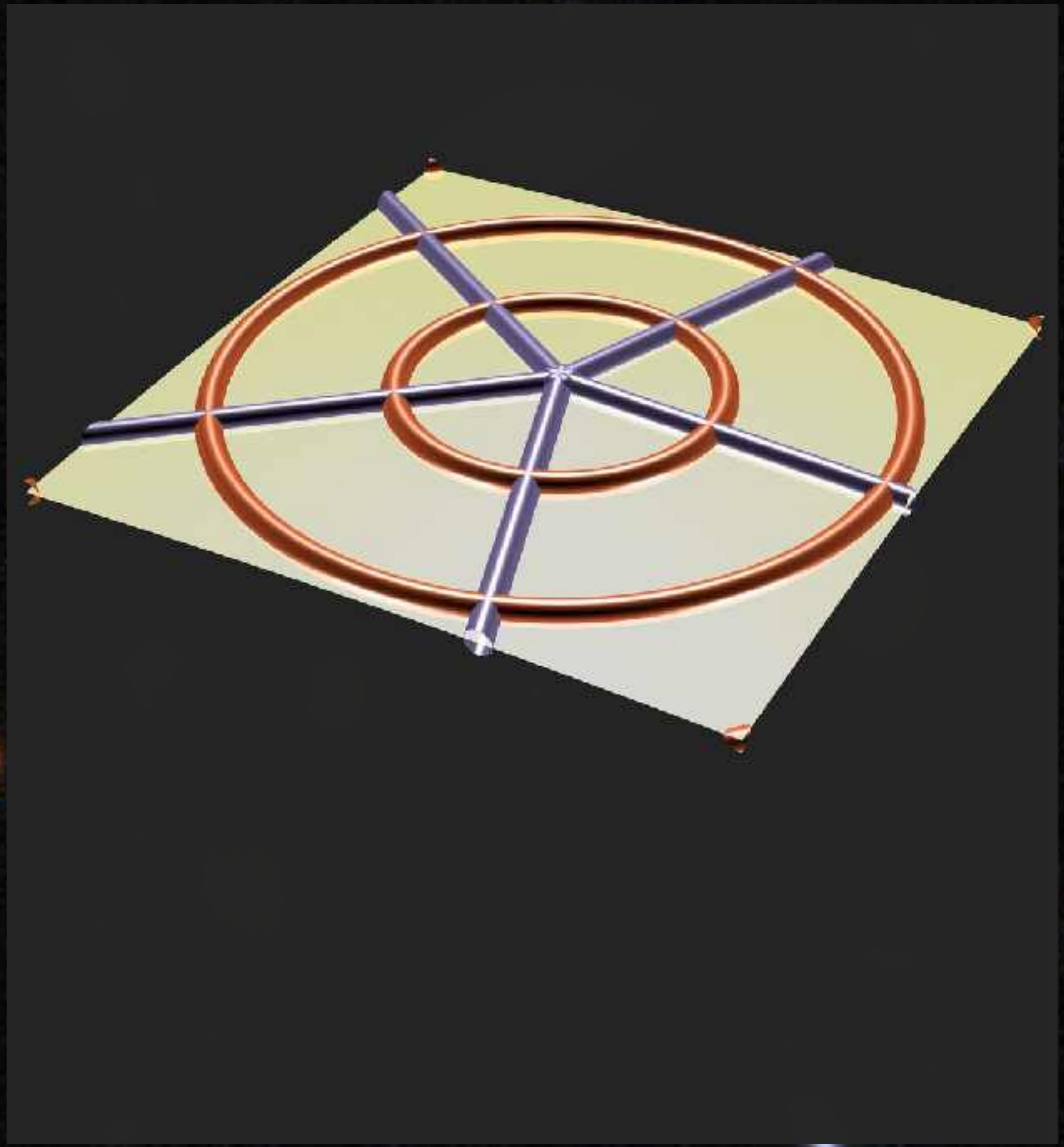
Cartan~Hadamard

In nonpositive curvature, our vision is diffeomorphic to reality

Exponential map in positive curvature need not be injective

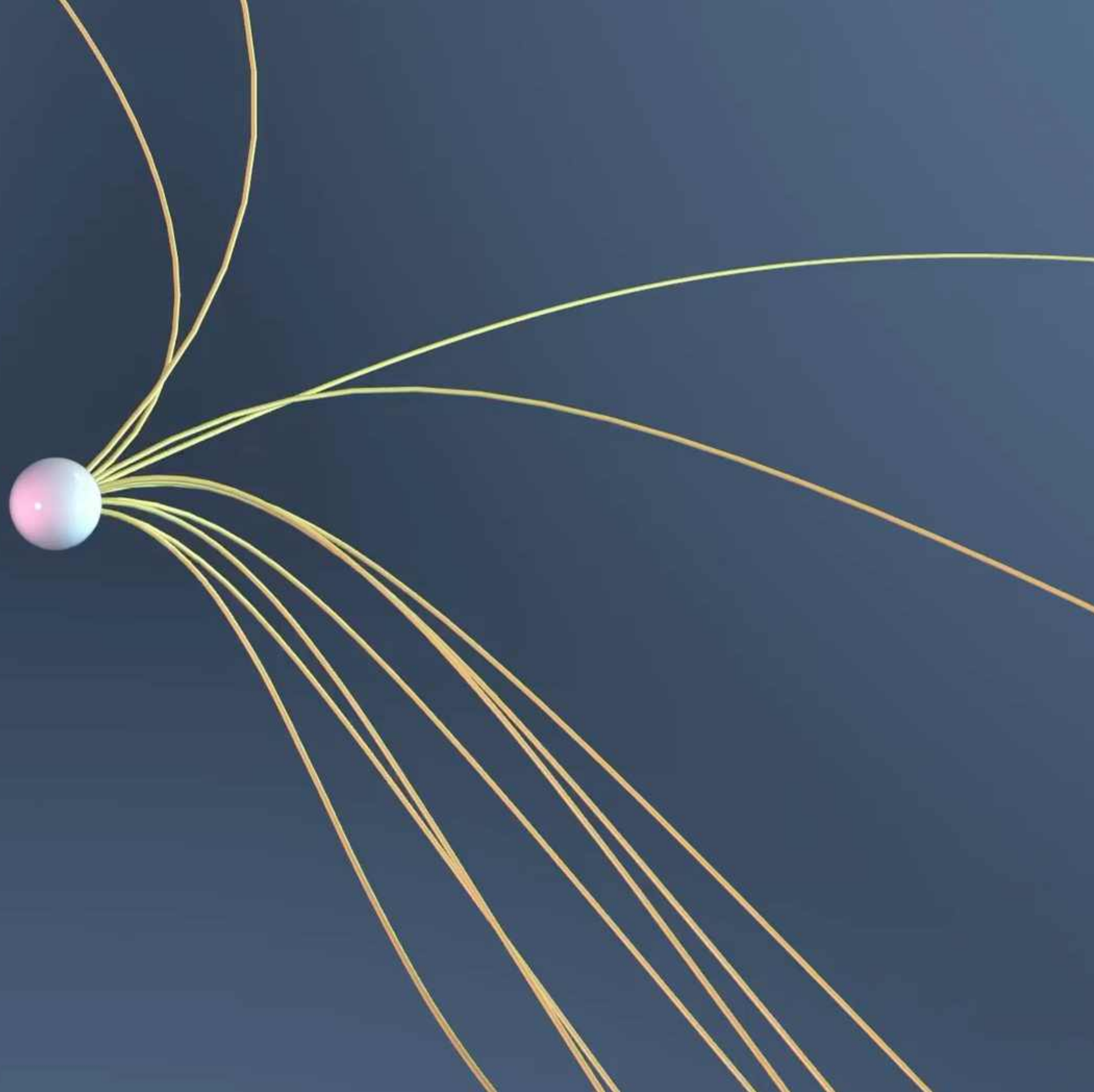
$$\exp : \mathbb{R}^2 \to \mathbb{S}^2$$





This can cause 'Lensing' Mirages

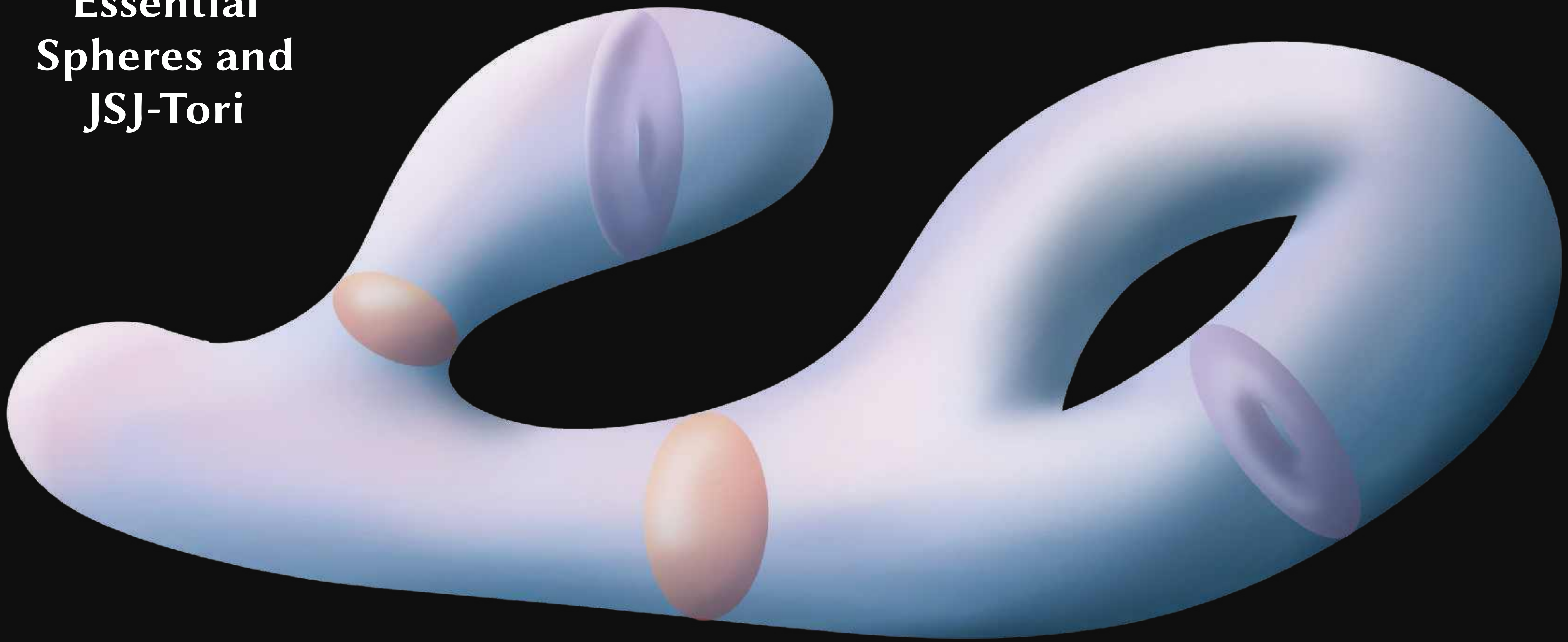
Nil Geometry



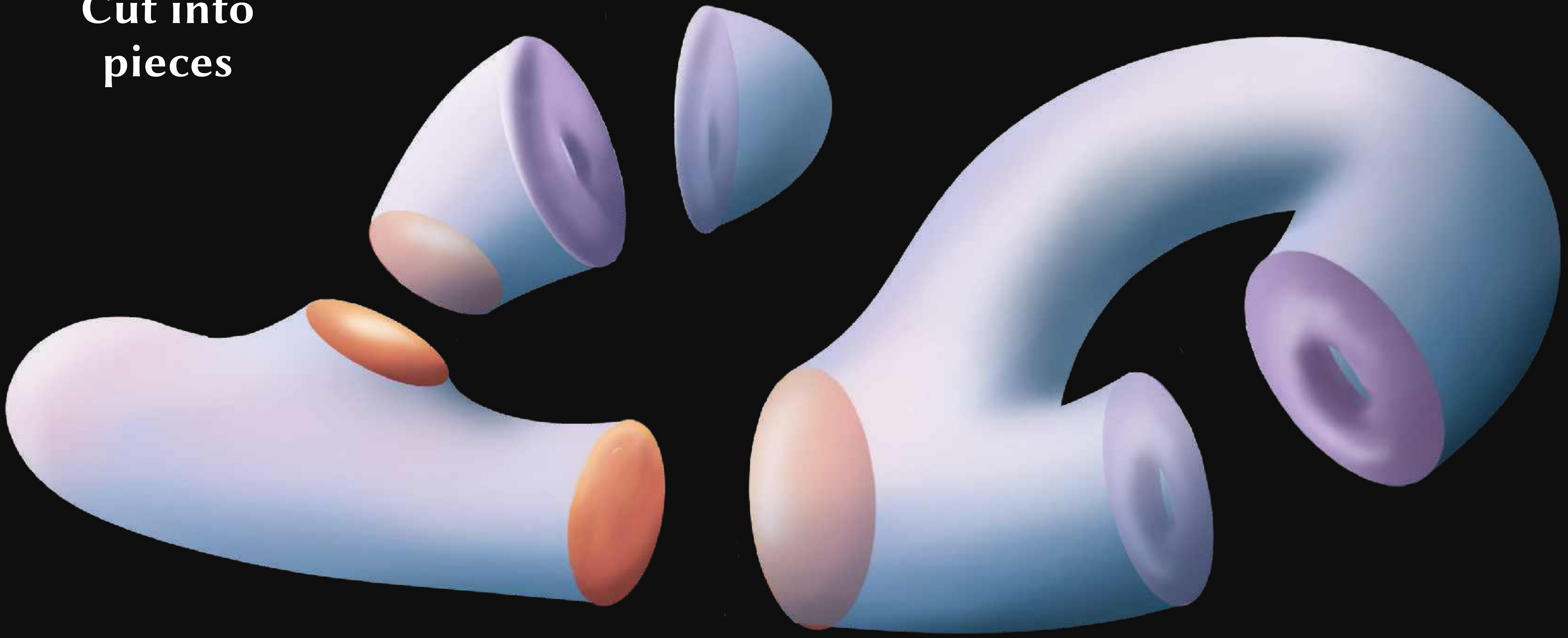
Seeing General 3 Manifolds

Using geometrization to build up a generic picture

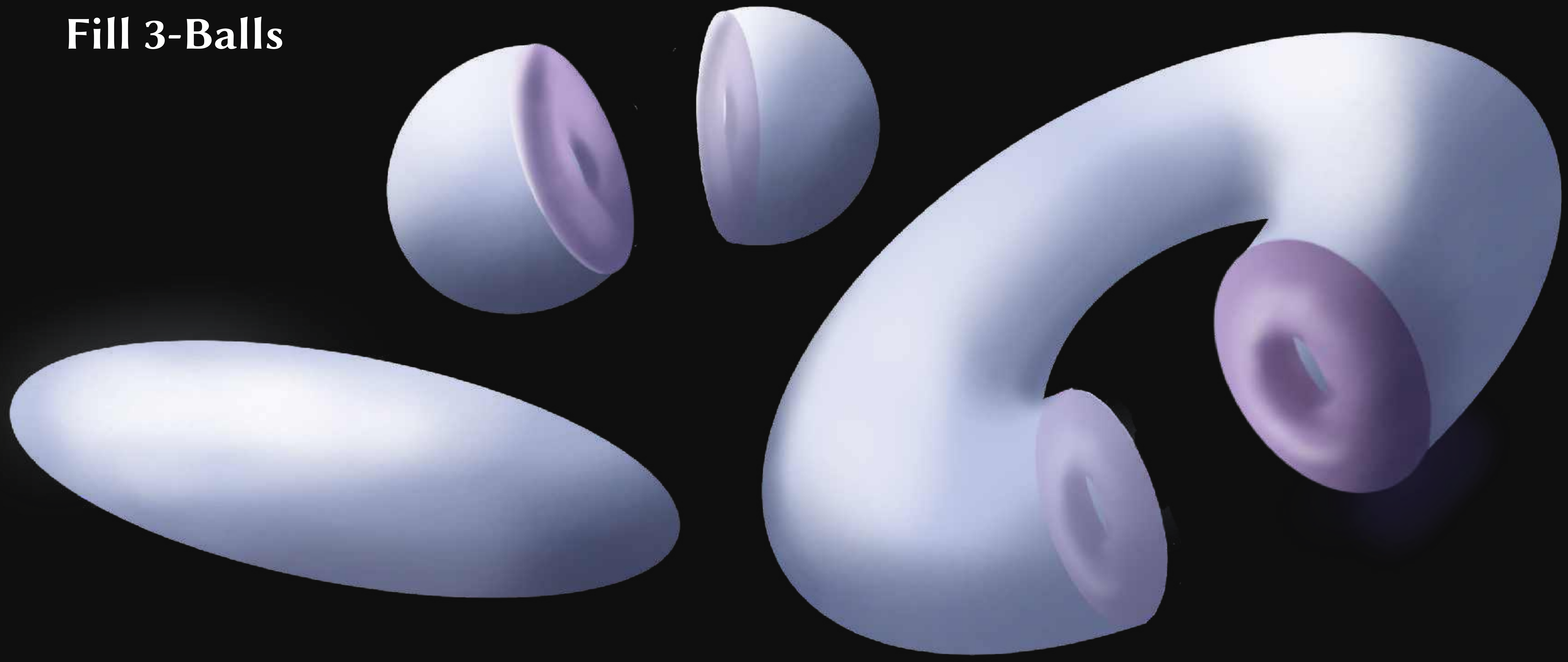
Essential Spheres and JSJ-Tori



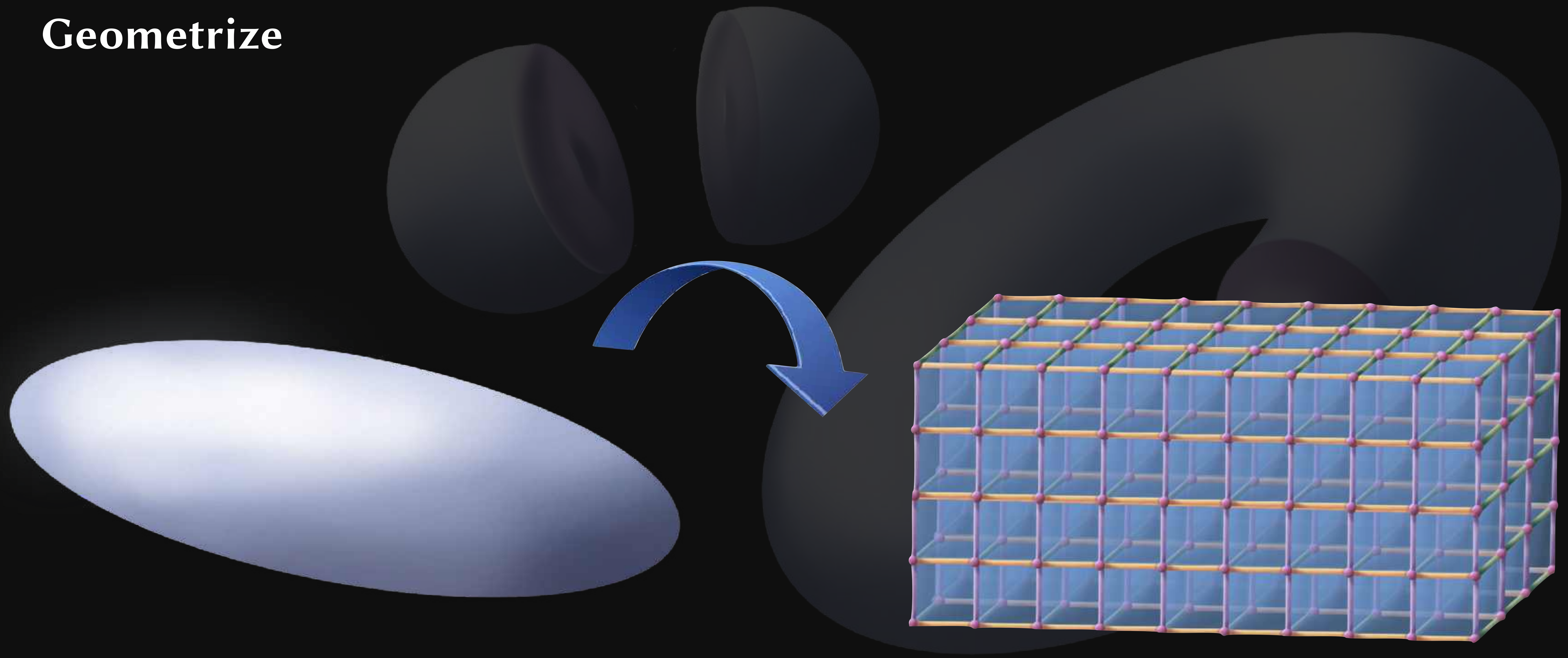
Cut into pieces



Fill 3-Balls

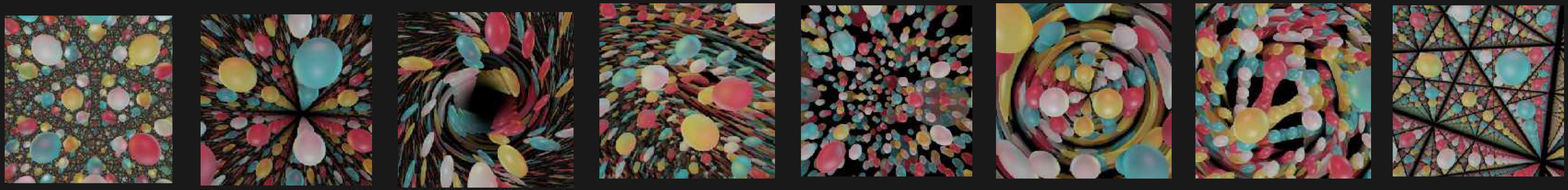


Geometrize

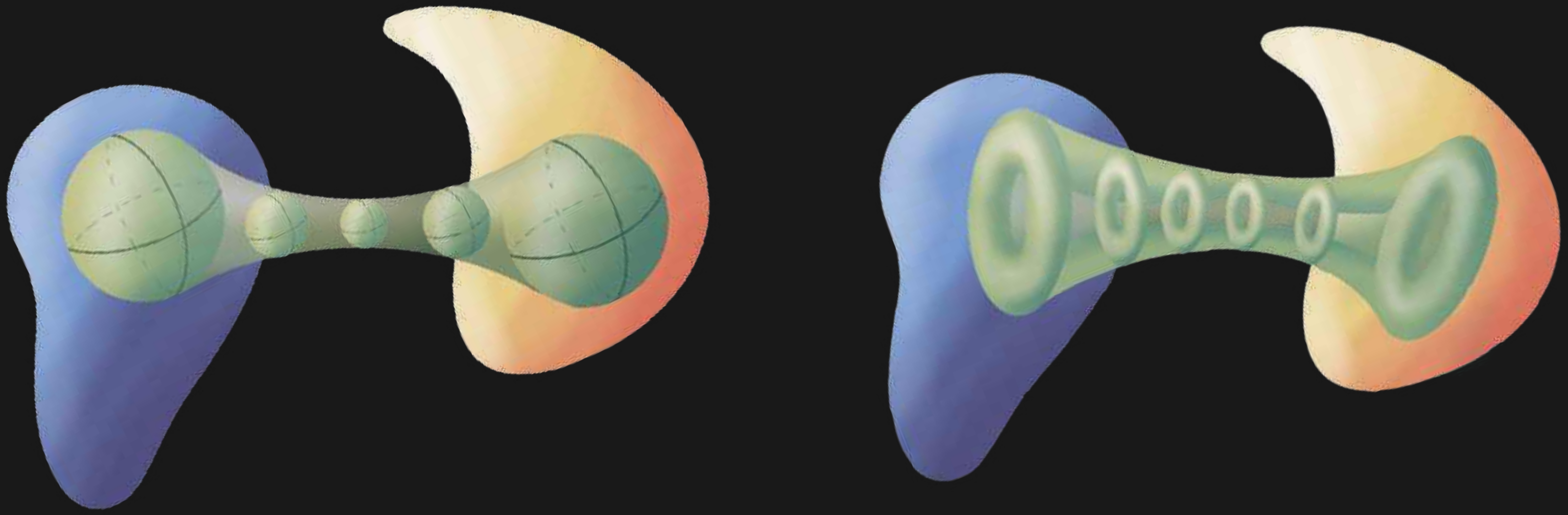


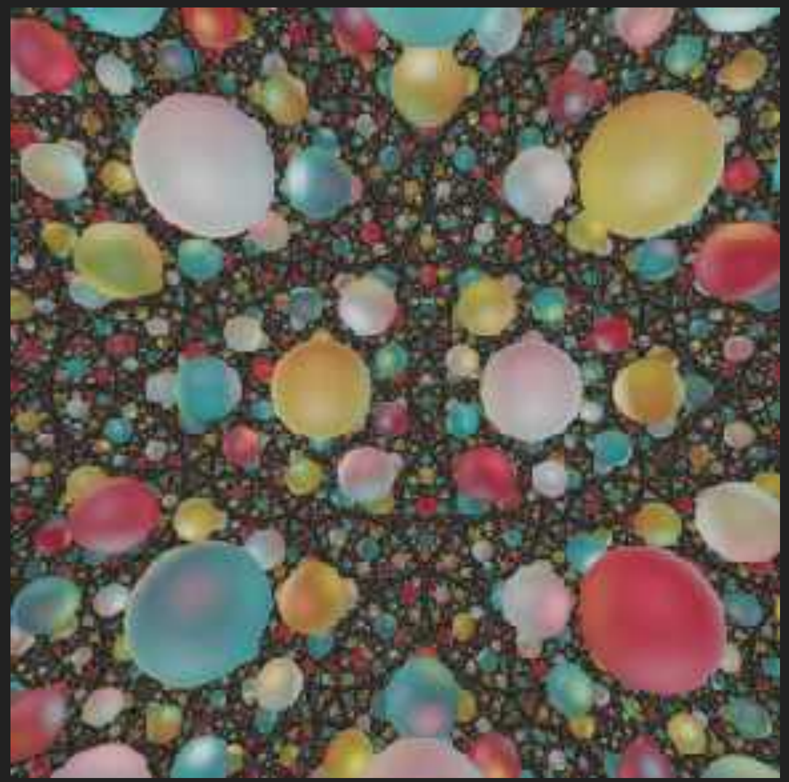
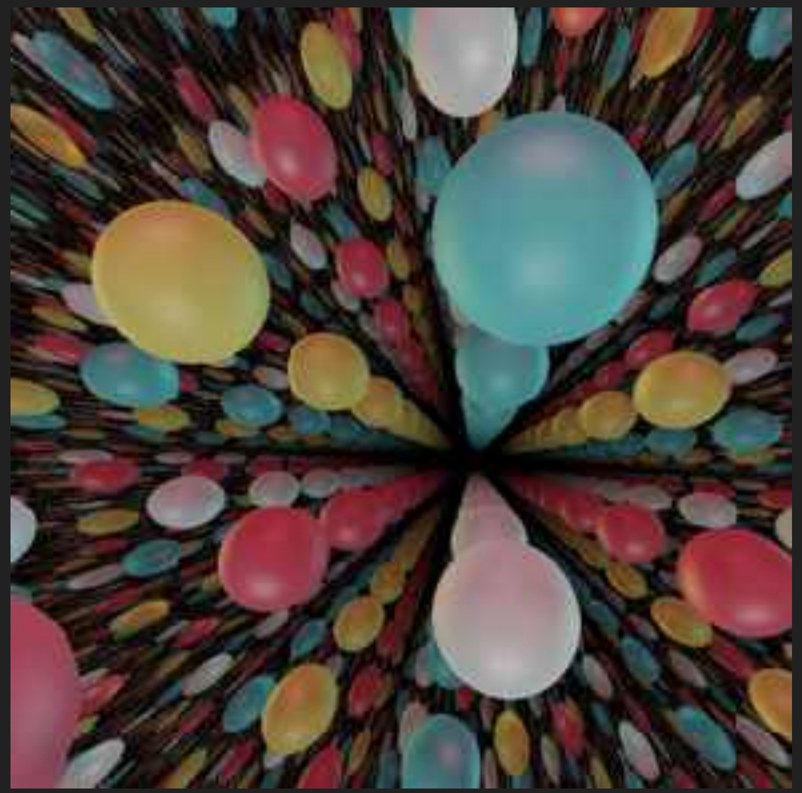
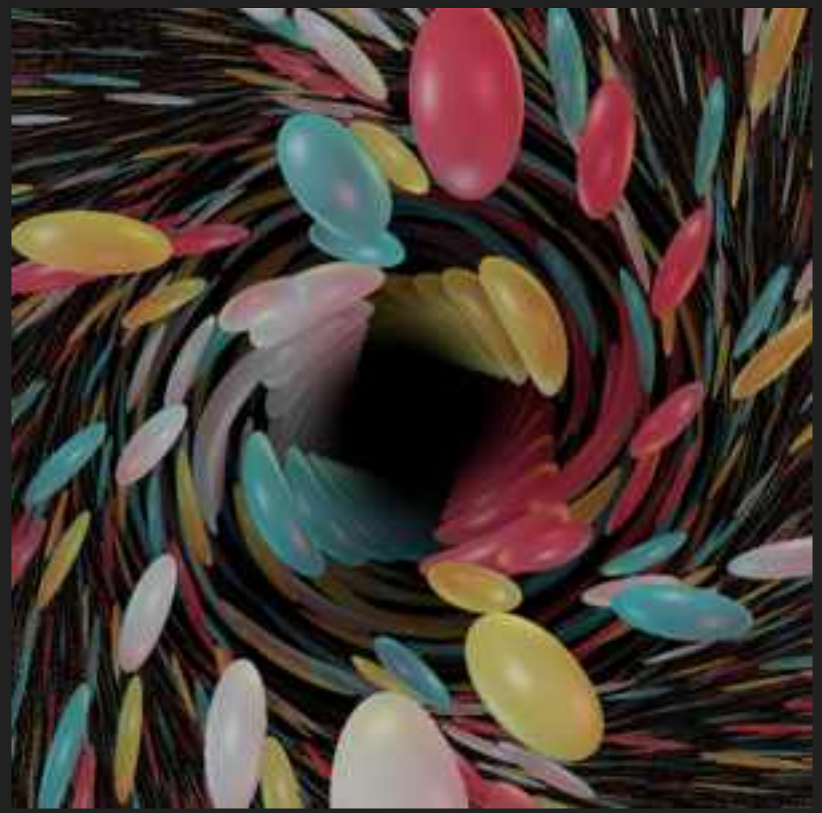
3 Manifolds are like Legos:

Eight general types of bricks

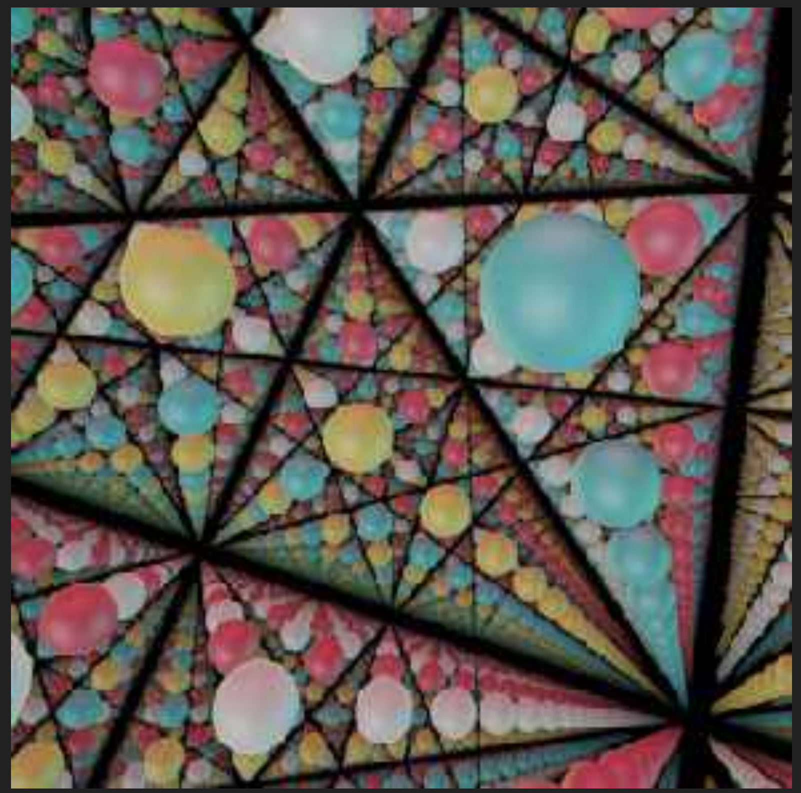


Two types of connectors



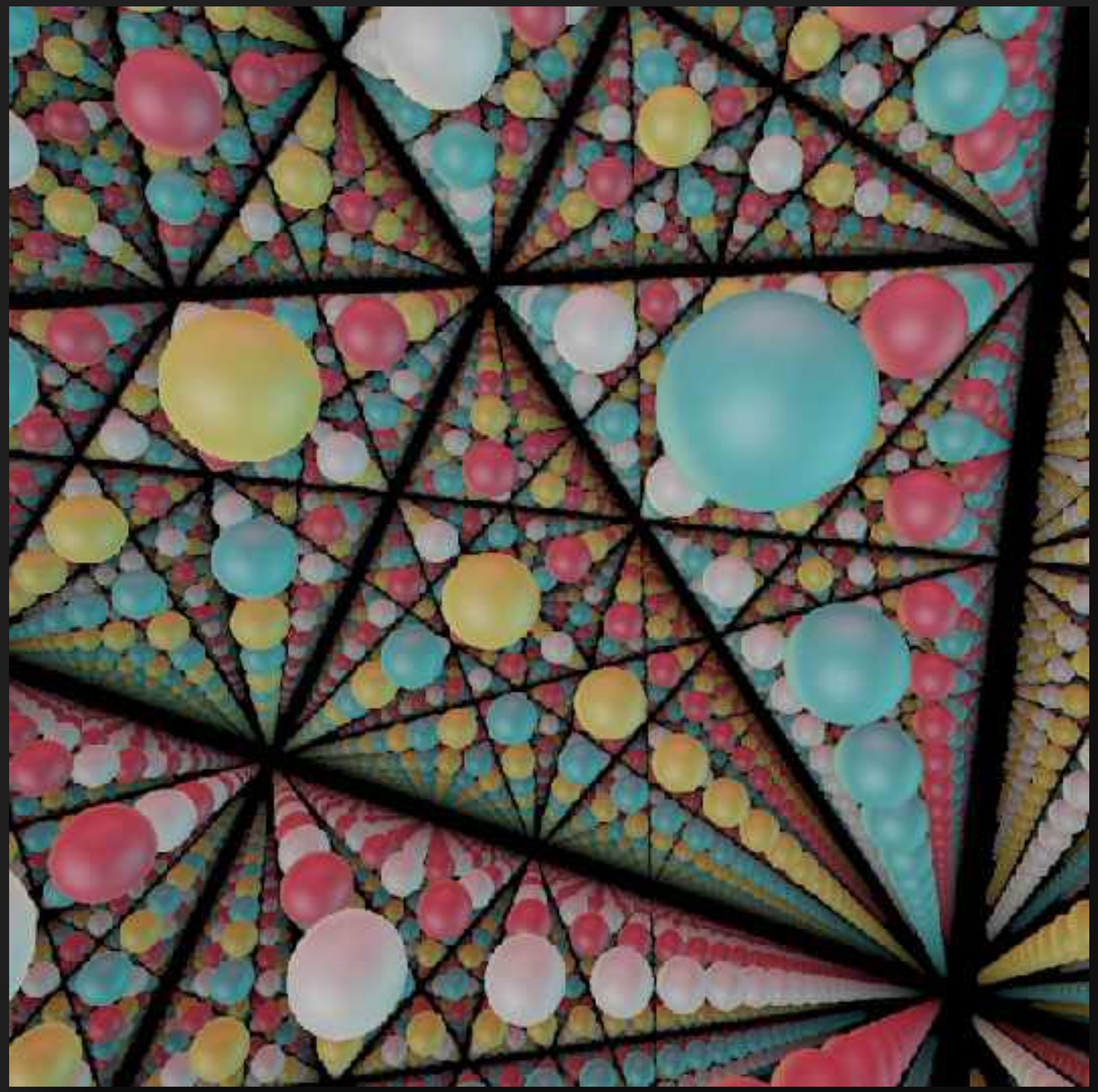


The Thurston Geometries

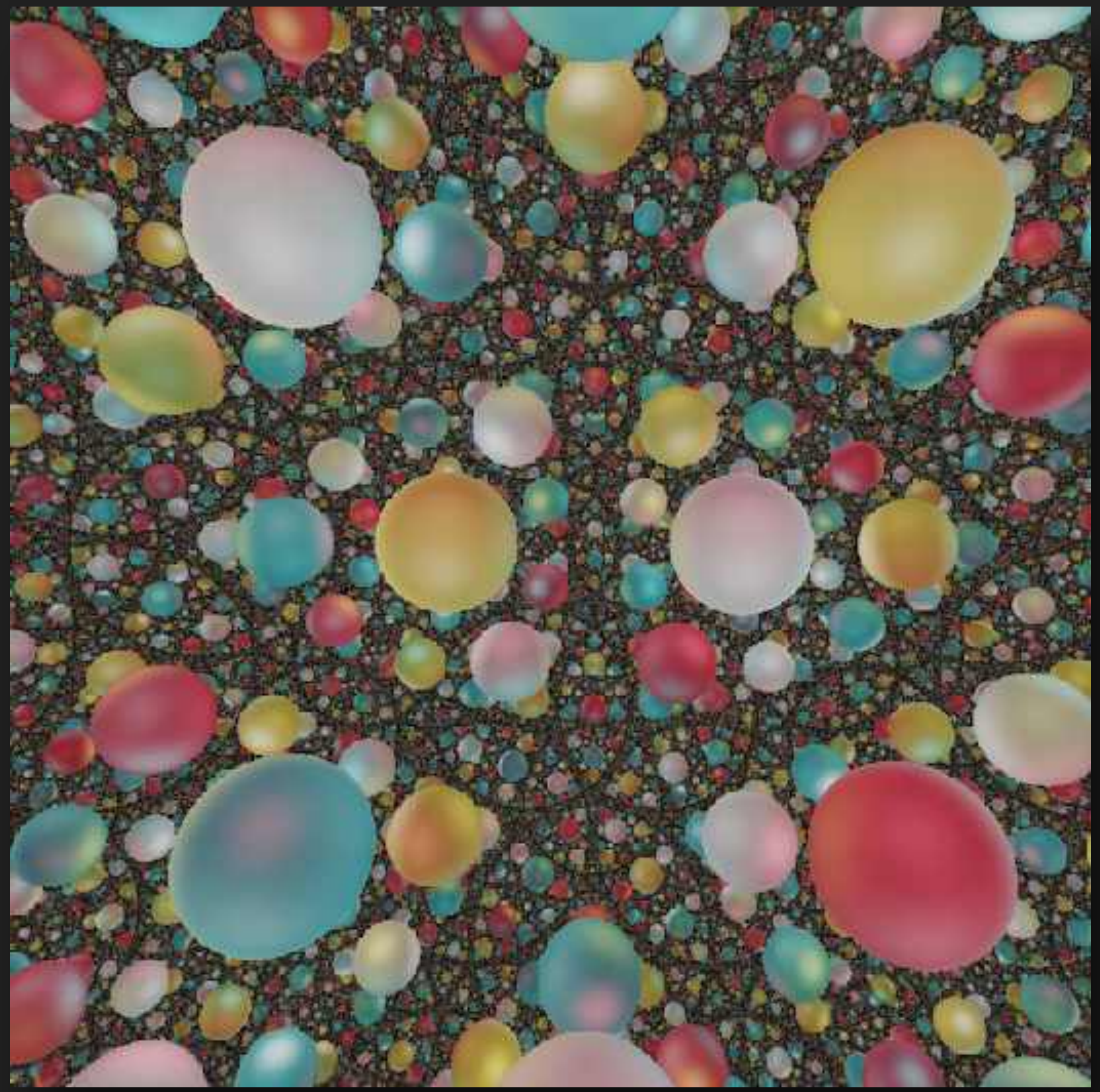
Lego Bricks of Geometrization

Constant Curvature



Euclidean

Ten closed manifolds, all finitely covered by 3-Torus.



Hyperbolic

Tons of closed manifolds!

Spherical

Manifolds determined by finite subgroups of SO(3)

Spherical

Poincare Homology Sphere

$\mathbb{S}^3 \cong \mathrm{SU}(2)$ **and** $\mathrm{SU}(2)$ **double covers** $\mathrm{SO}(3)$**.**

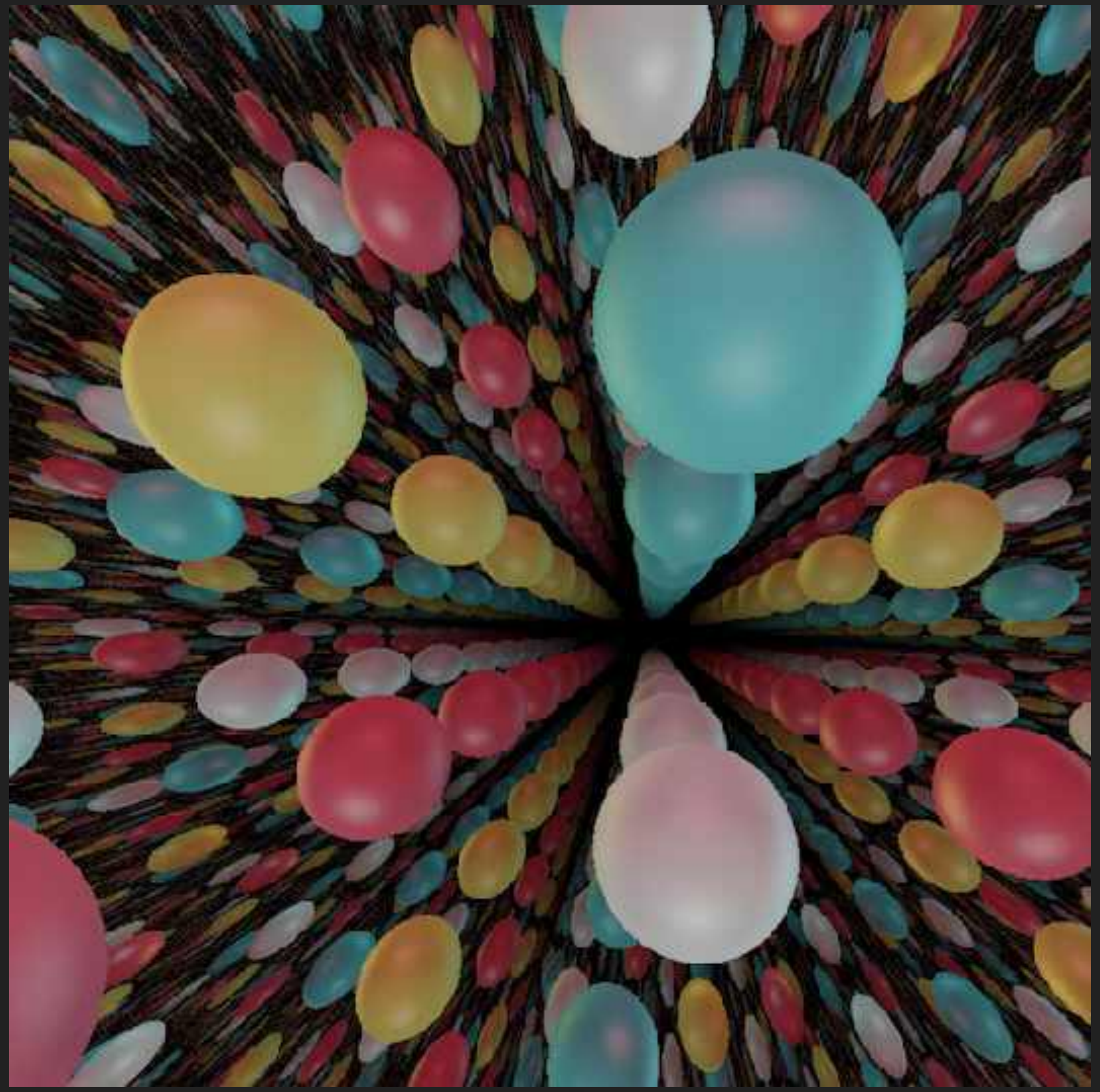
The finite subgroups of $\mathrm{SO}(3)$ **are:**

- **Cyclic**
- **Dice dihedral**
- **The Symmetries of a Platonic solid**

$$\frac{\mathrm{SO}(3)}{\mathrm{Sym}}$$


Poincare
Homology
Sphere

Product Geometries

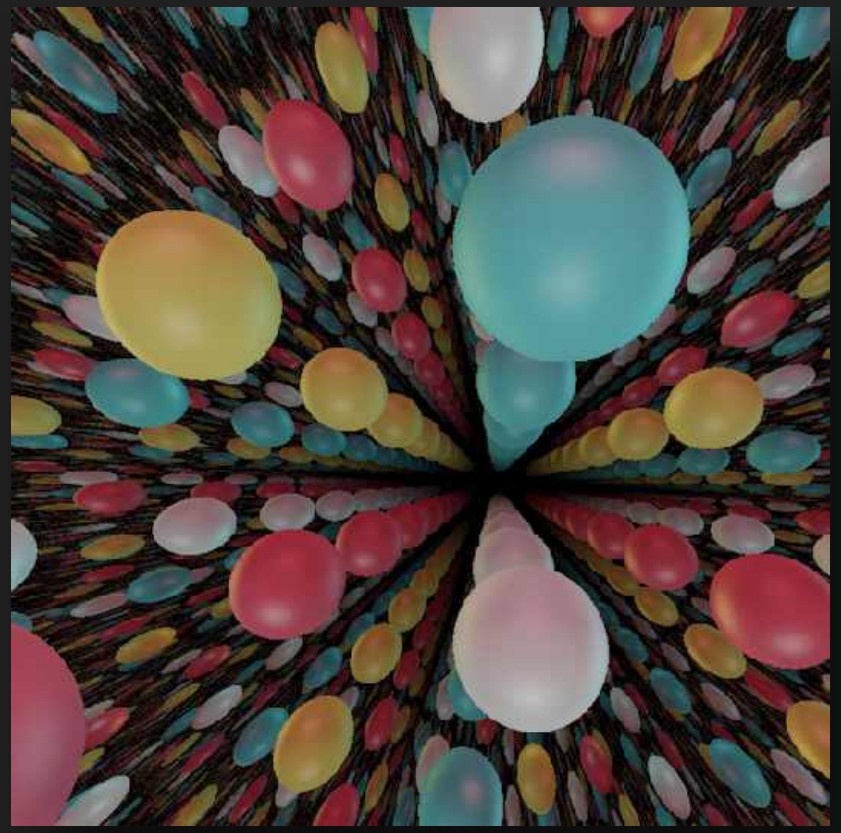


$\mathbb{H}^2 \times \mathbb{E}$

Finite quotients of
Hyperbolic Surface x $\mathbb{S}^1$

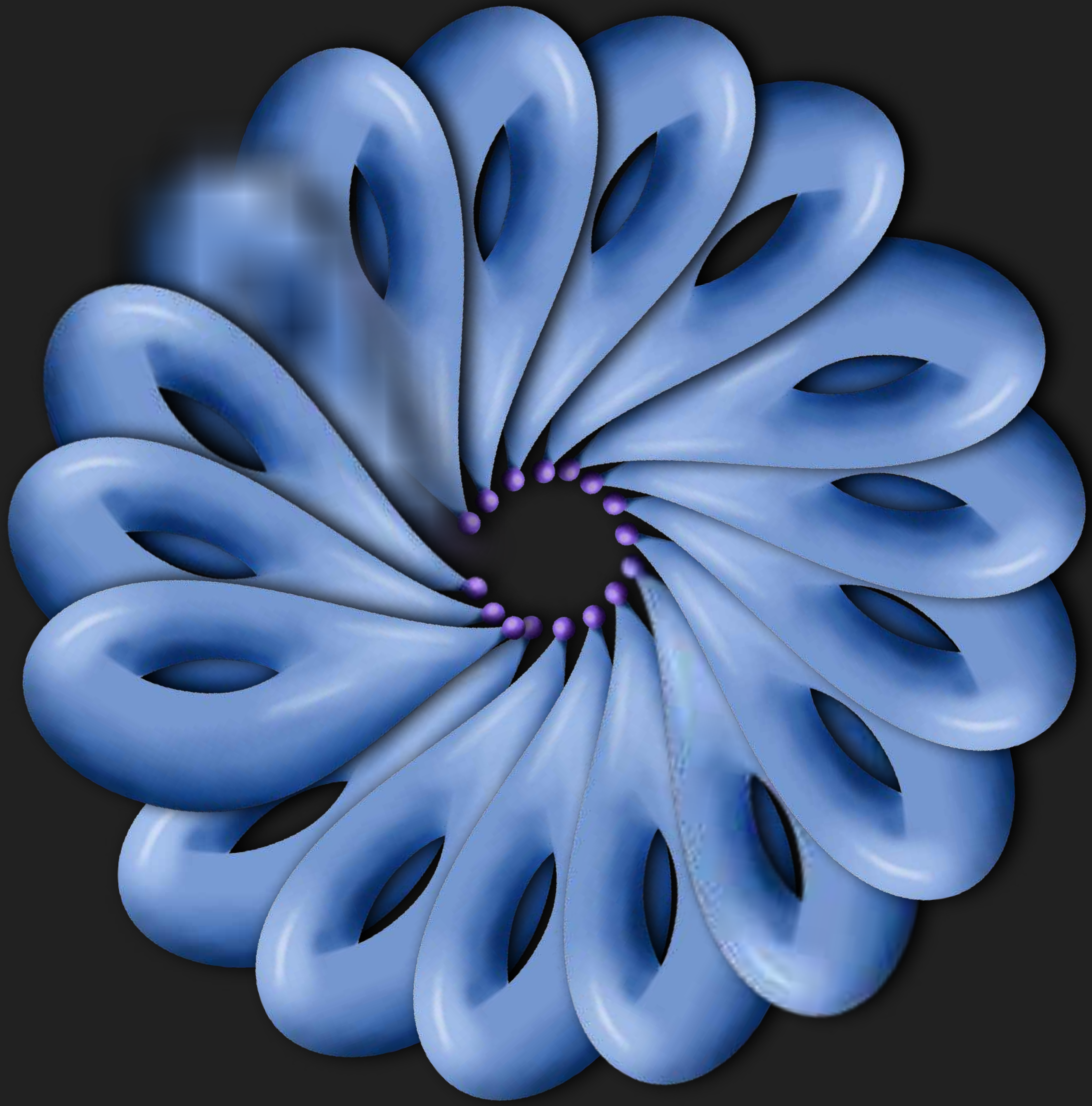
$\mathbb{S}^2 \times \mathbb{E}$

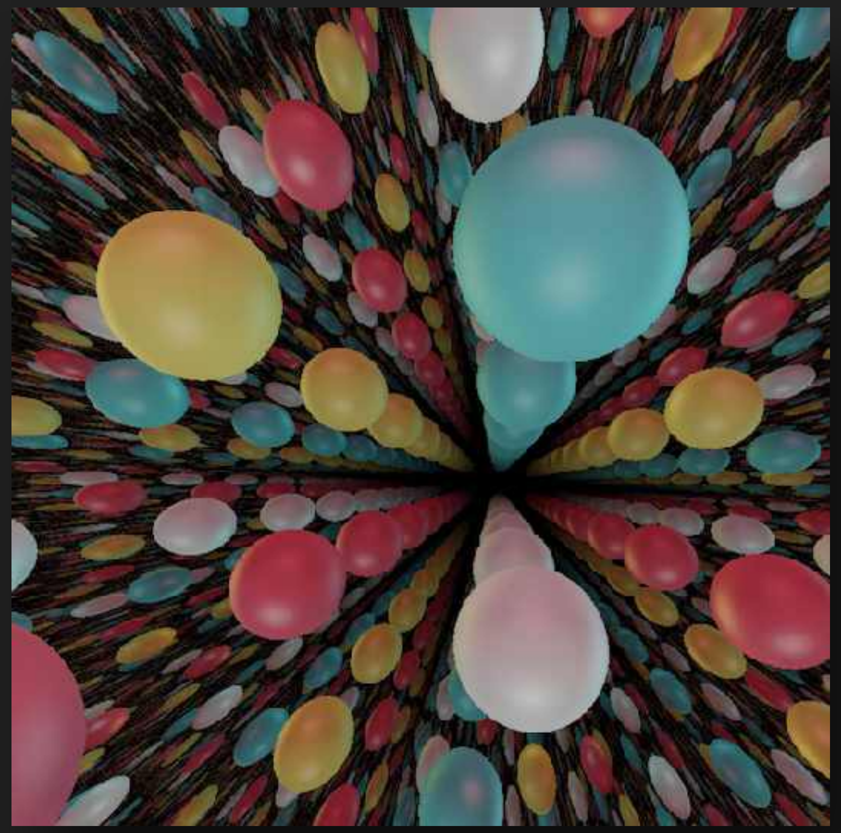
Only **FOUR** closed manifolds
with this geometry!



$$\mathbb{H}^2 \times \mathbb{E}$$

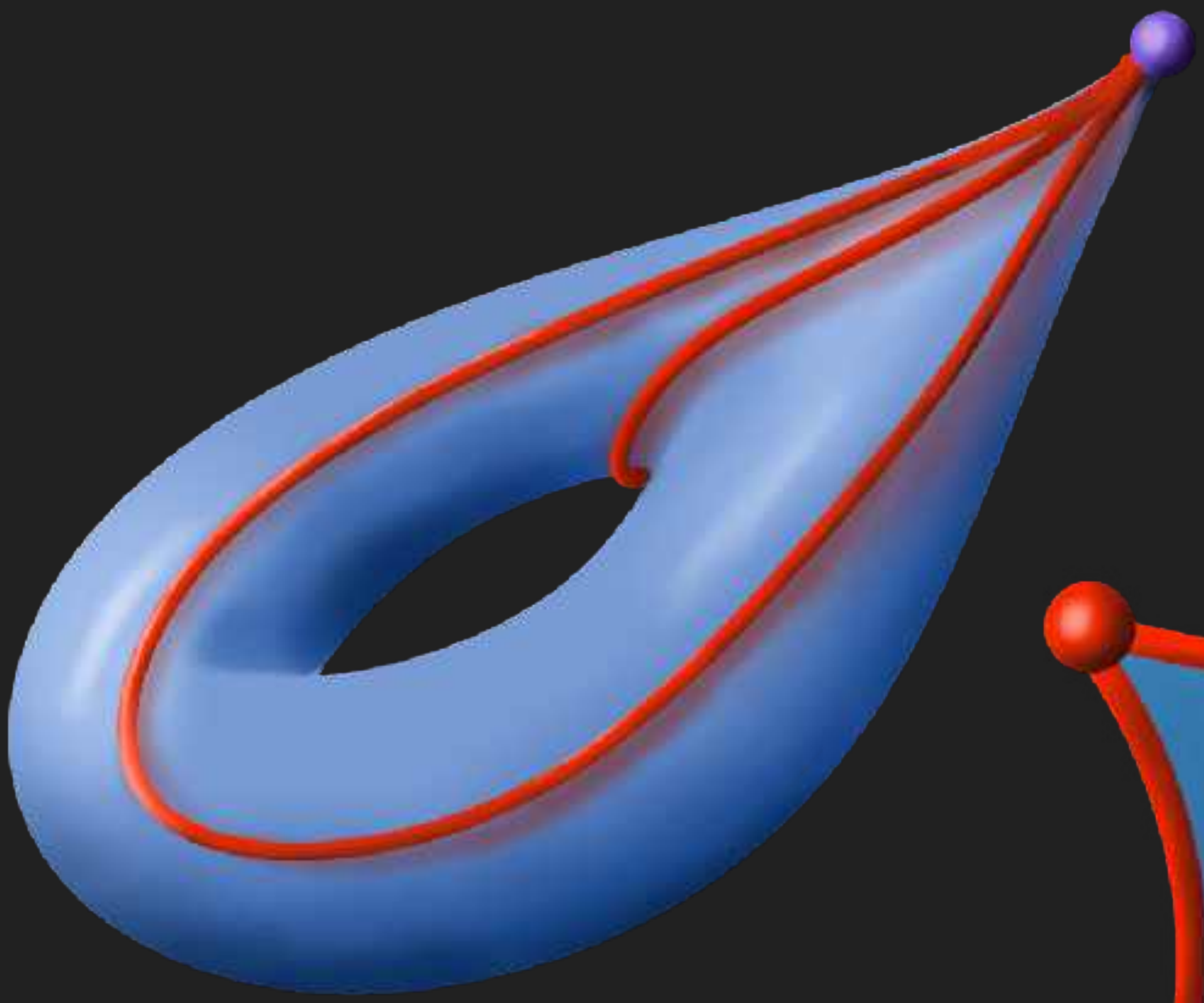
Surface $\times \ \mathbb{S}^1$

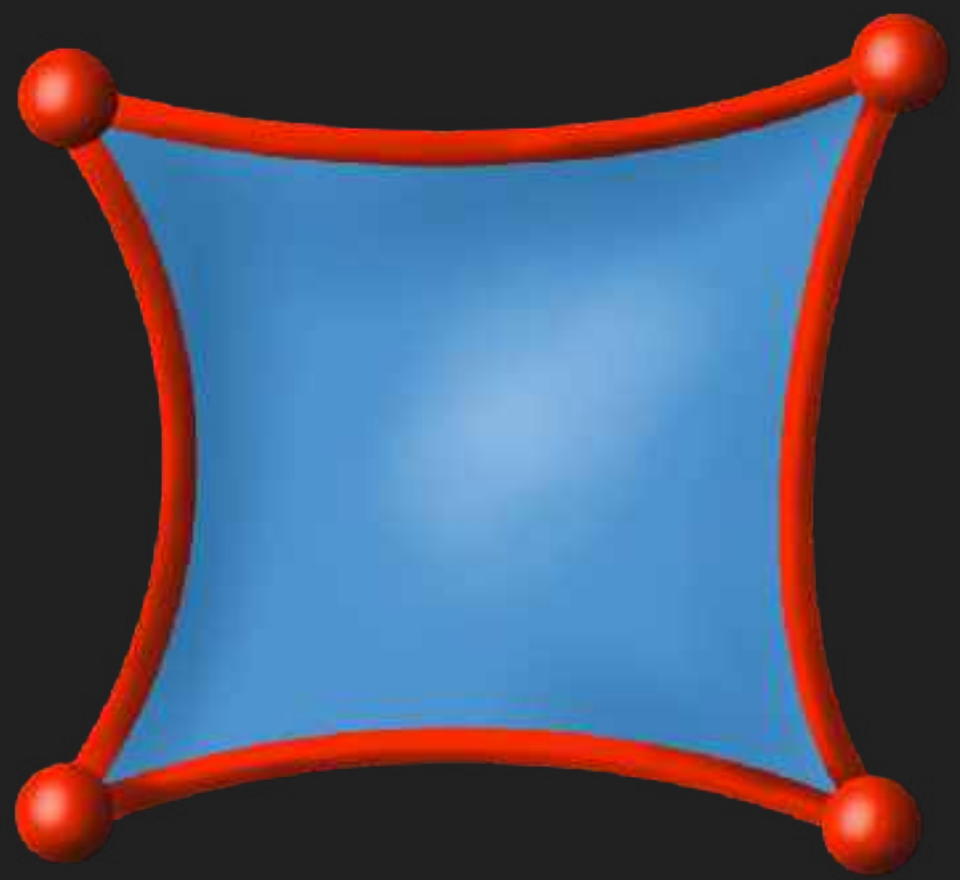


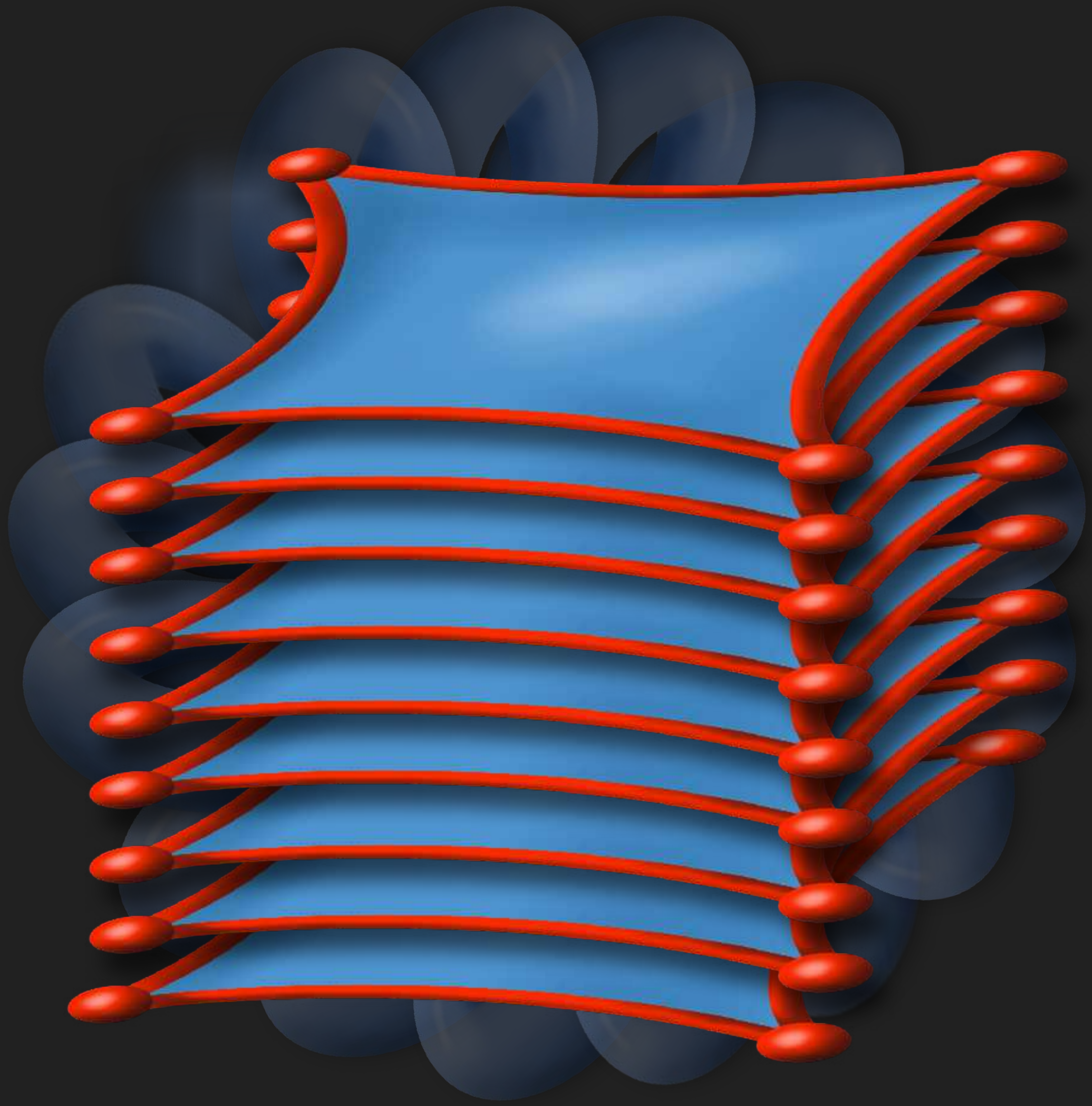


$\mathbb{H}^2 \times \mathbb{E}$

Surface $\times \mathbb{S}^1$



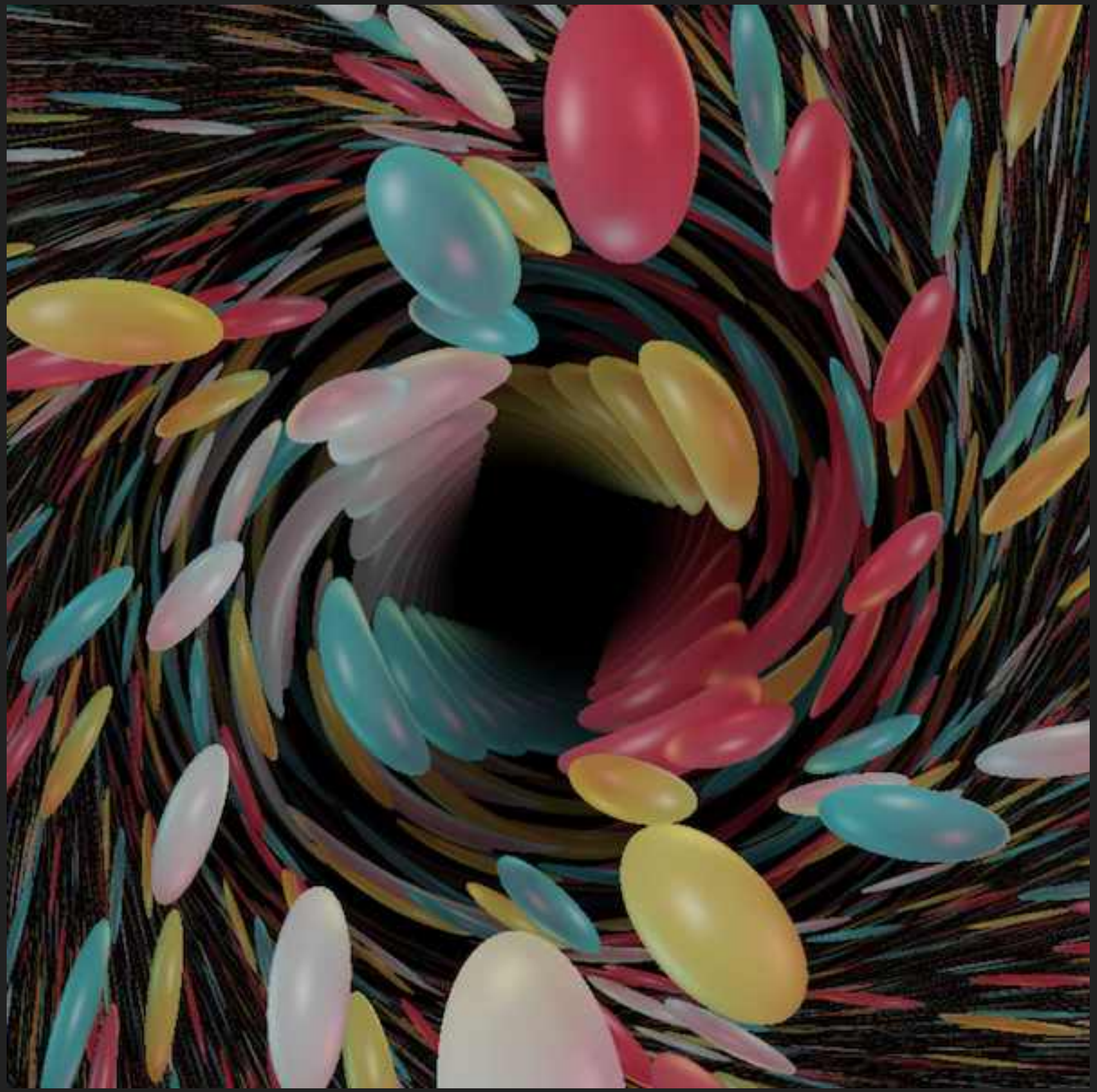





$\mathbb{H}^2$ **Orbifold**
$\times\ \mathbb{S}^1$

"Twisted" Products

Nil



SL2

Circle bundles over Euclidean/Hyperbolic orbifolds of nonzero Euler class

Nil

$$dx^2 + dy^2 + (dz - xdy)^2$$

Dehn Twist Mapping Torus

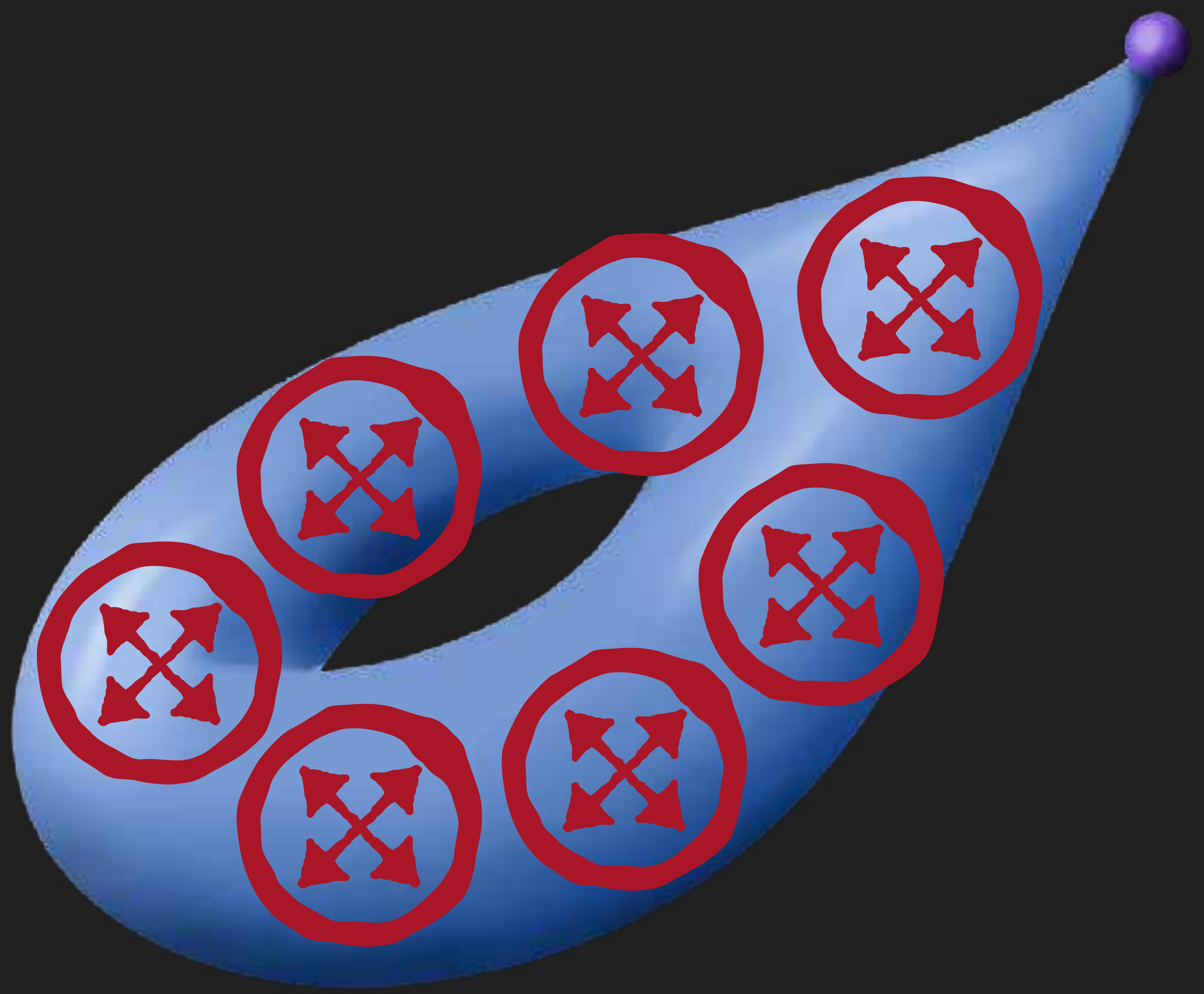
$$f\colon \pi_1 T^2 \to \pi_1 T^2 \quad (x, y) \mapsto (x + y, y)$$

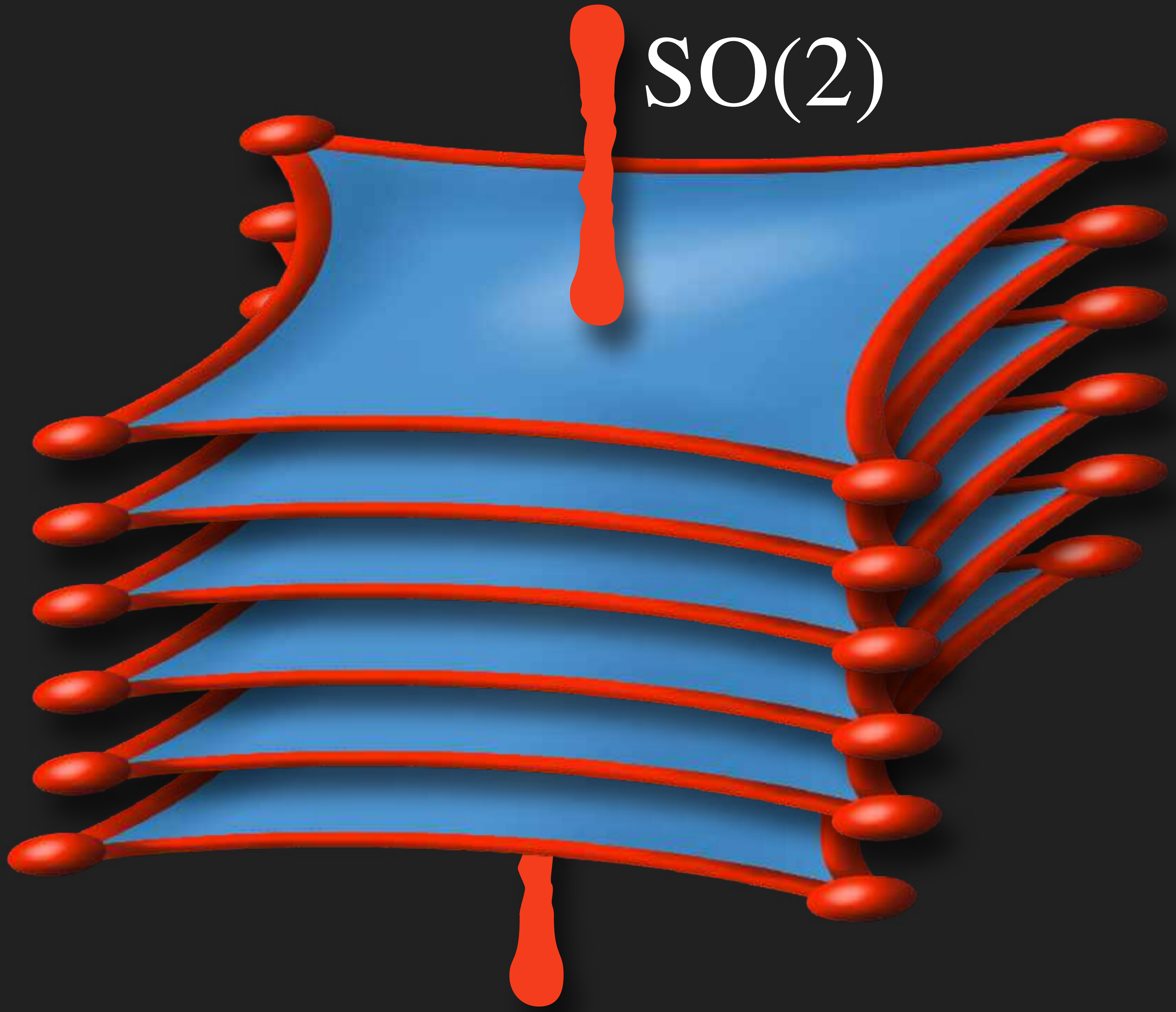

Dehn Twist
Torus Bundle


Dehn Twist
Torus Bundle

$SL_2\mathbb{R}$

Hyperbolic Unit Tangent Bundles

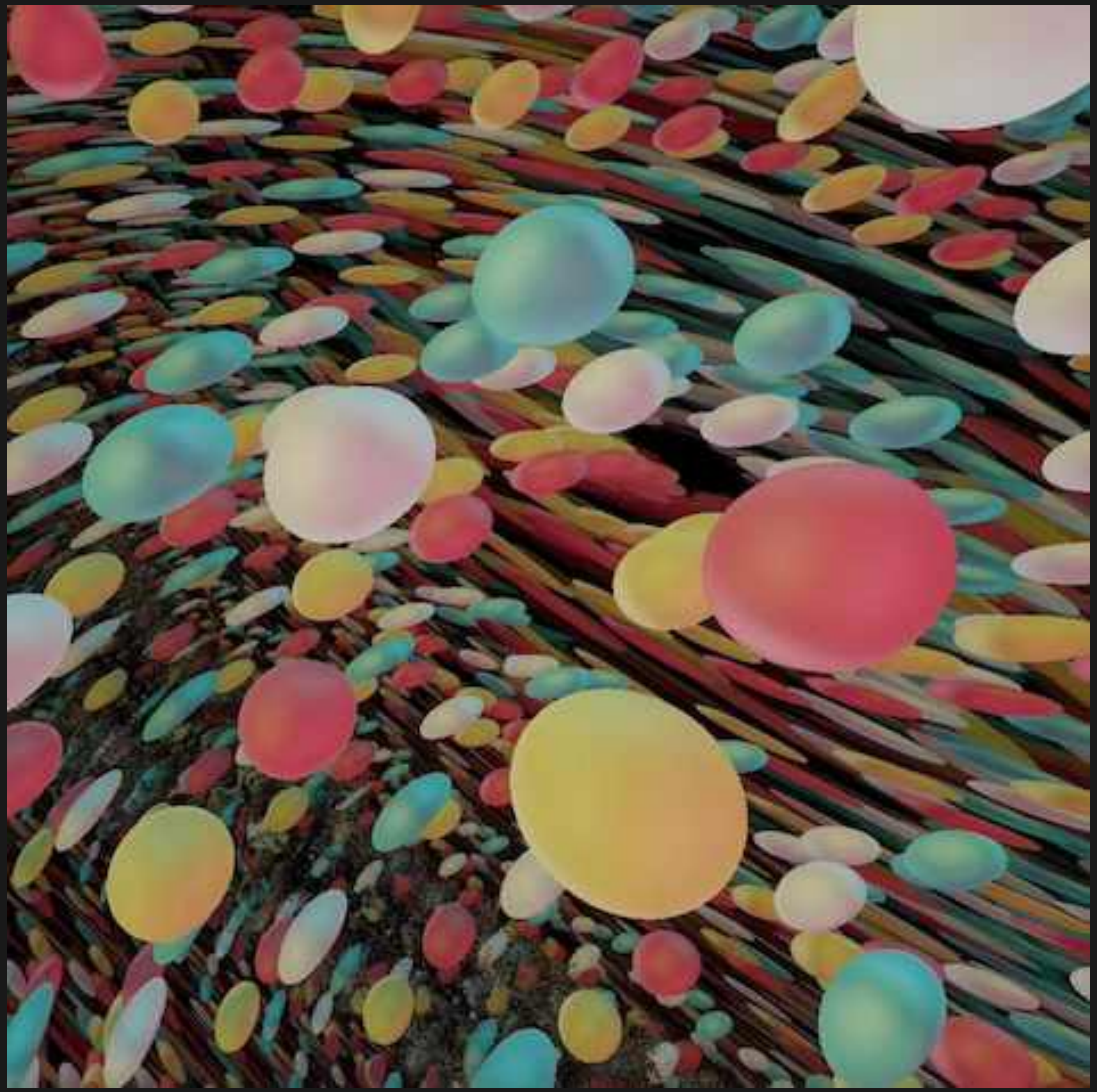




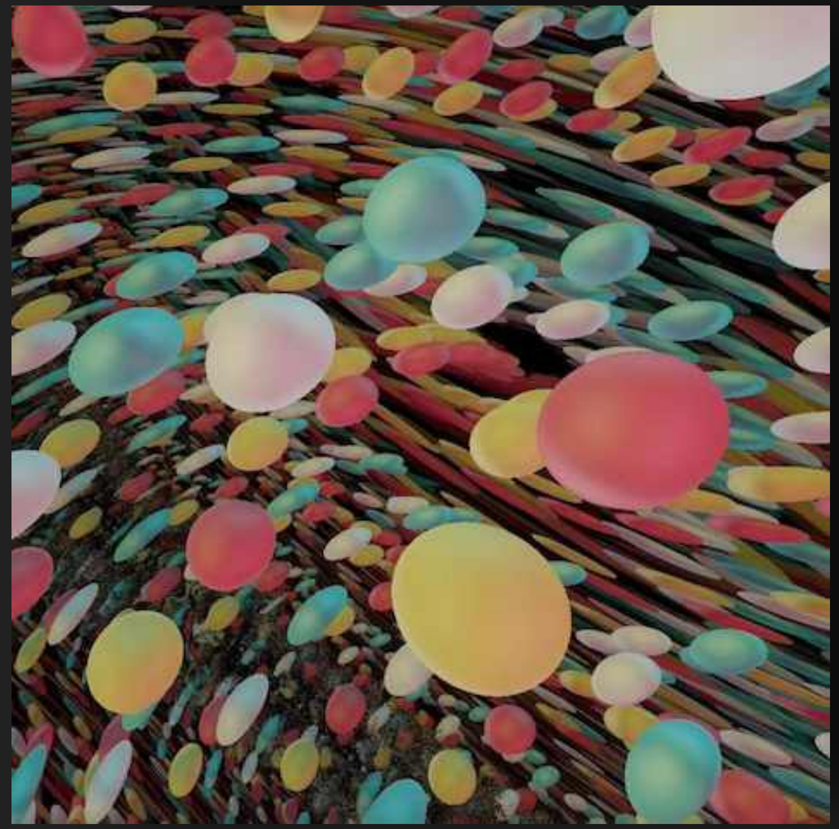

Unit Tangent
Bundle


Unit Tangent
Bundle

Sol Geometry



Finite quotients of Anosov mapping tori



Sol

$$e^{2z}dx^2 + e^{-2z}dy^2 + dz^2$$

Anosov Mapping Torus

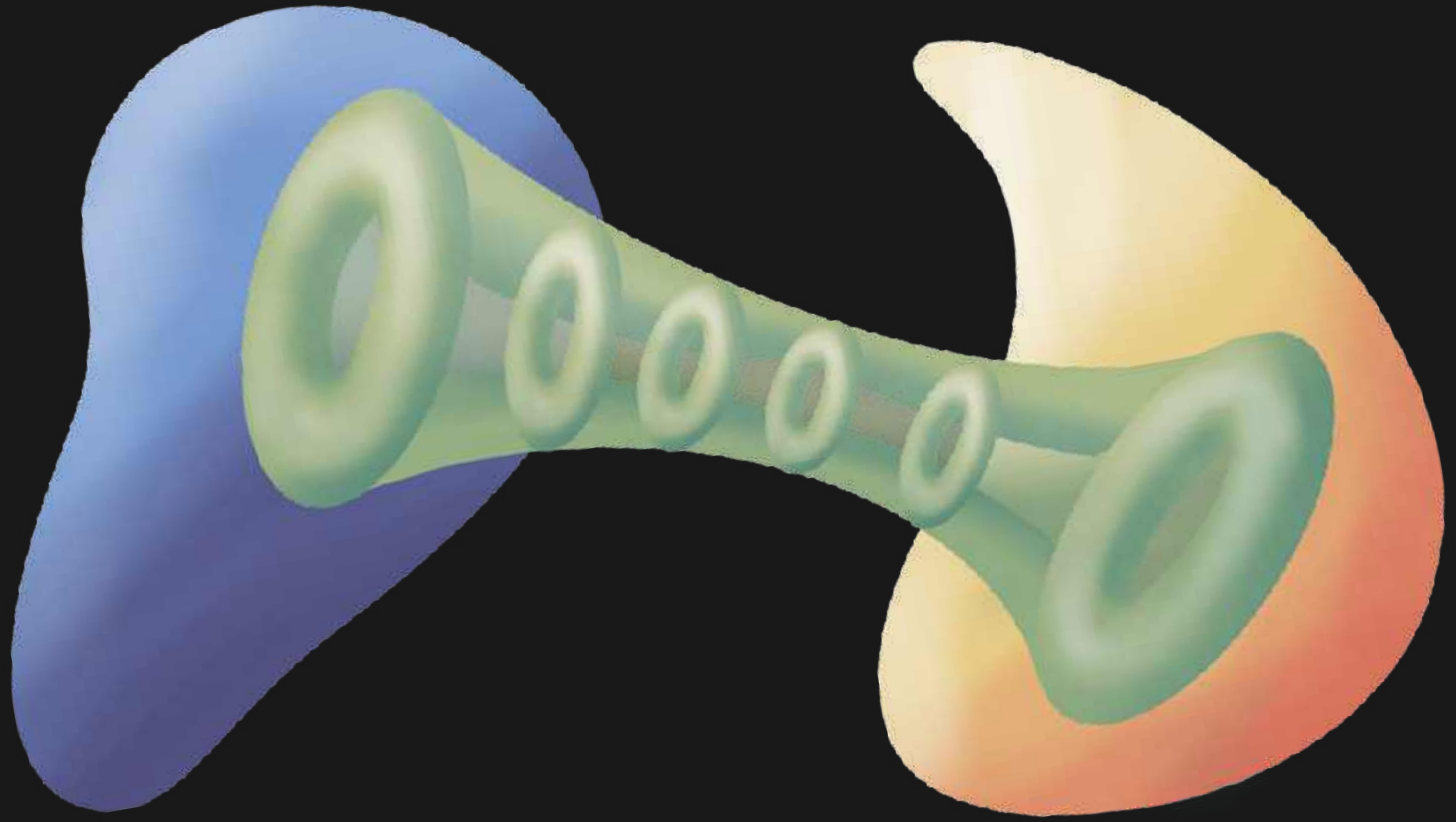
$$f\colon \pi_1 T^2 \to \pi_1 T^2 \quad (x, y) \mapsto (2x + y, x + y)$$

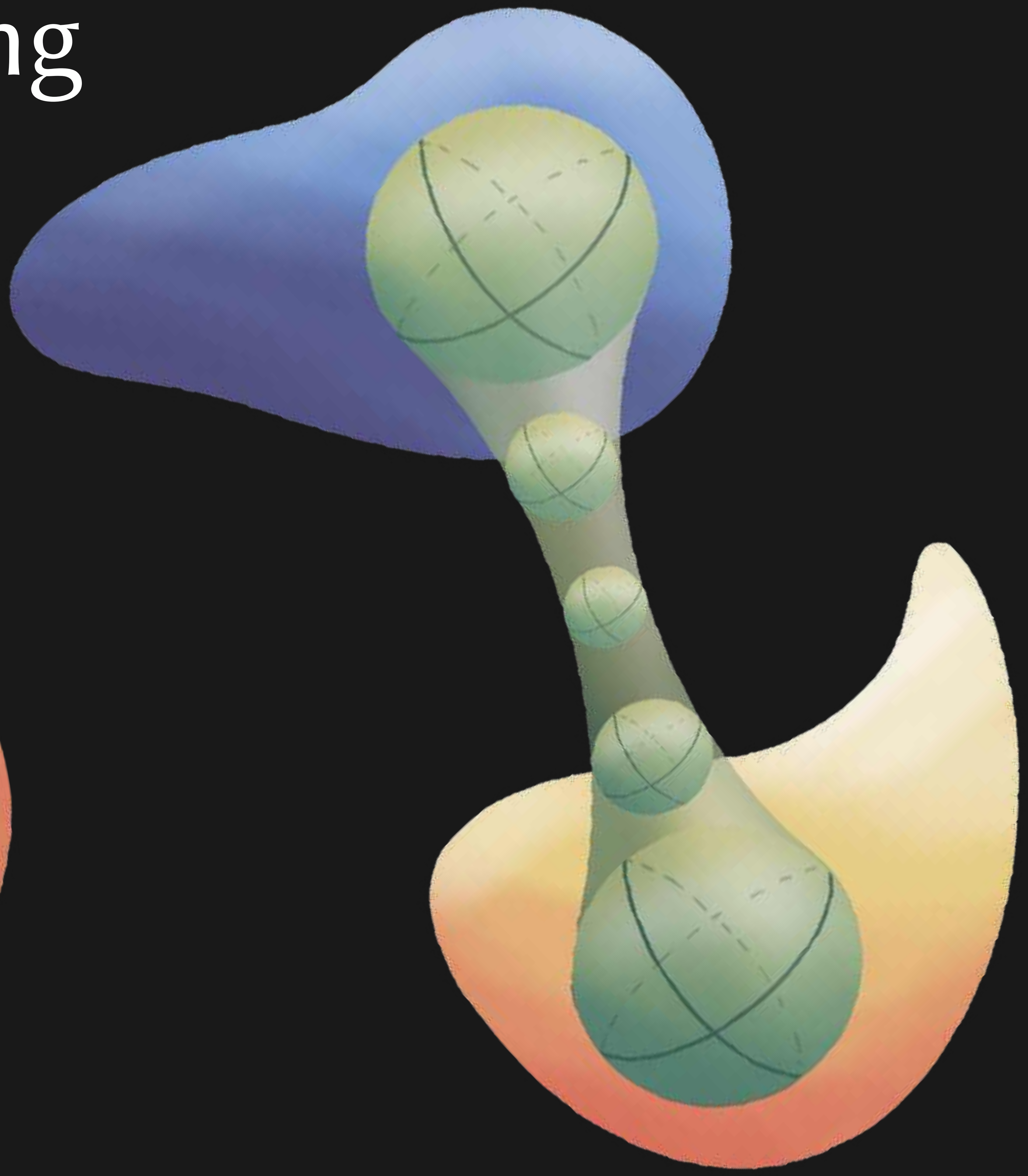

Anosov
Torus Bundle


Anosov
Torus Bundle

Putting it all together

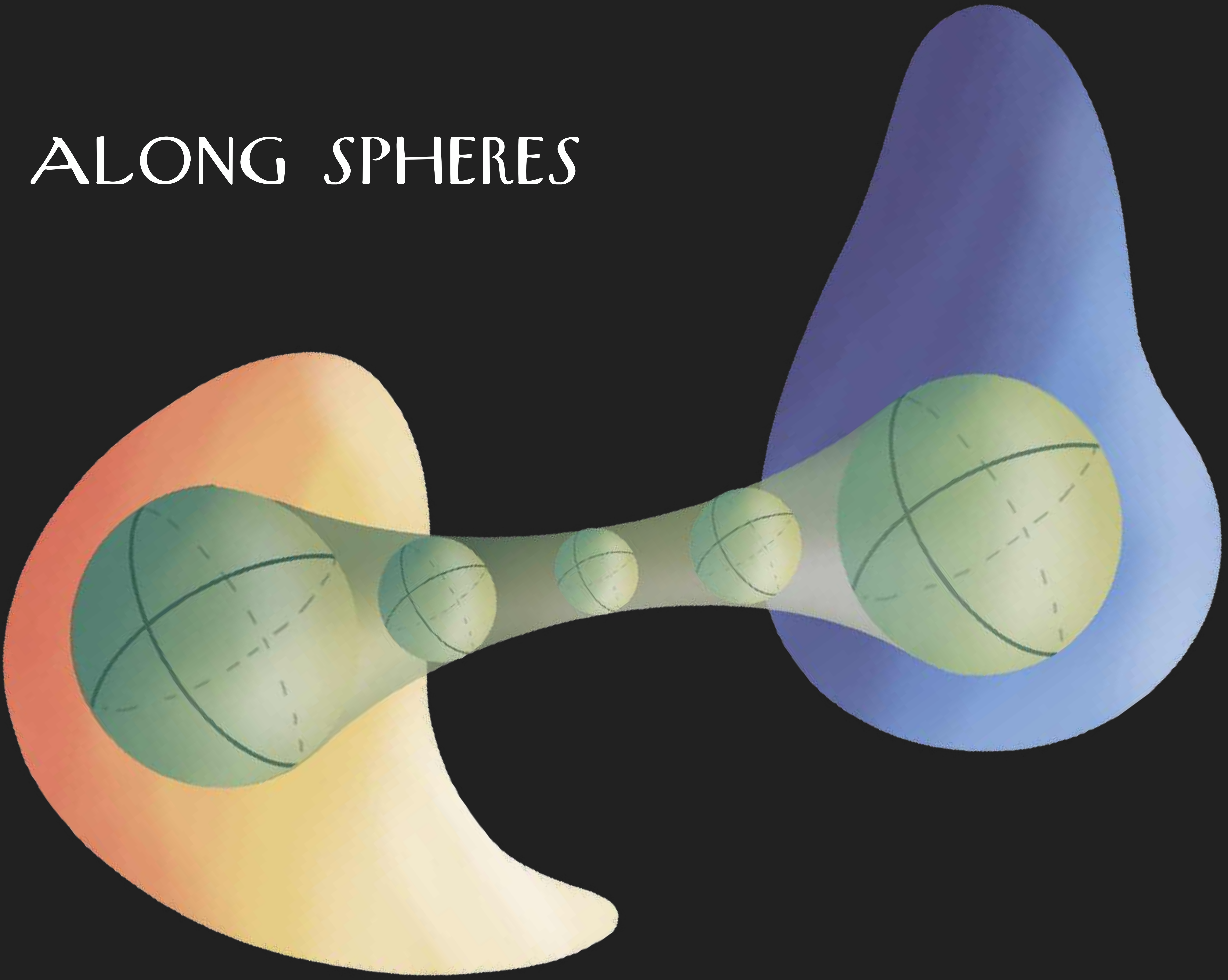
To build a general 3-Manifold, need to connect geometric pieces along spheres and tori.



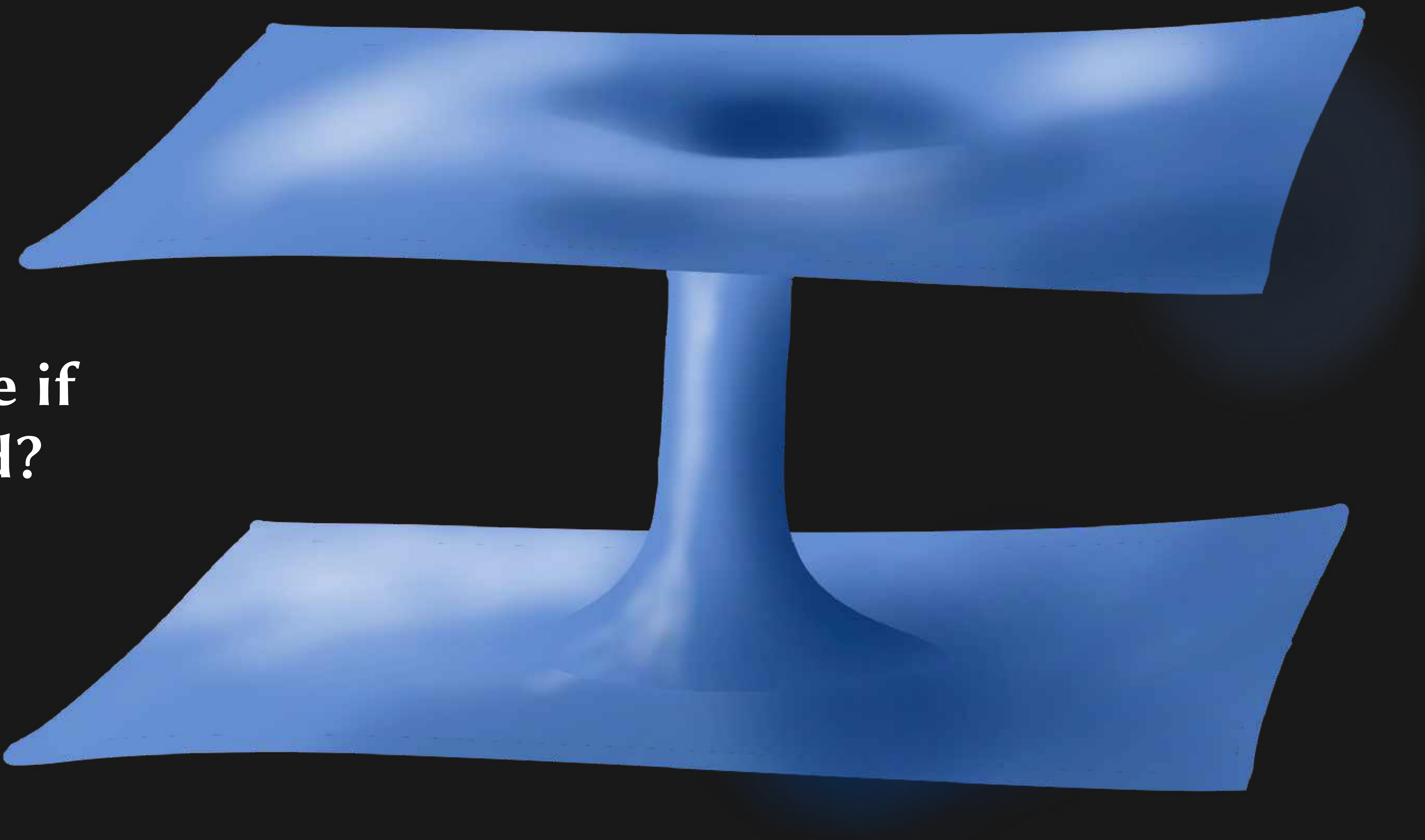


Connect sums along spheres

Connect via $\mathbb{S}^2 \times I$ while minimally distorting geometry



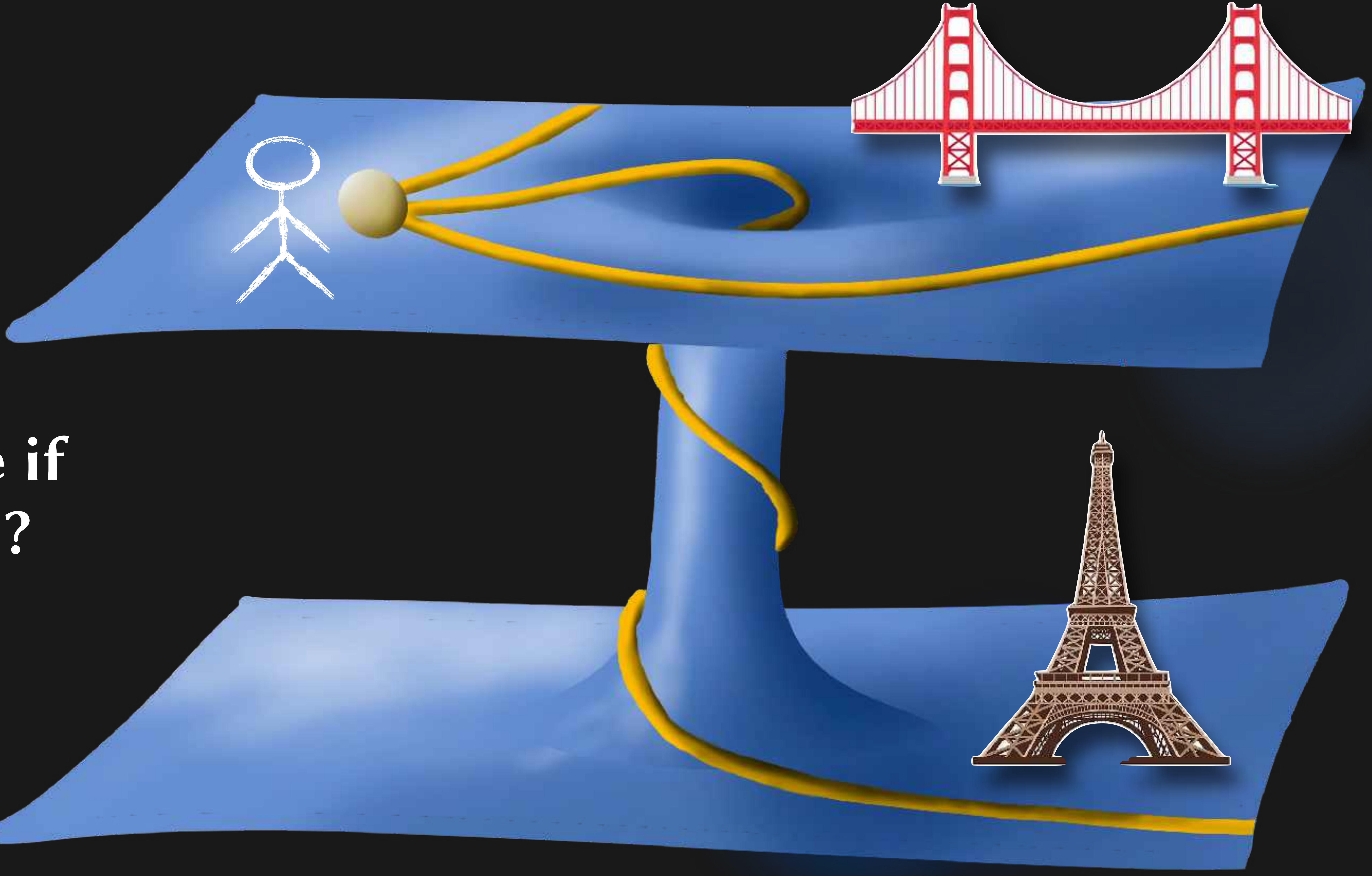
$\mathbb{E}^3 \# \mathbb{E}^3$



What do you see if you look around?

$\mathbb{E}^3 \# \mathbb{E}^3$

What do you see if you look around?




$\mathbb{E}^3 \# \mathbb{E}^3$

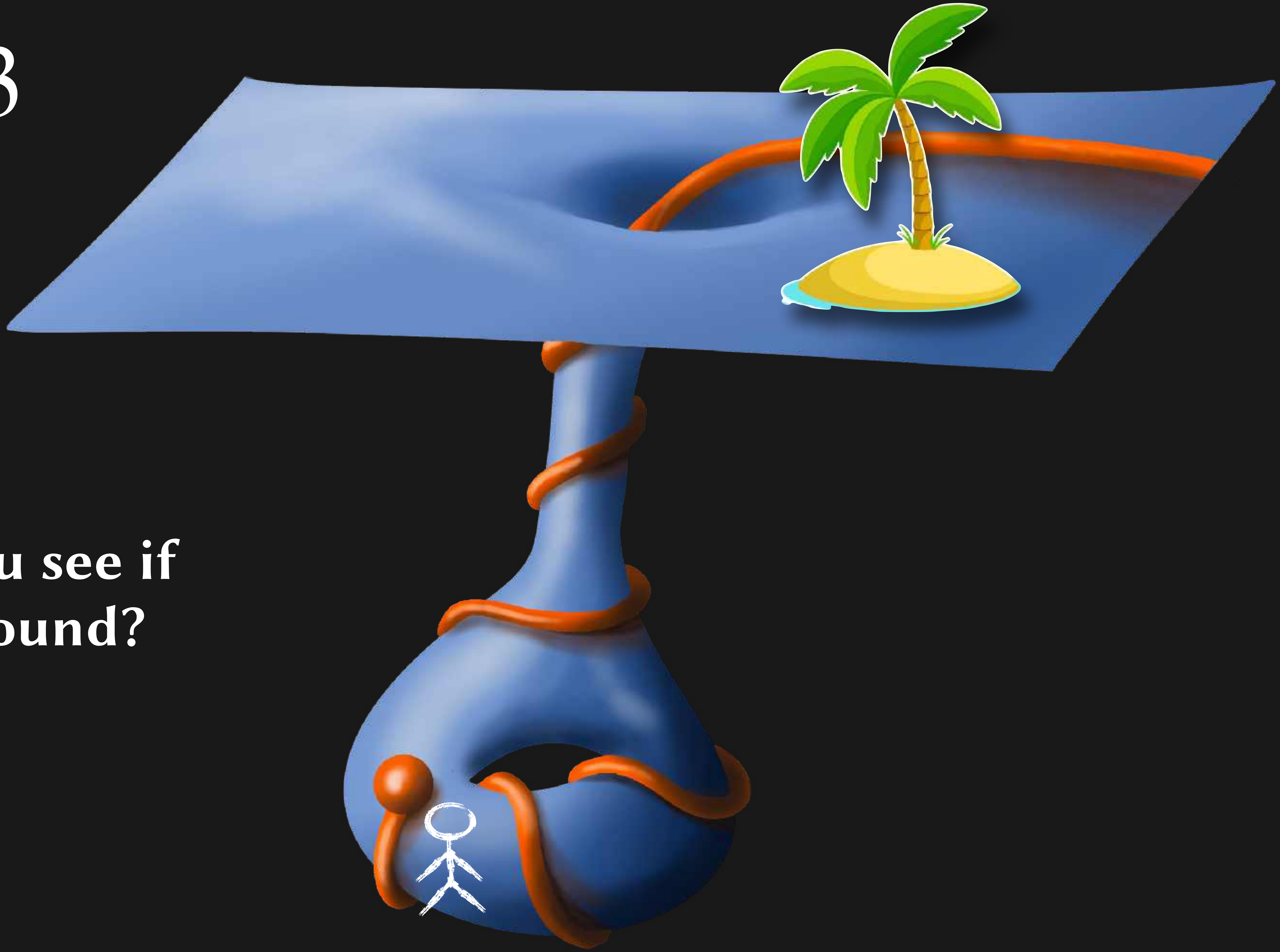
$\mathbb{E}^3 \# T^3$

What do you see if you look around?

$\mathbb{E}^3 \# T^3$

What do you see if you look around?

$\mathbb{E}^3 \# T^3$

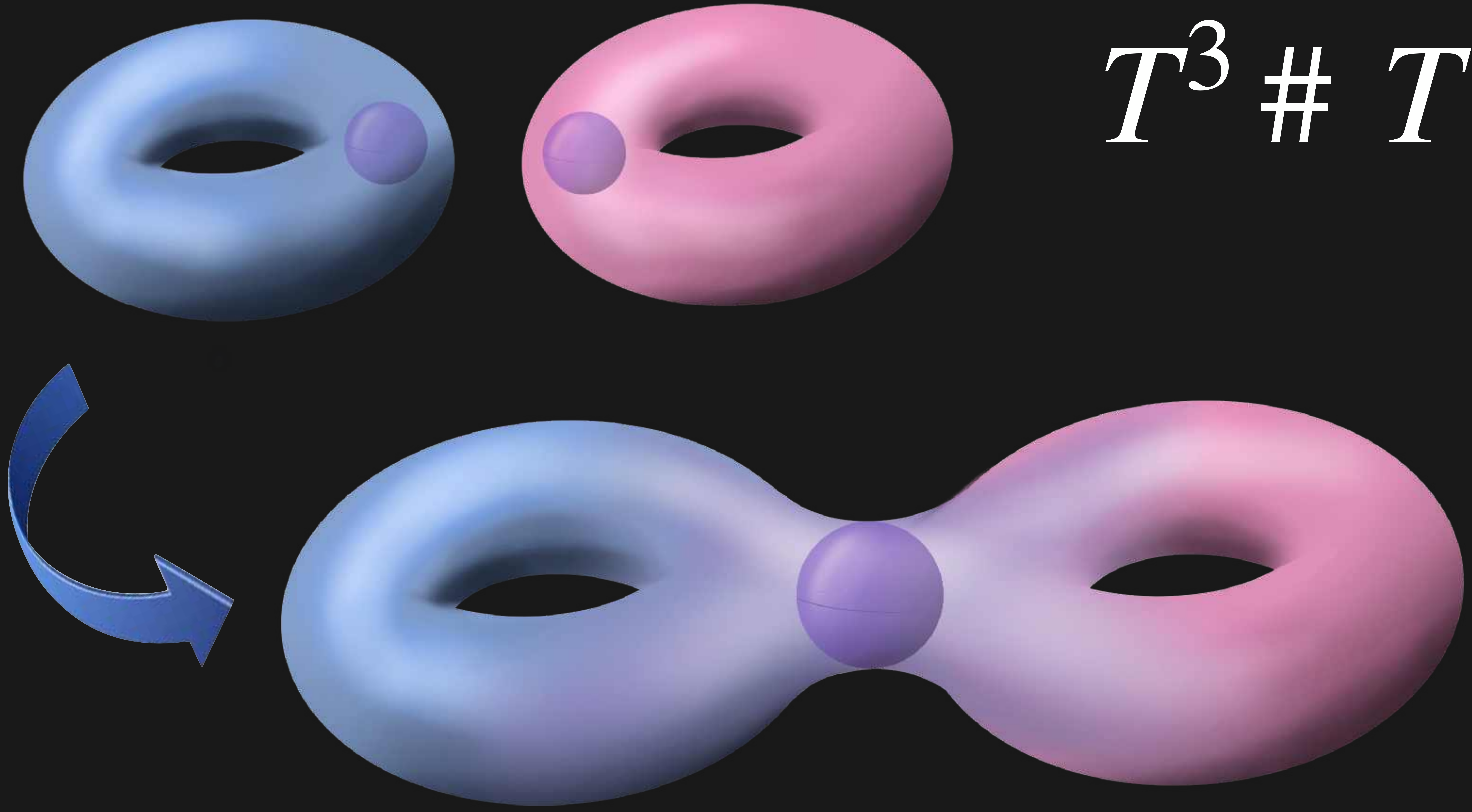


What do you see if you look around?

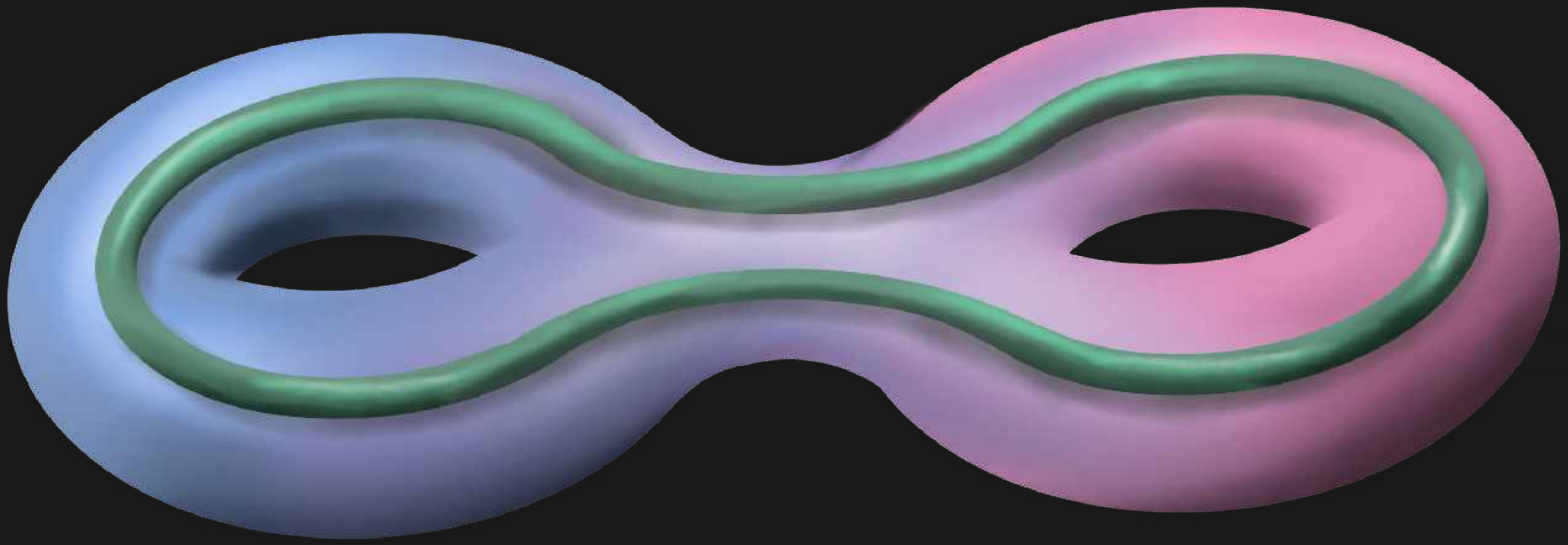

$\mathbb{E}^3 \# T^3$


$\mathbb{E}^3 \# T^3$

$$T^3 \# T^3$$

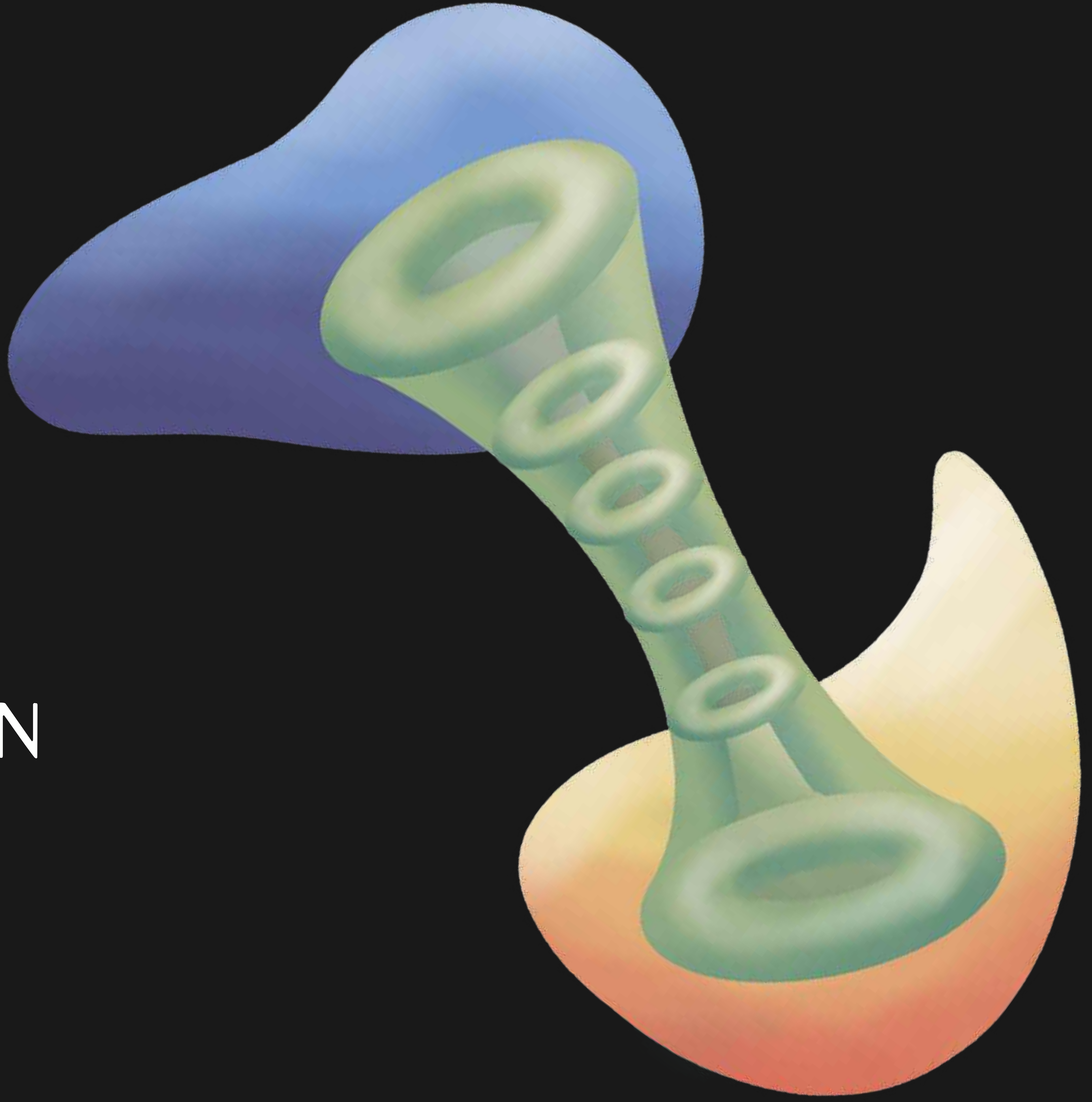


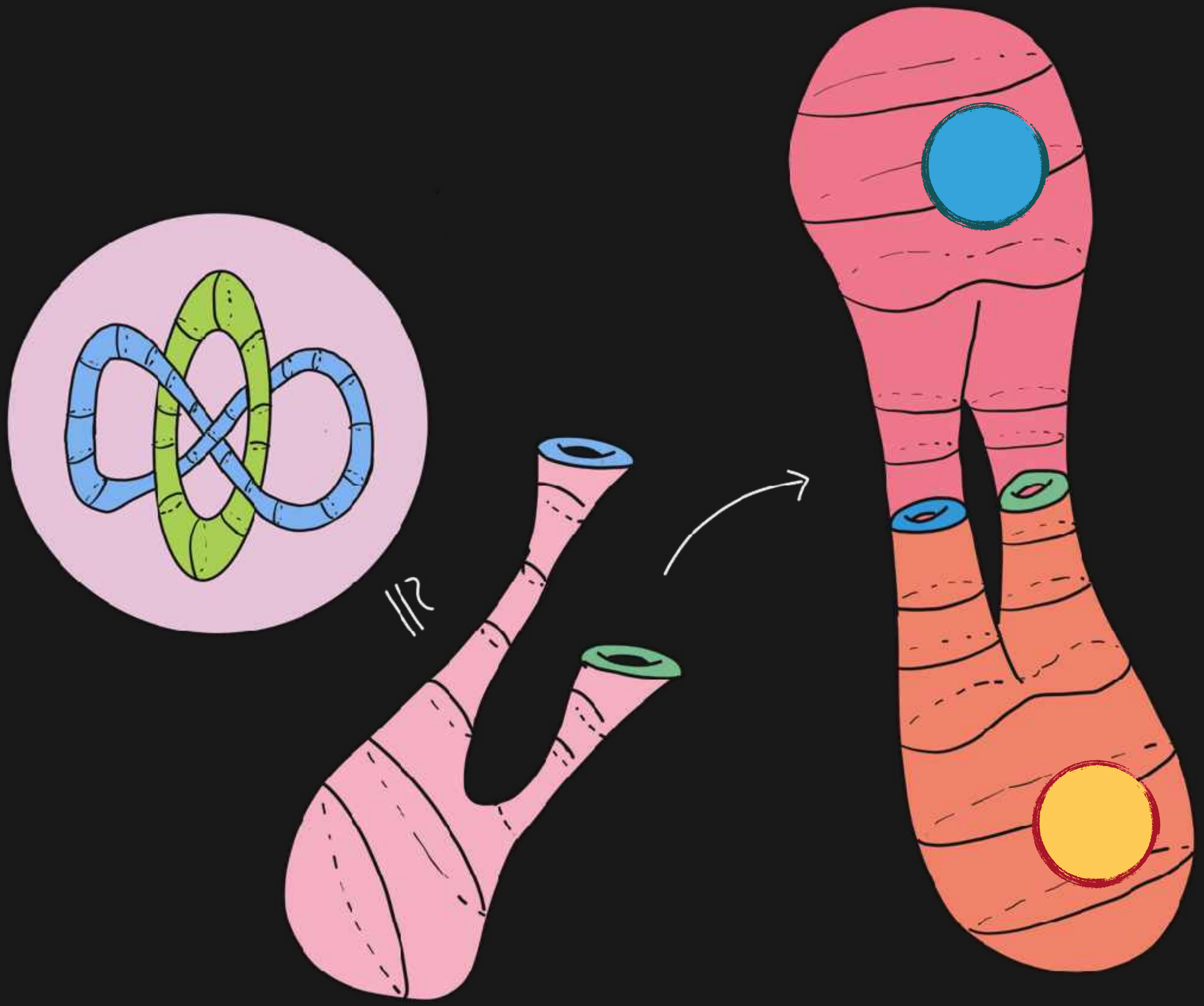

T3 # T3




$T^3 \# T^3$

Torus decomposition





The Double of the Whitehead Link

Q: What are we going to see?

Whitehead Link Double